<u>3.24Exmaple:</u> **Let** $X=\{1,2,3,4,5\}$
and $T=\{\phi, X, \{1,2,5\}, \{3,4\}, \{1\}, \{2,5\}, \{1,3,4\}, \{2,3,4,5\}\}$ **is a topology on** X **.Find**

$$C(A=\{3,4\},3), C(B=\{1,2,5\},1), C(B=\{1,2,5\},5), C(B=\{1,2,5\},2) .$$

Solution:

1) $C(A=\{3,4\},3)=\{3,4\}$ **.Since** A **is connected.**

2) $C(B=\{1,2,5\},1)=\{1\}$**.Since** $B=\{2,5\}/\{1\}$ **.**

3) $C(B=\{1,2,5\},5)=\{2,5\}$**.Since** $B=\{2,5\}/\{1\}$ **.**

4) $C(B=\{1,2,5\},2)=\{2,5\}$**.Since** $B=\{2,5\}/\{1\}$ **.**

<u>3.25Exercise:</u> **Let** $X=\{1,2,3,4\}$**and**

$T=\{\phi, X, \{1,2\}, \{3,4\}, \{3\}, \{1,2,3\}\}$ **is a topology on** X **.Find**

1) $C(A=\{1,3\},1)$
2) $C(B=\{1,2\},2)$
3) $C(C=\{3,4\},3)$
4) $C(D=\{1,3,4\},4)$

<u>3.26Theorem:</u> **The component of a topological space** (X,T) **are closed subsets of** X **.**

Proof: Suppose C **is a component of** X **and** $x \in C$ **.**

$\because C$ **is connected ,** $\therefore$ **by note 3.14**

$\overline{C}$ **is connected**

$\therefore \overline{C} \subset C$ **(Since is the largest connected subset of** X **containing** x **).**

$\therefore \overline{C} = C$

$\therefore C$ **is closed**

3.27Definition: A topological space (X,T) will be said to be locally connected iff $\forall x\in X$ *and* $\forall G\in T \ni x\in G \exists$ connected open set O $\ni x\in O$ and $O\subset G$ .Thus a space is locally connected iff the family of all open connected sets is a base for the topology for the space.

3.28Note:

- A subset A of a space (X,T) will be called locally connected iff (A,T_A) is locally connected.
- A locally connected set need not be connected, for example let $A=(1,2)\cup(2,3)$ subset of the usual topology (R,T) , then A is \locally connected but disconnected.

3.29Exmaple: Let $X=\{1,2,3,4\}$ and

$T=\{\phi, X, \{1,2\}, \{3,4\}, \{3\}, \{1,2,3\}\}$ is a topology on X .

Is (X,T) locally connected?

Solution:

1) Take $x=1$.

   Open sets containing $x=1$ are $X, \{1,2\}, \{1,2,3\}$.

   If $G=X\ \exists O=\{1,2\}\in T\ connected \quad \ni 1\in O\subset X$.

   If $G=\{1,2\}\exists O=\{1,2\}\in T\ connected \quad \ni 1\in O\subset \{1,2\}$.

   If $G=\{1,2,3\}\exists O=\{1,2\}\in T\ connected \quad \ni 1\in O\subset \{1,2,3\}$.

2) Take $x=2$.

   Open sets containing $x=2$ are $X, \{1,2\}, \{1,2,3\}$.

   If $G=X\ \exists O=\{1,2\}\in T\ connected \quad \ni 2\in O\subset X$.

   If $G=\{1,2\}\exists O=\{1,2\}\in T\ connected \quad \ni 2\in O\subset \{1,2\}$.

   If $G=\{1,2,3\}\exists O=\{1,2\}\in T\ connected \quad \ni 2\in O\subset \{1,2,3\}$.

3) Take $x=3$.

Open sets containing $x=3$ are $X,\{3,4\},\{3\},\{1,2,3\}$.

If $G=X\ \exists O=\{3\}\in T\ connected \quad \ni 3\in O\subset X$.

If $G=\{3,4\}\exists O=\{3\}\in T\ connected \quad \ni 3\in O\subset \{3,4\}$.

If $G=\{3\}\exists O=\{3\}\in T\ connected \quad \ni 3\in O\subset \{3\}$.

If $G=\{1,2,3\}\exists O=\{3\}\in T\ connected \quad \ni 3\in O\subset \{1,2,3\}$.

4) Take $x=4$.

Open sets containing $x=4$ are $X,\{3,4\}$.

If $G=X\ \exists O=\{3,4\}\in T\ connected \quad \ni 4\in O\subset X$.

If $G=\{3,4\}\exists O=\{3,4\}\in T\ connected \quad \ni 4\in O\subset \{3,4\}$.

From 1),2),3) and 4)

(X,T) **is locally connected.**

3.30Note:

- **A locally connected set need not be connected, example 3.29 ,** (X,T) **is \locally connected but disconnected.**

3.31Exercise:: **Let** $X=\{1,2,3\}$ **and**

$T=\{\phi,X,\{1,2\},\{1\},\{2\}\}$ **is a topology on** X **.Is** (X,T) **\locally connected?**

3.32Theorem: **The component of \locally connected** (X,T) **are open subsets of** X .

Proof: Suppose C **is a component of** X **and** $x\in C$.

$\therefore x\in X$,

$\because (X,T)$ **is locally connected,**

$\therefore \quad \forall G \in T \ni x \in G \exists$ **connected open set** $O_x \ni x \in O_x$ **and** $O_x \subset G$ **.**

$\therefore O_x \subset C$ **(Since is the largest connected subset of** X **containing** x **).**

$$\therefore C = \bigcup_{x \in C} O_x$$

$\therefore C$ **is open.**

Compactness

<u>3.33Definition:</u> Let (X,T) **be a topological space,** $E \subseteq X$ **.A covering of a set** E **is a collection of sets** $\{G_\lambda\}$ **such that** $E \subseteq \bigcup_{\lambda} G_\lambda$ **.**

<u>3.34Note:</u>

- **If all the sets of a covering are open sets, we say that we have an open covering.**
- **If** $E \subseteq \bigcup_{i=1}^{n} G_i$ **, then** $\{G_1, G_2, G_3, \ldots, G_n\}$ **is finite covering.**

<u>3.35Definition:</u> A subset E **of a topological space** (X,T) **is compact iff every open covering of** E **is reducible to a finite subcovering of** E **.**

<u>3.36Note:</u> If X **is compact, then** (X,T) **is compact space.**

<u>3.37Theorem:</u> **Let** (X,T) **be a topological space and** $A \subset X$ **. If** A **is finite set then** A **is compact.**

Proof: Let $A = \{a_1, a_2, a_3, ..., a_n\}$ **and** $\{G_\lambda\}$ **be open covering of** A

$$\therefore A \subseteq \bigcup_{\lambda} G_\lambda$$

$$\therefore \forall a \in A \exists G_{\lambda_a} \in \{G_\lambda\} \ni a \in G_{\lambda_a}$$

$$\therefore a_1 \in G_{\lambda_1}, a_2 \in G_{\lambda_2}, a_3 \in G_{\lambda_3}, ... a_n \in G_{\lambda_n}$$

$$\therefore A \subseteq \bigcup_{i=1}^{n} G_{\lambda_i}$$

$\therefore A$ **is compact.**

<u>3.38Exmaple:</u> **Let** $X \neq \phi$ **be infinite set and** $T = \{\phi\} \cup \{A : A \subseteq X \ and \quad A^C \ finite \quad set\}$**, prove that** X **is compact.**

Solution:

Let $\{G_\lambda\}$ **be open covering of** X

$$\therefore X \subseteq \bigcup_{\lambda} G_\lambda$$

Suppose $G_0 \in \{G_\lambda\}$

$$\therefore G_0 \in T$$

$\therefore G_0{}^C$ **is finite set and let** $G_0{}^C = \{a_1, a_2, a_3, ..., a_m\}$

$\because \{G_\lambda\}$ **open covering of** X **and** $G_0{}^C \subset X$

$\therefore \{G_\lambda\}$ **open covering of** $G_0{}^C$

$$\therefore a_1 \in G_{\lambda_1}, a_2 \in G_{\lambda_2}, a_3 \in G_{\lambda_3}, ... a_m \in G_{\lambda_m}$$

$$\therefore G_0^{\ C} \subseteq \bigcup_{i=1}^{m} G_{\lambda_i}$$

$$\therefore X = G_0 \cup G_0^{\ C} \subseteq G_0 \cup \left(\bigcup_{i=1}^{m} G_{\lambda_i} \right)$$

$\therefore X$ **is compact and** (X,T) **is compact space.**

<u>3.39Exmaple:</u> Let (R,T) **be usual topological space and** $A=(0,1)$ **.prove that** A **is not compact.**

Solution:

Let $\Psi = \left\{ \left(\frac{1}{n+2}, \frac{1}{n} \right); n \in N \right\}$ **be open covering of** A

$$\therefore \Psi = \left\{ \left(\frac{1}{3}, 1 \right), \left(\frac{1}{4}, \frac{1}{2} \right), \left(\frac{1}{5}, \frac{1}{3} \right), \left(\frac{1}{6}, \frac{1}{4} \right), ... \right\}$$

$$\therefore (0,1) \subseteq \bigcup_{n} \left(\frac{1}{n+2}, \frac{1}{n} \right)$$

Suppose $\Psi^* = \{(a_1,b_1),(a_2,b_2),(a_3,b_3),...,(a_n,b_n)\}$ **subcollection of** Ψ **and** $m = \min\{a_1, a_2, a_3,.., a_n\}$

$$\therefore (a_1,b_1) \cup (a_2,b_2) \cup (a_3,b_3) \cup ... \cup (a_n,b_n) \subseteq (m,1)$$

$\therefore \Psi^*$ **is not covering of** $(0,m)$

$\therefore \Psi^*$ **is not subcovering of** A

$\therefore \Psi$ **is open covering of** A **and** Ψ **is not reducible to a finite subcovering of** A**.**

$\therefore A$ **is not compact.**

<u>3.40Theorem:</u> **Let** (A,T_A) **be a subspace of** (X,T) **and** $E \subset A$**.Then** E **is compact in** (X,T) **iff** E **is compact in** (A,T_A)**.**

Proof: $\Leftarrow$

Suppose E **is compact in** (A,T_A)

Now we prove that E **is compact in** (X,T)

Let $\{G_\lambda\}$ **be open covering of** E **in** (X,T)

$$\therefore E \subseteq \bigcup_\lambda G_\lambda$$

$$\therefore E = E \cap A \subseteq \left(\bigcup_\lambda G_\lambda\right) \cap A = \bigcup_\lambda (G_\lambda \cap A) = \bigcup_\lambda G_\lambda^{\ *}$$

$\therefore \{G_\lambda^{\ *}\}$ **be open covering of** E **in** (A,T_A)

$\because E$ **is compact in** (A,T_A)

$$\therefore E \subseteq \bigcup_{i=1}^{n} G_i^{\ *}$$

$$\therefore E \subseteq \bigcup_{i=1}^{n} (G_i \cap A) = \bigcup_{i=1}^{n} G_i \cap A$$

$$\therefore E \subseteq \bigcup_{i=1}^{n} G_i$$

$\therefore E$ **is compact in** (X,T)

$\Rightarrow$

Suppose E **is compact in** (X,T)

Now we prove that E is compact in (A, T_A)

Let $\{G_\lambda^{\ *}\}$ **be open covering of** E **in** (A, T_A)

$$\therefore E \subseteq \bigcup_\lambda G_\lambda^{\ *}$$

$$\therefore E \subseteq \bigcup_\lambda (G_\lambda \cap A) = \bigcup_\lambda G_\lambda \cap A$$

$$\therefore E \subseteq \bigcup_\lambda G_\lambda$$

$\therefore \{G_\lambda\}$ **be open covering of** E **in** (X, T)

$\because E$ **is compact in** (X, T)

$$\therefore E \subseteq \bigcup_{i=1}^{n} G_i$$

$$\therefore E = E \cap A \subseteq \left(\bigcup_{i=1}^{n} G_i\right) \cap A = \bigcup_{i=1}^{n} (G_i \cap A) = \bigcup_{i=1}^{n} G_i^{\ *}$$

$$\therefore E \subseteq \bigcup_{i=1}^{n} G_i^{\ *}$$

$\therefore E$ **is compact in** (A, T_A)

3.41Note:

- **By Theorem3.40 compactness is absolute property.**

3.42Theorem: **Every closed subset of a compact space is compact.**

Proof: Let (X, T) **be compact space,** $F \subset X$ **and** F **is closed.**

Now we prove that F **is compact.**

Let $\{G_\lambda\}$ **be open covering of** F

$\therefore F \subseteq \bigcup_\lambda G_\lambda$ **.**

$\because F$ **is closed,** $\therefore F^C$ **is open.**

$$\therefore X = F \cup F^C \subseteq \left(\bigcup_\lambda G_\lambda\right) \cup F^C$$

$\therefore \{G_\lambda\}, \{F^C\}$ **be open covering of** X

$\because X$ **is compact .**

$$\therefore X \subseteq \left(\bigcup_{i=1}^{n} G_i\right) \cup F^C$$

$$\therefore F \subseteq \bigcup_{i=1}^{n} G_i$$

$\therefore F$ **is compact .**

<u>3.43Definition:</u> A family of sets $\{G_\lambda\}$ **will be said to have the finite intersection property iff every finite subfamily** $\{G_1, G_2, G_3, \ldots, G_n\}$ **of the family has** $\bigcap_{i=1}^{n} G_i \neq \phi$.

<u>3.44Exmaple:</u> Let (R,T) **be usual topological space and** $\Psi = \left\{(0,1), \left(0, \frac{1}{2}\right), \left(0, \frac{1}{3}\right), \left(0, \frac{1}{4}\right), \left(0, \frac{1}{n}\right), \ldots\right\}$ **.Then**

$$(0, a_1) \cap (0, a_2) \cap (0, a_3) \cap (0, a_4) \cap \ldots \cap (0, a_n) \neq \phi$$

$\therefore \Psi$ **have the finite intersection property.**

<u>3.45Exmaple:</u> **Let** (R,T) **be usual topological space and**
$\beta=\{\ldots,(-\infty,-3],(-\infty,-2],(-\infty,-1],(-\infty,0],(-\infty,1],(-\infty,2],(-\infty,3],\ldots\}$
.Then $(-\infty,a_1]\cap(-\infty,a_2]\cap(-\infty,a_3]\cap(-\infty,a_4]\cap\ldots\cap(-\infty,a_n]\neq\phi$
$\therefore\ \beta$ **have the finite intersection property.**

<u>3.46Theorem:</u> **A topological space** (X,T) **is compact iff any family of closed sets having the finite intersection property has nonempty intersection.**

Proof: $\Rightarrow$

Suppose (X,T) **is compact and** $\{F_\lambda\}$ **family of closed sets having the finite intersection property.**

Now we prove that $\bigcap_{\lambda} F_\lambda \neq \phi$

Suppose $\bigcap_{\lambda} F_\lambda = \phi$

$\therefore X=\phi^C=\bigcup_{\lambda} F_\lambda{}^C$

$\because F_\lambda$ **is closed** $\forall\lambda$

$\therefore F_\lambda{}^C$ **is open** $\forall\lambda$

$\therefore \{F_\lambda{}^C\}$ **be open covering of** X

$\because X$ **is compact**

$\therefore X=\bigcup_{i=1}^{n} F_i{}^C$

$\therefore X^C = \phi = \bigcap_{i=1}^{n} F_i$ **,and which is contradiction.(Since** $\{F_\lambda\}$ **having the finite intersection property)**

$$\therefore \bigcap_{\lambda} F_\lambda \neq \phi$$

$\Leftarrow$

Now we prove that X **is compact .Suppose** X **is not compact.**

$\exists\{G_\lambda\}$ **be open covering of** X **϶**

If $X = \bigcup_{\lambda} G_\lambda$ **then** $X \neq \bigcup_{i=1}^{n} G_i$

If $\phi = \bigcap_{\lambda} G_\lambda{}^C$ **then** $\phi \neq \bigcap_{i=1}^{n} G_i{}^C$

$\therefore \{G_\lambda{}^C\}$ **family of closed sets having the finite intersection property.**

but $\bigcap_{\lambda} G_\lambda{}^C = \phi$ **, and which is contradiction.**

$\therefore X$ **is compact**