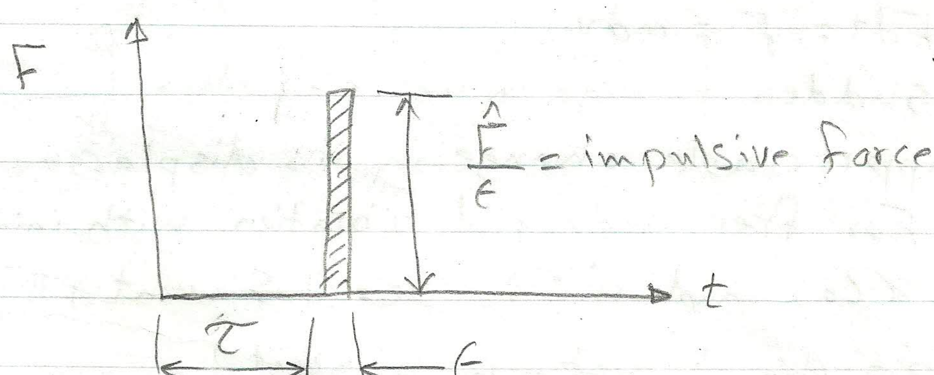
Transient Vibration   الإهتزاز العابر

It is the response of a dynamical system to a suddenly applied non-periodic excitation $F(t)$. Such oscillations occur at the natural frequencies of the system with the amplitude varying in a manner depends on the type of excitation.

Impulse excitation:- ($\hat{F}$)

$$\hat{F} = \int F(t)\,dt$$

Impulsive force:- It is a force of very high magnitude acts for a very short time but with a time integral that is finite.



ϵ - time duration

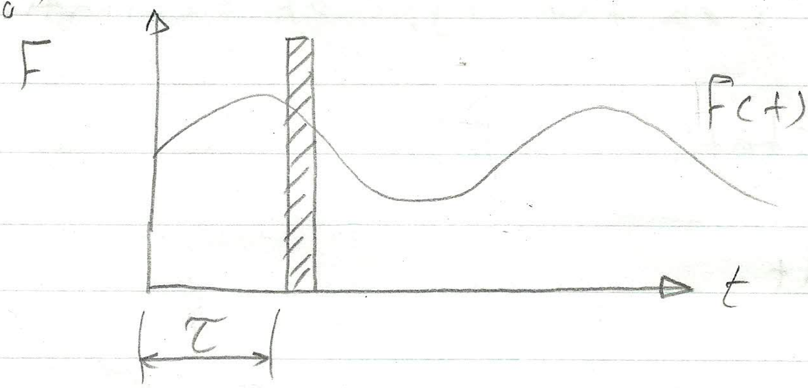
when $\left.\begin{matrix}\hat{F} \to 1 \\ \epsilon \to 0\end{matrix}\right\}$ unit impulse or delta function $\delta(t-\tau)$

Properties of delta function $\delta(t-\tau) = 0$ for all $t \neq \tau$

$$\int_0^\infty \delta(t-\tau)\,dt = 1 \qquad 0 < t < \infty \quad \text{or at } t = \tau$$

Now if $\delta(t-\tau)$ is multiplied by any time function $f(t)$, the product will be zero everywhere except at $t=\tau$ and its time integral will be

$$\int_0^\infty f(t)\,\delta(t-\tau)\,dt = f(\tau) \quad 0<\tau<\infty \quad \text{or at } t=\tau$$



$F\,dt$ = pulse = $m\,dv$ = momentum

$F\,dt = \hat{F} = m\,dv$

sudden change in velocity $dv = \dfrac{\hat{F}}{m}$ without an appreciable change in its displacement

For free undamped vibration with initial conditions $x(0)$ and $\dot{x}(0)$ $\quad x = A\sin\omega_n t + B\cos\omega_n t$

$\dot{x} = A\omega_n\cos\omega_n t - B\omega_n\sin\omega_n t$

at $t=0$ $\quad x(0) = B$

at $t=0$ $\quad \dot{x}(0) = A\omega_n \longrightarrow A = \dfrac{\dot{x}(0)}{\omega_n}$

$\therefore x = \dfrac{\dot{x}(0)}{\omega_n}\sin\omega_n t + x(0)\cos\omega_n t$

Hence the response of spring-mass system initial at rest and excited by an impulse $\hat{F}$

$x(0) =$ zero $\quad \dot{x}(0) = \dfrac{\hat{F}}{m}$

$$x = \frac{\hat{F}}{m\omega_n} \sin \omega_n t = \hat{F}\, h(t)$$

where $h(t) = \dfrac{\sin \omega_n t}{m\omega_n} \longrightarrow$ Response to a unit impulse.

For damped free vibration for $x(0) = 0$

$$x = A\, e^{-\zeta\omega_n t} \cos(\omega_d t - \phi)$$

$$\dot{x} = A\left[e^{-\zeta\omega_n t}\, \omega_d\left(-\sin(\omega_d t - \phi)\right) + \cos(\omega_d t - \phi)(-\zeta\omega_n)\, e^{-\zeta\omega_n t}\right]$$

$$\dot{x} = -A e^{-\zeta\omega_n t}\left[\omega_d \sin(\omega_d t - \phi) + \zeta\omega_n \cos(\omega_d t - \phi)\right]$$

$x(0) = 0 \longrightarrow 0 = A\cos(-\phi) \quad A \neq 0 \quad \cos(-\phi) = 0$

$\therefore \phi = \frac{\pi}{2} \qquad \cos(-\phi) = \cos\phi = 0$

$$\dot{x}(0) = -A\left[\omega_d \sin(-\phi) + \zeta\omega_n \cos(-\phi)\right]$$

$$\dot{x}(0) = -A\left[\omega_d \overset{-1}{\cancel{\sin\left(-\frac{\pi}{2}\right)}} + \zeta\omega_n \overset{0}{\cancel{\cos\left(\frac{\pi}{2}\right)}}\right]$$

$$A = \frac{\dot{x}(0)}{\omega_d} = \frac{\dot{x}(0)}{\omega_n\sqrt{1-\zeta^2}}$$

$$x = \frac{\dot{x}(0)}{\omega_n\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \cos\left(\omega_d t - \frac{\pi}{2}\right)$$

but $\cos\left(\omega_d t - \frac{\pi}{2}\right) = \sin \omega_d t = \sin\left(\omega_n\sqrt{1-\zeta^2}\right)t$

$$\therefore x = \frac{\dot{x}(0)}{\omega_n\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\right)t$$

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$$\dot{x}(0) = \frac{\hat{F}}{m}$$

The response to a unit impulse

$$\therefore x = \frac{\hat{F}}{m\omega_n\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\sqrt{1-\zeta^2}\,\omega_n t$$

$$x = \hat{F}h(t) \rightarrow h(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_n\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\,\omega_n t$$

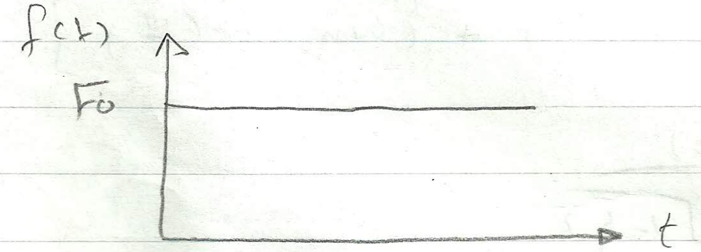
Arbitrary Excitation:

$f(\tau)$ - any arbitrary force

$$x(t) = \int_0^t f(\tau)h(t-\tau)\,d\tau$$

which is called convolution integral or superposition integral

For step function excitation



(a) For undamped system $h(t-\tau) = \frac{1}{m\omega_n}\sin\omega_n t$

$f(\tau) = F_0$

$$x(t) = \int_0^t F_0 \frac{1}{m\omega_n} \sin\omega_n(t-\tau)\,d\tau$$

$$x(t) = \frac{F_0}{m\omega_n}\int_0^t \sin\omega_n(t-\tau)\,d\tau$$

$$x = \frac{F_o}{m\omega_n} \int_0^t \sin \omega_n (t-\tau) \frac{-\omega_n}{-\omega_n} \, d\tau$$

$$x = \frac{F_o}{-m\omega_n^2} \Big[-\cos \omega_n (t-\tau) \Big]_0^t$$

$$x = \frac{F_o}{m\omega_n^2} \Big[\cos \omega_n (t-t) - \cos \omega_n (t-0) \Big]$$

$$x = \frac{F_o}{m\omega_n^2} \Big[\cos 0 - \cos \omega_n t \Big]$$

$$x = \frac{F_o}{m\omega_n^2} (1 - \cos \omega_n t) \qquad m\omega_n^2 = k$$

$x = \frac{F_o}{k} (1 - \cos \omega_n t)$ The peak response occurs when $\cos \omega_n t = -1$

$x = \frac{2F_o}{k}$ $\qquad \frac{F_o}{k} = \delta_{st}$ — statical deflection

$x = 2\delta_{st}$

That means, that the peak response to the step excitation of magnitude F_o is equal to the twice the statical deflection

(b) For damped system $V_o = \dot{x}(0) = \frac{\hat{E}}{m}$

$$h(t-\tau) = \frac{e^{-\zeta \omega_n (t-\tau)}}{m\omega_n \sqrt{1-\zeta^2}} \sin \sqrt{1-\zeta^2}\, \omega_n (t-\tau)$$

let $\omega_d = \omega_n \sqrt{1-\zeta^2}$

$$x(t) = \frac{F_o}{m\omega_d} \int_0^t e^{-\zeta \omega_n (t-\tau)} \sin \omega_d (t-\tau) \, d\tau$$

let $a = \zeta \omega_n$ $\quad b = \omega_d = \omega_n\sqrt{1-\zeta^2}$ $\quad t - \tau = x$

$-d\tau = dx \longrightarrow dx = -d\tau$

when $\tau = 0 \longrightarrow x = t$ $\quad$ when $\tau = t \longrightarrow x = 0$

$$\int_0^t e^{-\zeta\omega_n(t-\tau)} \sin \omega_d (t-\tau)\, d\tau = -\int_t^0 e^{-ax} \sin bx\, dx$$

To evaluate $\int_t^0 e^{-ax} \sin bx\, dx$

let $u = e^{-ax} \longrightarrow du = -a e^{-ax} dx$

$dv = \sin bx\, dx \longrightarrow v = \frac{-1}{b} \cos bx$

$$\int u\,dv = uv - \int v\,du$$

$$\int e^{-ax} \sin bx\, dx = -e^{-ax} \frac{\cos bx}{b} - \int \frac{-\cos bx}{b}(-a e^{-ax})\, dx$$

$$= \frac{-e^{-ax}}{b} \cos bx - \frac{a}{b} \int a^{-ax} \cos bx\, dx$$

let $u = e^{-ax} \longrightarrow du = -a e^{-ax}\, dx$

$dv = \cos bx\, dx \longrightarrow v = \frac{1}{b} \sin bx$

$$\int e^{-ax} \sin bx\, dx = -\frac{e^{-ax}}{b} \cos bx - \frac{a}{b}\left[e^{-ax} \frac{\sin bx}{b} - \int \frac{\sin bx}{b}(-a e^{-ax})\, dx \right]$$

$$= \frac{-e^{-ax}}{b} \cos bx - \frac{a}{b^2} e^{-ax} \sin bx - \frac{a^2}{b^2} \int e^{-ax} \sin bx\, dx$$

let $\int e^{-ax} \sin bx\, dx = M$

$$M\left(1 + \frac{a^2}{b^2}\right) = -\left[\frac{e^{-ax}}{b} \cos bx + \frac{a}{b^2} e^{-ax} \sin bx \right]$$

$$-M\left(\frac{b^2+a^2}{b^2}\right) = \frac{e^{-ax}\cos bx}{b} + \frac{a}{b^2}e^{-ax}\sin bx]$$

$$-M = \frac{1}{a^2+b^2}\left[b e^{-ax}\cos bx + a e^{-ax}\sin bx\right]_t^0$$

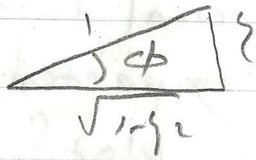
$$-M = \frac{1}{a^2+b^2}\left[b - \left\{b e^{-at}\cos bt + a e^{-at}\sin bt\right\}\right]$$

$$-M = \frac{b}{a^2+b^2}\left[1 - e^{-at}\cos bt - \frac{a}{b}e^{-at}\sin bt\right]$$

Now substitute $a = \zeta\omega_n$ $b = \omega_d = \sqrt{1-\zeta^2}\,\omega_n$

$$-M = \frac{\omega_d}{\zeta^2\omega_n^2 + \omega_n^2(1-\zeta^2)}\left[1 - e^{-\zeta\omega_n t}\left\{\cos\omega_d t + \frac{\zeta\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\zeta^2}}\sin\omega_d t\right\}\right]$$

$$M = \frac{\cancel{\omega_n}\sqrt{1-\zeta^2}}{\omega_n^{\cancel{2}}}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left\{\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right\}\right]$$



let $\cos\phi = \sqrt{1-\zeta^2}$ $\sin\phi = \zeta$

$\tan\phi = \frac{\zeta}{\sqrt{1-\zeta^2}}$

$$-M = \frac{\sqrt{1-\zeta^2}}{\omega_n}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left\{\cos\phi\cos\omega_d t + \sin\phi\sin\omega_d t\right\}\right]$$

$$-M = \frac{\sqrt{1-\zeta^2}}{\omega_n}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \phi)\right]$$

$$x(t) = \frac{F_0}{m\omega_n\sqrt{1-\zeta^2}} * M$$

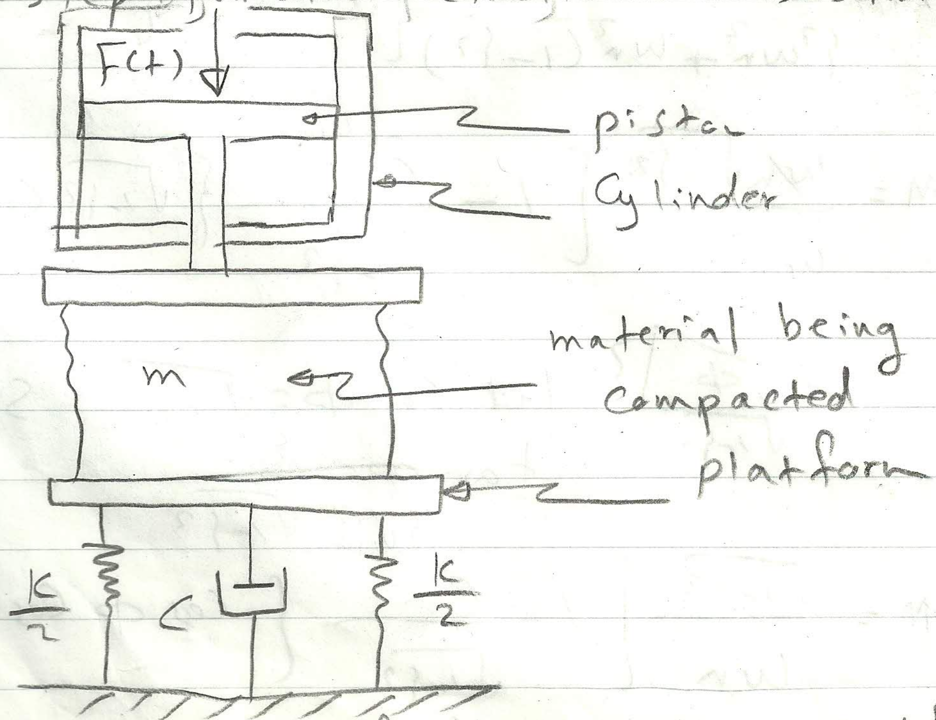
$$x(t) = \frac{F_0}{m\omega_n \cancel{\sqrt{1-\zeta^2}}} * \frac{\cancel{\sqrt{1-\zeta^2}}}{\omega_n}\left[1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t-\phi)\right]$$

$$m\omega_n^2 = k$$

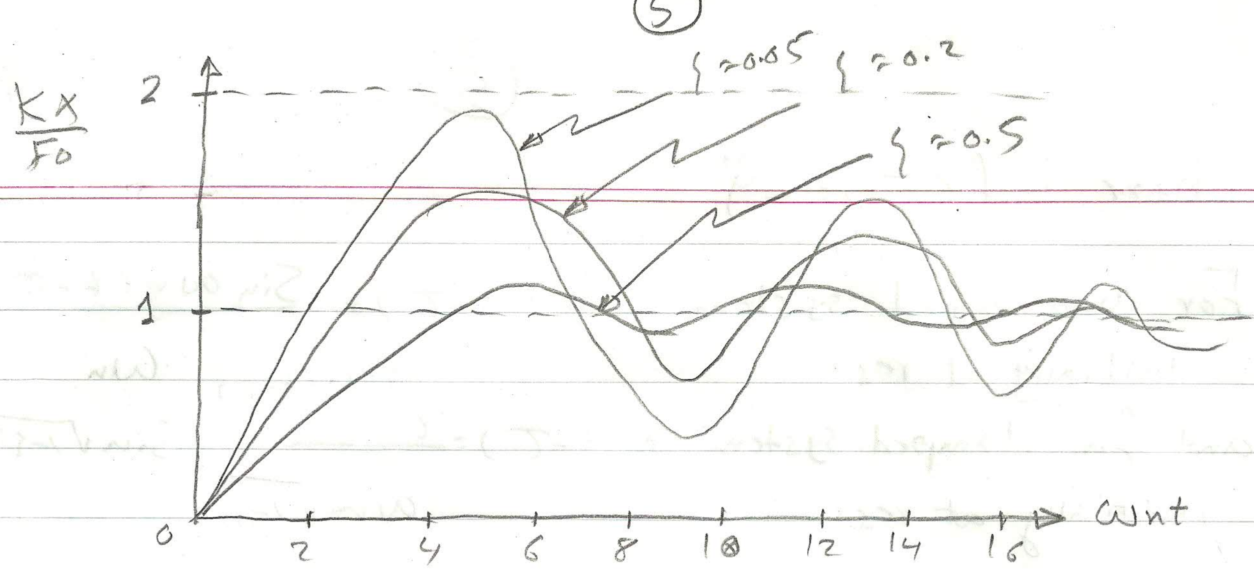
$$x(t) = \frac{F_0}{k}\left[1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t-\phi)\right]$$

$$\phi = \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}$$

Note: The compacting machine can be modeled as a step function excitation as shown



m includes the masses of the piston, platform and the material being compacted.



Response of unit step function

Basic excitation:

Many mechanical systems and structures are subjected to nonperiodic sudden base excitation (displacement, velocity and acceleration). A rigid wheel traveling along a road contour excites motion of vehicle through the suspension system. Earthquake excite structures through base motion. The governing equation for the relative displacement between a mass and its base when the mass is connected to the base through a spring and viscous damper in parallel

$$\ddot{z} + 2\zeta\omega_n \dot{z} + \omega_n^2 z = -\ddot{y} \qquad z = x - y$$

where y is the prescribed base motion

Applying convolution integral concept

$$z = \int_0^t f(\tau)\, h(t-\tau)\, d\tau$$

here $f(\tau) = -\ddot{y}$

For undamped system initially at rest $h(t-\tau) = \dfrac{\sin\omega_n(t-\tau)}{\omega_n}$

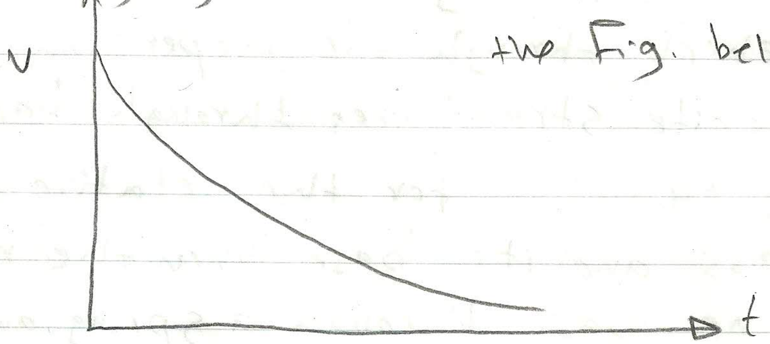
and for damped system initially at rest $h(t-\tau) = \dfrac{e^{-\xi\omega_n(t-\tau)}}{\omega_n\sqrt{1-\xi^2}} \sin\sqrt{1-\xi^2}\,\omega_n(t-\tau)$

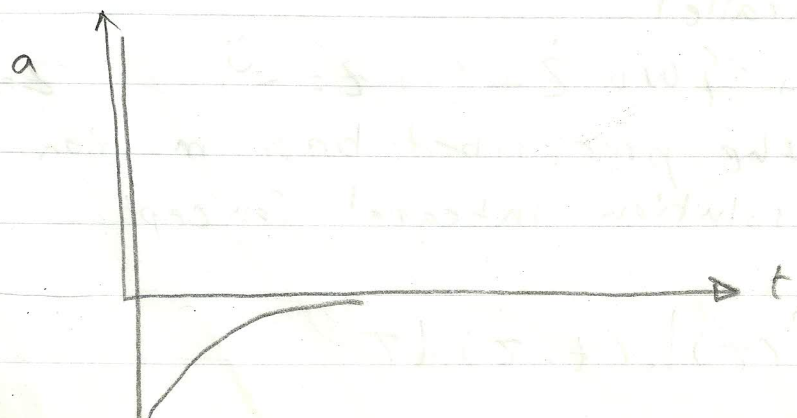
Ex. 4.2.2. page 93 (Thomson) Consider an undamped spring-mass system where the motion of the base is specified by the velocity impulse of the form

$$\dot{y}(t) = V_0 e^{-\frac{t}{\tau_0}} u(t)$$

where $u(t)$ is a unit step function.

The velocity together with its time rate of change is shown in the Fig. below





$\frac{d}{dt}u(t) = \delta(t)$ - delta function

$$\ddot{y} = V_0\left[e^{-\frac{t}{\tau_0}}\delta(t) + u(t)e^{-\frac{t}{t_0}}\left(-\frac{1}{t_0}\right)\right]$$

$$\ddot{y} = V_0 e^{-\frac{t}{t_0}}\delta(t) - \frac{V_0}{t_0}e^{-\frac{t}{t_0}}u(t) \qquad f(\tau) = -\ddot{y}$$

$$z(t) = \frac{-1}{\omega_n}\int_0^t \ddot{y}\sin\omega_n(t-\tau)\,d\tau$$

$$z(t) = \frac{-V_0}{\omega_n}\int_0^t\left[e^{-\frac{\tau}{t_0}}\delta(\tau) - \frac{e^{-\frac{\tau}{t_0}}}{t_0}u(\tau)\right]\sin\omega_n(t-\tau)\,d\tau$$

$$z(t) = \frac{-V_0}{\omega_n}\int_0^t e^{-\frac{\tau}{t_0}}\delta(\tau)\sin\omega_n(t-\tau)\,d\tau + \frac{V_0}{\omega_n}\int_0^t \frac{e^{-\frac{\tau}{t_0}}}{t_0}u(\tau)\sin\omega_n(t-\tau)\,d\tau$$

$u(\tau) = 1$

$$\int_0^t e^{-\frac{\tau}{t_0}}\delta(t)\sin\omega_n(t-\tau)\,d\tau = e^{-\frac{\tau}{t_0}}\sin\omega_n(t-\tau)\Big|_{\tau=0} = \sin\omega_n t$$

$$z(t) = \frac{V_0}{\omega_n}\int_0^t \frac{e^{-\frac{\tau}{t_0}}}{t_0}\sin\omega_n(t-\tau)\,d\tau - \frac{V_0}{\omega_n}\sin\omega_n t$$

$$z(t) = \frac{V_0}{\omega_n t_0}\int_0^t e^{-\frac{\tau}{t_0}}\sin\omega_n(t-\tau)\,d\tau - \frac{V_0}{\omega_n}\sin\omega_n t$$

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~~let $t-\tau=x$ $\quad -d\tau=dx$ $\quad \tau=t-x$~~

at $\tau=0$ $\quad x=t$ $\quad$ at $\tau=t$ $\quad x=0$

let $a=\frac{1}{t_0}$ $\quad b=\omega_n$

$$\int_0^t e^{-\frac{\tau}{t_0}} \sin \omega_n(t-\tau)\,d\tau = \int_t^0 e^{-(t-x)\cdot a} \sin bx\,(-dx)$$

$$= -\int_t^0 e^{-at} e^{ax} \sin bx\,dx = -e^{-at}\int_t^0 e^{ax} \sin bx\,dx$$

let $u=e^{ax} \longrightarrow du=ae^{ax}\,dx$

$dv=\sin bx\,dx \longrightarrow v=\frac{-1}{b}\cos bx$

$$\int_t^0 e^{ax}\sin bx\,dx = \int u\,dv = uv-\int v\,du$$

$$= -e^{ax}\cdot\frac{\cos bx}{b} - \int \frac{-1}{b}\cos bx\, a e^{ax}\,dx$$

$$= -\frac{e^{ax}}{b}\cos bx + \frac{a}{b}\int_t^0 \cos bx\, e^{ax}\,dx$$

Now let $u=e^{ax} \longrightarrow du=ae^{ax}\,dx$

$dv=\cos bx\,dx \longrightarrow v=\frac{1}{b}\sin x$

$$\int_t^0 e^{ax}\sin bx\,dx = \frac{-e^{ax}}{b}\cos bx + \frac{a}{b}\left[\frac{1}{b}e^{ax}\sin bx - \int_t^0 \frac{1}{b}\sin bx\, a e^{ax}\,dx\right]$$

$$= \frac{-e^{ax}}{b}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int_t^0 e^{ax}\sin bx\,dx$$

$$\text{let } \int_t^0 e^{ax} \sin bx \, dx = M$$

$$M\left(1 + \frac{a^2}{b^2}\right) = \frac{-e^{ax}}{b} \cos bx + \frac{a}{b^2} e^{ax} \sin bx$$

$$M\left(\frac{b^2 + a^2}{b^2}\right) = \frac{-e^{ax}}{b} \cos bx + \frac{a}{b^2} e^{ax} \sin bx$$

$$M = \frac{1}{a^2 + b^2} \Big[-b e^{ax} \cos bx + a e^{ax} \sin bx \Big]_t^0$$

$$\int_t^0 e^{ax} \sin bx \, dx = \frac{1}{a^2 + b^2} \Big[-b e^{ax} \cos bx + a e^{ax} \sin bx \Big]_t^0$$

$$= \frac{1}{a^2 + b^2} \Big[-b - \{ -b e^{at} \cos bt + a e^{at} \sin bt \} \Big]$$

$$= \frac{1}{a^2 + b^2} \Big[-b + b e^{at} \cos bt - a e^{at} \sin bt \Big]$$

$$\therefore \int = \frac{1}{\frac{1}{t_0^2} + \omega_n^2} \Big[-\omega_n + \omega_n e^{\frac{t}{t_0}} \cos \omega_n t - \frac{1}{t_0} e^{\frac{t}{t_0}} \sin \omega_n t \Big]$$

$${}^{ax} \sin bx \, dx = \frac{-t_0^2 \omega_n}{1 + (\omega_n t_0)^2} \Big[1 - e^{\frac{t}{t_0}} \cos \omega_n t + \frac{e^{\frac{t}{t_0}}}{\omega_n t_0} \sin \omega_n t \Big]$$

⑦

$$\therefore \int_0^t e^{-\frac{\tau}{t_0}} \sin \omega_n (t-\tau)\, d\tau = -e^{-at} \int_t^0 e^{ax} \sin bx\, dx$$

$$= -e^{-at}\left(\frac{-t_0^2 \omega_n}{1+(\omega_n t_0)^2}\right)\left[1 - e^{\frac{t}{t_0}} \cos \omega_n t + \frac{e^{\frac{t}{t_0}}}{\omega_n t_0} \sin \omega_n t\right]$$

$$= \frac{e^{-\frac{t}{t_0}}\, t_0^2 \omega_n \cancel{e^{\frac{t}{t_0}}}}{[1+(\omega_n t_0)^2]}\left[e^{-\frac{t}{t_0}} - \cos \omega_n t + \frac{1}{\omega_n t_0} \sin \omega_n t\right]$$

$$Z(t) = \frac{V_0}{\omega_n t_0} \cdot \frac{t_0^2 \omega_n}{(1+(\omega_n t_0)^2)}\left[e^{-\frac{t}{t_0}} + \frac{1}{\omega_n t_0} \sin \omega_n t - \cos \omega_n t\right] - \frac{V_0}{\omega_n} \sin \omega_n t$$

$$Z(t) = \frac{V_0 t_0}{1+(\omega_n t_0)^2}\left[e^{-\frac{t}{t_0}} - \cos \omega_n t + \frac{1}{\omega_n t_0} \sin \omega_n t\right] - \frac{V_0}{\omega_n} \sin \omega_n t$$

Take the term, $\dfrac{V_0 t_0}{1+(\omega_n t_0)^2} \dfrac{\sin \omega_n t}{\omega_n t_0} - \dfrac{V_0}{\omega_n} \sin \omega_n t$

$$= \frac{V_0 t_0}{1+\omega_n^2 t_0^2}\left[\frac{1}{\omega_n t_0} \sin \omega_n t - \frac{(1+\omega_n^2 t_0^2)}{t_0 \omega_n} \sin \omega_n t\right]$$

$$= \frac{V_0 t_0}{(1+\omega_n^2 t_0^2)(\omega_n t_0)}\left[\cancel{\sin \omega_n t} - \cancel{\sin \omega_n t} - \omega_n^2 t_0^2 \sin \omega_n t\right]$$

$$= \frac{-V_0 t_0 \omega_n^{\cancel{2}} t_0^{\cancel{2}}}{(\cancel{\omega_n t_0})[1+\omega_n^2 t_0^2]} \sin \omega_n t = \frac{-V_0 t_0}{1+\omega_n^2 t_0^2}\left[\omega_n t_0 \sin \omega_n t\right]$$

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$$\therefore\ Z(t) = \frac{V_o t_o}{1+(\omega_n t_o)^2}\left[e^{-\frac{t}{t_o}} - \cos\omega_n t\right] - \frac{V_o t_o}{1+\omega_n^2 t_o^2}(\omega_n t_o \sin\omega_n t)$$

$$Z(t) = \frac{V_o t_o}{1+(\omega_n t_o)^2}\left[e^{-\frac{t}{t_o}} - \cos\omega_n t - \omega_n t_o \sin\omega_n t\right]$$