Laplace Transform Formulation:

The Laplace transform method is a convenient method for finding the response of a system due to any excitation. The basic method is to use known properties of the transform to transform an ordinary differential equation into an algebric equation, using the initial conditions. The algebric equation is solved to find the transform of the solution. This transform is inverted by using properties of the transform and a table of known transform pairs.

The Laplace transform can be used to solve linear ordinary differential equations with constant or polynomial coefficients. The method easily handles excitations whose form change with time. Such excitations are written in a unified mathematical expression by using the unit step function. The shifting theorems help to perform the transform and evaluate inversions.

The Laplace transform is not as easy to apply as the convolution integrals unless one has extensive

experience in its use. The main drawback of the method is the difficulty in inverting the transform. let $\bar{x}(s)$ be the Laplace transform of the generalized coordinate $x(t)$ of one-degree of freedom system

$$\bar{x}(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

if $\mathcal{L}\, f(t) = f(s)$ then $f(t) = \mathcal{L}^{-1} f(s)$

Table of Laplace Transform,

	$f(s)$	$f(t)$
1-	1	$\delta(t)$ - unit impulse at $t=0$
2-	$\frac{1}{s}$	$u(t)$ - unit step function at $t=0$
3-	$\frac{1}{s^n}$ $(n=1,2,3..)$	$\frac{t^{n-1}}{(n-1)!}$
4-	$\frac{1}{s+a}$	e^{-at}
5-	$\frac{1}{(s+a)^2}$	$t e^{-at}$
6-	$\frac{1}{(s+a)^n}$ $n=1,2,..$	$\frac{1}{(n-1)!} t^{n-1} e^{-at}$
7-	$\frac{1}{s(s+a)}$	$\frac{1}{a}(1-e^{-at})$
8-	$\frac{1}{s^2(s+a)}$	$\frac{1}{a^2}(e^{-at}+at-1)$
9-	$\frac{s}{s^2+a^2}$	$\cos at$

	$F(s)$	$f(t)$
10-	$\dfrac{s}{s^2-a^2}$	$\cosh at$
11-	$\dfrac{1}{s^2+a^2}$	$\dfrac{1}{a}\sin at$
12-	$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\sinh at$
13-	$\dfrac{1}{s(s^2+a^2)}$	$\dfrac{1}{a^2}(1-\cos at)$
14-	$\dfrac{1}{s^2(s^2+a^2)}$	$\dfrac{1}{a^3}(at-\sin at)$
15-	$\dfrac{1}{(s^2+a^2)^2}$	$\dfrac{1}{2a^3}(\sin at-at\cos at)$
16-	$\dfrac{s}{(s^2+a^2)^2}$	$\dfrac{t}{2a}\sin at$
17.	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$
18.	$\dfrac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\dfrac{1}{\omega_n\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin \omega_n\sqrt{1-\zeta^2}\,t$
19.	$\dfrac{e^{-as}}{s}$	$u(t-a)$
20.	e^{-as}	$\delta(t-a)$

First shifting theorem:

If $\bar{f}(s) = \mathcal{L} f(t)$ then

$$\mathcal{L}\left(e^{-at} f(t)\right) = \bar{f}(s+a)$$

Ex.: Calculate $\mathcal{L}\left(e^{-\xi \omega_n t} \cos \omega_d t\right)$ where $\omega_d = \omega_n \sqrt{1-\xi^2}$ [here $a = \xi \omega_n$ $f(t) = \cos \omega_d t$

$$\mathcal{L}\, e^{-\xi \omega_n t} \cos \omega_d t = \left.\frac{s}{s^2 + \omega_d^2}\right|_{s \to s + \xi \omega_n}$$

$$\mathcal{L}\, e^{-\xi \omega_n t} \cos \omega_d t = \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2} = \frac{s+\xi\omega_n}{s^2 + 2\xi\omega_n + \xi^2\omega_n^2 + \omega_d^2}$$

$$= \frac{s+\xi\omega_n}{s^2 + 2\xi\omega_n + \xi^2\omega_n^2 + \omega_n^2(1-\xi^2)}$$

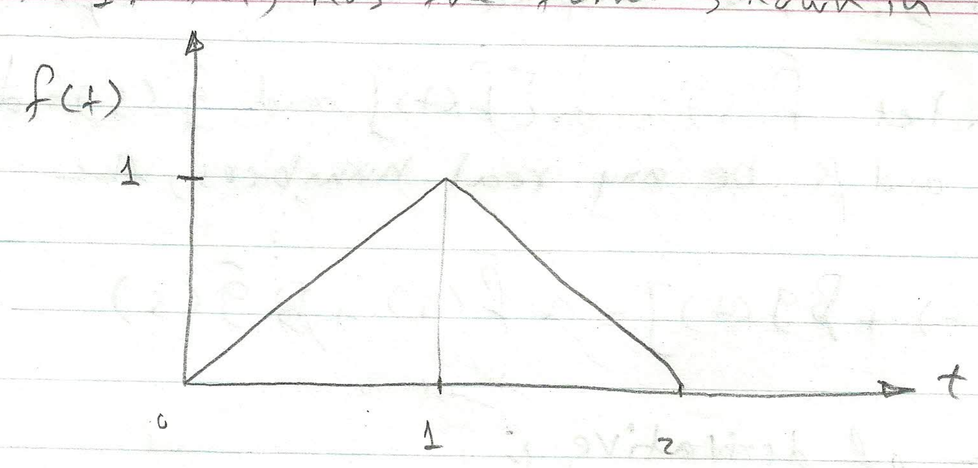
$$= \frac{s+\xi\omega_n}{s^2 + 2\xi\omega_n + \cancel{\xi^2\omega_n^2} + \omega_n^2 - \cancel{\omega_n^2\xi^2}} = \frac{s+\xi\omega_n}{s^2 + 2\xi\omega_n + \omega_n^2}$$

Second shifting theorem:

If $\bar{f}(s) = \mathcal{L} f(t)$ then

$$\mathcal{L}\left[f(t-a)\, u(t-a)\right] = e^{-as} \bar{f}(s).$$

Ex: If $f(t)$ has the form shown in the Fig. below



determine the Laplace transform of the above function.

The function of the above Fig. can be written using unit step functions as

$$f(t) = t\left[u(t) - u(t-1)\right] + (2-t)\left[u(t-1) - u(t-2)\right]$$

$$= tu(t) - tu(t-1) + 2u(t-1) - 2u(t-2) - tu(t-1) + tu(t-2)$$

$$= tu(t) - u(t-1)(2t-2) - u(t-2)(2-t)$$

$$= tu(t) - 2(t-1)u(t-1) + (t-2)u(t-2)$$

$$\mathcal{L}\, f(t) = \frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s} = \frac{1}{s}\left(1 - 2e^{-s} + e^{-2s}\right)$$

Linearity: The Laplace transform operator is linear. Let $\bar{f}(s) = \mathcal{L}[f(t)]$ and $\bar{g}(s) = \mathcal{L}[g(t)]$ and let α and β be any real numbers, the

$$\mathcal{L}[a f(t) + \beta g(t)] = a \bar{f}(s) + \beta \bar{g}(s)$$

Transform of derivatives:

The property of the Laplace transform of derivatives allows easy application of the Laplace transform to the solution of differential equations.

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-1} f'(0) - \dots - s f^{n-2}(0) - f^{n-1}(0)$$

Ex: Determine $\mathcal{L}\ \sin 2t$

$$\sin 2t = -\frac{1}{2} \frac{d}{dt} \cos 2t$$

Applying linearity and transform of derivative

$$\mathcal{L}\ \sin 2t = -\frac{1}{2}\left[s \mathcal{L} \cos 2t - \cos 0\right]$$

$$\mathcal{L}\ \sin 2t = -\frac{1}{2}\left[s \frac{s}{s^2+4} - 1\right] = -\frac{1}{2}\left(\frac{\cancel{s^2} - \cancel{s^2} - 4}{s^2+4}\right)$$

$\mathcal{L}\ \sin 2t = \dfrac{2}{s^2+4}$

Inversion of Transform:

If $\bar{f}(s) = \mathcal{L}[f(t)]$ then $f(t) = \mathcal{L}^{-1}[\bar{f}(s)]$ where

$$\mathcal{L}^{-1}[\bar{f}(s)] = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(s)e^{st}\,ds$$

It is often obtain by using the table in conjuction with transform properties.

Ex: If $e^{-2s}\dfrac{s+5}{s^2+2s+5} = \bar{f}(s)$ find $f(t)$

where $f(t) = \mathcal{L}^{-1}[\bar{f}(s)]$

$$\bar{f}(s) = e^{-2s}\frac{s+1+4}{s^2+2s+1+4} = e^{-2s}\left[\frac{s+1}{(s+1)^2+4} + \frac{4}{(s+1)^2+4}\right]$$

$$\bar{f}(s) = e^{-2s}\,\bar{g}(s)$$

Using linearity and the first shifting theorem

$$g(t) = \mathcal{L}^{-1}[\bar{g}(s)] = e^{-t}[\cos 2t + 2\sin 2t]$$

using the second shifting theorem

$$f(t) = u(t-2)\,g(t-2) = e^{-(t-2)}\left[\cos 2(t-2) + 2\sin 2(t-2)\right]$$

$(u(t-2))$

$$f(t) = e^{2-t}\left[\cos 2(t-2) + 2\sin 2(t-2)\right] u(t-2)$$

Ex Solve the equation of forced vibration of the damped system using Laplace transform

$$\ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = \frac{F(t)}{m}$$

$$s^2 \bar{x}(s) - s x(0) - \dot{x}(0) + 2\zeta\omega_n\left[s\bar{x}(s) - x(0)\right] + \omega_n^2 \bar{x}(s) = \frac{\bar{F}(s)}{m}$$

$$\bar{x}(s)\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right] = s x(0) + \dot{x}(0) + 2\zeta\omega_n x(0) + \frac{\bar{F}(s)}{m}$$

$$\bar{x}(s) = \frac{\frac{\bar{F}(s)}{m} + x_0(s + 2\zeta\omega_n) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$x(t) = \frac{1}{m}\mathcal{L}^{-1}\frac{\bar{F}(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \mathcal{L}^{-1}\frac{x(0)(s + 2\zeta\omega_n) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \text{--- (1)}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2\zeta^2 - \omega_n^2\zeta^2 + \omega_n^2$$
$$= (s + \omega_n\zeta)^2 + \omega_n^2(1 - \zeta^2)$$

$$\mathcal{L}^{-1}\frac{x_0(s + 2\zeta\omega_n) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \mathcal{L}^{-1}\frac{x_0(s + \zeta\omega_n) + x_0\zeta\omega_n + \dot{x}(0)}{(s + \omega_n\zeta)^2 + \omega_n^2(1 - \zeta^2)}$$

$$= \mathcal{L}^{-1}\left[\frac{x_0(s + \omega_n\zeta)}{(s + \omega_n\zeta)^2 + \omega_n^2(1 - \zeta^2)} + \frac{x_0\zeta\omega_n}{(s + \omega_n\zeta)^2 + \omega_n^2(1 - \zeta^2)} + \frac{\dot{x}(0)}{(s + \omega_n\zeta)^2 + \omega_n^2(1 - \zeta^2)}\right.$$

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~~Applying the first shifting theore~~

$$x(t) = e^{-\zeta\omega_n t}\left[x_0\cos(\sqrt{1-\zeta^2})\,\omega_n t + \frac{x_0\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right.$$

$$\left. + \frac{\dot{x}(0)}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right]$$

let $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$x(t) = e^{-\zeta\omega_n t}\left[x_0\cos\omega_d t + \frac{\sin\omega_d t}{\omega_d}\left\{x_0\zeta\omega_n + \dot{x}(0)\right\}\right]$$

The inverse transform of the first term of Eq. 1 is found by finding $\bar{F}(s)$ for the particular form of $F(t)$, forming $\dfrac{\bar{F}(s)}{s^2+2\zeta\omega_n s+\omega_n^2}$ and inverting.