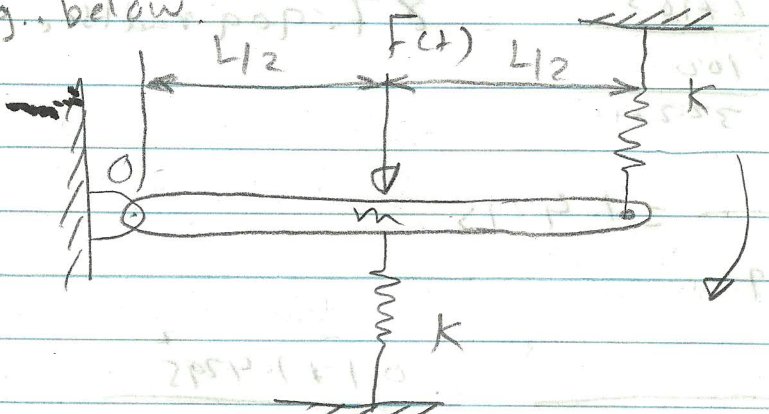
Energy method:

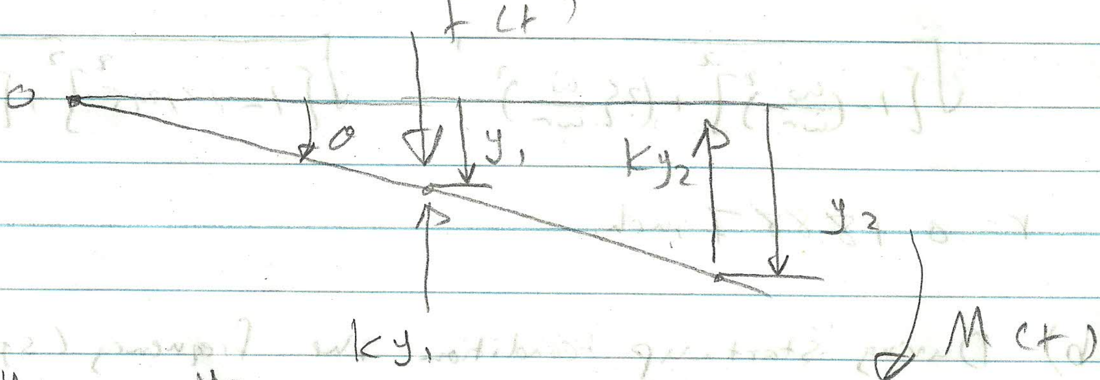
The excitation forces are nonconservative. The energy method can be applied in forced vibration with the following form

$$\frac{d}{dt}(K.E + P.E) = \frac{d}{dt}\text{work done by external forces and moment.}$$

it can be called equivalent systems method.

Ex1: Derive the differential equation governing the forced vibration of the system shown in the Fig. below.





$$\theta = \frac{y_1}{L/2} = \frac{y_2}{L}$$

$$y_1 = \theta\left(\frac{L}{2}\right) \quad y_2 = \theta L$$

Equivalent Systems method

1

$$PE = \frac{1}{2}ky_1^2 + \frac{1}{2}ky_2^2 = \frac{1}{2}k\left(\frac{L}{2}\right)^2\theta^2 + \frac{1}{2}kL^2\theta^2$$

$$\therefore PE = \frac{1}{8}kL^2\theta^2 + \frac{1}{2}kL^2\theta^2 = \frac{5kL^2}{8}\theta^2$$

$$KE = \frac{1}{2}m\dot{y}_1^2 + \frac{1}{2}J_{cg}\dot{\theta}^2 \qquad \dot{y}_1 = \frac{L}{2}\dot{\theta} \quad J_{cg} = \frac{mL^2}{12}$$

$$KE = \frac{1}{2}m\left(\frac{L^2}{4}\right)\dot{\theta}^2 + \frac{1}{2}\frac{mL^2}{12}\dot{\theta}^2$$

Work done by external forces and moment W_{ex}

$$W_{ex} = F(t)\,y_1 + M(t)\,\theta$$

$$W_{ex} = \left(\frac{L}{2}\right)F(t)\,\theta + M(t)\,\theta$$

$$\frac{d}{dt}(PE + KE) = \frac{d}{dt}(W_{ex})$$

$$\frac{\cancel{2}\times 5kL^2\theta\cancel{\dot{\theta}}}{\cancel{8}\,4} + m\frac{L^2}{4}\cancel{\dot{\theta}}\ddot{\theta} + \frac{mL^2}{12}\cancel{\dot{\theta}}\ddot{\theta}$$

$$= \frac{L}{2}F(t)\dot{\theta} + M(t)\cancel{\dot{\theta}}$$

$$\ddot{\theta}\,mL^2\left(\frac{3+1}{12}\right) + \frac{5}{4}kL^2\theta = M(t) + \frac{L}{2}F(t)$$

$$\frac{1}{3}mL^2\ddot{\theta} + \frac{5}{4}kL^2\theta = \frac{L}{2}F(t) + M(t)$$

Newton, Second law method

$$\sum M_o = J_o \ddot{\theta} \qquad J_o = \frac{mL^2}{3}$$

$$-Ky_1\left(\frac{L}{2}\right) - Ky_2 L + F(t)\frac{L}{2} + M(t) = \frac{mL^2}{3}\ddot{\theta}$$

$$\frac{mL^2}{3}\ddot{\theta} + K\frac{L^2}{4}\theta + kL^2\theta = M(t) + \frac{L}{2}F(t)$$

$$\frac{mL^2}{3}\ddot{\theta} + \frac{5}{4}kL^2\theta = M(t) + \frac{L}{2}F(t)$$

Note: In the frequency response

$$\frac{XK}{F_0} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}} \qquad \text{let } \frac{XK}{F_0} = M \quad \frac{\omega}{\omega_n} = r$$

$$\frac{dM}{dr} = 0 \longrightarrow r = \frac{\omega}{\omega_n} = \sqrt{1-2\zeta^2}$$

$$\therefore M_{max} = \frac{1}{\sqrt{\left[1-\left(\sqrt{1-2\zeta^2}\right)^2\right]^2 + \left(2\zeta\sqrt{1-2\zeta^2}\right)^2}}$$

$$M_{max} = \frac{1}{\sqrt{(1-1+2\zeta^2)^2 + 4\zeta^2(1-2\zeta^2)}} = \frac{1}{\sqrt{4\zeta^4 + 4\zeta^2 - 8\zeta^4}}$$

$$M_{max} = \frac{1}{\sqrt{4\zeta^2 - 4\zeta^4}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{1}{2\zeta(1-\zeta^2)^{\frac{1}{2}}}$$

(2)

Ex2: A machine of mass 25 kg is placed on an elastic foundation. A sinusoidal force of magnitude 25N is applied to the machine. A frequency sweep reveals that the maximum steady state amplitude of 1.3 mm occurs when the period of response is 0.22 sec. Determine the equivalent stiffness and damping ratio of the foundation

$m = 25\,kg \quad F_0 = 25N \quad X_{max} = 1.3\,mm \quad T = 0.22\,sec$

$K?\ \zeta?$ $\qquad \omega_n = \sqrt{\frac{k}{m}} \quad k = m\omega_n^2$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.22} = 28.56\ rad/sec$$

$$\frac{\omega}{\omega_n} = \sqrt{1-2\zeta^2} \quad \omega_n = \frac{28.56}{\sqrt{1-2\zeta^2}} \quad (1)$$

$$\frac{X_{max}\,k}{F_0} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad k = m\omega_n^2 = 25 \times \omega_n^2$$

$$\frac{1.3 \times 10^{-3} \times \cancel{25}\,\omega_n^2}{\cancel{25}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\frac{1.3 \times 10^{-3} \times \cancel{25}\ 28.56^2}{\cancel{25}\ \ 1-2\zeta^2} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\left[\frac{1060.3761 \times 10^{-3}}{1-2\zeta^2} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}\right] \text{ squaren}$$

$$\frac{1124396.583 \times 10^{-6}}{1-4\zeta^2+4\zeta^4} = \frac{1}{4\zeta^2(1-\zeta^2)}$$

③

$$1-4\zeta^2+4\zeta^4 = 4497586.372 \times 10^{-6}\,\zeta^2(1-\zeta^2)$$

$$1-4\zeta^2+4\zeta^4 = 4.5\zeta^2(1-\zeta^2)$$

$$1-4\zeta^2+4\zeta^4 = 4.5\zeta^2-4.5\zeta^4$$

$$8.5\zeta^4-8.5\zeta^2+1=0$$

$$\zeta^4-\zeta^2+0.1176=0$$

$$\zeta^2 = \frac{1 \mp \sqrt{1^2-4 \times 0.1176}}{2} = \frac{1 \mp 0.728}{2}$$

$$\zeta^2 = 0.864 \longrightarrow \zeta = 0.93$$

or

$$\zeta^2 = 0.136 \longrightarrow \zeta = 0.368$$

The larger value $\zeta = 0.93$ is disregarded because $\zeta < \frac{1}{\sqrt{2}} < 0.707$

is $\boxed{\zeta = 0.368}$

Substitute in eq. 1 $\omega_n = \dfrac{28.56}{\sqrt{1-2 \times 0.368^2}}$

$\therefore \omega_n = 33.446$ rad/sec

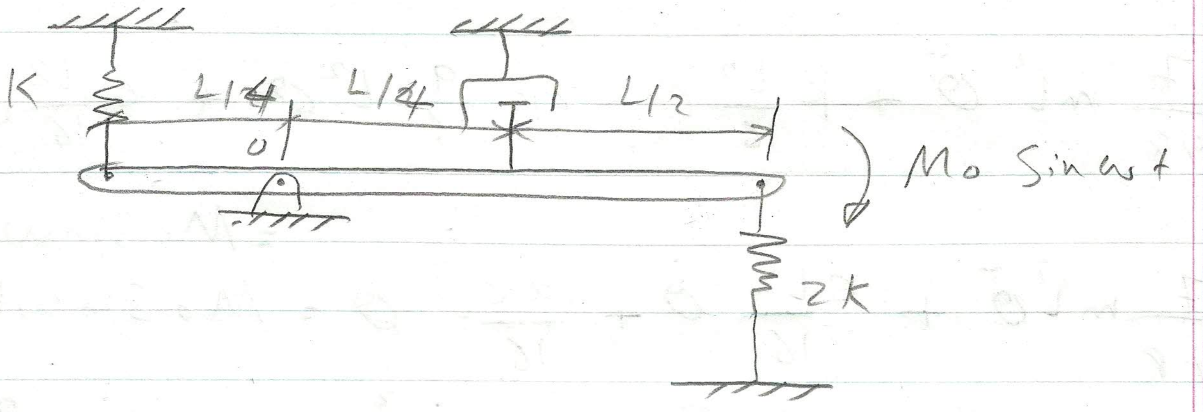
$\therefore K = m\omega_n^2 = 25 \times 33.446^2 = 27966.5145$ N/m

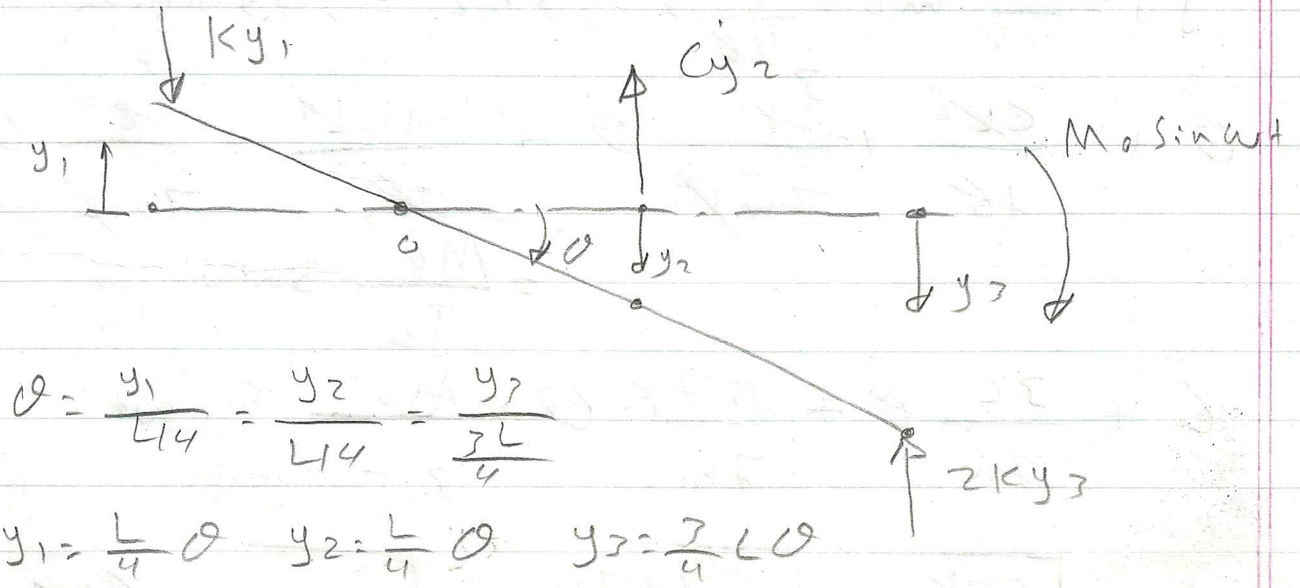
$K = 2.8 \times 10^4$ N/m

Ex: A moment $M_o \sin\omega t$ is applied to the end of the bar shown in the Fig. below. Determine the maximum value of M_o such that the steady-state amplitude of angular oscillation does not exceed 10° if $\omega = 500$ rpm, $K = 7000$ N/m, $C = 650$ N.sec/m, $L = 1.2$ m and the mass of the bar is 15 kg.

$M = M_o \sin\omega t$ $\quad M_{o\,max}?$ $\quad \theta_{max} = 10^\circ$ $\quad \omega = 500$ rpm $\quad K = 7000$ N/m

$C = 650$ N.sec/m $\quad L = 1.2$ m $\quad m = 15$ kg





$$\theta = \frac{y_1}{L/4} = \frac{y_2}{L/4} = \frac{y_3}{\frac{3L}{4}}$$

$$y_1 = \frac{L}{4}\theta \qquad y_2 = \frac{L}{4}\theta \qquad y_3 = \frac{3}{4}L\theta$$

④

$$\Sigma M_o = J_o \ddot{\theta}$$

$$J_o = J_{c.g} + md^2 \qquad J_{c.g} = \frac{mL^2}{12} \qquad d = \frac{L}{4}$$

$$J_o = \frac{mL^2}{12} + m\frac{L^2}{16} = mL^2\left(\frac{7}{48}\right)$$

$$-ky_1 \times \frac{L}{4} - c\dot{y}_2 \times \frac{L}{4} - 2ky_3 \times \frac{3L}{4} + M_o \sin\omega t = mL^2\left(\frac{7}{48}\right)\ddot{\theta}$$

$$\frac{7}{48}mL^2\ddot{\theta} + k\left(\frac{L}{4}\right)^2\theta + c\left(\frac{L}{4}\right)^2\dot{\theta} + 2k\left(\frac{3}{4}L\right)^2\theta = M_o \sin\omega t$$

$$\frac{7}{48}mL^2\ddot{\theta} + k\frac{L^2}{16}\theta + 2k\frac{9}{16}L^2\theta + c\frac{L^2}{16}\dot{\theta} = M_o \sin\omega t$$

$$\frac{7}{48}mL^2\ddot{\theta} + \frac{cL^2}{16}\dot{\theta} + \frac{19kL^2}{16}\theta = M_o \sin\omega t$$

$$J_o = \frac{7}{48}mL^2 = \frac{7}{48} \times 15 \times 1.2^2 = 3.15 \text{ kg.m}^2$$

$$\ddot{\theta} + \frac{c\cancel{L^2}}{\cancel{16}_1} \times \frac{\cancel{48}^3}{7m\cancel{L^2}}\dot{\theta} + \frac{19k\cancel{L^2}}{\cancel{16}_1} \times \frac{\cancel{48}^3}{7m\cancel{L^2}}\theta = \frac{M_o}{J_o}\sin\omega t$$

$$\ddot{\theta} + \frac{3c}{7m}\dot{\theta} + \frac{57k}{7m}\theta = \frac{M_o}{3.15}\sin\omega t$$

$$\therefore \omega_n = \sqrt{\frac{57k}{7m}} = \sqrt{\frac{57 \times 7000}{7 \times 15}} = \sqrt{3800} = 61.64 \text{ rad/sec.}$$

(4)

$2\xi\omega_n = \frac{3c}{7m} \longrightarrow \xi = \frac{3c}{14m\omega_n}$

$\xi = \frac{3 \times 656}{14 \times 15 \times 61.64} = 0.151$

$r = \frac{\omega}{\omega_n} \qquad \omega = \frac{2\pi N}{60} = \frac{2\pi \times 500}{60} = 52.36$

$r = \frac{52.36}{61.64} = 0.85$

$$\frac{Xk}{F_o} = \frac{\theta \times \omega_n^2}{\frac{M_o}{J_o}} = \frac{\theta \omega_n^2 J_o}{M_o} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$\frac{\theta_{max}\, \omega_n^2 J_o}{M_{o max}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$M_{o max} = \theta_{max} J_o \omega_n^2 \sqrt{(1-r^2)^2 + (2\xi r)^2}$$

$$M_{o max} = \left(\frac{10 \times \pi}{360}\right) \times 3.15 \times (61.64)^2 \sqrt{(1-0.85^2)^2 + (2 \times 0.151 \times 0.85)^2}$$

$$M_{o max} = 2088.88 \sqrt{0.1429} = 789.644 \text{ N.m}$$