Matched filters :- The matched filter is the optimal filter for maximizing the signal to noise Ratio (SNR) in the persence of additive white gaussian noise.

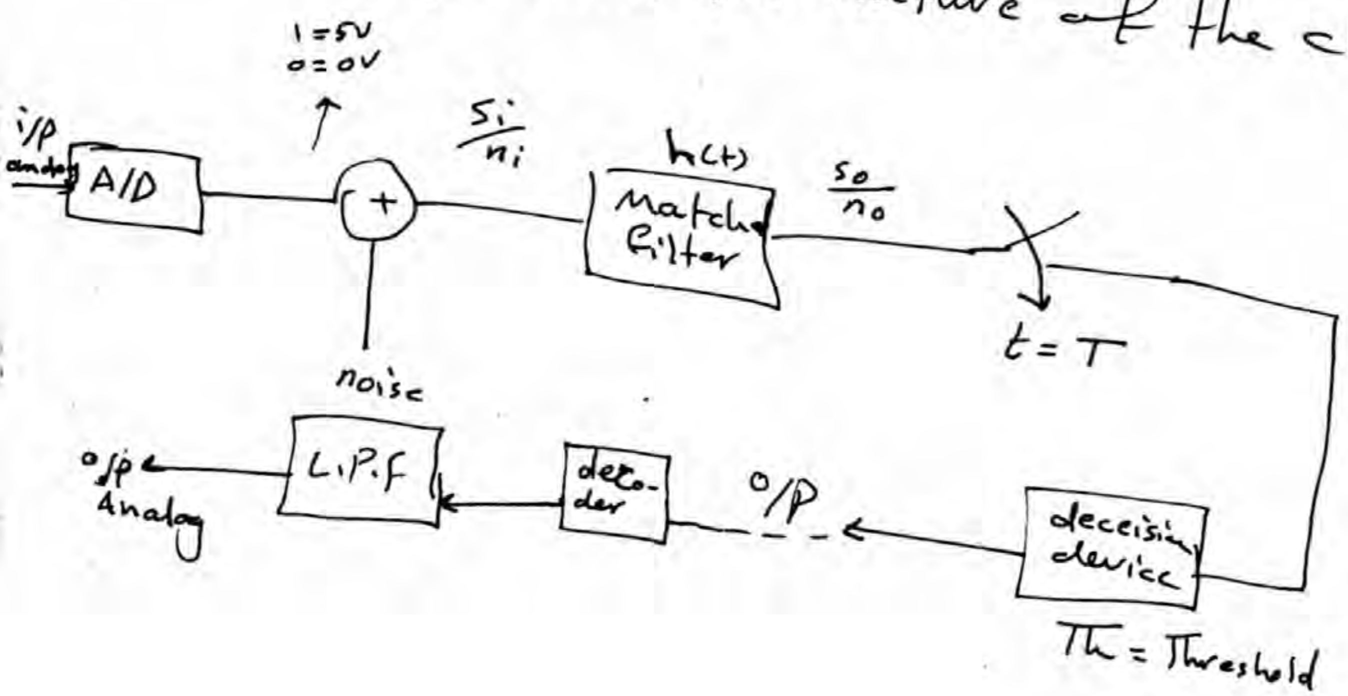
Matched filters are commenly used in Radar in wh a signal is sentout, and we measure the reflected signal, looking for something similar to what was sent out.

Two dimension filters are used commonly in image pro- cessing to improve SNR for, X-Ray Pictures.

communcation II
11\3\2013
Lecture 8

Two Problems occur in channel
① noise
② ISI due to limited BW.

The noise depend on the nature of the channel



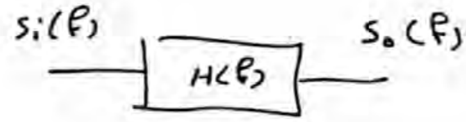
The purpose of this, is to design a filter its impulse response is h(t). The filter improve $\frac{S_o}{N_o}$ in such that

$$\frac{S_o}{N_o} \gg \frac{S_i}{N_i}$$

The Ratio can be increased by increasing S_o or making N_o less.

Design h(t) :- Find h(t) to improve S_o/N_o, let us define $\eta = \frac{S_o}{N_o}$

$$\eta = \frac{|s_o(T)|^2}{E(N_o(T))} \quad \text{using Fourier transform}$$



$$S_o(F) = H(f)\, S_i(f)$$

$$S_o(t) = \int_{-\infty}^{\infty} S_i(f)\, H(f)\, e^{j2\pi f t}\, .df \leftarrow \text{IFT}$$

$$S_o(T) = \int_{-\infty}^{\infty} S(f)\, H(f)\, e^{j2\pi f T}\, .df$$

$$|S_o(T)|^2 = \left| \int_{-\infty}^{\infty} S(f)\, H(f)\, e^{j2\pi f T}\, .df \right|^2$$

$$E(N_o^2(t)) = \int_{-\infty}^{\infty} \frac{N_o}{2} \, |H(f)|^2 .df = \text{Total Power of noise in the output}$$

$$\eta = \frac{S_o}{N_o} = \frac{\left| \int_{-\infty}^{\infty} S(f)\, H(f)\, e^{j2\pi f T} . df \right|^2}{\int_{-\infty}^{\infty} \frac{N_o}{2} \left| H(f) \right|^2 . df}$$

we use Schwartiz inequality

$$\left| \int_{-\infty}^{\infty} \phi_1(x) . \phi_2(x) . dx \right|^2 \leq \int_{-\infty}^{\infty} \phi_1^2(x) . dx . \int_{-\infty}^{\infty} \phi_2^2(x) . dx$$

maximum is obtaied when the inequlity is equal, this is acheived if

$$\phi_1(x) = K\, \phi_2^*(x)$$

$$\therefore \eta_{max} = \frac{\int_{-\infty}^{\infty} |H(f)|^2 . df . \int_{-\infty}^{\infty} \left| S(f)\, e^{j2\pi f T} . df \right|^2}{\frac{N_o}{2} \int_{-\infty}^{\infty} |H(f)|^2 . df}$$

$$\phi_1(x) = H(f)$$

$$\phi_2(x) = S(f)\, e^{+j2\pi f T}$$

$$\phi_1(x) = K\, \phi_2^*(x)$$

$$H(f) = K\, S^*(f)\, e^{-j2\pi f T}$$

but $h(t) = \int_{-\infty}^{\infty} H(f)\, e^{j2\pi f t} . df$ [IFT]

optimum to obtain $\frac{S_o}{N_o}$ max

(25) Impulse Response is

$$h_{optimum}(t) = \int_{-\infty}^{\infty} K\, S^{*}(f)\, e^{-j2\pi f T}\, e^{j2\pi f t} \cdot df$$

$$h(t) = \int_{-\infty}^{\infty} K\, S(-f)\, e^{-j2\pi f (T-t)} \cdot df$$

Using the complex conjugate Property of FT we know that $X^{*}(f) = X(-f)$ which has been substituted in the above equation and $X(-f) = x(-t)$

∴ The impulse Response is

$$\boxed{h(t) = K\, S(T-t)}$$

if $s(t)$ is considered $s(t) = s_1(t) - s_2(t)$

$h(t) = K\left[x_1(T-t) - x_2(T-t)\right]$

K: real Positive constant $\boxed{K = \frac{2k}{N_o}}$

The frequency transfer function is:

$$\boxed{H(f) = K\, X^{*}(f)\, e^{-j2\pi f T}}$$ Transfer function

The equation of $h(t)$ shows that the impulse response of the matched filter is signal waveshape that is "Played backward" and translated by an amount.

③

$$\eta_{max} = \frac{2}{N_0} \cdot \frac{\cancel{\int_{-\infty}^{\infty} |H(f)|^2 \cdot df} \; \left|\int_{-\infty}^{\infty} S(f)\, e^{j2\pi fT} \cdot df\right|^2}{\cancel{\int_{-\infty}^{\infty} |H(f)|^2 \cdot df}}$$

$$\eta_{max} = \frac{S_o}{N_o} = \frac{2}{N_o} \; \frac{\left|\int_{-\infty}^{\infty} S(f)\, e^{j2\pi fT} \cdot df\right|^2}{1}$$

$$|e^{j2\pi fT}| = 1 \quad : |\cos\theta + j\sin\theta| = \cos^2\theta + \sin^2\theta = 1$$

ich

$$\eta_{max} = \frac{2}{N_o} \cdot \underbrace{\int_{-\infty}^{\infty} |S(f)| \cdot df}_{\text{Parsseval's theorem}}$$

$$\boxed{\eta_{max} = \frac{2\, E_s}{N_o}} \rightarrow \text{Maximum SNR}$$

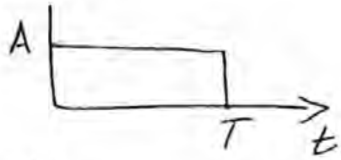
Properties of matched filters:

① The SNR of matched filter depends only on ratio of the energy signal to the PSD of white noise at

② The Maximum signal component occurs at $t = T$ (time of Sampling) and has E energy of the Signal

③ The output signal of a matched filter is Proportional to a shifted version of the auto-correlation function of the auto-correlation function of the input signal to which the filter is matched to.

Ex: Given s(t)

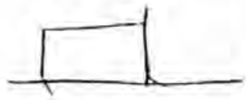


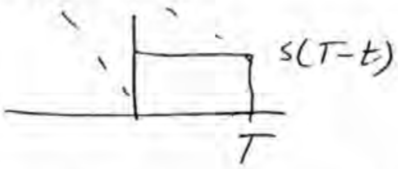
① Find h(t)
② Find the output of the filter
③ The max output
④ The SNR_{max} if PSD $\frac{N_o}{2} = 10^{-10}$ W/KHz

Sol:-

① $h(t) = C\, s(T-t)$ $\qquad C = 1$

$s(-t)$

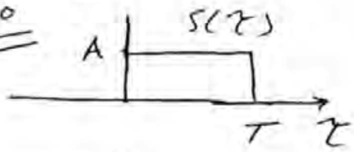


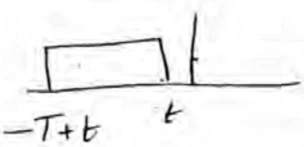


② $s_o(t) = s(t) * h(t)$

$$s_o(t) = \int_{-\infty}^{\infty} s(\tau)\, h(t-\tau)\, d\tau$$

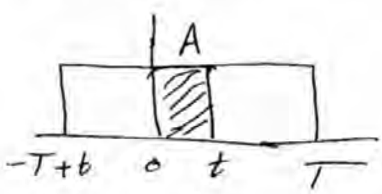
t<0





$s_o(t) = 0$

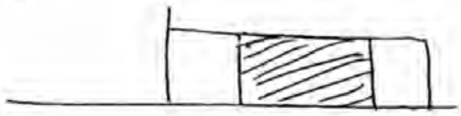
if $t > 0$



$$s_o(t) = \int_0^t A \cdot A \cdot d\tau$$

$s_o(t) = A^2 t \quad T > t > 0$

if $t > T$



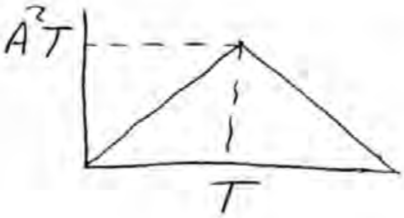
$$S_o(t) = \int_{-T+t}^{T} A \cdot A \cdot d\tau$$

$$s_o(t) = A^2 (2T - t) \quad 2T > t > T$$

if $t > 2T$

$$s_o(t) = 0 \quad T < t < 2T$$

The total output



Max output $= A^2 T$

$$E_s = \int_{-\infty}^{\infty} s_i^2(t) \cdot dt = \int_0^T A^2 \cdot dt = A^2 T \quad [E_s = S_o(T)] \text{ Property } 2$$

Power at T Equals the sum of the energy

4 $$\left(\frac{S_o}{N_o}\right)_{max} = \frac{2E_s}{N_o}$$
$$= \frac{2A^2T}{2 * 10^{-10}}$$

$$SNR_{max} = A^2 T * 10^{10}$$