3.9Definition: A property P is called "absolute" iff for all subspaces $B \subseteq A \subseteq X$ of a space X , B fulfills P a subspace of A iff B fulfills P as a subspace of X .

3.10Note:

- **By Theorem3.8** connectedness is absolute property**.**

3.11Theorem: **Let (X,T) be a topological space, then $X = H/K$ iff H, K are two non –empty disjoint open and closed subsets of X .**

Proof: $\Longrightarrow$

Suppose $X = H/K$ **.**

1. $H \neq \phi, K \neq \phi$
2. $H \bigcup K = X$
3. $\left(H \bigcap \overline{K}\right) \bigcup \left(K \bigcap \overline{H}\right) = \phi$

$$\because \left(H \bigcap \overline{K}\right) = \phi, \quad H \bigcap K = \phi \, and \quad H \bigcup K = X$$

$\therefore \overline{K} \subset K$

$\therefore \overline{K} = K$

$\therefore K$ **is closed**

In the same way from $\left(K \bigcap \overline{H}\right) = \phi$ **.**

H **is closed**

But $H = K^C$

$\therefore H$ **is open**

And $K = H^C$

$\therefore K$ **is open**

$\therefore H, K$ **are two non –empty disjoint open and closed subsets of** X **.**

$\Leftarrow$

$\because H, K$ **are two non –empty disjoint open and closed subsets of** X $\therefore$

1. $H \neq \phi, K \neq \phi$
2. $H \bigcup K = X$
3. $H \bigcap K = \phi$

$\because H$ **is closed**

$\therefore H = \overline{H}$

$\therefore \left(K \bigcap \overline{H}\right) = \phi$

$\because K$ **is closed**

$\therefore K = \overline{K}$

$\therefore \left(H \bigcap \overline{K}\right) = \phi$

$\therefore X = H/K$

<u>3.12Theorem:</u> **Let** (X,T) **be a topological space and** C **is connected subset of** X **,if** $X=H/K$ **then** $C\subset H\, or \quad C\subset K$ **.**

Proof: Take $C\bigcap H\, and \quad C\bigcap K$ **.Now**

1. $C=C\bigcap X=C\bigcap(H\bigcup K)=(C\bigcap H)\bigcup(C\bigcap K)$
2. $\left(C\bigcap H\bigcap\overline{C\bigcap K}\right)\bigcup\left(C\bigcap K\bigcap\overline{C\bigcap H}\right)\subseteq\left(H\bigcap\overline{K}\right)\bigcup\left(K\bigcap\overline{H}\right)$

$\because X=H/K$

$\therefore\left(H\bigcap\overline{K}\right)\bigcup\left(K\bigcap\overline{H}\right)=\phi$

$\therefore\left(C\bigcap H\bigcap\overline{C\bigcap K}\right)\bigcup\left(C\bigcap K\bigcap\overline{C\bigcap H}\right)=\phi$

If $C\bigcap H\neq\phi\, and \quad C\bigcap K\neq\phi$

$\therefore C=C\bigcap H/C\bigcap K$ **,and which is contradiction.**

$\therefore C\bigcap H=\phi\, or \quad C\bigcap K=\phi$

$\therefore C\subset K\, or \quad C\subset H$

<u>3.13Corollary:</u> **Let** (X,T) **be a topological space and** C **is connected subset of** X **,if** $C\subset E\subset\overline{C}$ **then** E **is connected.**

Proof: Suppose $E=H/K$

$\because C\subset E$ **and** C **is connected,** $\therefore$ **By theorem 3.12**

$C\subset H\, or \quad C\subset K$

Let $C\subset H$

$\therefore \quad C\bigcap K=\phi$**...** $\qquad (1)$

$\because C \subset H$

$\therefore \overline{C} \subset \overline{H}$

$\therefore \overline{C} \bigcap K = \phi ... \quad (2)$

$\because K \subset E \subset \overline{C}$

$\therefore \overline{C} \bigcap K = K ... \quad (3)$

From $(1),(2) and \quad (3)$

$K = \phi$ **and which is contradiction.**

$\therefore E$ **is connected**

3.14Note:

- **By Corollary3.13** If C **is connected** then $\overline{C}$ **is connected.**
  Since $C \subset \overline{C} \subset \overline{C}$**.**

3.15Corollary: **Let** (X,T) **be a topological space and** $E \subseteq X$ **.If** $\forall a,b \in E$ $\exists C$ **is connected and** $a,b \in C \subset E$ **,then** E **is connected.**

Proof: Suppose $E = H/K$

$\because H \neq \phi$

$\therefore \exists \quad a \in H$

$\because K \neq \phi$

$\therefore \exists \quad b \in K$

By hypothesis $\exists C$ **is connected and** $a,b \in C \subset E$ **.**

$\therefore$ **By theorem 3.12**

$C \subset H \, or \quad C \subset K$

$\therefore a,b \in H \, or \quad a,b \in K$ **,and which is contradiction.**

$\therefore E$ **is connected.**

<u>3.15Corollary:</u> Let (X,T) **be a topological space and** $\{C_\lambda\}$**.be a collection of connected subsets of** X **and** $\bigcap_\lambda C_\lambda \neq \phi$ **,then** $\bigcup_\lambda C_\lambda$ **is connected.**

Proof: Suppose $\bigcup_\lambda C_\lambda = H/K$

$\because \bigcap_\lambda C_\lambda \neq \phi$

$\therefore \exists \quad x \in \bigcap_\lambda C_\lambda$

$\because \bigcup_\lambda C_\lambda = H/K$

$\therefore x \in H \, or \quad x \in K$

Suppose $x \in H$ **.**

$\because C_\lambda$ **is connected** $\forall \lambda$**,** $x \in C_\lambda \; \forall \lambda \quad and \; x \in H$

$\therefore$ **By theorem 3.12**

$C_\lambda \subset H \qquad \forall \lambda$

$\therefore \bigcup_\lambda C_\lambda \subset H$

$\therefore \bigcup_\lambda C_\lambda = H$

$\therefore K = \phi$ **,and which is contradiction.** $\therefore \bigcup_\lambda C_\lambda$ **is connected.**

<u>3.16Theorem:</u> **Let** (X,T) **be a topological space,** $E \subseteq X$ **and** C **is connected subset of** $X \ni C \cap E \neq \phi \quad and \quad C \cap E^C \neq \phi$**,**
then $C \cap b(E) \neq \phi$**.**

Proof: Suppose $C \cap b(E) = \phi$ **.Now we prove that** $C = C \cap E / C \cap E^C$ **.**

1. $C \cap E \neq \phi, C \cap E^C \neq \phi$
2. $C = C \cap X = C \cap (E \cup E^C) = (C \cap E) \cup (C \cap E^C)$
3. $(C \cap E) \cap (C \cap E^C) = C \cap (E \cap E^C) = C \cap \phi = \phi$
4.

$$(C \cap E) \cap \overline{C \cap E^C} \subset (C \cap E) \cap \overline{E^C} = C \cap (E \cap \overline{E^C}) = C \cap b(E) = \phi$$
$$\therefore (C \cap E) \cap \overline{C \cap E^C} = \phi$$

In the same way prove $(C \cap E^C) \cap \overline{(C \cap E)} = \phi$**.**

From 1,2,3 and 4 $\quad C = C \cap E / C \cap E^C$ **, and which is contradiction.**

$$\therefore C \cap b(E) \neq \phi$$

<u>3.17Theorem:</u> **Let** (X,T) **be a topological space, then** X **is connected iff the only subsets of** X **that are both open and closed in** X **are** ϕ **and** X **.**

Proof: $\Rightarrow$ **Suppose** X **is connected and** $H \neq X, \quad H \neq \phi$ **is open and closed.**

$\therefore H^C, \quad H^C \neq \phi$ **is open and closed .Such that**

1. $H \cup H^C = X$
2. $H \cap \overline{H^C} = H \cap H^C = \phi \, and \qquad H^C \cap \overline{H} = H^C \cap H = \phi$

$\therefore X = H/H^C$ **, and which is contradiction.**

$\therefore$ **the only subsets of** X **that are both open and closed in** X **are** ϕ **and** X **.**

$\Leftarrow$

the only subsets of X **that are both open and closed in** X **are** ϕ **and** X **. Suppose** $X = H/K$ **.** $\therefore$ **By theorem3.11**

H, K **Are both open and closed in** X **, and which is contradiction.**

$\therefore X$ **is connected.**

<u>3.18Exercise:</u> Find all connected topologies on $X = \{1,2,3\}$**.**

<u>3.l9Definition:</u> Let (X,T) **be a topological space,** $E \subseteq X$ **and** $x \in E$ **,then the union of all connected sets containing** x **and contained in** E **will be called component of** E **corresponding to** x **and will be denoted by** $C(E,x)$**.**

<u>3.20Notes:</u>

- **By Corollary 3.15** $C(E,x)$ is the largest connected subset of E containing x **.**
- **The components corresponding to different points of are either equal or disjoint.**

<u>3.21Exmaple:</u> Let $X = \{1,2,3,4,5\}$ **and** $T = \{\phi, X, \{2,5\}, \{3\}, \{2,3,5\}\}$ **is a topology on** X **.Find**

1) $C(X,1)$.
2) $C(X,2)$.
3) $C(X,3)$.
4) $C(X,4)$.
5) $C(X,5)$.

Solution: Since X **is connected, then**

1) $C(X,1)=X$.
2) $C(X,2)=X$.
3) $C(X,3)=X$.
4) $C(X,4)=X$.
5) $C(X,5)=X$.

3.22Exmaple: **Let** $X=\{1,2,3,4,\}$ **and** $T=\{\phi, X, \{1,2\}, \{3,4\}, \{3\}, \{1,2,3\}\}$ **is a topology on** X **.Find**

1) $C(X,1)$
2) $C(X,2)$
3) $C(X,3)$
4) $C(X,4)$

Solution: Since $X=H=\{1,2\}/K=\{3,4\}$**,then**

1) $C(X,1)=\{1,2\}$
2) $C(X,2)=\{1,2\}$
3) $C(X,3)=\{3,4\}$
4) $C(X,4)=\{3,4\}$

3.23Exercise: **Let** $X=\{1,2,3,4,5\}$ **and** $T=\{\phi, X, \{1,2,5\}, \{3,4\}, \{1\}, \{2,5\}, \{1,3,4\}, \{2,34,5\}\}$ **is a topology on** X **.Find**

1) $C(X,1), C(X,2), C(X,3), C(X,4), C(X,5)$.