2.15Notes:

- **Open set in** T_A **denote by** G^* **.**
- **Closed set in** T_A **denote by** F^* **.**
- **Closure set of** E **in** T_A **denote by** $\overline{E}^*$ **.**
- **Interior set of** E **in** T_A **denote by** E^{0^*} **.**

2.16Theorem: (A,T_A) **is a topological space.**

Proof: Now we prove that T_A **is a topology on** A **.**

1. **a.** $\because X \in T$

$\therefore A = A \bigcap X \in T_A$

b. $\because \phi \in T$

$\therefore \phi = A \bigcap \phi \in T_A$

2. **If** $G_\alpha{}^* \in T_A \quad \forall \alpha$ **.**

$$\therefore \exists G_\alpha \in T \ni G_\alpha{}^* = \quad A \bigcap G_\alpha \ \forall \alpha$$

$$\therefore \bigcup_\alpha G_\alpha{}^* = \bigcup_\alpha (A \bigcap G_\alpha) = A \bigcap \left(\bigcup_\alpha G_\alpha \right)$$

$$\because \bigcup_\alpha G_\alpha \in T$$

$$\therefore \bigcup_\alpha G_\alpha{}^* \in T_A$$

3. If $G_i^* \in T_A \quad \forall 1 \le i \le n$

$$\therefore \exists G_i \in T \ni \quad G_i^* = A \bigcap G_i \qquad \forall 1 \le i \le n$$

$$\therefore \quad \bigcap_{i=1}^{n} G_i^* = \bigcap_{i=1}^{n} (A \cap G_i) = A \bigcap \left(\bigcap_{i}^{n} G_i \right)$$

$$\because \bigcap_{i=1}^{n} G_i \in T \qquad \therefore \bigcap_{i=1}^{n} G_i^* \in T_A$$

$\therefore T_A$ **is a topology on** (X,T).

2.17Note:

- (A,T_A) **is called a subspace of** (X,T). **Or** A **is a subspace of** X .

2.18Theorem: **Let** (A,T_A) **be a subspace of** (X,T) **and** $E \subset A$.

Then $\overline{E}^* = A \bigcap \overline{E}$.

Proof:

$$A \bigcap \overline{E} = A \bigcap \left(\bigcap_{i} F_i \right)$$

$$\ni \quad F_i \quad closed \quad set in \quad X \quad and \quad E \subseteq F_i \quad \forall \quad i$$

$$\therefore A \bigcap \overline{E} = \bigcap_{i} (A \cap F_i) = \bigcap_{i} F_i^* = \overline{E}^*$$

$$\ni \quad F_i^* \quad closed \quad set in \quad A \quad and \quad E \subseteq F_i^* \quad \forall \quad i$$

2.19Exercise: **Let** $X = \{1,2,3,4,5\}$,

$T = \{\phi, X, \{2,5\}, \{5\}, \{2,3,5\}, \{2,3\}, \{2\}, \{3\}, \{3,5\}\}$ **is a topology on** X **and** $A = \{2,3,5\}$.

Find

1) $\overline{B}^*$ **If** $B=\{2,3\}$.

2) $\overline{C}^*$ **If** $C=\{3,5\}$.

3) $\overline{D}^*$ **If** $D=\{3\}$.

<u>2.20Theorem:</u> Let (A,T_A) **be a subspace of** (X,T) **and** $B\subset A$ **.If** B **is open in** X **, then** B **is open in** A **.**

Proof: $\because B\in T \quad \therefore B=A\bigcap B\in T_A$

<u>2.21Note:</u>

The converse of theorem 2.20 is not true for example Let $X=\{1,2,3\}$ **and** $T=\{\phi, X, \{1,2\}, \{2,3\}, \{2\}\}$ **is a topology on** X **,if** $A=\{1,3\}$ **,then**

$T_A=\{\phi, A, \{1\}, \{3\}\}$

$\therefore \{1\}\in T_A\, but \quad \{1\}\notin T$

<u>2.22Exercises:</u>

1. Let (A,T_A) **be a subspace of** (X,T) **and** $H\subset A$**. If** H **is closed in** X **,**

Prove that H **is closed in** A **,and The converse is not true.**

2 .If β **is basis of** T **on** X **,and** $\beta\subset\Psi\subset T$ **.Prove that also** Ψ **is basis of** T **.**

3. Let (X,T) **is the usual topological space. Find the relative topology on** $N(T_N)$**.**

Chapter Three

Connectedness and Compactness

3.1Definition: **Let** (X,T) **be a topological space and** $E \subseteq X$ **, then** H,K **is said to be a separation of** E **and denoted by** $E = H/K$ **iff**

1. $H \neq \phi, K \neq \phi$
2. $H \bigcup K = E$
3. $H \bigcap K = \phi$
4. $H \bigcap K' = \phi, K \bigcap H' = \phi$

3.2Notes:

- **We can replace conditions 3and 4 by** $\left(H \cap \overline{K}\right) \cup \left(K \cap \overline{H}\right) = \phi$.
- E **is said to be connected if there does not exist a separation of** E **.**
- **By definition** ϕ **is connected and also singleton set** $\{x\}$ **.**

3.3Exmaple: **Let** $X = \{1,2,3,4\}$ **and** $T = \{\phi, X, \{1,3\}, \{2,4\}, \{4\}, \{1,3,4\}\}$ **is a topology on** X **.Is** X **connected?**

Solution:

Let $H = \{1,3\}, K = \{2,4\}$ **,then**

1. $H \neq \phi, K \neq \phi$
2. $H \bigcup K = X$
3. $\left(H \cap \overline{K}\right) \cup \left(K \cap \overline{H}\right) = (\{1,3\} \cap \{2,4\}) \cup (\{2,4\} \cap \{1,3\}) = \phi$

$\therefore X = H/K$

$\therefore X$ **is disconnected.**

<u>3.4Exmaple:</u> Let $X=\{1,2,3,4\}$ **and** $T=\{\phi, X, \{1\}, \{4\}, \{1,4\}, \{1,3,4\}\}$ **is a topology on** X **.Is** X **connected?**

Solution:

1. Let $H=\{1,2\}, K=\{3,4\}$***, then***

1. $H \neq \phi, K \neq \phi$
2. $H \cup K = X$
3. $$\left(H \cap \overline{K}\right) \cup \left(K \cap \overline{H}\right) = (\{1,2\} \cap \{2,3,4\}) \cup (\{3,4\} \cap \{1,2,3\})$$
$$= \{2,3\} \neq \phi$$

$$\therefore X \neq H/K$$

2. Let $H=\{1,3\}, K=\{2,4\}$***, then***

1. $H \neq \phi, K \neq \phi$
2. $H \cup K = X$
3. $$\left(H \cap \overline{K}\right) \cup \left(K \cap \overline{H}\right) \neq \phi$$

$$\therefore X \neq H/K$$

3. Let $H=\{1,4\}, K=\{2,3\}$***, then***

1. $H \neq \phi, K \neq \phi$
2. $H \cup K = X$

3.

$$\left(H \cap \overline{K}\right) \cup \left(K \cap \overline{H}\right) \neq \phi$$

$$\therefore X \neq H/K$$

4. Let $H = \{1,2,3\}, K = \{4\}$***, then***

1. $H \neq \phi, K \neq \phi$
2. $H \cup K = X$
3.

$$\left(H \cap \overline{K}\right) \cup \left(K \cap \overline{H}\right) \neq \phi$$

$$\therefore X \neq H/K$$

5. Let $H = \{1,2,4\}, K = \{3\}$***, then***

1. $H \neq \phi, K \neq \phi$
2. $H \cup K = X$
3.

$$\left(H \cap \overline{K}\right) \cup \left(K \cap \overline{H}\right) \neq \phi$$

$$\therefore X \neq H/K$$

6. Let $H = \{2,3,4\}, K = \{1\}$***, then***

1. $H \neq \phi, K \neq \phi$
2. $H \cup K = X$
3.

$$\left(H \cap \overline{K}\right) \cup \left(K \cap \overline{H}\right) \neq \phi$$

$$\therefore X \neq H/K$$

7. Let $H = \{1,3,4\}, K = \{2\}$***, then***

1. $H \neq \phi, K \neq \phi$
2. $H \cup K = X$
3. $$\left(H \cap \overline{K}\right) \cup \left(K \cap \overline{H}\right) \neq \phi$$

$$\therefore X \neq H/K$$

From 1,2,3,,4,5,6and 7 there does not exist a separation of X **.**

$\therefore X$ **is connected.**

3.5Notes:

- **A space** X **is connected iff the only subsets of** X **that are both open and closed in** X **are the empty set** ϕ **and** X **itself.**
- A **is connected subset of** X **iff** (A, T_A) **is connected space.**

3.6Exmaple: **Let** $X = \{1,2,3,4,5\}$ **and** $T = \{\phi, X, \{1\}, \{1,2,3\}, \{1,4,4\}\}$ **is a topology on** X **.**

a. Is $A = \{1,2,3\}$ **connected?**

b. Is $B = \{2,3,4\}$ **connected?**

Solution:

a. $\because \quad T_A = \{\phi, A, \{1\}\}$

$\therefore$ **the only subsets of** A **that are both open and closed in** A **are the empty set** ϕ

and A itself.

$\therefore$ **there does not exist a separation of A.**

$\therefore A$ **is connected.**

b. $\because \quad T_B = \{\phi, B, \{2.3\}, \{4\}\}$

$\therefore \{2,3\}, \{4\}$ **are open and closed in B.**

$\therefore B = H = \{2,3\} / K = \{4\}$

$\therefore B$ **is disconnected.**

<u>3.7Exercise:</u> Let $X = \{1,2,3,4,5\}$,

$T = \{\phi, X, \{2,5\}, \{5\}, \{2,3,5\}, \{2,3\}, \{2\}, \{3\}, \{3,5\}\}$ **is a topology on X.**

a. **Is X connected?**
b. **Is $A = \{1,2,3,5\}$ connected?**
c. **Is $B = \{2,3,4\}$ connected?**
d. **Is $C = \{2,5\}$ connected?**

<u>3.8Theorem:</u> Let (A, T_A) be a subspace of (X, T) and $E \subset A$. Then E is connected in (X, T) iff E is connected in (A, T_A).

Proof: $\Rightarrow$

E **is connected in (X, T). Now we prove that E is connected in (A, T_A).**

Suppose E is disconnected in (A, T_A).

$\therefore E = H/K \ni \quad H \subset E \subset A, \quad K \subset E \subset A$

But

$$\left(H \cap \overline{K}\right) \cup \left(K \cap \overline{H}\right) = \left(H \cap A \cap \overline{K}\right) \cup \left(K \cap A \cap \overline{H}\right) = \left(H \cap \overline{K}^*\right) \cup \left(K \cap \overline{H}^*\right) \ldots \otimes$$

$\because E = H/K \quad in(A, T_A)$

$\therefore \quad \left(H \cap \overline{K}^*\right) \cup \left(K \cap \overline{H}^*\right) = \phi$

From $\otimes$

$\left(H \cap \overline{K}\right) \cup \left(K \cap \overline{H}\right) = \phi$

$\therefore E = H/K \quad in(X,T)$

$\therefore$ E **is disconnected in** (X,T)**.and which is contradiction.**

$\therefore$ E **is connected in** (A,T_A)**.**

$\Leftarrow$

E **is connected in** (A,T_A)**.**

Now we prove that E **is connected in** (X,T)**.**

Suppose E **is disconnected in** (X,T)**.**

$\therefore E = H/K \ni \quad H \subset E \subset A, \quad K \subset E \subset A$

But

$\left(H \cap \overline{K}\right) \cup \left(K \cap \overline{H}\right) = \left(H \cap A \cap \overline{K}\right) \cup \left(K \cap A \cap \overline{H}\right) = \left(H \cap \overline{K}^*\right) \cup \left(K \cap \overline{H}^*\right) \ldots \otimes$

$\because E = H/K \quad in(X,T)$

$\therefore \quad \left(H \cap \overline{K}\right) \cup \left(K \cap \overline{H}\right) = \phi$

From $\otimes$

$\left(H \cap \overline{K}^*\right) \cup \left(K \cap \overline{H}^*\right) = \phi$

$\therefore E = H/K \quad in(A,T_A)$

$\therefore$ E **is disconnected in** (A,T_A)**.and which is contradiction.**

$\therefore$ E **is connected in** (X,T)**.**