1.43Theorem: Let (X,T) be a topological space and $A,B \subseteq X$.

1) A^0 is the biggest open subset of A.
2) $A^0 = A$ iff A is open.
3) $A^{0^0} = A^0$.
4) $(A \cap B)^0 = A^0 \cap B^0$.
5) $a-\quad \phi^0 = \phi \qquad b-\quad X^0 = X$.

Proof:

1) $\because \quad A^0$ is the open subset of A (By definition)

Now we prove A^0 is the biggest open subset of A.

Suppose G is open and $G \subset A$.

Now prove $G \subset A^0$.

Suppose $x \in G$

$$\therefore \quad x \in \bigcup_i G_i \ni \quad G_i \text{ open set and } G_i \subseteq A \quad \forall \quad i$$

$$\therefore \quad x \in A^0$$

$$\therefore \quad G \subset A^0$$

$\therefore \quad A^0$ is the biggest open subset of A.

2) $\Rightarrow$ Suppose $A^0 = A$

$\therefore \quad A$ is open (since A^0 is open)

Suppose A is open, we must prove

a- $A^0 \subset A$ b- $A \subset A^0$

a- $A^0 \subset A$ **By definition 1.38**

b- **Let** $x \in A$

$\because A$ **is open set and** $A \subset A$

$\therefore \quad x \in \bigcup_i G_i \ni \quad G_i \quad open \quad set \quad and \quad G_i \subseteq A \quad \forall \quad i$

$\therefore \quad x \in A^0$

$\therefore A \subset A^0$

From (a) and(b) , we get $A^0 = A$

3) $\because A^0$ **is open**

$\therefore A^{0^0} = A^0$

4)

$$(A \cap B)^0 = (A \cap B)^{C^{-C}} = \left(\overline{A^C \cup B^C}\right)^C = \left(\overline{A^C} \cup \overline{B^C}\right)^C$$

$$= \overline{A^C}^C \cap \overline{B^C}^C = A^0 \cap B^0$$

5)Exercise.

<u>1.44Exercise:</u> Let (X, T) **be a topological space and** $A, B \subseteq X$ **.**

Is $(A \cup B)^0 = A^0 \cup B^0$ **?**

1.45Definition: Let (X,T) be a topological space and $A \subseteq X$, then the exterior of A denoted by A^e or $ext(A)$ and we define the exterior of A by

$$A^e = A^{C0}$$

1.46Exmaple: Let $X = \{1,2,3,4,5\}$ and $T = \{\phi, X, \{2,5\}, \{3\}, \{2,3,5\}\}$ is a topology on X .Find

1) A^e If $A = \{2,3\}$.
2) B^e If $B = \{1\}$.
3) C^e If $C = \{1,2,3,4\}$.
4) D^e If $D = \{1,2,3\}$.
5) E^e If $E = \{3,4,5\}$.

Solution:

1) $A^e = A^{C0} = \{1,4,5\}^0 = \phi$.

2) $B^e = B^{C0} = \{2,3,4,5\}^0 = \{2,5\} \cup \{3\} \cup \{2,3,5\} = \{2,3,5\}$.

3) $C^e = C^{C0} = \{5\}^0 = \phi$.

4) Exercise.

5) Exercise.

1.47Theorem: Let (X,T) be a topological space and $A, B \subseteq X$.

1) $a-\quad \phi^e = X \qquad b-\quad X^e = \phi$
2) $A^e \subset A^C$.
3) $A^e = A^{e^{C^e}}$.
4) $(A \cup B)^e = A^e \cap B^e$.

Proof:

1) $a-\quad \phi^{e}=\phi^{C^{0}}=X^{0}=X \qquad b-\quad X^{e}=X^{C^{0}}=\phi^{0}=\phi$

2) $A^{e}=A^{C^{0}}\subset A^{C}$.

3) $A^{e^{C^{e}}}=A^{C^{0^{C^{C^{0}}}}}=A^{C^{0^{0}}}=A^{C^{0}}=A^{e}$

4) $(A\cup B)^{e}=(A\cup B)^{C^{0}}=\left(A^{C}\cap B^{C}\right)^{0}=A^{C^{0}}\cap B^{C^{0}}=A^{e}\cap B^{e}$

<u>1.48Exercise:</u> **Let** (X,T) **be a topological space and** $A,B\subseteq X$ **.**

Is $(A\cap B)^{e}=A^{e}\cup B^{e}$ **?**

<u>1.49Exercise:</u> **Let** (R,T) **be a usual topological space .Find** Q^{e} **?**

<u>1.50Definition:</u> **Let** (X,T) **be a topological space and** $A\subseteq X$ **, then the boundary of** A **denoted by** $b(A)$ **or** $\partial(A)$ **and we define the boundary of** A **by** $b(A)=\left(A^{0}\cup A^{C^{0}}\right)^{C}=A^{0^{C}}\cap A^{C^{0^{C}}}=A^{0^{C}}\cap\overline{A}=\overline{A}/A^{0}$

<u>1.51Exmaple:</u> **Let** $X=\{1,2,3,4\}$ **and** $T=\{\phi,X,\{1\},\{3\},\{4\},\{1,3\},\{1,4\},\{3,4\},\{1,3,4\}\}$ **is a topology on** X **.Find**

1) $b(A)$ **If** $A=\{2,3\}$.
2) $b(B)$ **If** $B=\{4\}$.
3) $b(C)$ **If** $C=\{1,3,4\}$.
4) $b(D)$ **If** $D=\{1,3\}$.
5) $b(E)$ **If** $E=\{1,4\}$.

Solution: Closed sets are

$X, \phi, \quad \{2,3,4\}, \quad \{1,2,4\}, \quad \{1,2,3\}, \quad \{2,4\}, \quad \{2,3\} \quad \{1,2\} \quad \{2\}$

1)

$$A^0 = \{3\}, \quad \overline{A} = X \cap \{2,3,4\} \cap \{1,2,3\} \cap \{2,3\} = \{2,3\}$$

$$b(A) = \overline{A} \big/ A^0 = \{2,3\} \big/ \{3\} = \{2\}$$

.

2)

$$B^0 = \{4\}, \quad \overline{B} = X \cap \{2,3,4\} \cap \{1,2,4\} \cap \{2,4\} = \{2,4\}$$

$$b(B) = \overline{B} \big/ B^0 = \{2,4\} \big/ \{4\} = \{2\}$$

.

3)

$$C^0 = \{1\} \cup \{3\} \cup \{4\} \cup \{1,3\} \cup \{1,4\} \cup \{3,4\} \cup \{1,3,4\} = \{1,3,4\}, \quad \overline{C} = X$$

$$b(C) = \overline{C} \big/ C^0 = X \big/ \{1,3,4\} = \{2\}$$

4) Exercise.

5) Exercise.

1.52Exercises:

(1) Let $X = \{1,2,3,4,5\}$ **and** $T = \{\phi, X, \{2,5\}, \{3\}, \{2,3,5\}\}$ **is a topology on** X **.Find**

1) A^e **If** $A = \{1,2,5\}$.

2) B^e **If** $B = \{5\}$.

3) C^e **If** $C = \{1,4,5\}$.

4) D^e **If** $D = \{1,2\}$.

5) E^e **If** $E = \{3,4\}$.

(2) Let $X=\{1,2,3,4\}$ **and** $T=\{\phi, X, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\}\}$ **is a topology on** X **.Find**

1) $b(A)$ **If** $A=\{2,4\}$.

2) $b(B)$ **If** $B=\{2,3\}$.

3) $b(C)$ **If** $C=\{1,2,4\}$.

4) $b(D)$ **If** $D=\{2,3\}$.

5) $b(E)$ **If** $E=\{3,4,\}$.

(3) Prove that

- $\overline{A}=A\bigcup b(A)$
- $\overline{A}=A^0\bigcup b(A)$
- $\overline{A}-b(A)=A^0$
- $b(A)=b(A^C)$

(4)Let (R,T) **be a usual topological space .Find** $b(Q)$**?**

<u>1.53Definition:</u> **Let** $X \neq \phi$ **and** T_1, T_2 **are two topologies on** X **.** T_2 **is said to be stronger then** T_1 **and** T_1 **is weaker than** T_2 **and we write** $T_1 \subseteq T_2$ **iff every open subset of** X **with respect to** T_1 **is an open subset of** X **with respect to** T_2 **.**

<u>1.54Notes:</u>

- T_1 **and** T_2 **are said to be not compare if** $T_1 \not\subset T_2$ **and** $T_2 \not\subset T_1$**.**
- **The indiscrete topology is weaker than any topology defined on** X **.**
- **The discrete topology is stronger than any topology defined on** X **.**

<u>1.55Exmaple:</u> **Let** $X = \{1,2,3\}$ **and**

$$T_1 = \{\phi, X, \{1\}\}, \qquad T_2 = \{\phi, X, \{2\}\}$$
$$T_3 = \{\phi, X, \{1\}, \{1,2\}\}$$

, then $T_1 \subseteq T_3$**. Hence** T_1 **is weaker than** T_3 **(or** T_3 **is stronger than** T_1 **).** T_1 **and** T_2 **are not compare since** $T_1 \not\subset T_2$ **and** $T_2 \not\subset T_1$**. Also** T_2 **and** T_3 **are not compare since** $T_2 \not\subset T_3$ **and** $T_3 \not\subset T_2$**.**

<u>1.56Exercises:</u> Let $X = \{1,2,3,4\}$ **and**

$$T_1 = \{\phi, X, \{4\}, \{1\}\text{و}\}, \qquad T_2 = \{\phi, X, \{4\}\}$$
$$T_3 = \{\phi, X, \{1\}, \{1,2\}\}$$

- **Is** T_1 **stronger than** T_2 **?**
- **Is** T_1 **is weaker than** T_3 **?**
- **Are** T_2 **and** T_3 **not compare?**

Chapter two

Bases and Relative Topologies

2.1Definition: **Let** (X,T) **be a topological space and** $\beta \subseteq T$ **,then** β **is basis of** T **iff** $\forall G \in T, x \in G \quad \exists B \in \beta \quad \ni x \in B \subseteq G$ **. i.e Every element of** T **is a union of elements of** β **.**

- **Not necessary** $\phi \in \beta$ **since** $\phi = \bigcup_{\lambda \in \phi} B_\lambda$ **.**

2.2Exmaple: **Let** $X = \{1,2,3\}$ **and** $\beta = \{X,\{1\},\{2\}\}$**.Is** β **basis of** T **?**

Solution: The topology T **generated by** β **is** $T = \{,\phi, X,\{1\},\{2\},\{1,2\}\}$**,**

$\therefore$ T **is a topology on** X **.**

$\therefore$ β **is basis of** T **. Since**

$$1 \in G = \{1\} \in T \; \exists 1 \in B = \{1\} \quad \ni 1 \in B \subset G$$
$$2 \in G = \{2\} \in T \; \exists 2 \in B = \{2\} \quad \ni 2 \in B \subset G$$
$$3 \in G = X \in T \; \exists 3 \in B = X \quad \ni 3 \in B \subset G$$
$$1 \in G = X \in T \; \exists 1 \in B = X \quad \ni 1 \in B \subset G$$
$$2 \in G = X \in T \; \exists 2 \in B = X \quad \ni 2 \in B \subset G$$
$$1 \in G = \{1,2\} \in T \; \exists 1 \in B = \{1\} \quad \ni 1 \in B \subset G$$
$$2 \in G = \{1,2\} \in T \; \exists 2 \in B = \{2\} \quad \ni 2 \in B \subset G$$

2.3Exmaple: **Let** $X = \{1,2,3\}$ **and** $\beta = \{\phi,\{1\},\{2\}\}$**.Is** β **basis of** T **?**

Solution: The topology T **generated by** β **is** $T = \{,\phi,\{1\},\{2\},\{1,2\}\}$**,**

$\therefore$ T **is not topology on** X **(Since** $X \notin T$ **).**

$\therefore \quad \beta$ **is not basis of** T **. Since**

$$3 \in G = X \in T \, but \, not \quad B \in \beta \quad \ni 3 \in B \subset G = X$$

<u>2.4Exmaple:</u> Let $X = \{1,2,3\}$ **and** $\beta = \{X, \{1,3\}, \{2,3\}\}$**.Is** β **basis of** T **?**

Solution: The topology T **generated by** β **is** $T = \{, \phi, X, \{1,3\}, \{2,3\}\}$**,**

$\therefore \quad T$ **is not topology on** X **(Since** $\{1,3\} \cap \{2,3\} = \{3\} \notin T$ **).**

$\therefore \quad \beta$ **is not basis of** T **.**

<u>2.5Exmaple:</u> Let $X \neq \phi$ **and** $\beta = \{\{x\} : x \in X\}$**.Is** β **basis of** T **?**

Solution:

$$\forall G \in T, x \in G \quad \exists B = \{x\} \in \beta \quad \ni x \in B \subseteq G$$

$\therefore \quad \beta$ **is basis of** T **on X.**

$\therefore \quad \beta$ **is basis of discrete topology on X.**

<u>2.6Exmaple:</u> Let $X = \{1,2,3\}$ **and** $\beta = \{\{1\}, \{2\}, \{3\}\}$**.Is** β **basis of** T **?**

Solution: The topology T **generated by** β **is** $T = \{, \phi, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$**,**

$\therefore \quad T$ **is a topology on** X **.**

$\therefore \quad \beta$ **is basis of** T **.**

<u>2.7Exmaple:</u> Let $X = R$ **and** $\beta = \{(a,b) : a,b \in R, a < b\}$**.Is** β **basis of** T **?**

Solution:

$$\forall G \in T, x \in G \quad \exists B = (c,d) \in \beta \quad \ni x \in B = (c,d) \subseteq G$$

$\therefore\ T$ **is a topology on** X **.**

$\therefore\ \beta$ **is basis of usual topology** T **on** R **.**

<u>2.8Theorem:</u> Let $X \neq \phi$ **and** β **be a family of subsets of** X **. Then** β **is basis of topology on** X **iff**

1. $X = \bigcup_{B\in\beta} B$

2. $\forall B_1, B_2 \in \beta, \forall x \in B_1 \bigcap B_2 \quad \exists B \in \beta \quad \ni x \in B \subseteq B_1 \bigcap B_2$

Proof: $\Rightarrow$

Suppose β **is basis of topology on** X **.**

1. $\because X$ **is an open set and** β **is basis of topology on** X **.**

   $\therefore X$ **is union of elements of** β

   / $\therefore X$ **is union all elements of** β

   i.e $X = \bigcup_{B\in\beta} B$

2. $\because B_1, B_2 \in \beta$ **and** β **is basis of topology on** X **.**

   $\therefore B_1, B_2 \in \beta$ **are open sets**

   $\therefore B_1 \bigcap B_2$ **is open set**

   $\therefore B_1 \bigcap B_2$ **is union of elements of** β

   $\therefore \forall x \in B_1 \bigcap B_2 \quad \exists B \in \beta \quad \ni x \in B \subseteq B_1 \bigcap B_2$

$\Longleftarrow$

Let $T = \{G \subseteq X : G \; is \quad union \quad of \quad elements \quad of \quad \beta \quad \}$

Now to prove T **is a topology on** X **,then** β **is basis of** T **on** X

3. **a.** $\because X = \bigcup_{B \in \beta} B$ **by hypothesis**

$\therefore X \in T$

b. $\because \phi = \bigcup_{\lambda \in \phi} B_\lambda$

$\phi \in T$

4. **If** $G_\alpha \in T \quad \forall \alpha$ **.**

$\therefore G_\alpha$ **is union of elements of** $\beta \quad \forall \alpha$

$\therefore \bigcup_\alpha G_\alpha$ **is union of elements of** β

$\therefore \bigcup_\alpha G_\alpha \in T$

5. **If** $G_1, G_2 \in T$ **and** $x \in G_1 \bigcap G_2$

$\therefore x \in G_1 , \quad x \in G_2$

$\because x \in G_1 \, and \quad G_1 \in T$

$\therefore \quad \exists B_1 \in \beta \quad \ni x \in B_1 \subseteq G_1$

$\because x \in G_2 \, and \quad G_2 \in T$

$\therefore \quad \exists B_2 \in \beta \quad \ni x \in B_2 \subseteq G_2$

$\because B_1 \bigcap B_2 \in \beta$ **by hypothesis**

$$\therefore \quad \exists B_1 \cap B_2 \in \beta \quad \ni x \in B_1 \cap B_2 \subseteq G_1 \cap G_2$$

$\therefore G_1 \cap G_2$ **is union of elements of** β

$$\therefore G_1 \cap G_2 \in T$$

$\therefore T$ **is a topology on** X **,since the conditions are satisfied.**

$\therefore \beta$ **is basis of** T **on** X

<u>2.9Exmaple:</u> Let $X = \{1,2,3\}$ **and** $\beta = \{\{1,2\}, \{2,3\}, \{2\}\}$**.Is** β **basis of topology on** X **?**

Solution: By theorem 2.8 β **is basis of topology on** X **, since**

1. $\{1,2\} \cup \{2,3\} \cup \{2\} = X$
2. $$\{1,2\} \cap \{2,3\} = \{2\} \in \beta$$
$$\{1,2\} \cap \{2\} = \{2\} \in \beta$$
$$\{2\} \cap \{2,3\} = \{2\} \in \beta$$

<u>2.10Exmaple:</u> Let $X = R$ **and** $\beta = \{(a,b] : a,b \in R, a < b\}$**. Is** β **basis of topology on** X **?**

Solution: By theorem 2.8 β **is basis of topology on** X **. The topology** T **generated by** β **is called upper limit topology.**

<u>2.11Exercise:</u> Let $X = R$ **and** $\beta = \{[a,b) : a,b \in R, a < b\}$**. Prove that** β **is basis of topology on** X **? The topology** T **generated by** β **is called lower limit topology.**

2.12Definition: Let (X,T) be a topological space and $A \subseteq X$, then the topology defined on A is called relative topology denoted by T_A such that $T_A = \{A \bigcap G : G \in T\}$

2.13Exmaple: Let $X = \{1,2,3,4,5\}$ and

$T = \{\phi, X, \{2,5\}, \{5\}, \{2,3,5\}, \{2,3\}, \{2\}, \{3\}, \{3,5\}\}$ is a topology on X .Find

1) T_A If $A = \{2,3,5\}$.
2) T_B If $B = \{1,2,4,5\}$.
3) T_C If $C = \{1,3,4\}$.
4) T_D If $D = \{3,5\}$.
5) T_E If $E = \{5\}$.

Solution:

1) $T_A = \{\phi, A, \{2,5\}, \{5\}, \{2,3\}, \{2\}, \{3\}, \{3,5\}\}$.
2) $T_B = \{\phi, B, \{2,5\}, \{5\}, \{2\}\}$.
3) $T_C = \{\phi, C, \{3\}\}$.
4) Exercise
5) Exercise

2.14Exercise:

Let $X = \{1,2,3,4,5\}$ and $T = \{\phi, X, \{2,3,4,5\}, \{2\}, \{3,4\}, \{2,5\}, \{2,3,4\}\}$ is a topology on X .Find

1) T_A If $A = \{1,3,4\}$.
2) T_B If $B = \{1,3,4,5\}$.
3) T_C If $C = \{1,2,4\}$.
4) T_D If $D = \{2,4\}$.
5) T_E If $E = \{1\}$.