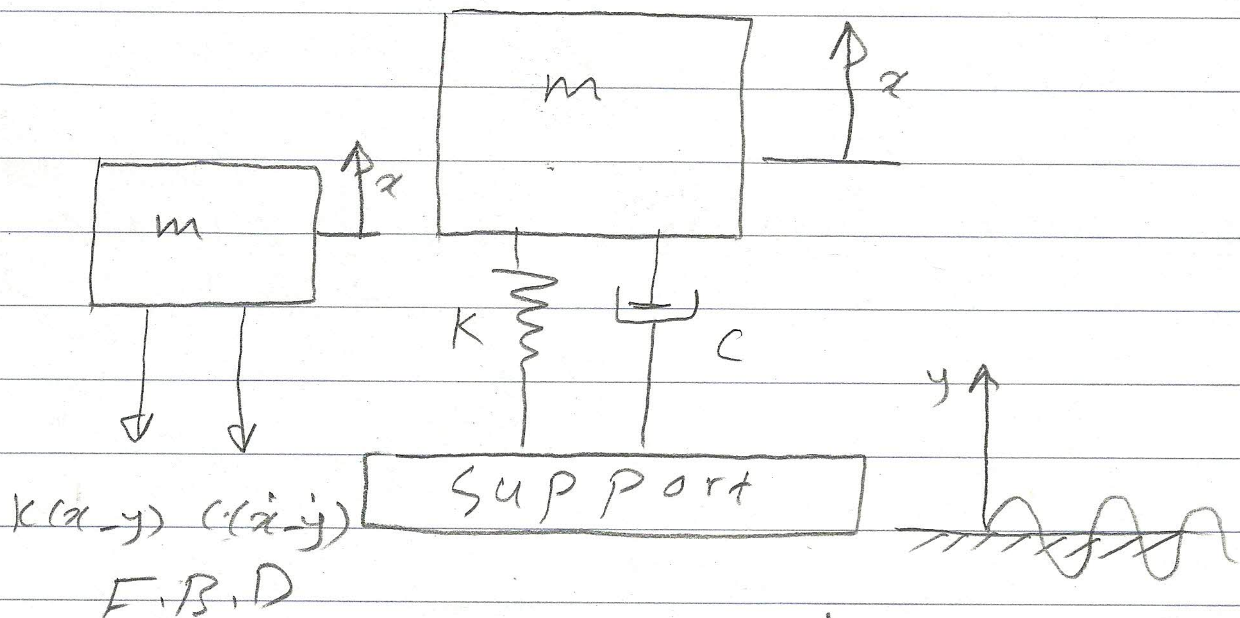
Support Motion:-

In many dynamical systems, the excitation may cause by the motion of the support point.



let y - the harmonic displacement of the support point

x - the displacement of the mass m from a fixed reference

Applying Newton's Second law $\sum F = m\ddot{x}$

$$-K(x-y) - C(\dot{x}-\dot{y}) = m\ddot{x}$$

$$m\ddot{x} + K(x-y) + C(\dot{x}-\dot{y}) = 0$$

$$m\ddot{x} + Kx - Ky + C\dot{x} - C\dot{y} = 0$$

$$m\ddot{x} + C\dot{x} + Kx = Ky + C\dot{y} \quad \text{--- (1)}$$

Now let $y = Y e^{i\omega t}$ $\quad x = X e^{i(\omega t - \psi)}$

$$x = e^{i\omega t} e^{-i\psi}$$

which allows the displacement x to differ in phase from the displacement y by the

(1)

angle (ψ)
Substitute in Eq. (1)

$$-mXe^{-i\psi}\omega^2 e^{i\omega t} + kXe^{-i\psi}e^{i\omega t} + ic\omega Xe^{-i\psi}e^{i\omega t}$$

$$= kYe^{i\omega t} + ic\omega Ye^{i\omega t}$$

$$e^{-i\psi} X\cancel{e^{i\omega t}}\left[-m\omega^2 + ic\omega + k\right] = \cancel{e^{i\omega t}}\left[k + ic\omega\right]$$

$$\left[-m\omega^2 + ic\omega + k\right]Xe^{-i\psi} = Y\left[k + ic\omega\right]$$

$$\frac{X}{Y}e^{-i\psi} = \frac{k + ic\omega}{(k - m\omega^2) + ic\omega} \quad \text{--- (2)}$$

The absolute value of the amplitude ratio

$$\left|\frac{X}{Y}\right| = \sqrt{\frac{k + (c\omega)^2}{\left[k - m\omega^2\right]^2 + (c\omega)^2}}$$

$$\left|\frac{X}{Y}\right| = \frac{K}{K}\sqrt{\frac{1 + \left(\frac{c\omega}{k}\right)^2}{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c\omega}{k}\right)^2}}$$

As previously proved

$$\left|\frac{X}{Y}\right| = \sqrt{\frac{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2}}$$

To find the phase angle ψ, the real and imaginary parts of Eq. 2 are equated to determine $\sin\psi$ and $\cos\psi$. The ratio then results in the equation for the phase angle

$$\frac{X}{Y}(\cos\psi - i\sin\psi) = \frac{(k - i\omega c)\left[k - m\omega^2 - ic\omega\right]}{\left[(k - m\omega^2) + ic\omega\right]\left[(k - m\omega^2) - ic\omega\right]}$$

$$+ (\omega c)^2$$

$$\frac{X}{Y}(\cos\psi - i\sin\psi) = \frac{k^2 - mk\omega^2 - ic\omega k + ic\omega k - i\omega cm)}{(k - m\omega^2)^2 + (\omega c)^2}$$

$$\frac{X}{k}(\cos\psi - i\sin\psi) = \frac{k^2 - mk\omega^2 + (\omega c)^2}{(k - m\omega^2)^2 + (\omega c)^2} - \frac{i\,\omega^3 cm}{(k - m\omega^2)^2 + (\omega c)^2}$$

$$\therefore \cos\psi = \frac{k^2 - mk\omega^2 + (\omega c)^2}{(k - m\omega^2)^2 + (\omega c)^2}$$

and

$$\sin\psi = \frac{\omega^3 cm}{(k - m\omega^2)^2 + (\omega c)^2}$$

Knowing that $\frac{m}{k} = \frac{1}{\omega_n^2}$, $\frac{\omega c}{k} = 2\zeta \frac{\omega}{\omega_n}$

It can be concluded after some mathematical manipulations that

$$\cos\psi = \frac{1-(\frac{\omega}{\omega_n})^2 + (2\zeta\frac{\omega}{\omega_n})^2}{[1-(\frac{\omega}{\omega_n})^2]^2 + [2\zeta\frac{\omega}{\omega_n}]^2}$$

and

$$\sin\psi = \frac{2\zeta(\frac{\omega}{\omega_n})^3}{[1-(\frac{\omega}{\omega_n})^2]^2 + [2\zeta\frac{\omega}{\omega_n}]^3}$$

Then

$$\tan\psi = \frac{\sin\psi}{\cos\psi} = \frac{2\zeta(\frac{\omega}{\omega_n})^3}{1-(\frac{\omega}{\omega_n})^2 + (2\zeta\frac{\omega}{\omega_n})^2}$$

when $|\frac{X}{Y}| = 1$, then

$$1 + \cancel{(2\zeta\frac{\omega}{\omega_n})^2} = [1-(\frac{\omega}{\omega_n})^2]^2 + \cancel{[2\zeta\frac{\omega}{\omega_n}]^2}$$

$1-(\frac{\omega}{\omega_n})^2 = \mp 1 \rightarrow$ either $1-(\frac{\omega}{\omega_n})^2 = 1$

$\frac{\omega}{\omega_n} = 0 \rightarrow$ neglected or $1-(\frac{\omega}{\omega_n})^2 = -1$

$(\frac{\omega}{\omega_n})^2 = 2 \longrightarrow \frac{\omega}{\omega_n} = \sqrt{2}$

let $\left|\frac{X}{Y}\right| = T \qquad \frac{\omega}{\omega_n} = r$

$$\therefore \left|\frac{X}{Y}\right| = \sqrt{\frac{1+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$T(r,\zeta) = \frac{\left[1+(2r\zeta)^2\right]^{\frac{1}{2}}}{\left[(1-r^2)^2+(2\zeta r)^2\right]^{\frac{1}{2}}} \cdots ① \text{ For all } 0<\zeta<1$$

$$\frac{dT}{dr} = \frac{\left[(1-r^2)^2+(2\zeta r)^2\right]^{\frac{1}{2}}\left[\frac{1}{2}\left\{1+4\zeta^2r^2\right\}^{-\frac{1}{2}}8\zeta^2 r\right] - \left[1+4\zeta^2r^2\right]^{\frac{1}{2}}\left[\frac{1}{2}\left\{(1-r^2)^2+4\zeta^2r^2\right\}^{-\frac{1}{2}}\left\{2(1-r^2)(-2r)+8\zeta^2 r\right\}\right]}{(1-r^2)^2+(2\zeta r)^2} = 0$$

$$\sqrt{(1-r^2)^2+4\zeta^2r^2}\left[\frac{4\zeta^2 r}{\sqrt{1+4\zeta^2r^2}}\right] = \frac{1}{2}\frac{\sqrt{1+4\zeta^2r^2}}{\sqrt{(1-r^2)^2+4\zeta^2r^2}}\left[8\zeta^2 r-4r(1-r^2)\right]$$

$$\left[(1-r^2)^2+4\zeta^2r^2\right]\cancel{4}\zeta^2\cancel{r} = \frac{1}{2}\left[1+4\zeta^2r^2\right]\cancel{4r}\left[2\zeta^2-1+r^2\right]$$

$$2\zeta^2\left[(1-r^2)^2+4\zeta^2r^2\right] = \left[1+4\zeta^2r^2\right]\left[2\zeta^2-1+r^2\right]$$

$$2\zeta^2\left[1-2r^2+r^4+4\zeta^2r^2\right] = 2\zeta^2-1+r^2+8\zeta^4r^2-4\zeta^2r^2+4\zeta^2r^4$$

$$\cancel{2\zeta^2}-\cancel{4r^2\zeta^2}+2\zeta^2r^4+\cancel{8\zeta^4r^2} = \cancel{2\zeta^2}-1+r^2+\cancel{8\zeta^4r^2}-\cancel{4\zeta^2r^2}+4\zeta^2r^4$$

$$2\zeta^2r^4-r^2+1=0 \rightarrow r^2 = \frac{-1\mp\sqrt{1+8\zeta^2}}{4\zeta^2}$$

$$r^2 = \frac{-1}{4\zeta^2} \mp \frac{\sqrt{1+8\zeta^2}}{4\zeta^2} = \frac{1}{4\zeta^2}\left[\sqrt{1+8\zeta^2} - 1\right]$$

(-ve) sign of $\sqrt{1+8\zeta^2}$ is neglected

$$\therefore r_{max} = \frac{1}{2\zeta}\left[\sqrt{1+8\zeta^2} - 1\right]^{\frac{1}{2}} \quad \text{---} \ (2)$$

Substitute eq. (2) into eq. (1)

$$T_{max} = \frac{\left[1 + \cancel{4\zeta^2} \cdot \frac{1}{\cancel{4\zeta^2}}\left\{\sqrt{1+8\zeta^2} - 1\right\}\right]^{\frac{1}{2}}}{\left[\left\{1 - \frac{1}{4\zeta^2}\left(\sqrt{1+8\zeta^2} - 1\right)\right\}^2 + \cancel{4\zeta^2} \cdot \frac{1}{\cancel{4\zeta^2}}\left(\sqrt{1+8\zeta^2} - 1\right)\right]^{\frac{1}{2}}}$$

$$T_{max} = \frac{\left[\cancel{1} + \sqrt{1+8\zeta^2} - \cancel{1}\right]^{\frac{1}{2}}}{\left[\left\{\frac{4\zeta^2 - \left(\sqrt{1+8\zeta^2} - 1\right)}{4\zeta^2}\right\}^2 + \sqrt{1+8\zeta^2} - 1\right]^{\frac{1}{2}}}$$

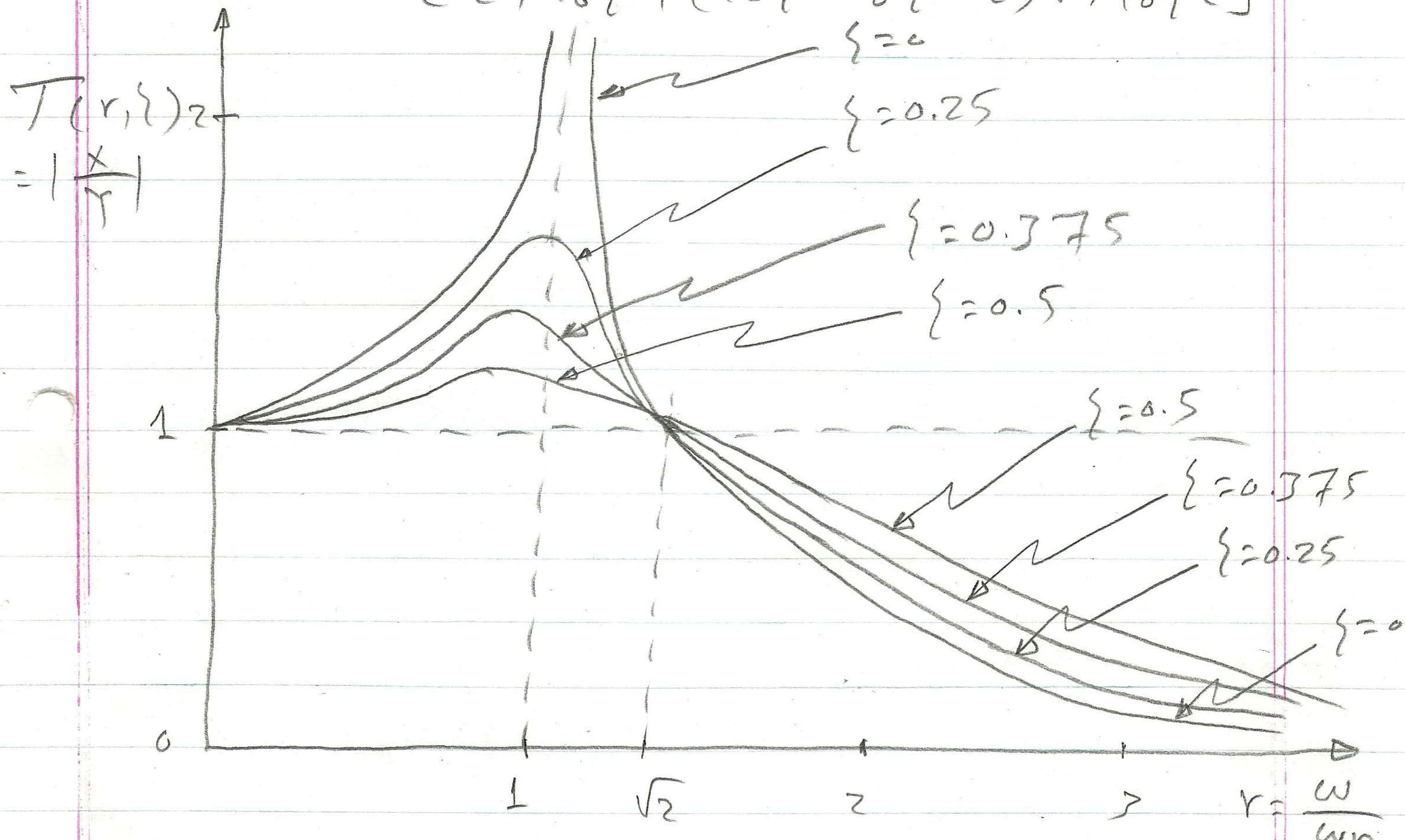
$$T_{max} = \frac{\left[\sqrt{1+8\zeta^2}\right]^{\frac{1}{2}}}{\left[\frac{\left(4\zeta^2 - \sqrt{1+8\zeta^2} + 1\right)^2 + 16\zeta^4\left(\sqrt{1+8\zeta^2} - 1\right)}{16\zeta^4}\right]^{\frac{1}{2}}}$$

$$T_{max} = \frac{4\zeta^2\left[\sqrt{1+8\zeta^2}\right]^{\frac{1}{2}}}{\left[\cancel{16\zeta^4} + 1 + 8\zeta^2 + 1 - 8\zeta^2\sqrt{1+8\zeta^2} + 8\zeta^2 - 2\sqrt{1+8\zeta^2} - \cancel{16\zeta^4} + 16\zeta^4\sqrt{1+8\zeta^2}\right]^{\frac{1}{2}}}$$

$$T_{max} = \frac{4\zeta^2\left[\sqrt{1+8\zeta^2}\right]^{\frac{1}{2}}}{\left[2 + 16\zeta^2 + \sqrt{1+8\zeta^2}\,(16\zeta^4 - 8\zeta^2 - 2\right]^{\frac{1}{2}}}$$

④

$$\therefore T_{max} = 4\zeta^2 \left[\frac{\sqrt{1+8\zeta^2}}{2+16\zeta^2+(16\zeta^4-8\zeta^2-2)\sqrt{1+8\zeta^2}} \right]^{\frac{1}{2}}$$



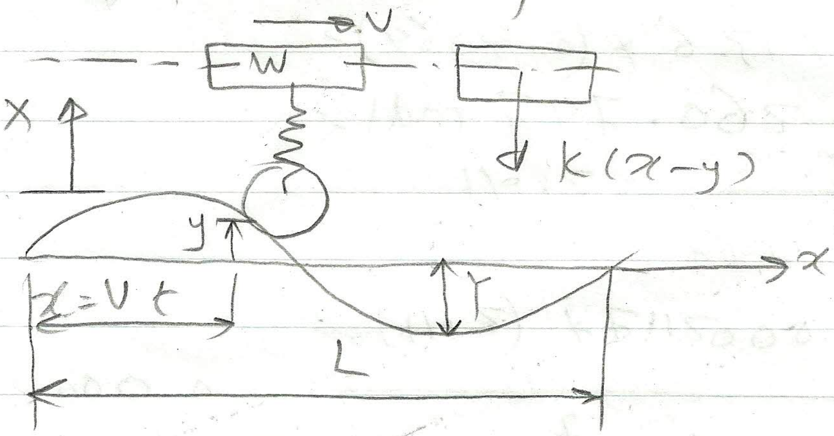
① $T(\sqrt{2},\zeta)=1$ is independent on the value of ζ.

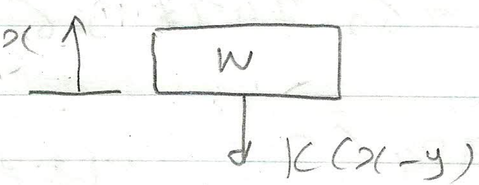
② For $r<\sqrt{2}$, $T(r,\zeta)$ is larger for smaller value of ζ, however for $r>\sqrt{2}$, $T(r,\zeta)$ is smaller for smaller value of ζ.

③ For all values of ζ, $T(r,\zeta)$ is less than one when only $r>\sqrt{2}$

(Thomson)

Prob. 3.20 page 84) The Fig. below represents a simplified diagram of a spring-supported vehicle traveling over a rough road. Determine the equation for the amplitude of W as a function of the speed and determine the most unfavorable speed.





$\Sigma F = m\ddot{x}$

$-K(x-y) = m\ddot{x}$

$m\ddot{x} + Kx = Ky \qquad y = Y\sin\omega t \qquad \omega = \frac{2\pi}{T}$

$T = L \qquad x = Vt \qquad y = Y\sin\frac{2\pi}{L}Vt$

$\therefore \ddot{x} + \frac{k}{m}x = \frac{k}{m}Y\sin\left(\frac{2\pi}{L}V\right)t \qquad \omega_n = \sqrt{\frac{k}{m}}$

$\sin\frac{2\pi}{L}Vt = \sin\omega t \longrightarrow \omega = \frac{2\pi}{L}V$

The unfavorable speed occurs when

$\omega = \omega_n \qquad \frac{2\pi V}{L} = \sqrt{\frac{k}{m}} \qquad \therefore V = \frac{L}{2\pi}\sqrt{\frac{K}{m}}$

(Thomson) 5

Prob. 3.21 page 84) The springs of an automobile trailer are compressed 10.16 cm under its weight. Find the critical speed when the trailer is traveling over a road with a profile approximated by a sine wave of amplitude 7.62 cm and wavelength of 14.63 m. What will be the amplitude of vibration at 64.4 km/hr? (Neglect damping)

$\delta_{st} = 10.16 \times 10^{-2}$ m $\quad \omega_n = \sqrt{\frac{g}{\delta_{st}}} = \sqrt{\frac{9.8}{10.16 \times 10^{-2}}}$

$\omega_n = \sqrt{96.4567} = 9.821$ rad/sec

$y = 7.62 \times 10^{-2} \sin \overset{\omega}{\frac{2\pi}{L} v} t \qquad Y = 7.62 \times 10^{-2}$ m

$\therefore \omega = \frac{2\pi}{L} v = \frac{2\pi}{14.63} v$

The critical speed occurs when $\omega = \omega_n$

$\frac{2\pi}{14.63} v = \omega_n = 9.821 \longrightarrow v = \frac{9.821 \times 14.63}{2\pi}$

$v = 22.86813$ m/sec

$v = 22.86813 \times 3.6 = 82.325$ km/hr

$v = \frac{64.4}{3.6} = 17.89$ m/sec

$\therefore \omega = \frac{2\pi v}{14.63} = \frac{2\pi \times 17.89}{14.63} = 7.683$ rad/sec

5

$$\frac{\omega}{\omega_n} = \frac{7.683}{9.821} = 0.7823$$

$$\left|\frac{X}{Y}\right| = \sqrt{\frac{1 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}}$$

$\zeta = $ zero

$$\left|\frac{X}{Y}\right| = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \longrightarrow X = \frac{Y}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

$$X = \frac{7.62 \times 10^{-2}}{1-(0.7823)^2} = 19.639 \times 10^{-2} m$$

$X = 19.639$ cm