1.30 Theorem:

Let (X,T) **be a topological space and** $A \subseteq X$ **, then** $\overline{A} = A \bigcup A'$.

Proof: We must prove

a) $\overline{A} \subset A \bigcup A'$ **b)** $A \bigcup A' \subset \overline{A}$.

a) Suppose $x \in \overline{A}$

$$x \in \bigcap_i F_i$$

$\ni \quad F_i \quad closed \quad set \quad and \quad A \subseteq F_i \quad \forall \quad i$

$\therefore \quad x \in F_i \quad \forall i \qquad and \quad A \subseteq F_i \quad \forall \quad i$

$\therefore \quad x \in A \qquad or \quad x \notin A$

If $x \in A \qquad \therefore \qquad x \in A \bigcup A'$

$\therefore \qquad \overline{A} \subset A \bigcup A'$

If $x \notin A$

Suppose x **is not limit point to** A. $\therefore \quad \exists G \in T \quad , x \in G \ni \quad G - \{x\} \bigcap A = \phi$

$\because \qquad x \notin A \qquad \therefore \quad G \bigcap A = \phi$

$\therefore \quad A \subset G^C \quad , G^C \quad closed \quad set and \quad x \notin G^C$

$\therefore \qquad x \notin \overline{A}$

This is contradiction

$\therefore$ x **is a limit point to** A

$\therefore \quad x \in A'$

$\therefore \quad x \in A \bigcup A'$

$\therefore \quad \overline{A} \subset A \bigcup A'$

b) Suppose $\quad x \in A \bigcup A'$

$\therefore \quad x \in A \, or \quad x \in A'$

If $\quad x \in A$

$\therefore \quad x \in F_i \quad \forall i \quad \ni \quad F_i \quad closed \quad set \quad and \quad A \subseteq F_i \quad \forall \quad i$

$$\therefore \quad x \in \bigcap_i F_i$$

$\ni \quad F_i \quad closed \quad set \quad and \quad A \subseteq F_i \quad \forall \quad i$

$\therefore \quad x \in \overline{A}$

If $\quad x \in A'$

$\because \quad A \subset F_i \quad \forall i \quad \ni \quad F_i \quad closed \quad set$

$\therefore \quad A' \subset F_i' \quad \forall i$

$\because \quad F_i \, closed \quad set \quad \forall i$

$\therefore \quad F_i' \subset F_i \quad \forall i$

$$\therefore \quad A' \subset F_i \quad \forall i$$

$$\therefore \quad x \in F_i \quad \forall i \quad \ni \quad F_i \quad closed \quad set \quad and \quad A \subseteq F_i \quad \forall \quad i$$

$$\therefore \quad x \in \bigcap_i F_i$$

$$\ni \quad F_i \quad closed \quad set \quad and \quad A \subseteq F_i \quad \forall \quad i$$

$$\therefore \quad x \in \overline{A}$$

$$\therefore \quad A \bigcup A' \subset \overline{A}$$

From a) and b) we get $\overline{A} = A \bigcup A'$

1.31Exercise:

Let $X = \{1,2,3,4\}$ **and** $T = \{\phi, X, \{1,2,3\}, \{1,3,4\}, \{1,3\}, \{1\}, \{3\}\}$ **is a topology on** X **.Find**

1) A' **If** $A = \{1,3,4\}$.
2) B' **If** $B = \{2,4\}$.
3) C' **If** $C = \{1,2,3\}$.

1.32Exercise:

Let $X = \{1,2,3,4,5\}$ **and** $T = \{\phi, X, \{1,2,3,5\}, \{1,3,5\}, \{3,5,4\}, \{3,5\}, \{1,3,4,5\}\}$ **is a topology on** X **.Find**

1) $\overline{A}$ **If** $A = \{1,3,4\}$.
2) $\overline{B}$ **If** $B = \{1,2,3,5\}$.
3) $\overline{C}$ **If** $C = \{1,2,4\}$.
4) $\overline{D}$ **If** $D = \{1,2,3\}$.
5) $\overline{E}$ **If** $E = \{1,2\}$.

1.33Exmaple:

Let (R,T) **be a usual topological space and**
$A=\left\{\frac{1}{n} \quad : \quad n\in N\right\}=\left\{1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \ldots\right\}$**.Find** $\overline{A}$**.**

Solution: $\overline{A}=A\bigcup A'$

Now find A'

Suppose $x\in R$

$\therefore \quad x\in A or \quad x\notin A$

1) If $x\in A$

a) If $x=1$

$$\therefore \quad \exists\left(\frac{1}{2},2\right)\in T,1\in\left(\frac{1}{2},2\right)\ni \quad \left(\frac{1}{2},2\right)-\{1\}\bigcap A=\phi$$

$\therefore \quad x=1$ **is not limit point to** A**.**

b) If $x=\frac{1}{n}\ni \quad n\in N-\{1\}$

$$\therefore \quad \exists\left(\frac{1}{n+1},\frac{1}{n-1}\right)\in T,\frac{1}{n}\in\left(\frac{1}{n+1},\frac{1}{n-1}\right)\ni \quad \left(\frac{1}{n+1},\frac{1}{n-1}\right)-\left\{\frac{1}{n}\right\}\bigcap A=\phi$$

$\therefore \quad x=\frac{1}{n}$ **is not limit point to** A**.**

2) If $x\notin A$

a) If $x > 1$

$$\therefore \quad \exists (1, x+1) \in T, x \in (1, x+1) \ni \quad (1, x+1) - \{x\} \cap A = \phi$$

$\therefore$ x **is not limit point to** A.

b) If $x < 0$

$$\therefore \quad \exists (x-1, 0) \in T, x \in (x-1, 0) \ni \quad (x-1, 0) - \{x\} \cap A = \phi$$

$\therefore$ x **is not limit point to** A.

c) If $\frac{1}{n+1} < x < \frac{1}{n} \ni \quad n \in N$

$$\therefore \quad \exists \left(\frac{1}{n+1}, \frac{1}{n} \right) \in T, x \in \left(\frac{1}{n+1}, \frac{1}{n} \right) \ni \quad \left(\frac{1}{n+1}, \frac{1}{n} \right) - \{x\} \cap A = \phi$$

$\therefore$ x **is not limit point to** A.

a) If $x = 0$

$$\therefore \quad \forall (0-r, 0+r) \in T, r > 0 \, and \quad 0 \in (-r, r) \ni \quad (-r, r) - \{0\} \cap A \neq \phi$$

$\therefore$ $x = 0$ **is a limit point to** A.

$$\therefore \quad A' = \{0\}$$

$$\therefore \quad \overline{A} = A \cup A' = A \cup \{0\}$$

1.34 Exercises:

1- Let (R,T) **be a usual topological space and** $B=(0,3]$**.Find** $\overline{B}$ **.**

2- Let (R,T) **be a usual topological space and** $C=\{1,2\}$**.Find** $\overline{C}$ **.**

1.35Theorem: **Let** (X,T) **be a topological space and** $A,B\subseteq X$ **.**

1) $a-\quad \overline{\phi}=\phi \qquad b-\quad \overline{X}=X$ **.**
2) $\overline{A}$ **is the smallest closed subset of** X **contains** A**.**
3) $\overline{A}=A$ **iff** A **is closed.**
4) $\overline{\overline{A}}=\overline{A}$**.**
5) $\overline{A\cup B}=\overline{A}\cup\overline{B}$**.**

Proof:

1)

$$a-\quad \overline{\phi}=\phi\cup\phi'=\phi\cup\phi=\phi$$

$$b-\quad \overline{X}=X\cup X'=X\cup X=X$$

2) $\overline{A}$ **is closed by definition.**

Now we prove that is the smallest set contains A**.**

Suppose F **is closed set contains** A **and we must prove** $\overline{A}\subset F$ **.**

$$\because \quad A\subset F$$

$$\therefore \quad A'\subset F' \; By \quad theorem 1.22$$

$$\therefore \quad A\cup A'\subset F\cup F'$$

$$\therefore \quad A\cup A'\subset F \quad since \quad F \quad is \quad closed \quad and \quad F'\subset F$$

$$\therefore \quad \overline{A} \subset F$$

$\therefore \quad \overline{A}$ **is the smallest closed subset of** X **contains** A **.**

3)

$$A = \overline{A} \Leftrightarrow A = A \bigcup A' \Leftrightarrow A' \subset A \Leftrightarrow A is \quad closed$$

4) $\because \quad \overline{A}$ **is closed by definition**

$$\therefore \quad \overline{\overline{A}} = \overline{A} By \quad part 3 \text{.}$$

5)

$$\begin{aligned} \overline{A \cup B} &= (A \cup B) \cup (A \cup B)' \\ &= (A \cup B) \cup A' \cup B' \\ &= (A \cup A') \cup (B \cup B') \\ &= \overline{A} \cup \overline{B} \end{aligned}$$

__**1.36Exercise:**__ **Let** (X, T) **be a topological space and** $A, B \subseteq X$ **.**

Is $\overline{(A \cap B)} = \overline{A} \cap \overline{B}$ **?**

__**1.37Corollary:**__ **Let** (X, T) **be a topological space and** $A \subseteq X$ **,**
if $F' \subset A \subset F$ **, then** A **is closed.**

Proof:

$\because \quad A \subset F \quad by \quad$ hypothesis

$\therefore \quad A' \subset F' \quad by \quad$ thorem1.22

$\because \quad F' \subset A \quad by \quad \text{hypothesis}$

$\therefore \quad A' \subset A$

$\therefore \quad A$ **is closed.**

<u>1.38Definition:</u> Let (X,T) **be a topological space and** $A \subseteq X$ **, then the interior of** A **denoted by** A^0 **or** $\text{int}(A)$ **and we define the interior of** A **by** $A^0 = \bigcup_i G_i$

$G_i \quad open \quad set \quad and \quad G_i \subseteq A \quad \forall \quad i$

- A^0 **Open set since** (X,T)**be a topological space.**
- $A^0 \subset A$**By definition**

<u>1.39Exmaple:</u> Let $X = \{1,2,3,4\}$ **and** $T = \{\phi, X, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1\}, \{3\}\}$ **is a topology on** X **.Find**

1) A^0 **If** $A = \{1,3,4\}$.
2) B^0 **If** $B = \{1,2,3\}$.
3) C^o **If** $C = \{2,4\}$.
4) D^0 **If** $D = \{4\}$.
5) E^0 **If** $E = \{1,2\}$.

Solution:

1) $A^0 = \{1\} \cup \{3\} \cup \{1,3\} = \{1,3\}$.

2) $B^0 = \{1,2\} \cup \{1\} \cup \{1,3\} \cup \{1,2,3\} = \{1,2,3\}$.

3) $C^0 = \phi$.

4) Exercise.

5) **Exercise.**

1.40Exercise:

Let $X = \{1,2,3,4,5\}$ **and** $T = \{\phi, X, \{2,3,4,5\}, \{2\}, \{3,4\}, \{2,5\}, \{2,3,4\}\}$ **is a topology on** X **.Find**

1) A^0 **If** $A = \{1,3,4\}$.
2) B^0 **If** $B = \{1,2,3,5\}$.
3) C^0 **If** $C = \{1,2,4\}$.
4) D^0 **If** $D = \{1,2,3\}$.
5) E^0 **If** $E = \{1,2\}$.

1.41Theorem: **Let** (X,T) **be a topological space and** $A \subseteq X$ **. Then** $A^0 = \overline{A^C}^{\,C}$ **.**

Proof: We must prove **a-** $\overline{A^C}^{\,C} \subset A^0$ **b-** $A^0 \subset \overline{A^C}^{\,C}$

a-Suppose $x \in \overline{A^C}^{\,C}$ **.**

$\therefore \quad x \notin \overline{A^C}$

$\therefore \quad x \notin A^C \cup A^{C'}$

$\therefore \quad x \notin A^C and\, x \notin A^{C'}$

$\therefore \quad x \in A$ **and** $\therefore \quad x$ **is not limit point to** A^C **.**

$\therefore \quad x \in A and \quad \exists G \in T, x \in G \quad \ni G - \{x\} \cap A^C = \phi$

$$\therefore \quad x \in A \, and \quad G \subset A$$

$$\therefore \quad \exists G \in T, x \in G \quad \ni G \subset A$$

$$\therefore \quad x \in A^0$$

$$\therefore \quad \overline{A^C}^{\,C} \subset A^0$$

b- Suppose $x \in A^0$ **.**

$$\therefore \quad \exists G \in T, x \in G \quad \ni G \subset A$$

$$\therefore \quad x \notin A^C \; and \qquad G - \{x\} \bigcap A^C = \phi$$

$$\therefore \quad x \notin A^C and \, x \notin A^{C'}$$

$$\therefore \quad x \notin A^C \bigcup A^{C'}$$

$$\therefore \quad x \notin \overline{A^C}$$

$$\therefore \quad x \in \overline{A^C}^{\,C}$$

$$\therefore \quad A^0 \subset \overline{A^C}^{\,C}$$

From (a) and (b) we get $A^0 = \overline{A^C}^{\,C}$

<u>1.42Corollary:</u> Let (X, T) **be a topological space and** $A \subseteq X$ **. Then** $\overline{A} = A^{C^{0^{C}}}$ **.**

Proof: Exercise.