An Important Property of Linear Time-Invariant Systems:

For the unit-ramp input the output c(t)is

$$c(t) = t - T + Te^{-t/T}, \qquad \text{for } t \geq 0$$

For the unit-step input, which is the derivative of unit-ramp input, the output c(t)is

$$c(t) = 1 - e^{-t/T}, \qquad \text{for } t \geq 0$$

For the unit-impulse input, which is the derivative of unit-step input, the output c(t)is

$$c(t) = \frac{1}{T} e^{-t/T}, \qquad \text{for } t \geq 0$$

Comparing the system responses to these three inputs clearly indicates that the response to the derivative of an input signal can be obtained by differentiating the response of the system to the original signal.This is a property of linear time-invariant systems only.

Poles of a Transfer Function:

The poles of a transfer function are (1) the values of the Laplace transform variable, **s**, that cause the transfer function to become infinite or (2) any roots of the denominator of the transfer function that are common to roots of the numerator, even though the transfer function will not be infinite at this value.

Zeros of a Transfer Function:

The zeros of a transfer function are (1) the values of the Laplace transform variable, **s**, that cause the transfer function to become zero, or (2) any roots of the numerator of the transfer function that are common to roots of the denominator, even though the transfer function will not be zero at this value.

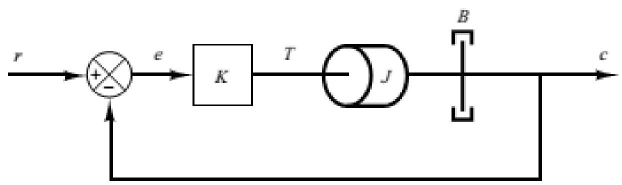
The effect of poles and zeros can be listed as follow:

1. A pole of the input function generates the form of the forced response.
2. A pole of the transfer function generates the form of the natural response.
3. A pole on the real axis generates an exponential response. Thus, the farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero.

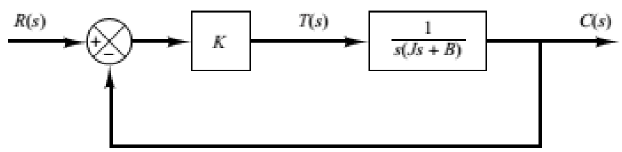
4. The zeros and poles generate the amplitudes for both the forced and natural responses.

Second order systems:

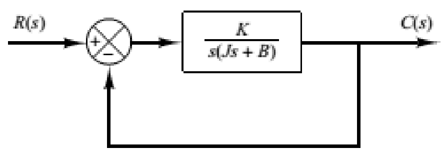
As example for second -order system shown in figure below:



(a)



(b)



(c)

The servo system shown in consists of a proportional controller and load elements (inertia and viscous-friction elements). The output position and the input position are $c(t)$ and $r(t)$. T is the torque produced by the proportional controller whose gain is K. The equation for the load elements is:

$$J\ddot{c} + B\dot{c} = T$$

By taking Laplace transforms

$$Js^2C(s) + BsC(s) = T(s)$$

The feedforward transfer function is

$$F.F.T = \frac{K}{s(Js+B)}$$

The closed-loop transfer function is then obtained as

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K} = \frac{K/J}{s^2 + (B/J)s + (K/J)}$$

The closed-loop transfer function has two poles and no zeros.

$$s = \frac{-B \pm \sqrt{(B^2 - 4JK)}}{2J}$$

1. The closed-loop poles are complex conjugates if $(B^2 - 4JK) < 0$.
2. The closed-loop poles are real if $(B^2 - 4JK) \geq 0$.

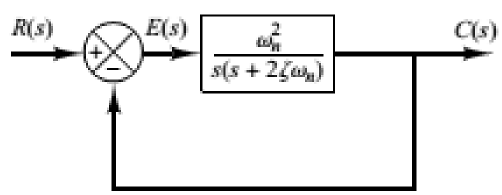
$\frac{K}{J} = \omega_n^2$ $\frac{B}{J} = 2\zeta\omega_n = 2\sigma$

Where σ is called the attenuation, ω_n natural frequency , ζ damping ratio of the system.

The damping ratio is the ratio of the actual damping B to the critical damping $B_c = 2\sqrt{JK}$

$$\zeta = \frac{B}{2\omega_n J} = \frac{B}{2\sqrt{JK}}$$

In terms of ζ and ω_n , the system shown in



And the closed-loop transfer function form(it is called the standard formof the second-order system) as :

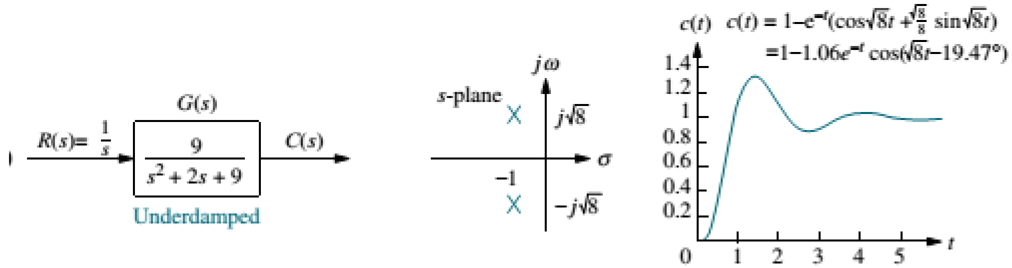
$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The dynamic behavior of the second-order system can then be described in terms of two parameters

$\boldsymbol{\zeta}$ **and** $\boldsymbol{\omega_n}$**.**

1. If $0 < \boldsymbol{\zeta} < 1$ the closed-loop poles are complex conjugates and lie in the left-half splane. The system is then called **underdamped**, and the transient response is oscillatory.

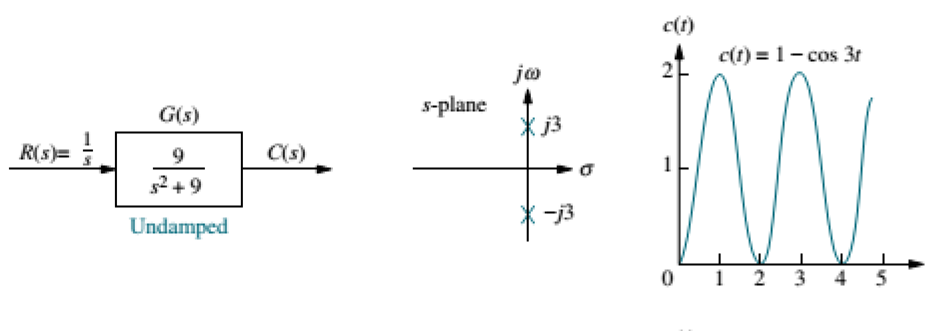
Example for a system with unit-step input:



2. If ζ = 0, the closed-loop poles are lie complex axis of s plane. The system is then called

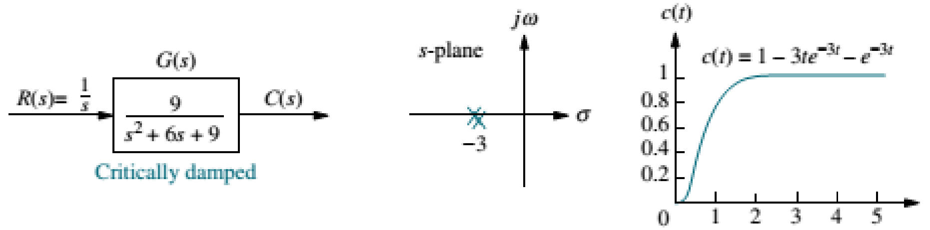
undamped, and the transient response does not die out.

Example for a system with unit-step input:



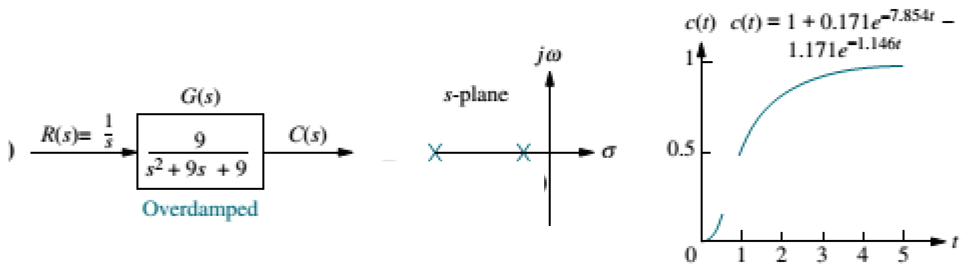
3. If ζ = 1, the closed-loop poles are two equal negative real poles in s plane, the system is called **critically damped**.

Example for a system with unit-step input:

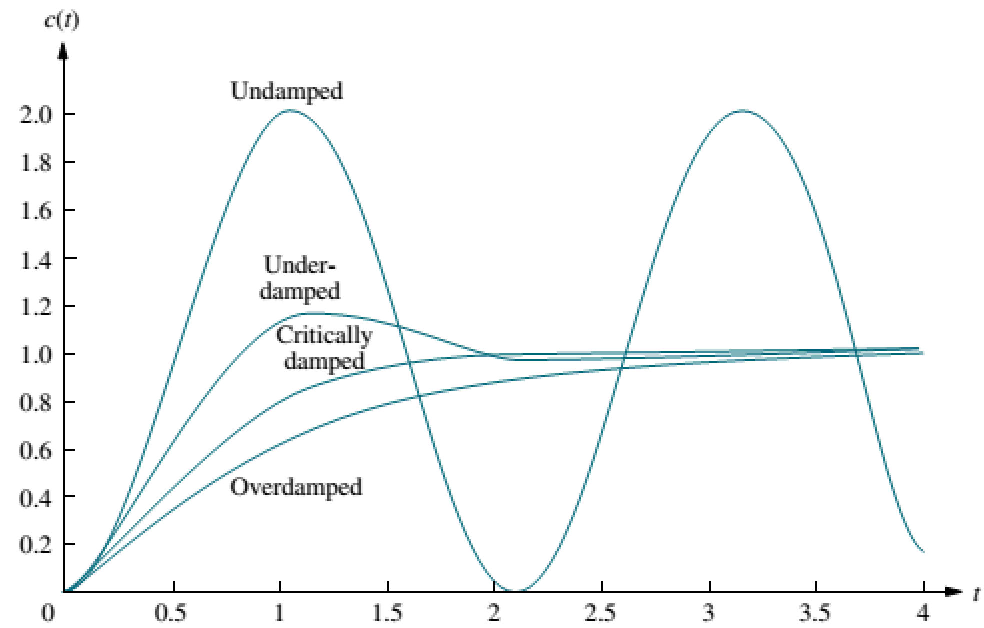


4. If $\zeta > 1$, the closed-loop poles are two unequal negative real poles in s plane ,the system is called

   Overdamped.

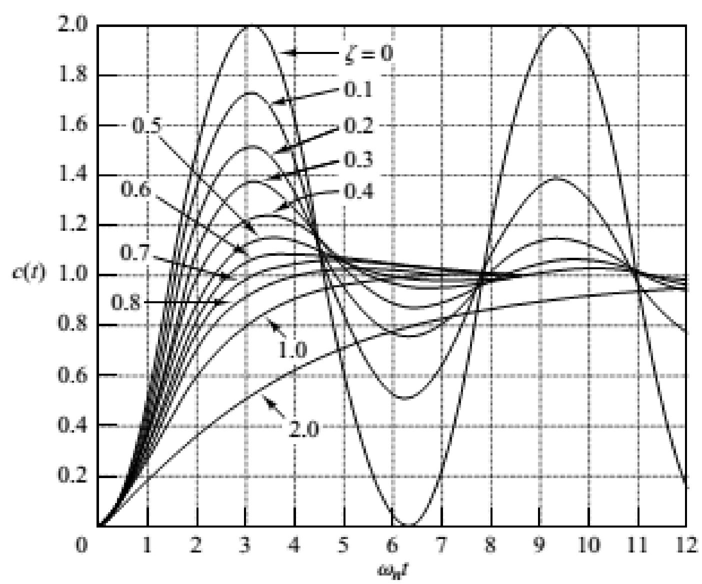


The step responses for the four cases of damping discussed in this section are superimposed in figure below. Notice that the critically damped case is the division between the overdamped cases and the underdamped cases .



Step responses for second-order system damping cases

A family of unit-step response curves **c(t)** with various values of ζ is shown in figure below where the abscissa is the dimensionless variable $\boldsymbol{\omega_n t}$.



The curves are functions only of ζ we see that an underdamped system with ζ between 0.5 and 0.8 gets close to the final value more rapidly than a critically damped or overdamped system. Among the systems responding without oscillation, a critically

damped system exhibits the fastest response.An overdamped system is always sluggish in responding to any inputs.