2 Lagrange's Equations

Example 2.1. The planar mechanical system considered is shown in Fig. 2.1 has a slider 1 of mass m and a mathematical pendulum 2 with the mass M concentrated at the point B. The length of AB is L and the elastic constant of the spring R is k. The spring deflects only horizontally. Find the equations of motion using Lagrange's method.

Solution.

To characterize the instantaneous configuration of the system the generalized coordinate are employed. The generalized coordinates are quantities associated with the position of the system. The first generalized coordinate $q_1(t)$ denotes the linear displacement of the slider. The second generalized coordinate $q_2(t)$ is the radian measure of the angle between the vertical axis and the line AB.

Kinematics

A cartesian reference frame $xOyz$ with the unit vectors $[\boldsymbol{\imath}, \boldsymbol{\jmath}, \mathbf{k}]$ is selected, Fig. 2.1.

The position vector of mass 1 is

$$\mathbf{r}_1 = \mathbf{r}_A = q_1(t)\,\boldsymbol{\imath}. \tag{2.1}$$

The position vector of mass 2 is

$$\mathbf{r}_2 = \mathbf{r}_B = [q_1(t) + L\sin q_2(t)]\,\boldsymbol{\imath} + L\cos q_2(t)\boldsymbol{\jmath}. \tag{2.2}$$

The velocity of the slider 1 is

$$\mathbf{v}_A = \frac{d\,\mathbf{r}_A}{d\,t} = \dot{\mathbf{r}}_A = \dot{q}_1\,\boldsymbol{\imath}, \tag{2.3}$$

and the velocity of the particle at B is

$$\mathbf{v}_B = \frac{d\,\mathbf{r}_B}{d\,t} = \dot{\mathbf{r}}_B = (\dot{q}_1 + L\dot{q}_2\cos q_2)\,\boldsymbol{\imath} - L\dot{q}_2\sin q_2\boldsymbol{\jmath}. \tag{2.4}$$

Kinetic energy

The kinetic energy of the slider 1 is

$$T_1 = \frac{1}{2}m\mathbf{v}_A\cdot\mathbf{v}_A = \frac{1}{2}m\dot{q}_1{}^2, \tag{2.5}$$

and the the kinetic energy of the mass 2 is

$$T_2 = \frac{1}{2} M \mathbf{v}_B \cdot \mathbf{v}_B = \frac{1}{2} M \left(\dot{q_1}^2 + 2L\dot{q_1}\dot{q_2}\cos q_2 + L^2\dot{q_2}^2 \right). \tag{2.6}$$

The total kinetic energy is

$$T = T_1 + T_2. \tag{2.7}$$

Generalized forces

The forces that act on 1 at A are the spring force and the gravity force

$$\mathbf{F}_A = -kq_1\,\imath + mg\,\jmath, \tag{2.8}$$

where g=9.81 m/s^2 is the gravity acceleration. The gravity force acts on mass 2 at B

$$\mathbf{F}_B = Mg\,\jmath. \tag{2.9}$$

There are two generalized forces. The generalized force associated to q_1 is

$$\begin{aligned} &Q_1 = \mathbf{F}_A \cdot \frac{\partial \mathbf{r}_A}{\partial q_1} + \mathbf{F}_B \cdot \frac{\partial \mathbf{r}_B}{\partial q_1} = \\ &(-kq_1\,\imath + mg\,\jmath) \cdot \imath + Mg\,\jmath \cdot \imath = -kq_1. \end{aligned} \tag{2.10}$$

The generalized force associated to q_2 is

$$\begin{aligned} &Q_2 = \mathbf{F}_A \cdot \frac{\partial \mathbf{r}_A}{\partial q_2} + \mathbf{F}_B \cdot \frac{\partial \mathbf{r}_B}{\partial q_2} = \\ &(-kq_1\,\imath + mg\,\jmath) \cdot \mathbf{0} + Mg\,\jmath \cdot (L\cos q_2\,\imath - L\sin q_2\,\jmath) = \\ &-MgL\sin q_2. \end{aligned} \tag{2.11}$$

Lagrange's equations

The two Lagrange's equations are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_1}}\right) - \frac{\partial T}{\partial q_1} &= Q_1, \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_2}}\right) - \frac{\partial T}{\partial q_2} &= Q_2. \end{aligned} \tag{2.12}$$

One can calculate for q_1

$$\begin{aligned}
&\frac{\partial T}{\partial \dot{q}_1} = (m+M)\dot{q}_1 + LM\dot{q}_2 \cos q_2,\\
&\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = (m+M)\ddot{q}_1 + LM\ddot{q}_2 \cos q_2 - LM{\dot{q}_2}^2 \sin q_2,\\
&\frac{\partial T}{\partial q_1} = 0.
\end{aligned} \tag{2.13}$$

For the generalized coordinate q_2 the left hand side of Lagrange's equation is

$$\begin{aligned}
&\frac{\partial T}{\partial \dot{q}_2} = LM\left(\dot{q}_1 \cos q_2 + L\dot{q}_2\right),\\
&\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = LM\left(\ddot{q}_1 \cos q_2 - \dot{q}_1\dot{q}_2 \sin q_2 + L\ddot{q}_2\right),\\
&\frac{\partial T}{\partial q_2} = -LM\dot{q}_1\dot{q}_2 \sin q_2.
\end{aligned} \tag{2.14}$$

The equations of motion are

$$\begin{aligned}
&(m+M)\ddot{q}_1 + LM\ddot{q}_2 \cos q_2 - LM{\dot{q}_2}^2 \sin q_2 = -kq_1,\\
&LM\left(\ddot{q}_1 \cos q_2 - \dot{q}_1\dot{q}_2 \sin q_2 + L\ddot{q}_2\right) + LM\dot{q}_1\dot{q}_2 \sin q_2 = -MgL \sin q_2,
\end{aligned} \tag{2.15}$$

or

$$\begin{aligned}
&(m+M)\ddot{q}_1 + LM\ddot{q}_2 \cos q_2 - LM{\dot{q}_2}^2 \sin q_2 + kq_1 = 0,\\
&LM\ddot{q}_1 \cos q_2 + ML^2\ddot{q}_2 + MgL \sin q_2 = 0.
\end{aligned} \tag{2.16}$$

For small oscillations of the pendulum $\sin q_2 \approx q_2$ and $\cos q_2 \approx 1$, the equations of motion are

$$\begin{aligned}
&(m+M)\ddot{q}_1 + LM\ddot{q}_2 - LM{\dot{q}_2}^2 q_2 + kq_1 = 0,\\
&LM\ddot{q}_1 + ML^2\ddot{q}_2 + MgLq_2 = 0.
\end{aligned} \tag{2.17}$$

The *Mathematica*^{TM} program with the equations of motion are given in Program 2.1. The equations of motion are numerically solved for $m = M = 1$ kg, $L = 1$ m, $k = 1$ N/m, and $g = 10$ m/s^2, and the following initial conditions

$$q_1(0) = q_2(0) = 0.1,\ \dot{q}_1(0) = \dot{q}_2(0) = 0.$$

Example 2.2. A double compound pendulum is considered in Fig. 2.2. The bars 1 and 2 are homogenuos and have the lengths $OA = AB = L$ and the masses $m_1 = m_2 = m$. At O and A there are pin joints. The mass centers of links 1 and 2 are at C_1 and at C_2. Find the Lagrange's equations of motion.

Solution.

To characterize the instantaneous configuration of the system, two generalized coordinates $q_1(t)$ and $q_2(t)$ are employed. The generalized coordinates q_1 and q_2 denote the radian measure of the angles between the link 1 and 2 and the vertical y axis.

Kinematics

The position vector of the mass center of link 1 is

$$\mathbf{r}_{C_1} = 0.5\,L\,\sin q_1\,\mathbf{ı} + 0.5\,L\,\cos q_1\,\mathbf{ȷ}, \tag{2.18}$$

and the position vector of the mass center of link 2 is

$$\mathbf{r}_{C_2} = (L\,\sin q_1 + 0.5\,L\,\sin q_2)\,\mathbf{ı} + (L\,\cos q_1 + 0.5\,L\,\cos q_2)\,\mathbf{ı}. \tag{2.19}$$

The velocity of C_1 is

$$\mathbf{v}_{C_1} = \frac{d\,\mathbf{r}_{C_1}}{d\,t} = \dot{\mathbf{r}}_{C_1} = 0.5\,L\,\dot{q}_1\,\cos q_1\,\mathbf{ı} - 0.5\,L\,\dot{q}_1\,\sin q_1\,\mathbf{ȷ}, \tag{2.20}$$

and the velocity of C_2 is

$$\mathbf{v}_{C_2} = \frac{d\,\mathbf{r}_{C_2}}{d\,t} = \dot{\mathbf{r}}_{C_2} =$$
$$(L\,\dot{q}_1\,\cos q_1 + 0.5\,L\,\dot{q}_2\,\cos q_2)\,\mathbf{ı} - (L\,\dot{q}_1\,\sin q_1 + 0.5\,L\,\dot{q}_2\,\sin q_2)\,\mathbf{ȷ}. \tag{2.21}$$

Kinetic energy

The kinetic energy of the link 1 which is in rotational motion is

$$T_1 = \frac{1}{2}\,I_0\,\dot{q}_1^2 = \frac{1}{2}\,\frac{mL^2}{3}\,\dot{q}_1^2 = \frac{ML^2}{6}\,\dot{q}_1^2, \tag{2.22}$$

where I_0 is the mass moment of inertia about the center of rotation O, $I_O = mL^2/3$.

The kinetic energy of the bar 2 is due to the translation and rotation and can be expressed as

$$T_2 = \frac{1}{2} I_{C_2} \dot{q}_2^2 + \frac{1}{2} m_2 \mathbf{v}_{C_2}^2, \tag{2.23}$$

where I_{C_2} is the mass moment of inertia about the center of mass C_2, $I_{C_2} = mL^2/12$, and

$$\mathbf{v}_{C_2}^2 = \mathbf{v}_{C_2} \cdot \mathbf{v}_{C_2} = L^2\,\dot{q}_1^2 + \frac{1}{4}\,L^2\,\dot{q}_2^2 + L^2\,\dot{q}_1\,\dot{q}_2\cos(q_2 - q_1). \tag{2.24}$$

Equation (2.23) becomes

$$T_2 = \frac{1}{2}\,\frac{mL^2}{12}\,\dot{q}_2^2 + \frac{1}{2}mL^2\left[\dot{q}_1^2 + \frac{1}{4}\,\dot{q}_2^2 + \dot{q}_1\,\dot{q}_2\cos(q_2 - q_1)\right]. \tag{2.25}$$

The total kinetic energy of the system is

$$T = T_1 + T_2 = \frac{mL^2}{6}\,\left[4\dot{q}_1^2 + 3\,\dot{q}_1\,\dot{q}_2\,\cos(q_2 - q_1) + \dot{q}_2^2\right]. \tag{2.26}$$

The left hand sides of Lagrange's equations are

$$\frac{\partial T}{\partial \dot{q}_1} = \frac{mL^2}{6}\,\left[8\dot{q}_1 + 3\dot{q}_2\,\cos(q_2 - q_1)\right],$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = \frac{mL^2}{6}\,\left[8\ddot{q}_1 + 3\ddot{q}_2\,\cos(q_2 - q_1) - 3\dot{q}_2\,(\dot{q}_2 - \dot{q}_1)\,\sin(q_2 - q_1)\right],$$
$$\frac{\partial T}{\partial q_1} = \frac{mL^2}{6}\,3\,\dot{q}_1\,\dot{q}_2\,\sin(q_2 - q_1) = \frac{mL^2}{2}\,\dot{q}_1\,\dot{q}_2\,\sin(q_2 - q_1);$$

$$\frac{\partial T}{\partial \dot{q}_2} = \frac{mL^2}{6}\,\left[3\dot{q}_1\,\cos(q_2 - q_1) + 2\dot{q}_2\right],$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = \frac{mL^2}{6}\,\left[3\ddot{q}_1\,\cos(q_2 - q_1) - 3\dot{q}_1\,(\dot{q}_2 - \dot{q}_1)\,\sin(q_2 - q_1) + 2\ddot{q}_2\right],$$
$$\frac{\partial T}{\partial q_2} = -\frac{mL^2}{6}\,3\,\dot{q}_1\,\dot{q}_2\,\sin(q_2 - q_1) = -\frac{mL^2}{2}\,\dot{q}_1\,\dot{q}_2\,\sin(q_2 - q_1). \tag{2.27}$$

Generalized forces

The gravity forces on links 1 and 2 at the mass centers C_1 and C_2

$$\mathbf{F}_{C_1} = \mathbf{G}_1 = m_1 g\,\boldsymbol{\jmath} = mg\,\boldsymbol{\jmath} \quad \text{and} \quad \mathbf{F}_{C_2} = \mathbf{G}_2 = mg\,\boldsymbol{\jmath}. \tag{2.28}$$

There are two generalized forces. The generalized force associated to q_1 is

$$Q_1 = \mathbf{F}_{C_1} \cdot \frac{\partial \mathbf{r}_{C_1}}{\partial q_1} + \mathbf{F}_{C_2} \cdot \frac{\partial \mathbf{r}_{C_2}}{\partial q_1} =$$
$$mg\,\boldsymbol{\jmath} \cdot (0.5\,L\,\cos q_1\,\boldsymbol{\imath} - 0.5\,L\,\sin q_1\,\boldsymbol{\jmath}) + mg\,\boldsymbol{\jmath} \cdot (L\,\cos q_1\,\boldsymbol{\imath} - L\,\sin q_1\,\boldsymbol{\jmath})$$
$$= -1.5mgL\,\sin q_1. \tag{2.29}$$

The generalized force associated to q_2 is

$$\begin{aligned} Q_2 = \mathbf{F}_{C_1} \cdot \frac{\partial \mathbf{r}_{C_1}}{\partial q_2} + \mathbf{F}_{C_2} \cdot \frac{\partial \mathbf{r}_{C_2}}{\partial q_2} = \\ mg\,\mathbf{j} \cdot \mathbf{0} + mg\,\mathbf{j} \cdot (0.5\,L\,\cos q_2\,\mathbf{\imath} - 0.5\,L\,\sin q_2\,\mathbf{j}) \\ = -0.5mgL\,\sin q_2. \end{aligned} \tag{2.30}$$

The two Lagrange's equations are

$$\begin{aligned} &\frac{d}{d\,t}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1, \\ &1.333mL^2\ddot{q}_1 + 0.5mL^2\ddot{q}_2\cos(q_2 - q_1) - 0.5mL^2\dot{q}_2^2\sin(q_2 - q_1) \\ &+1.5mgL\sin q_1 = 0; \\ \\ &\frac{d}{d\,t}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} = Q_2, \\ &0.5mL^2\ddot{q}_1\cos(q_2 - q_1) + 0.333mL^2\ddot{q}_2 + 0.5mL^2\dot{q}_1^2\sin(q_2 - q_1) \\ &+0.5mgL\sin q_2 = 0. \end{aligned} \tag{2.31}$$

The *Mathematica*^{TM} program with the equations of motion are given in Program 2.2. The equations of motion are numerically solved for $m_1 = m_2 = m = 1$ kg, $L_1 = L_2 = 1$ m, and $g = 10$ m/s^2, and the following initial conditions

$$q_1(0) = q_2(0) = 0.1,\ \dot{q}_1(0) = \dot{q}_2(0) = 0.$$

Example 2.3. The mechanism shown in Fig. 2.3(a) is considered. The lengths of the links are $L_1 = 0.001$ m, $L_2 = 0.470$ m, and $L_3 = 0.047$ m. The links 1 and 2 are rectangular prisms with the depth $d = 0.001$ m and height $h = 0.01$ m. Link 3 has the height $h_3 = 0.02$ m, and the depth $d_3 = 0.05$ m. The mass density of the links is $\rho = 7850$ Kg/m^3. The angle of the driver link with the horizontal axis is $\theta(t) = \angle BAC$ and the angular velocity of the driver link is $\omega(t) = \dot{\theta}$. A motor moment acts on link 1 and is given by $\mathbf{M}_m = M\mathbf{k}$ [Fig. 2.39(b)]. For a D.C. electric motor, $M = M_0\left(1 - \dfrac{\omega}{\omega_0}\right)$, where M_0 and ω_0 are given in catalogues. For the considered mechanism $M_0 = 1$ N m and $\omega_0 = 4$ rad/s. The initial conditions $\theta(0) = \pi/6$ rad and $\omega(0) = \dot{\theta}(0) = 0$ rad/s are given. Find the Lagrange's equation of motion of the mechanism.

Solution.

A fixed reference frame xyz is chosen with the origin at A. The center of mass locations of the links $i = 1,\ 2,\ 3$ are designated by $C_i(x_{Ci}, y_{Ci}, 0)$. The mechanism has one degree of freedom and the angle $\theta(t)$ is selected as the generalized coordinate.

Kinematics

The position vector of the center of the mass C_1 of the link 1 is

$$\mathbf{r}_{C_1} = x_{C_1}\mathbf{\imath} + y_{C_1}\mathbf{\jmath}, \tag{2.32}$$

where x_{C_1} and y_{C_1} are the coordinates of C_1

$$x_{C_1} = \frac{L_1}{2}\cos\theta,\ \ y_{C_1} = \frac{L_1}{2}\sin\theta. \tag{2.33}$$

The position vector of the center of the mass C_2 of the link 2 is

$$\mathbf{r}_{C_2} = x_{C_2}\mathbf{\imath} + y_{C_2}\mathbf{\jmath}, \tag{2.34}$$

where x_{C_2} and y_{C_2} are the coordinates of C_2

$$x_{C_2} = L_1\cos\theta + \frac{L_2}{2}\cos\theta_2,\ \ y_{C_2} = L_1\sin\theta + \frac{L_2}{2}\sin\theta_2, \tag{2.35}$$

where $\theta_2 = \arctan\dfrac{L_1\sin\theta}{L_1\cos\theta - AC}$.

The position vector of the center of the mass C_3 of the link 3 is

$$\mathbf{r}_{C_3} = AC\boldsymbol{\imath}. \tag{2.36}$$

The velocity vector of C_1 is the derivative with respect to time of the position vector of C_1

$$\mathbf{v}_{C_1} = \dot{\mathbf{r}}_{C_1} = \dot{x}_{C_1}\boldsymbol{\imath} + \dot{y}_{C_1}\boldsymbol{\jmath}, \tag{2.37}$$

where

$$\dot{x}_{C_1} = -\frac{L_1}{2}\dot{\theta}\sin\theta,\ \dot{y}_{C_1} = \frac{L_1}{2}\dot{\theta}\cos\theta. \tag{2.38}$$

The velocity vector of C_2 is the derivative with respect to time of the position vector of C_2

$$\mathbf{v}_{C_2} = \dot{\mathbf{r}}_{C_2} = \dot{x}_{C_2}\boldsymbol{\imath} + \dot{y}_{C_2}\boldsymbol{\jmath}, \tag{2.39}$$

where

$$\begin{aligned} \dot{x}_{C_2} &= -L_1\dot{\theta}\sin\theta - \frac{L_2}{2}\dot{\theta}_2\sin\theta_2, \\ \dot{y}_{C_2} &= L_1\dot{\theta}\cos\theta + \frac{L_2}{2}\dot{\theta}_2\cos\theta_2. \end{aligned} \tag{2.40}$$

The velocity vector of C_3 is zero

$$\mathbf{v}_{C_3} = \mathbf{0}. \tag{2.41}$$

The acceleration vector of C_1 is the double derivative with respect to time of the position vector of C_1

$$\mathbf{a}_{C_1} = \ddot{\mathbf{r}}_{C_1} = \ddot{x}_{C_1}\boldsymbol{\imath} + \ddot{y}_{C_1}\boldsymbol{\jmath}, \tag{2.42}$$

where

$$\begin{aligned} \ddot{x}_{C_1} &= -\frac{L_1}{2}\ddot{\theta}\sin\theta - \frac{L_1}{2}\dot{\theta}^2\cos\theta, \\ \ddot{y}_{C_1} &= \frac{L_1}{2}\ddot{\theta}\cos\theta - \frac{L_1}{2}\dot{\theta}^2\sin\theta. \end{aligned} \tag{2.43}$$

The acceleration vector of C_2 is the double derivative with respect to time of the position vector of C_2

$$\mathbf{a}_{C_2} = \ddot{\mathbf{r}}_{C_2} = \ddot{x}_{C_2}\boldsymbol{\imath} + \ddot{y}_{C_2}\boldsymbol{\jmath}, \tag{2.44}$$

where

$$\ddot{x}_{C_2} = -L_1\ddot{\theta}\sin\theta - L_1\dot{\theta}^2\cos\theta - \frac{L_2}{2}\ddot{\theta}_2\sin\theta_2 - \frac{L_2}{2}\dot{\theta}_2^2\cos\theta_2,$$
$$\ddot{y}_{C_2} = L_1\ddot{\theta}\cos\theta - L_1\dot{\theta}^2\sin\theta + \frac{L_2}{2}\ddot{\theta}_2\cos\theta_2 - \frac{L_2}{2}\dot{\theta}_2^2\sin\theta_2. \tag{2.45}$$

The acceleration vector of C_3 is zero

$$\mathbf{a}_{C_3} = \mathbf{0}. \tag{2.46}$$

The angular velocity vectors of the links 1, 2, and 3 are

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{k},$$
$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_3 = \dot{\theta}_2\mathbf{k}. \tag{2.47}$$

The angular acceleration vectors of the links 1, 2, and 3 are

$$\boldsymbol{\alpha} = \ddot{\theta}\mathbf{k},$$
$$\boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_3 = \ddot{\theta}_2\mathbf{k}. \tag{2.48}$$

Kinetic energy

The masses of the links 1, 2 and 3 are

$$m_1 = \rho L_1 h d,\ m_2 = \rho L_2 h d,\ m_3 = m_{3a} - m_{3b}, \tag{2.49}$$

where $m_{3a} = \rho L_3 h_3 d_3$ and $m_{3b} = \rho L_3 h d$.

The mass moment of inertia of the link 1 with respect to the center of mass C_1 is

$$I_{C_1} = \frac{m_1}{12}\left(L_1^2 + h^2\right).$$

The mass moment of inertia of the link 2 with respect to the center of mass C_2 is

$$I_{C_2} = \frac{m_2}{12}\left(L_2^2 + h^2\right).$$

The mass moment of inertia of the link 3 with respect to the center of mass C_3 is

$$I_{C_3} = \frac{m_{3a}}{12}\left(L_3^2 + h_3^2\right) - \frac{m_{3b}}{12}\left(L_3^2 + h^2\right).$$

The kinetic energy T_1 for the link 1 is

$$T_1 = \frac{1}{2}m_1\mathbf{v}_{C_1}\cdot\mathbf{v}_{C_1} + \frac{1}{2}I_{C_1}\boldsymbol{\omega}\cdot\boldsymbol{\omega}. \tag{2.50}$$

The kinetic energy T_2 for the link 2 is

$$T_2 = \frac{1}{2} m_2 \mathbf{v}_{C_2} \cdot \mathbf{v}_{C_2} + \frac{1}{2} I_{C_2} \boldsymbol{\omega}_2 \cdot \boldsymbol{\omega}_2. \tag{2.51}$$

The kinetic energy T_3 for the link 3 is

$$T_3 = \frac{1}{2} I_{C_3} \boldsymbol{\omega}_3 \cdot \boldsymbol{\omega}_3. \tag{2.52}$$

The total kinetic energy is

$$T = \sum_{i=1}^{3} T_i = T_+ T_2 + T_3. \tag{2.53}$$

Generalized force

The gravitational forces on links 1, 2, and 3 are

$$\mathbf{G}_1 = -m_1 g \mathbf{J},\ \ \mathbf{G}_2 = -m_2 g \mathbf{J},\ \ \mathbf{G}_3 = -m_3 g \mathbf{J}. \tag{2.54}$$

The generalized force Q_i associated with the gravitational force $\mathbf{G}_i$ is

$$Q_i = \frac{\partial \mathbf{r}_{C_i}}{\partial \theta} \cdot \mathbf{G}_i. \tag{2.55}$$

The generalized force Q_m associated to the motor moment is

$$Q_m = \frac{\partial \boldsymbol{\omega}}{\partial \dot{\theta}} \cdot \mathbf{M}_m = M_0 \left(1 - \frac{\dot{\theta}}{\omega_0}\right). \tag{2.56}$$

The total generalized force Q for the mechanism is

$$Q = \sum_{i=1}^{3} Q_i + Q_m = \sum_{i=1}^{3} \frac{\partial \mathbf{r}_{C_i}}{\partial \theta} \cdot \mathbf{G}_i + \frac{\partial \boldsymbol{\omega}}{\partial \dot{\theta}} \cdot \mathbf{M}_m. \tag{2.57}$$

Lagrange's equation

The Lagrange's differential equation for the mechanism with one degree of freedom is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q, \tag{2.58}$$

where T is the total kinetic energy of the system, and Q is the generalized force.

For the link 1 some calculations are given

$$
\begin{aligned}
&T_1 = \frac{1}{2}(I_{C1} + \frac{1}{4}L_1^2 m_1)\dot{\theta}^2, \\
&\frac{\partial T_1}{\partial \dot{\theta}} = (I_{C1} + \frac{1}{4}L_1^2 m_1)\dot{\theta}, \\
&\frac{\partial \mathbf{r}_{C1}}{\partial \dot{\theta}} = -\frac{1}{2}L_1(\sin\theta\mathbf{\imath} + \cos\theta\mathbf{\jmath}), \\
&Q_1 = \frac{\partial \mathbf{r}_{C1}}{\partial \dot{\theta}} \cdot (-m_1 g\mathbf{\jmath}) = -\frac{1}{2}m_1 g L_1 \cos\theta.
\end{aligned}
$$

The *Mathematica*^{TM} program with the equations of motion are given in Program 2.3.
Remark: Lagrange's method does not require the calculation of the joint forces.

Example 2.4. Figure 2.4(a) is a schematic representation of an open kinematic chain (robot arm) consisting of three links 1, 2, 3, and a rigid body RB. Link 1 can be rotated at A in a "fixed" cartesian reference frame (0) of unit vectors $[\mathbf{\imath}_0, \mathbf{\jmath}_0, \mathbf{k}_0]$ about a vertical axis $\mathbf{\imath}_0$. The unit vector $\mathbf{\imath}_0$ is fixed in link 1. Link 1 is connected to link 2 through pin joints B and B'. The link 2 rotates relative to 1 about an axis fixed in both 1 and 2, passing through B, and B'. The link 3 is connected to 2 by means of a slider joint 2'. The slider joint is rigidly attached to link 2. The last link 3 holds rigidly the rigid body RB. The mass centers of links 1, 2, 2' and 3 are C_1, $C_2 = C_{2'}$, and C_3, respectively. The mass center of RB is at C_R. The length of link 2 is l and the length of link 3 is L. The mass of the link 1 is m_1, the masses of the bars 2 and 3 are m_2 and m_3, the mass of the slider 2' is $m_{2'}$, and the mass of RB is m_R. Find the equations of motion for the robotic system.

Solution

A reference frame (1) of unit vectors $[\mathbf{\imath}_1, \mathbf{\jmath}_1, \mathbf{k}_1]$ is attached to body 1, with $\mathbf{\imath}_1 = \mathbf{\imath}_0$.

A reference frame (2) of unit vectors $[\mathbf{\imath}_2, \mathbf{\jmath}_2, \mathbf{k}_2]$ is attached to link 2, as it is shown in Fig. 2.4. The unit vector $\mathbf{\jmath}_2$ is parallel to the axis of link 2, BB', and $\mathbf{\jmath}_2 = \mathbf{\jmath}_1$. The unit vector $\mathbf{k}_2$ is parallel to the axis of link 3, C_2C_R.

To characterize the instantaneous position of the arm, the generalized coordinates $q_1(t)$, $q_2(t)$, $q_3(t)$ are employed. The first generalized coordinate q_1 denotes the radian measure of the angle between the axes of (1) and (0) [Fig. 2.4(b)]. The second generalized coordinate q_2 designates the radian measure of rotation of the angle between (1) and (2) [Fig. 2.4(c)]. The last generalized coordinate q_3 is the distance from C_2 to C_3.

Angular velocities

Next the angular velocities of the links and the rigid body will be expressed in the fixed reference frame (0). The angular velocity of link 1 in (0) is

$$\boldsymbol{\omega}_{10} = \dot{q}_1\, \mathbf{\imath}_1 = \dot{q}_1\, \mathbf{\imath}_0. \tag{2.59}$$

The angular velocity of link 2 with respect to (1) is

$$\boldsymbol{\omega}_{21} = \dot{q}_2\, \mathbf{\jmath}_2 = \dot{q}_2\, \mathbf{\jmath}_1, \tag{2.60}$$

and the angular velocity of link 2 with respect to the fixed reference frame (0) is

$$\boldsymbol{\omega}_{20} = \boldsymbol{\omega}_{10} + \boldsymbol{\omega}_{21} = \dot{q}_1\, \mathbf{\imath}_1 + \dot{q}_2\, \mathbf{\jmath}_2. \tag{2.61}$$

The unit vectors $\mathbf{\imath}_1$, $\mathbf{\jmath}_1$, and $\mathbf{k}_1$ can be expressed as functions of $\mathbf{\imath}_0$, $\mathbf{\jmath}_0$, and $\mathbf{k}_0$ [Fig. 2.4(b)]

$$\begin{aligned}
\mathbf{\imath}_1 &= \mathbf{\imath}_0, \\
\mathbf{\jmath}_1 &= c_1\,\mathbf{\jmath}_0 + s_1\,\mathbf{k}_0, \\
\mathbf{k}_1 &= -s_1\,\mathbf{\jmath}_0 + c_1\,\mathbf{k}_0,
\end{aligned} \tag{2.62}$$

where $s_1 = \sin q_1$ and $c_1 = \cos q_1$.

The unit vectors $\mathbf{\imath}_2$, $\mathbf{\jmath}_2$ and $\mathbf{k}_2$ can be expressed as [Fig. 2.4(c)]

$$\begin{aligned}
\mathbf{\imath}_2 &= c_2\mathbf{\imath}_1 - s_2\mathbf{k}_1 \\
&= c_2\mathbf{\imath}_0 + s_1 s_2\,\mathbf{\jmath}_0 - c_1 s_2\,\mathbf{k}_0, \\
\mathbf{\jmath}_2 &= \mathbf{\jmath}_1, \\
&= c_1\,\mathbf{\jmath}_0 + s_1\,\mathbf{k}_0, \\
\mathbf{k}_2 &= s_2\mathbf{\imath}_1 + c_2\mathbf{k}_1 \\
&= s_2\mathbf{\imath}_0 - c_2 s_1\,\mathbf{\jmath}_0 + c_1 c_2\,\mathbf{k}_0,
\end{aligned} \tag{2.63}$$

where $s_2 = \sin q_2$ and $c_2 = \cos q_2$.

The angular velocity of link 2 in (0) can be written in terms of the unit vectors of the reference frame (2) as

$$\boldsymbol{\omega}_{20} = \dot{q}_1 c_2 \mathbf{\imath}_2 + \dot{q}_2\,\mathbf{\jmath}_2 + \dot{q}_1 s_2\,\mathbf{k}_2, \tag{2.64}$$

and in terms of the unit vectors of the reference frame (0) as

$$\boldsymbol{\omega}_{20} = \dot{q}_1 \mathbf{\imath}_0 + \dot{q}_2 c_1\,\mathbf{\jmath}_0 + \dot{q}_2 s_1\,\mathbf{k}_0. \tag{2.65}$$

The link 3 and the rigid body RB have the same rotation motion as link 2, i.e.,

$$\boldsymbol{\omega}_{30} = \boldsymbol{\omega}_{R0} = \boldsymbol{\omega}_{20},$$

where $\boldsymbol{\omega}_{30}$ is the angular velocity of link 3 in (0) and $\boldsymbol{\omega}_{R0}$ is the angular velocity of RB in (0).

Linear velocities

The position vector of C_1, the mass center of link 1 is

$$\mathbf{r}_{C_1} = L_1\mathbf{\imath}_1 = L_1\mathbf{\imath}_0, \tag{2.66}$$

and the velocity of C_1 in (0) is

$$\mathbf{v}_{C_1} = \frac{d}{dt}\mathbf{r}_{C_1} = \dot{\mathbf{r}}_{C_1} = \mathbf{0}. \tag{2.67}$$

The position vector of C_2, the mass center of link 2, is

$$\mathbf{r}_{C_2} = L_2\mathbf{\imath}_1 = L_2\mathbf{\imath}_0,$$

or written in terms of the unit vectors of the reference frame (2)

$$\mathbf{r}_{C_2} = L_2 c_2 \mathbf{\imath}_2 + L_2 s_2 \mathbf{k}_2.$$

The velocity of C_2 in (0) is

$$\mathbf{v}_{C_2} = \frac{d}{dt}\mathbf{r}_{C_2} = \frac{d}{dt}\left(L_2\mathbf{\imath}_0\right) = \mathbf{0}.$$

The position vector of C_3 with respect to reference frame (0) is

$$\begin{aligned} \mathbf{r}_{C_3} &= \mathbf{r}_{C_2} + q_3\mathbf{k}_2 \\ &= L_2\mathbf{\imath}_0 + q_3\mathbf{k}_2, \end{aligned} \tag{2.68}$$

or expressing $\mathbf{k}_2$ in terms of reference (0) unit vectors yields

$$\mathbf{r}_{C_3} = (L_2 + q_3 s_2)\mathbf{\imath}_0 - q_3 c_2 s_1\, \mathbf{\jmath}_0 + q_3 c_2 c_1\, \mathbf{k}_0.$$

The position vector of C_3 with respect to reference frame (0) written in terms of the unit vectors of the reference frame (2) is

$$\mathbf{r}_{C_3} = L_2 c_2 \mathbf{\imath}_2 + (q_3 + L_2 s_2)\mathbf{k}_2.$$

The velocity of the mass center C_3 in (0), written in terms of the unit vectors of the reference frame (0), can be calculated taking the derivative with respect to time of Eq. (2.69)

$$\begin{aligned} \mathbf{v}_{C_3} = \frac{d}{dt}\mathbf{r}_{C_3} &= (c_2 q_3 \dot{q}_2 + s_2 \dot{q}_3)\mathbf{\imath}_0 + \\ &\quad (s_1 s_2 \dot{q}_2 q_3 - c_1 c_2 q_3 \dot{q}_1 - s_1 c_2 \dot{q}_3)\, \mathbf{\jmath}_0 + \\ &\quad (c_1 c_2 \dot{q}_3 - s_1 c_2 q_3 \dot{q}_1 - c_1 s_2 q_3 \dot{q}_2)\, \mathbf{k}_0. \end{aligned} \tag{2.69}$$

The velocity of C_3 in (0) can be computed using the derivation formula for the moving vector $\mathbf{r}_{C_3}$

$$\mathbf{v}_{C_3} = \frac{d}{dt}\mathbf{r}_{C_3} = \frac{{}^{(2)}d}{dt}\mathbf{r}_{C_3} + \boldsymbol{\omega}_{20} \times \mathbf{r}_{C_3}, \tag{2.70}$$

where $\frac{{}^{(2)}d}{dt}$ represents the partial derivative with respect to time in reference frame (2), $[\imath_2, \jmath_2, \mathbf{k}_2]$,

$$\frac{{}^{(2)}d}{dt}\mathbf{r}_{C_3} = \frac{{}^{(2)}d}{dt}\left[L_2 c_2 \imath_2 + (q_3 + L_2 s_2)\mathbf{k}_2\right] = -\dot{q}_2 L_2 s_2 \imath_2 + (\dot{q}_3 + \dot{q}_2 L_2 c_2)\mathbf{k}_2 \tag{2.71}$$

Using Eqs. (2.65)(2.69)(2.70)(2.71) the velocity of C_3 in (0), written in terms of the unit vectors of the reference frame (2) is

$$\begin{aligned} \mathbf{v}_{C_3} &= -\dot{q}_2 L_2 s_2 \imath_2 + (\dot{q}_3 + \dot{q}_2 L_2 c_2)\mathbf{k}_2 + \begin{vmatrix} \imath_2 & \jmath_2 & \mathbf{k}_2 \\ \dot{q}_1 c_2 & \dot{q}_2 & \dot{q}_1 s_2 \\ L_2 c_2 & 0 & q_3 + L_2 s_2 \end{vmatrix} \\ &= \dot{q}_2 q_3 \imath_2 - \dot{q}_1 q_3 c_2 \jmath_2 + \dot{q}_3 \mathbf{k}_2. \end{aligned} \tag{2.72}$$

The position vector of the mass center C_R of the rigid body RB is

$$\begin{aligned} \mathbf{r}_{C_R} &= \mathbf{r}_{C_3} + \mathbf{r}_{C_3 C_R} \\ &= \mathbf{r}_{C_3} + \frac{L}{2}\mathbf{k}_2, \end{aligned} \tag{2.73}$$

or expressed in terms of the reference frame (0) is

$$\mathbf{r}_{C_R} = \left[L_2 + \left(q_3 + \frac{L}{2}\right) s_2\right] \imath_0 - \left(q_3 + \frac{L}{2}\right) c_2 s_1 \, \jmath_0 + \left(q_3 + \frac{L}{2}\right) c_1 c_2 \, \mathbf{k}_0.$$

The velocity of C_R in (0) is

$$\begin{aligned} \mathbf{v}_{C_R} = \frac{d}{dt}\mathbf{r}_{C_R} &= \left[\left(q_3 + \frac{L}{2}\right) c_2 \dot{q}_2 + s_2 \dot{q}_3\right] \imath_0 + \\ &\left[s_1 s_2 \dot{q}_2 \left(q_3 + \frac{L}{2}\right) - s_1 c_2 \dot{q}_3 - c_1 c_2 \dot{q}_1 \left(q_3 + \frac{L}{2}\right)\right] \jmath_0 + \\ &\left[-c_2 s_1 \dot{q}_1 \left(q_3 + \frac{L}{2}\right) - c_1 s_2 \dot{q}_2 \left(q_3 + \frac{L}{2}\right) + c_1 c_2 \dot{q}_3\right] \mathbf{k}_0. \end{aligned} \tag{2.74}$$

The velocity of C_R in (0) can be computed much easier using mobile reference frame (2)

$$\mathbf{v}_{C_R} = \frac{d}{dt}\mathbf{r}_{C_R} = \frac{{}^{(2)}d}{dt}\mathbf{r}_{C_R} + \boldsymbol{\omega}_{20} \times \mathbf{r}_{C_R}, \tag{2.75}$$

where

$$\mathbf{r}_{C_R} = L_2 c_2 \boldsymbol{\imath}_2 + (q_3 + L_2 s_2 + L/2)\mathbf{k}_2.$$

The velocity of C_R is

$$\begin{aligned} \mathbf{v}_{C_R} &= -\dot{q}_2 L_2 s_2 \boldsymbol{\imath}_2 + (\dot{q}_3 + \dot{q}_2 L_2 c_2)\mathbf{k}_2 + \begin{vmatrix} \boldsymbol{\imath}_2 & \boldsymbol{\jmath}_2 & \mathbf{k}_2 \\ \dot{q}_1 c_2 & \dot{q}_2 & \dot{q}_1 s_2 \\ L_2 c_2 & 0 & q_3 + L_2 s_2 + L/2 \end{vmatrix} \\ &= (L/2 + q_3)\dot{q}_2 \boldsymbol{\imath}_2 - c_2 \dot{q}_1 (q_3 + L/2)\boldsymbol{\jmath}_2 + \dot{q}_3 \mathbf{k}_2. \end{aligned} \tag{2.76}$$

Remark: The angular velocity $\boldsymbol{\omega}_{10}$ was expressed in terms of unit vectors $[\boldsymbol{\imath}_1, \boldsymbol{\jmath}_1, \mathbf{k}_1]$ and $\boldsymbol{\omega}_{20}$ expressed in terms of unit vectors $[\boldsymbol{\imath}_2, \boldsymbol{\jmath}_2, \mathbf{k}_2]$ This will facilitate later work, where it will be assumed that the central principal axes of inertia of link 1 are parallel to $[\boldsymbol{\imath}_1, \boldsymbol{\jmath}_1, \mathbf{k}_1]$ and the central principal axis of inertia of links 2 and 3 are parallel to $[\boldsymbol{\imath}_2, \boldsymbol{\jmath}_2, \mathbf{k}_2]$. When it comes to dealing with the velocities of C_1, C_2 C_3, and C_R it is best to use whatever vector basis permits one to write the simplest expression.

Kinetic energy

The kinetic energy of a rigid body is

$$T = \frac{1}{2}m\mathbf{v}_C \cdot \mathbf{v}_C + \frac{1}{2}\boldsymbol{\omega} \cdot (\bar{I} \cdot \boldsymbol{\omega}), \tag{2.77}$$

where m is the mass, $\mathbf{v}_C$ is the velocity of the mass center, $\boldsymbol{\omega} = \omega_x \boldsymbol{\imath} + \omega_y \boldsymbol{\jmath} + \omega_z \mathbf{k}$ is the angular velocity of the rigid body in (0), and $\bar{I} = (I_x \boldsymbol{\imath})\boldsymbol{\imath} + (I_y \boldsymbol{\jmath})\boldsymbol{\jmath} + (I_z \mathbf{k})\mathbf{k}$ is the central inertia *dyadic* of the rigid body. The central principal axes of the rigid body are parallel to $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ and the associated moments of inertia have the values I_x, I_y, I_z, respectively. The inertia matrix associated to $\bar{I}$ is

$$\bar{I} \rightarrow \mathbf{I} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}. \tag{2.78}$$

The dot product of the vector $\boldsymbol{\omega}$ with the central inertia dyadic $\bar{I}$ is

$$\boldsymbol{\omega} \cdot \bar{I} = \bar{I} \cdot \boldsymbol{\omega} = \omega_x I_x \,\mathbf{\imath} + \omega_y I_y \,\mathbf{\jmath} + \omega_z I_z \,\mathbf{k}. \tag{2.79}$$

The total kinetic energy of the robot arm is

$$T = T_1 + T_2 + T_{2'} + T_3 + T_R, \tag{2.80}$$

where T_1 is the kinetic energy of link 1, T_2 is the kinetic energy of bar 2, $T_{2'}$ is the kinetic energy of slider 2', T_3 is the kinetic energy of bar 3, and T_R is the kinetic energy of RB.

The kinetic energy of link 1 is

$$T_1 = \frac{1}{2} m_1 \mathbf{v}_{C_1} \cdot \mathbf{v}_{C_1} + \frac{1}{2} \boldsymbol{\omega}_{10} \cdot (\bar{I}_1 \cdot \boldsymbol{\omega}_{10}) = \frac{1}{2} \boldsymbol{\omega}_{10} \cdot (\bar{I}_1 \cdot \boldsymbol{\omega}_{10}), \tag{2.81}$$

where m_1 is the mass of the link, $\bar{I}_1 = (I_{1x}\mathbf{\imath}_1)\mathbf{\imath}_1 + (I_{1y}\mathbf{\jmath}_1)\mathbf{\jmath}_1 + (I_{1z}\mathbf{k}_1)\mathbf{k}_1$ is the central inertia dyadic of link 1, and $\boldsymbol{\omega}_{10} = \dot{q}_1 \,\mathbf{\imath}_1$. Using the above relation the kinetic energy of link 1 is

$$T_1 = \frac{1}{2} I_{1x} \dot{q}_1^2. \tag{2.82}$$

The kinetic energy of bar 2 is

$$T_2 = \frac{1}{2} m_2 \mathbf{v}_{C_2} \cdot \mathbf{v}_{C_2} + \frac{1}{2} \boldsymbol{\omega}_{20} \cdot (\bar{I}_2 \cdot \boldsymbol{\omega}_{20}) = \frac{1}{2} \boldsymbol{\omega}_{20} \cdot (\bar{I}_2 \cdot \boldsymbol{\omega}_{20}), \tag{2.83}$$

where m_2 is the mass of the bar and

$$\bar{I}_2 = (I_{2x}\mathbf{\imath}_2)\mathbf{\imath}_2 + (I_{2y}\mathbf{\jmath}_2)\mathbf{\jmath}_2 + (I_{2z}\mathbf{k}_1)\mathbf{k}_1 = (\frac{m_2 l^2}{12}\mathbf{\imath}_2)\mathbf{\imath}_2 + (\frac{m_2 l^2}{12}\mathbf{k}_2)\mathbf{k}_2,$$

is the central inertia dyadic of bar 2 with the length l. The kinetic energy is

$$T_2 = \frac{m_2 l^2}{24} \dot{q}_1^2. \tag{2.84}$$

The kinetic energy of slider 2' is

$$T_{2'} = \frac{1}{2} m_{2'} \mathbf{v}_{C2'} \cdot \mathbf{v}_{C2'} + \frac{1}{2} \boldsymbol{\omega}_{20} \cdot (\bar{I}_{2'} \cdot \boldsymbol{\omega}_{20}) = \frac{1}{2} \boldsymbol{\omega}_{20} \cdot (\bar{I}_{2'} \cdot \boldsymbol{\omega}_{20}), \tag{2.85}$$

where $m_{2'}$ is the mass of the slider and

$$\bar{I}_{2'} = (I_{2'x}\mathbf{\imath}_2)\mathbf{\imath}_2 + (I_{2'y}\mathbf{\jmath}_2)\mathbf{\jmath}_2 + (I_{2'z}\mathbf{k}_2)\mathbf{k}_2,$$

is the central inertia dyadic of the slider. The kinetic energy is

$$T_{2'} = \frac{1}{2}\left[(I_{2'x}c_2^2 + I_{2'z}s_2^2)\dot{q}_1^2 + I_{2'y}\dot{q}_2^2\right]. \tag{2.86}$$

The kinetic energy of bar 3 is

$$T_3 = \frac{1}{2}m_3\mathbf{v}_{C_3}\cdot\mathbf{v}_{C_3} + \frac{1}{2}\boldsymbol{\omega}_{20}\cdot(\bar{I}_3\cdot\boldsymbol{\omega}_{20}), \tag{2.87}$$

where m_3 is the mass of the bar and

$$\bar{I}_3 = (I_{3x}\mathbf{\imath}_2)\mathbf{\imath}_2 + (I_{3y}\mathbf{\jmath}_2)\mathbf{\jmath}_2 + (I_{3z}\mathbf{k}_2)\mathbf{k}_2 = (\frac{m_3L^2}{12}\mathbf{\imath}_2)\mathbf{\imath}_2 + (\frac{m_3L^2}{12}\mathbf{\jmath}_2)\mathbf{\jmath}_2,$$

is the central inertia dyadic of bar 3.

The rigid body RB is considered as a particle with the mass m_R concentrated at C_R. The kinetic energy of RB is

$$T_R = \frac{1}{2}m_R\mathbf{v}_{C_R}\cdot\mathbf{v}_{C_R} = \frac{m_R}{2}\left[(L/2+q_3)^2\dot{q}_2^2 + c_2^2(q_3+L/2)^2\dot{q}_1^2 + \dot{q}_3^2\right]. \tag{2.88}$$

Generalized forces

In the case of the robot arm, there are two kinds of forces that contribute to the generalized forces Q_1, Q_2, Q_3 namely, contact forces applied in order to drive 1, 2, 3 and RB, and gravitational forces exerted on 1, 2, 3, and RB by the Earth. The contact forces are neglected for this example. The gravitational forces exerted on 1, 2, 3, and RB by the Earth, are denoted by $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{G}_3$, $\mathbf{G}_R$, respectively, and can be expressed as

$$\begin{aligned}
\mathbf{G}_1 &= -m_1\, g\, \mathbf{\imath}_0,\\
\mathbf{G}_2 &= -(m_2 + m_{2'})\, g\, \mathbf{\imath}_0,\\
\mathbf{G}_3 &= -m_3\, g\, \mathbf{\imath}_0,\\
\mathbf{G}_R &= -m_R\, g\, \mathbf{\imath}_0.
\end{aligned} \tag{2.89}$$

One can express the contribution to the generalized force of all forces and torques acting on the system, as

$$Q_r = \frac{\partial\mathbf{r}_{C_1}}{\partial q_r}\cdot\mathbf{G}_1 + \frac{\partial\mathbf{r}_{C_2}}{\partial q_r}\cdot\mathbf{G}_2 + \frac{\partial\mathbf{r}_{C_3}}{\partial q_r}\cdot\mathbf{G}_3 + \frac{\partial\mathbf{r}_{CB}}{\partial q_r}\cdot\mathbf{G}_R,\ \ r = 1,2,3. \tag{2.90}$$

The vectors in Eq. (2.90) must be expressed in terms of the fixed reference frame (0). The generalized forces are

$$\begin{aligned} Q_1 &= 0, \\ Q_2 &= -gc_2(m_r L/2 + m_r q_3 + m_3 q_3), \\ Q_3 &= -g(m_R + m_3)s_2. \end{aligned}$$

The same results can be obtained using the relations

$$Q_r = \frac{\partial \mathbf{v}_{C_1}}{\partial \dot{q}_r} \cdot \mathbf{G}_1 + \frac{\partial \mathbf{v}_{C_2}}{\partial \dot{q}_r} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r} \cdot \mathbf{G}_3 + \frac{\partial \mathbf{v}_{CB}}{\partial \dot{q}_r} \cdot \mathbf{G}_R, \quad r = 1, 2, 3. \quad (2.91)$$

In Eq. (2.91) the vectors $\mathbf{v}_{C_1}$, $\mathbf{G}_1$ are expressed in terms of the mobile reference frame (1), and the vectors $\mathbf{v}_{C_2}$, $\mathbf{G}_2$, $\mathbf{v}_{C_3}$, $\mathbf{G}_3$, $\mathbf{v}_{C_R}$, $\mathbf{G}_R$ are expressed in terms of the mobile reference frame (2).

The Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r,$$

where $r = 1, 2, 3$.

The *Mathematica*^{TM} program with the equations of motion are given in Program 2.4.

Example 2.5. Figure 2.5(a) is a schematic representation of an open kinematic chain (robot arm) consisting of three links 1, 2, and 3. Let m_1, m_2, m_3 be the masses of 1, 2, 3, respectively. Link 1 can be rotated at A in a "fixed" reference frame (0) of unit vectors $[\mathbf{\imath}_0, \mathbf{\jmath}_0, \mathbf{k}_0]$ about a vertical axis $\mathbf{\imath}_0$. The unit vector $\mathbf{\imath}_0$ is fixed in 1. The link 1 is connected to link 2 at the pin joint B. The element 2 rotates relative to 1 about an axis fixed in both 1 and 2, passing through B, and perpendicular to the axis of 1. The last link 3 is connected to 2 by means of a slider joint. The mass centers of links 1, 2, and 3 are C_1, C_2, and C_3, respectively. The distances $L_1 = AC_1$, $L_2 = BC_2$, and $L_B = AB$ are indicated in Fig. 2.5. The reference frame (1) of the unit vectors $[\mathbf{\imath}_1, \mathbf{\jmath}_1, \mathbf{k}_1]$ is attached to link 1, and the reference frame (2) of the unit vectors $[\mathbf{\imath}_2, \mathbf{\jmath}_2, \mathbf{k}_2]$ is attached to link 2, as shown in Fig. 2.5(b). Find and solve the Lagrange's equations of motion.

Solution

The generalized coordinates (quantities associated with the the instantaneous position of the system) are $q_1(t)$, $q_2(t)$, $q_3(t)$.

The first generalized coordinate q_1 denotes the radian measure of the angle between the axes of (1) and (0). The unit vectors $\mathbf{\imath}_1$, $\mathbf{\jmath}_1$, and $\mathbf{k}_1$ can be expressed as functions of $\mathbf{\imath}_0$, $\mathbf{\jmath}_0$, and $\mathbf{k}_0$

$$\begin{aligned}
\mathbf{\imath}_1 &= \mathbf{\imath}_0,\\
\mathbf{\jmath}_1 &= c_1\,\mathbf{\jmath}_0 + s_1\,\mathbf{k}_0,\\
\mathbf{k}_1 &= -s_1\,\mathbf{\jmath}_0 + c_1\,\mathbf{k}_0,
\end{aligned}$$

or

$$\begin{bmatrix}\mathbf{\imath}_1\\ \mathbf{\jmath}_1\\ \mathbf{k}_1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & c_1 & s_1\\ 0 & -s_1 & c_1\end{bmatrix}\begin{bmatrix}\mathbf{\imath}_0\\ \mathbf{\jmath}_0\\ \mathbf{k}_0\end{bmatrix},$$

where $s_1 = \sin q_1$ and $c_1 = \cos q_1$. The transformation matrix from (1) to (0) is

$$R_{10} = \begin{bmatrix}1 & 0 & 0\\ 0 & c_1 & s_1\\ 0 & -s_1 & c_1\end{bmatrix}.$$

The second generalized coordinate designates also a radian measure of the rotation angle between (1) and (2). The unit vectors $\mathbf{\imath}_2$, $\mathbf{\jmath}_2$ and $\mathbf{k}_2$ can be

expressed as

$$\begin{aligned}
\mathbf{\imath}_2 &= c_2\mathbf{\imath}_1 - s_2\mathbf{k}_1 \\
&= c_2\mathbf{\imath}_0 + s_1 s_2\,\mathbf{\jmath}_0 - c_1 s_2\,\mathbf{k}_0, \\
\mathbf{\jmath}_2 &= \mathbf{\jmath}_1, \\
&= c_1\,\mathbf{\jmath}_0 + s_1\,\mathbf{k}_0, \\
\mathbf{k}_2 &= s_2\mathbf{\imath}_1 + c_2\mathbf{k}_1 \\
&= s_2\mathbf{\imath}_0 - c_2 s_1\,\mathbf{\jmath}_0 + c_1 c_2\,\mathbf{k}_0,
\end{aligned}$$

where $s_2 = \sin q_2$ and $c_2 = \cos q_2$. The transformation matrix from (2) to (1) is

$$R_{21} = \begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix}.$$

The last generalized coordinate q_3 is the distance from C_2 to C_3.

Angular velocities

Next the angular velocity of the links 1, 2, and 3 will be expressed in the fixed reference frame (0). The angular velocity of 1 in (0) is

$$\boldsymbol{\omega}_{10} = \dot{q}_1\,\mathbf{\imath}_1.$$

The angular velocity of the link 2 with respect to (1) is

$$\boldsymbol{\omega}_{21} = \dot{q}_2\,\mathbf{\jmath}_2.$$

The angular velocity of the link 2 with respect to the fixed reference frame (0) is

$$\boldsymbol{\omega}_{20} = \boldsymbol{\omega}_{10} + \boldsymbol{\omega}_{21} = \dot{q}_1\,\mathbf{\imath}_1 + \dot{q}_2\mathbf{\jmath}_2.$$

With $\mathbf{\imath}_0 = \mathbf{\imath}_1 = c_1\,\mathbf{\imath}_2 + s_1\,\mathbf{k}_2$ the angular velocity of the link 2 in the reference frame (0) written in terms of the reference frame (2) is

$$\boldsymbol{\omega}_{20} = \dot{q}_1(c_2\,\mathbf{\imath}_2 + s_2\,\mathbf{k}_2) + \dot{q}_2\,\mathbf{\jmath}_2 = \dot{q}_1\,c_2\,\mathbf{\imath}_2 + \dot{q}_2\,\mathbf{\jmath}_2 + \dot{q}_1\,s_2\,\mathbf{k}_2.$$

The link 3 has the same rotational motion as link 2, i.e., $\boldsymbol{\omega}_{30} = \boldsymbol{\omega}_{20}$.

The angular acceleration of the link 1 in the reference frame (0) can be expressed as

$$\boldsymbol{\alpha}_{10} = \ddot{q}_1\,\mathbf{\imath}_1.$$

The angular acceleration of the link 2 with respect to the reference frame (0) is

$$\boldsymbol{\alpha}_{20} = \frac{d}{dt}\boldsymbol{\omega}_{20} = \frac{{}^{(2)}d}{dt}\boldsymbol{\omega}_{20}.$$

where $\frac{{}^{(2)}d}{dt}$ represents the derivative with respect to time in reference frame (2), $[\mathbf{\imath}_2, \mathbf{\jmath}_2, \mathbf{k}_2]$. The link 3 has the same angular acceleration as link 2, i.e., $\boldsymbol{\alpha}_{30} = \boldsymbol{\alpha}_{20}$.

Linear velocities

The position vector of C_1, the mass center of link 1, is

$$\mathbf{r}_{C_1} = L_1\mathbf{k}_1,$$

and the velocity of C_1 in (0) is

$$\begin{aligned}\mathbf{v}_{C_1} &= \frac{d}{dt}\mathbf{r}_{C_1} = \frac{{}^{(1)}d}{dt}\mathbf{r}_{C_1} + \boldsymbol{\omega}_{10} \times \mathbf{r}_{C_1} \\ &= \mathbf{0} + \begin{vmatrix} \mathbf{\imath}_1 & \mathbf{\jmath}_1 & \mathbf{k}_1 \\ \dot{q}_1 & 0 & 0 \\ 0 & 0 & L_1 \end{vmatrix} = -\dot{q}_1 L_1 \mathbf{\jmath}_1.\end{aligned}$$

The position vector of C_2, the mass center of link 2, is

$$\begin{aligned}\mathbf{r}_{C_2} &= L_B\mathbf{k}_1 + L_2\mathbf{k}_2 = L_B(-s_2\mathbf{\imath}_2 + c_2\mathbf{k}_2) + L_2\mathbf{k}_2 \\ &= -L_B s_2 \mathbf{\imath}_2 + (L_B c_2 + L_2)\mathbf{k}_2.\end{aligned}$$

The velocity of C_2 in (0) is

$$\begin{aligned}\mathbf{v}_{C_2} &= \frac{d}{dt}\mathbf{r}_{C_2} = \frac{{}^{(2)}d}{dt}\mathbf{r}_{C_2} + \boldsymbol{\omega}_{20} \times \mathbf{r}_{C_2} \\ &= -L_B c_1 \dot{q}_2 \mathbf{\imath}_2 - L_B c_2 \dot{q}_2 \mathbf{k}_2 + \begin{vmatrix} \mathbf{\imath}_2 & \mathbf{\jmath}_2 & \mathbf{k}_2 \\ \dot{q}_1 c_2 & \dot{q}_2 & \dot{q}_1 s_2 \\ -L_B s_2 & 0 & L_B c_2 + L_2 \end{vmatrix} \\ &= L_2 \dot{q}_2 \mathbf{\imath}_2 - (L_B + L_2 c_2)\dot{q}_1 \mathbf{\jmath}_2.\end{aligned}$$

The position vector of C_3 with respect to reference frame (0) is

$$\begin{aligned}\mathbf{r}_{C_3} &= \mathbf{r}_{C_2} + q_3\mathbf{k}_2 \\ &= -L_B s_2 \mathbf{\imath}_2 + (L_B c_2 + L_2 + q_3)\mathbf{k}_2,\end{aligned}$$

and the velocity of this mass center in (0) is

$$\begin{aligned}
\mathbf{v}_{C_3} &= \frac{d}{dt}\mathbf{r}_{C_3} = \frac{{}^{(2)}d}{dt}\mathbf{r}_{C_3} + \boldsymbol{\omega}_{20} \times \mathbf{r}_{C_3} \\
&= -L_B c_2 \dot{q}_2 \boldsymbol{\imath}_2 - (L_B c_2 \dot{q}_2 + \dot{q}_3)\mathbf{k}_2 + \begin{vmatrix} \boldsymbol{\imath}_2 & \boldsymbol{\jmath}_2 & \mathbf{k}_2 \\ \dot{q}_1 c_2 & \dot{q}_2 & \dot{q}_1 s_2 \\ -L_B s_2 & 0 & L_B c_2 + L_2 + q_2 \end{vmatrix} \\
&= (L_2 + q_3)\dot{q}_2 \boldsymbol{\imath}_2 - (L_B + L_2 c_2 + c_2 q_2)\dot{q}_1 \boldsymbol{\jmath}_2 + \dot{q}_3 \mathbf{k}_2.
\end{aligned}$$

Kinetic energy

The total kinetic energy of the robot arm in the reference frame (0) is

$$T = \sum_{i=1}^{3} T_i.$$

The kinetic energy of the link i, $i = 1, 2, 3$, is

$$T_i = \frac{1}{2} m_i \mathbf{v}_{C_i} \cdot \mathbf{v}_{C_i} + \frac{1}{2} \boldsymbol{\omega}_{i0} \cdot (\bar{I}_i \cdot \boldsymbol{\omega}_{i0}).$$

Remark: The kinetic energy for a rigid body is

$$T_{\text{rigid body}} = \frac{1}{2} m \mathbf{v}_C \cdot \mathbf{v}_C + \frac{1}{2} \boldsymbol{\omega} \cdot (\bar{I}_C \cdot \boldsymbol{\omega}),$$

where m is the mass of the rigid body, $\mathbf{v}_C$ is the velocity of the mass center of the rigid body in (0), $\boldsymbol{\omega} = \omega_x \boldsymbol{\imath} + \omega_y \boldsymbol{\jmath} + \omega_z \mathbf{k}$ is the angular velocity of the rigid body in (0), and $\bar{I} = (I_x \boldsymbol{\imath})\boldsymbol{\imath} + (I_y \boldsymbol{\jmath})\boldsymbol{\jmath} + (I_z \mathbf{k})\mathbf{k}$ is the central inertia dyadic of the rigid body. The central principal axes of the rigid body are parallel to $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ and the associated moments of inertia have the values I_x, I_y, I_z, respectively. The inertia matrix associated with $\bar{I}$ is

$$\bar{I} \to \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}.$$

The dot product of the vector $\boldsymbol{\omega}$ with the dyadic $\bar{I}$ is

$$\boldsymbol{\omega} \cdot \bar{I} = \bar{I} \cdot \boldsymbol{\omega} = \omega_x I_x \, \boldsymbol{\imath} + \omega_y I_y \, \boldsymbol{\jmath} + \omega_z I_z \, \mathbf{k}.$$

The links 1, 2, and 3 have the following mass distribution properties. The central principal axes of 1 are parallel to $\mathbf{\imath}_1$, $\mathbf{\jmath}_1$, $\mathbf{k}_1$, Fig. 2.5, and the associated moments of inertia have the values I_{1x}, I_{1y}, I_{1z}, respectively. The central inertia dyadic of 1 is

$$\bar{I}_1 = (I_{1x}\mathbf{\imath}_1)\mathbf{\imath}_1 + (I_{1y}\mathbf{\jmath}_1)\mathbf{\jmath}_1 + (I_{1z}\mathbf{k}_1)\mathbf{k}_1.$$

The central principal axes of 2 and 3 are parallel to $\mathbf{\imath}_2$, $\mathbf{\jmath}_2$, $\mathbf{k}_2$ and the associated moments of inertia have values I_{2x}, I_{2y}, I_{2z}, and I_{3x}, I_{3y}, I_{3z} respectively. The central inertia dyadic of 2 is

$$\bar{I}_2 = (I_{2x}\mathbf{\imath}_2)\mathbf{\imath}_2 + (I_{2y}\mathbf{\jmath}_2)\mathbf{\jmath}_2 + (I_{2z}\mathbf{k}_2)\mathbf{k}_2,$$

and the central inertia dyadic of 3 is

$$\bar{I}_3 = (I_{3x}\mathbf{\imath}_2)\mathbf{\imath}_2 + (I_{3y}\mathbf{\jmath}_2)\mathbf{\jmath}_2 + (I_{3z}\mathbf{k}_2)\mathbf{k}_2.$$

The central inertia dyadics of links 1 and 2 are calculated using Fig. 2.5(c). The kinetic energy is given in Program 2.5.

The left hand side of Lagrange's equations is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r}, \quad r = 1,\ 2,\ 3.$$

Generalized forces

Remark: If a set of contact and/or body forces acting on a rigid body is equivalent to a couple of torque $\mathbf{T}$ together with force $\mathbf{R}$ applied at a point P of the rigid body, then the contribution of this set of forces to the generalized force, Q_r, is given by

$$Q_r = \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_r} \cdot \mathbf{T} + \frac{\partial \mathbf{v}_P}{\partial \dot{q}_r} \cdot \mathbf{R}, \quad r = 1, 2, ...,$$

where $\boldsymbol{\omega}$ is the angular velocity of the rigid body in (0), $\mathbf{v}_P$ is the velocity of P in (0), and r represents the generalized coordinates.

In the case of the robotic arm, there are two kinds of forces that contribute to the generalized forces Q_1, Q_2, and Q_3 namely, contact forces applied in order to drive the links 1, 2, and 3, and gravitational forces exerted on 1, 2, and 3 by the Earth.

The set of contact forces transmitted from 0 to 1 can be replaced with a couple of torque $\mathbf{T}_{01}$ applied to 1 at A.

Similarly, the set of contact forces transmitted from 1 to 2 can be replaced with a couple of torque $\mathbf{T}_{12}$ applied to 2 at B. The law of action and reaction then guarantees that the set of contact forces transmitted from 1 to 2 is equivalent to a couple of torque $-\mathbf{T}_{12}$ to 1 at B.

Next, the set of contact forces exerted by link 2 on link 3 can be replaced with a force $\mathbf{F}_{23}$ applied to 3 at C_3. The law of action and reaction guarantees that the set of contact forces transmitted from 3 to 2 is equivalent to a force $-\mathbf{F}_{23}$ applied to 2 at C_{32}.

The point C_{32} ($C_{32} \in$ link2) instantaneously coincides with C_3, ($C_3 \in$ link3).

The expressions $\mathbf{T}_{01}$, $\mathbf{T}_{12}$, and $\mathbf{F}_{23}$ are

$$\mathbf{T}_{01} = T_{01x}\mathbf{\imath}_1 + T_{01y}\mathbf{\jmath}_1 + T_{01z}\mathbf{k}_1, \quad \mathbf{T}_{12} = T_{12x}\mathbf{\imath}_2 + T_{12y}\mathbf{\jmath}_2 + T_{12z}\mathbf{k}_2, \quad \text{and}$$
$$\mathbf{F}_{23} = F_{23x}\mathbf{\imath}_2 + F_{23y}\mathbf{\jmath}_2 + F_{23z}\mathbf{k}_2.$$

The external gravitational forces exerted on the links 1, 2, and 3 by the Earth, can be denoted by $\mathbf{G}_1$, $\mathbf{G}_2$, and $\mathbf{G}_3$ respectively, and can be expressed as

$$\begin{aligned}
\mathbf{G}_1 &= -m_1\, g\mathbf{\imath}_1, \\
\mathbf{G}_2 &= -m_2\, g\, \mathbf{\imath}_1 = -m_2\, g\,(c_2\, \mathbf{\imath}_2 + s_2\, \mathbf{k}_2), \\
\mathbf{G}_3 &= -m_3\, g\, \mathbf{\imath}_1 = -m_3\, g\,(c_2\, \mathbf{\imath}_2 + s_2\, \mathbf{k}_2).
\end{aligned}$$

The reason for replacing $\mathbf{\imath}_1$ with $c_2\, \mathbf{\imath}_2 + s_2\, \mathbf{k}_2$ in connection with the forces $\mathbf{G}_2$ and $\mathbf{G}_3$ is that they are soon to be dot-multiplied with $\dfrac{\partial \mathbf{v}_{C_2}}{\partial \dot{q}_r}$ and $\dfrac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r}$ which have been expressed in terms of $\mathbf{\imath}_2, \mathbf{\jmath}_2,$ and $\mathbf{k}_2$.

One can express $(Q_r)_1$, the contribution to the generalized active force Q_r of all the forces and torques acting on the particles of the link 1, as

$$(Q_r)_1 = \frac{\partial \boldsymbol{\omega}_{10}}{\partial \dot{q}_r} \cdot (\mathbf{T}_{01} - \mathbf{T}_{12}) + \frac{\partial \mathbf{v}_{C_1}}{\partial \dot{q}_r} \cdot \mathbf{G}_1, \ \ r = 1, 2, 3.$$

The contribution $(Q_r)_2$ to the generalized active force of all the forces and torques acting on the link 2 is

$$(Q_r)_2 = \frac{\partial \boldsymbol{\omega}_{20}}{\partial \dot{q}_r} \cdot \mathbf{T}_{12} + \frac{\partial \mathbf{v}_{C_2}}{\partial \dot{q}_r} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{v}_{C_{32}}}{\partial \dot{q}_r} \cdot (-\mathbf{F}_{23})\,, \ \ r = 1, 2, 3,$$

where

$$\mathbf{v}_{C_{32}} = \mathbf{v}_{C_2} + \boldsymbol{\omega}_{20} \times \mathbf{r}_{C_2C_3} = \mathbf{v}_{C_2} + \boldsymbol{\omega}_{20} \times q_3\mathbf{k}_2.$$

The contribution $(Q_r)_3$, to the generalized active force of all the forces and torques acting on the link 3 is

$$(Q_r)_3 = \frac{\partial \boldsymbol{\omega}_{20}}{\partial \dot{q}_r} \cdot \mathbf{T}_{23} + \frac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r} \cdot \mathbf{G}_3 + \frac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r} \cdot \mathbf{F}_{23}, \quad r = 1, 2, 3.$$

The generalized active force Q_r of all the forces and torques acting on the links 1, 2, and 3 are

$$Q_r = (Q_r)_1 + (Q_r)_2 + (Q_r)_3, \quad r = 1, 2, 3,$$

or

$$\begin{aligned} Q_1 &= T_{01x}, \\ Q_2 &= T_{12y} - g\, m_2\, L_2\, c_2 - g\, m_3\, c_2\, (L_2 + q_3), \\ Q_3 &= F_{23z} - g\, m_3\, s_2. \end{aligned}$$

To arrive at the dynamical equations governing the robot arm, all that remains to be done is to substitute into Lagrange's equations, namely,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r, \quad r = 1,\ 2,\ 3.$$

The Lagrange's equations are calculated in Program 2.5.

Numerical simulation

The robot arm is characterized by the following geometry: $L_1 = 0.4$ m, $L_2 = 0.7$ m, $L_B = 0.8$ m, $I_{3x} = 5$ kg·m^2, $I_{3y} = 4$ kg·m^2, $I_{3z} = 1$ kg·m^2. The masses of the rigid bodies are $m_1 = 90$ kg, $m_2 = 60$ kg, $m_3 = 40$ kg, and the gravitational acceleration is $g = 9.81$ m/s^2.

The initial conditions, at $t = 0$ s, are $q_1(0) = \pi/18$ rad, $q_2(0) = \pi/6$ rad, $q_3(0) = 0.1$ m, and $\dot{q}_1(0) = \dot{q}_2(0) = \dot{q}_3(0) = 0$.

The robot arm can be brought from an initial state of rest in reference frame (0) to a final state of rest in (0) in such a way that q_1, q_2, and q_3 have specified

values q_{1f}, q_{2f}, and q_{3f}, respectively, by using the following feedback control laws

$$\begin{aligned}
T_{01x} &= -\beta_{01}\dot{q}_1 - \gamma_{01}(q_1 - q_{1f}), \\
T_{12y} &= -\beta_{12}\dot{q}_2 - \gamma_{12}(q_2 - q_{2f}) + g\, m_2\, L_2\, c_2 + g\, m_3\, c_2\, (L_2 + q_3), \\
F_{23z} &= -\beta_{23}\dot{q}_3 - \gamma_{23}(q_3 - q_{3f}) + g\, m_3\, s_2.
\end{aligned}$$

The constant gains are: $\beta_{01} = 450$ N·m·s/rad, $\gamma_{01} = 300$ N·m/rad, $\beta_{12} = 200$ N·m·s/rad, $\gamma_{12} = 300$ N·m/rad, $\beta_{23} = 150$ N·s/m, $\gamma_{23} = 50$ N/m. The values specified for the generalized coordinates are $q_{1f} = \pi/3$ rad, $q_{2f} = \pi/3$ rad, and $q_{3f} = 0.25$ m.

The *Mathematica*^{TM} calculations that were used to compute and solve Lagrange's equations are given in Program 2.5.

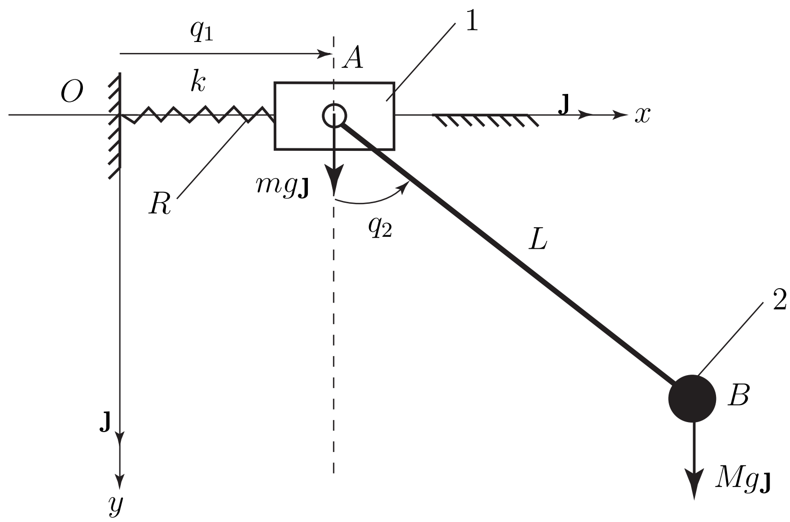


Figure 1

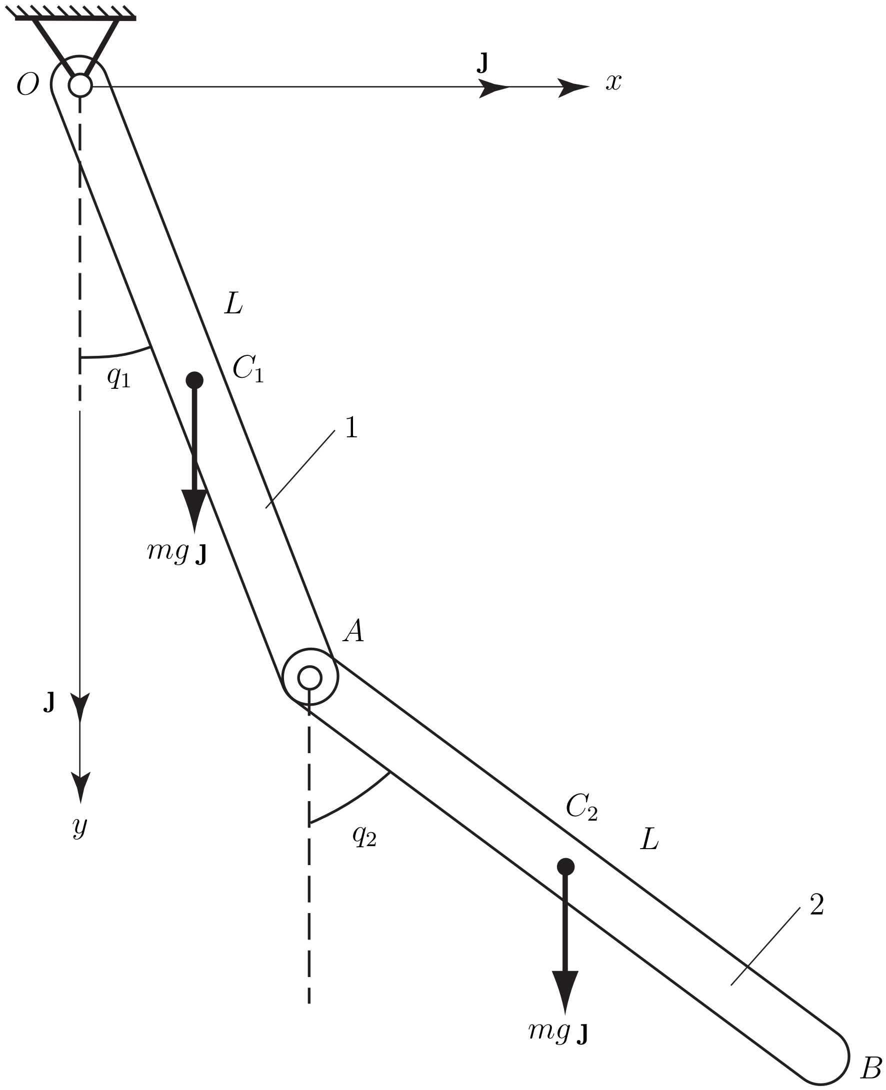
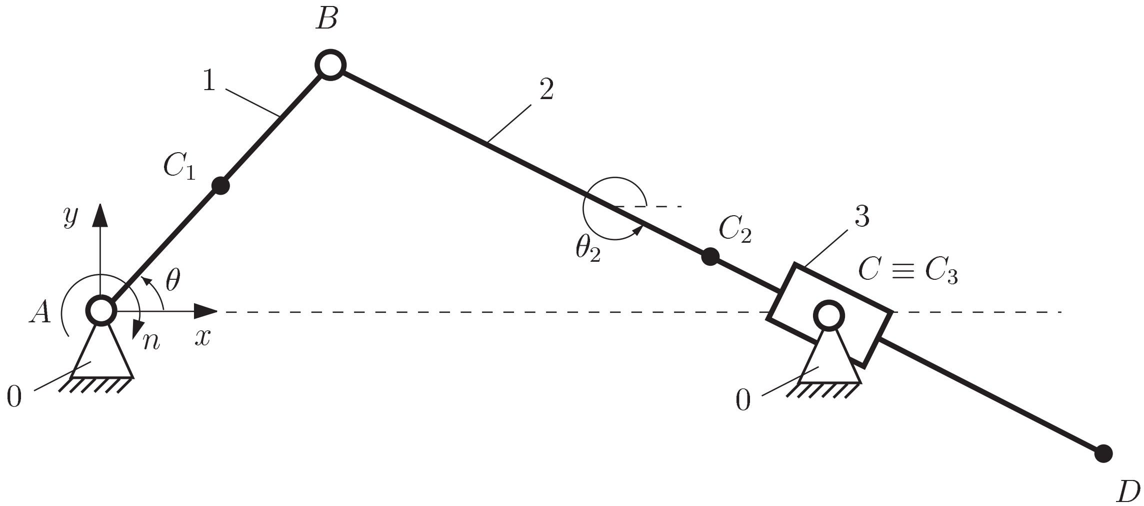


Figure 2


B
1
2
C_1
y
θ_2
C_2
3
$C \equiv C_3$
A
θ
n
x
0
0
D
(a)

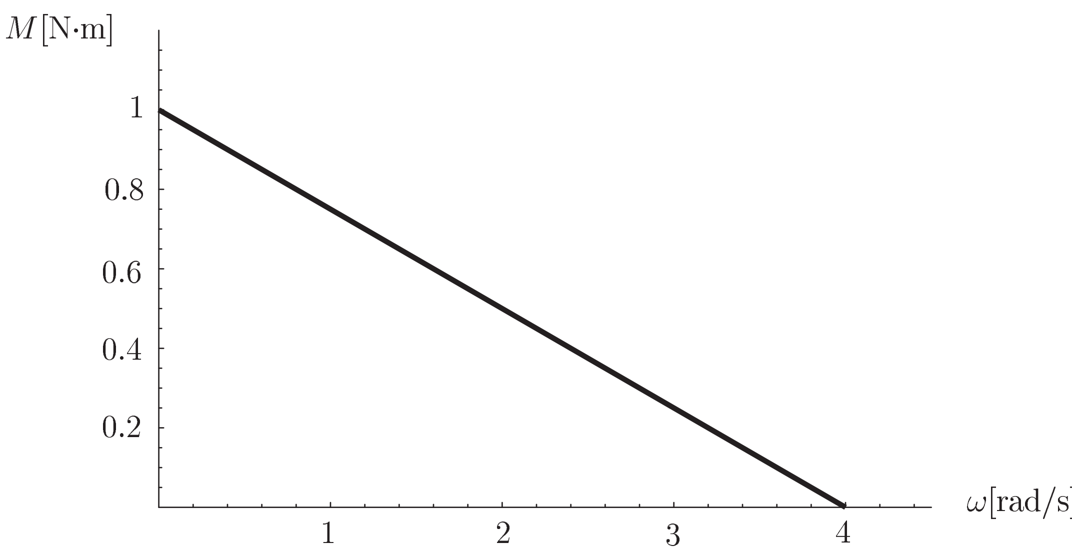
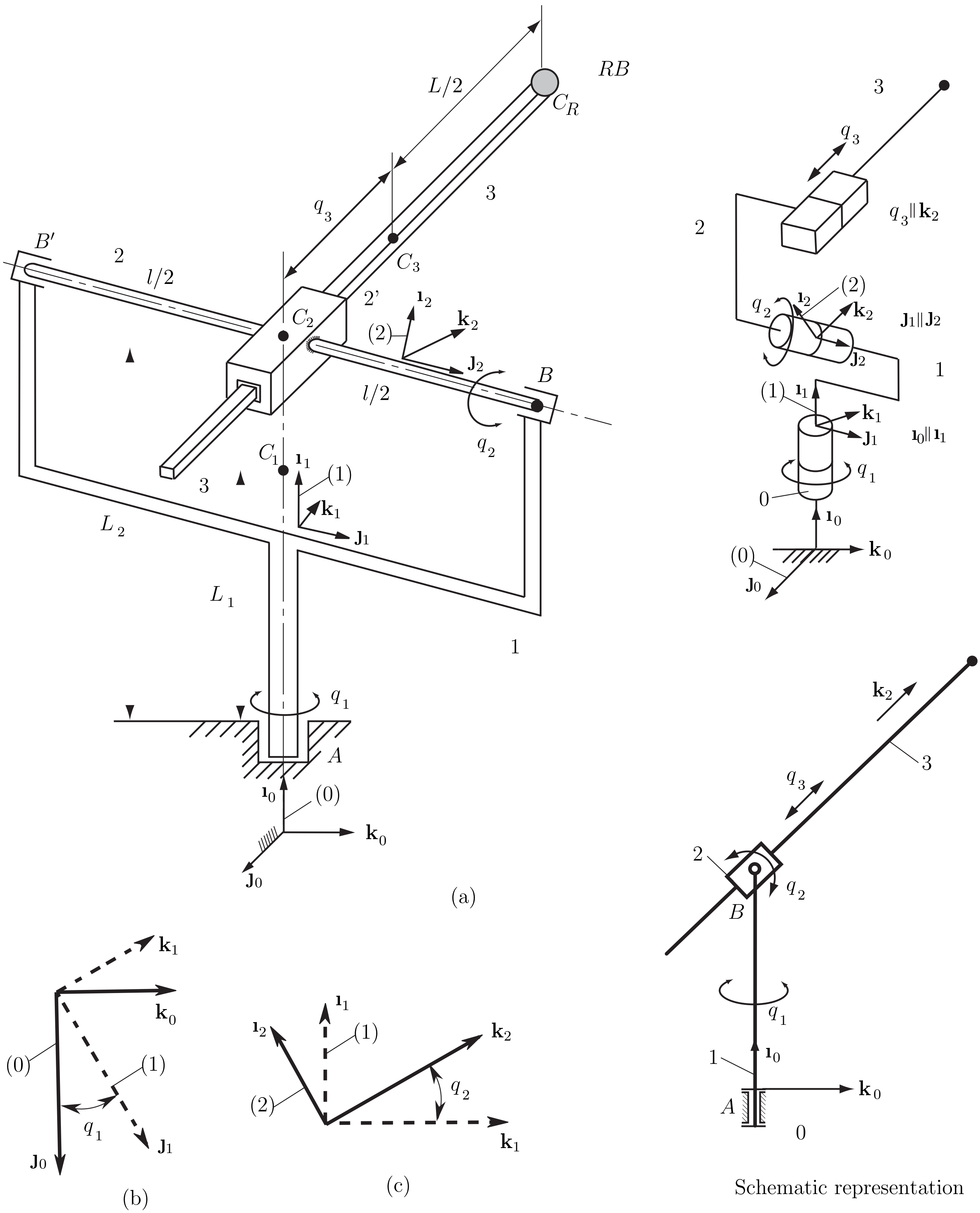
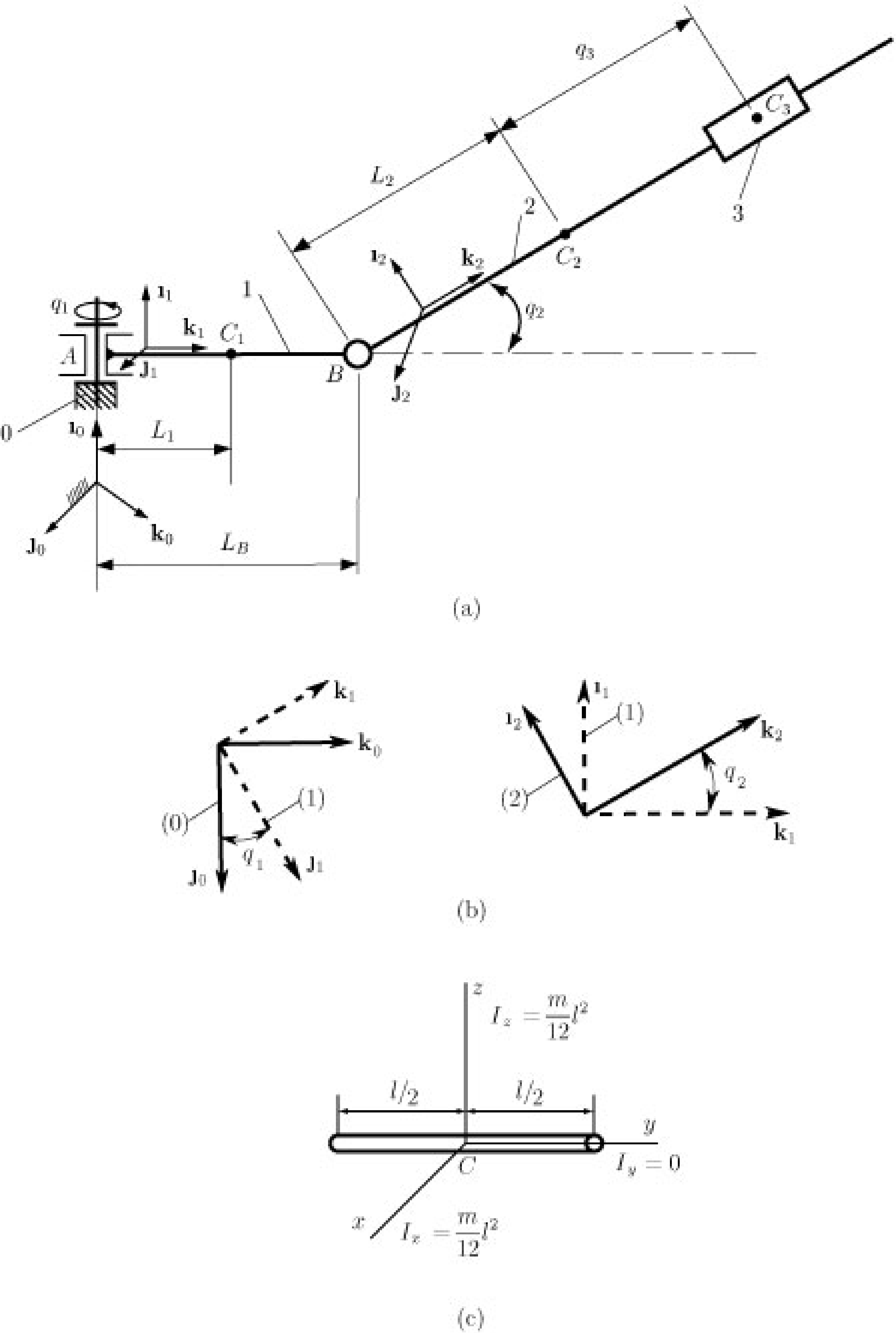

M [N·m]
1
0.8
0.6
0.4
0.2
ω[rad/s]
1
2
3
4
(b)

Figure 3



RB
$L/2$
C_R
3
q_3
C_3
B'
2
$l/2$
C_2
2'
(2)
$\mathbf{\imath}_2$
$\mathbf{k}_2$
$\mathbf{j}_2$
B
$l/2$
q_2
C_1
$\mathbf{\imath}_1$
(1)
$\mathbf{k}_1$
$\mathbf{j}_1$
3
L_2
L_1
1
q_1
A
$\mathbf{\imath}_0$
(0)
$\mathbf{k}_0$
$\mathbf{j}_0$
(a)
$q_3 \parallel \mathbf{k}_2$
$\mathbf{j}_1 \parallel \mathbf{j}_2$
$\mathbf{\imath}_0 \parallel \mathbf{\imath}_1$
0
(b)
(c)
Schematic representation

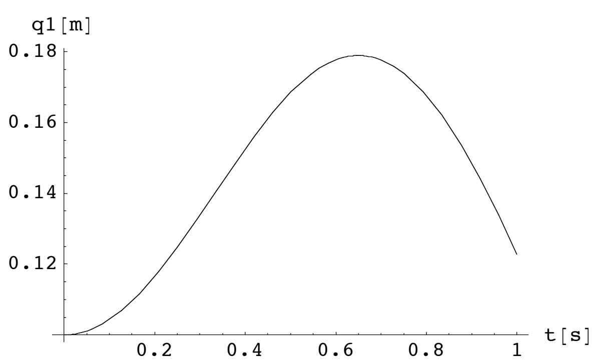
Figure 4

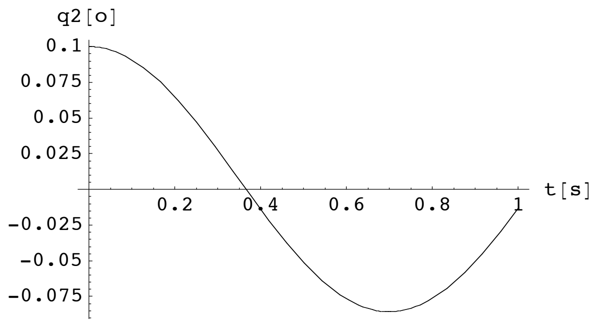


q3
C3
L2
3
2
i2
k2
C2
q2
q1
i1
k1
C1
1
A
J1
B
J2
0
i0
L1
J0
k0
LB
(a)
k1
k0
(0)
(1)
q1
J0
J1
i1
(1)
i2
k2
q2
(2)
k1
(b)
z
$I_z = \frac{m}{12}l^2$
l/2
l/2
y
C
$I_y = 0$
x
$I_x = \frac{m}{12}l^2$
(c)

Figure–5

```
(*Lagrange's equations - example 1*)
Off[General::spell1];
Off[General::spell];
rA={q1[t],0,0};
rB={q1[t]+L*Sin[q2[t]],
    L*Cos[q2[t]],0};
vA=D[rA,t];
vB=D[rB,t];
T1=m vA.vA/2;
T2=M vB.vB/2;
T=T1+T2;
(*External forces*)
FA={-k q1[t],m g,0};
FB={0,M g,0};
(*Generalized forces*)
Q1=FA.D[rA,q1[t]]+FB.D[rB,q1[t]];
Q2=FA.D[rA,q2[t]]+FB.D[rB,q2[t]];
(*Lagrange's Equations*)
eq1=D[D[T,q1'[t],t]]-D[T,q1[t]]-Q1;
eq2=D[D[T,q2'[t],t]]-D[T,q2[t]]-Q2;
(*Small oscilations*)
rule={Sin[q2[t]]->q2[t],
      Cos[q2[t]]->1};
(*input data*)
rule1={m->1.,M->1.,L->1.,
       k->1.,g->10.};
equation1=eq1/.rule/.rule1;
equation2=eq2/.rule/.rule1;
sol=NDSolve[
  {equation1==0,equation2==0,
  q1[0]==.1,q2[0]==.1,
  q1'[0]==0.,q2'[0]==0.},
  {q1,q2},{t,0.,1.}];
Plot[Evaluate[q1[t]]/.sol,
 {t,0.,1.}, PlotRange->All,
    AxesLabel->{"t[s]","q1[m]"}];
Plot[Evaluate[q2[t]]/.sol,
{t,0.,1.}, PlotRange->All,
    AxesLabel->{"t[s]","q2[o]"}]
```

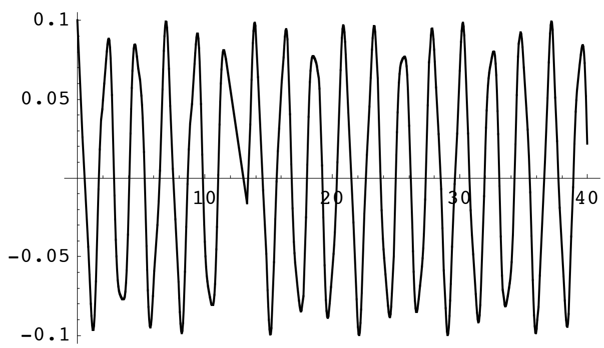




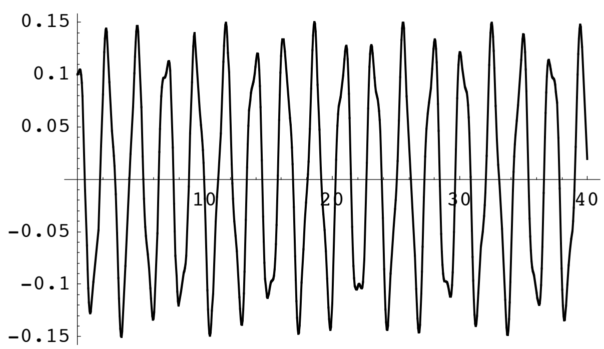
- Graphics -

```
(*Lagrange's equations - example 2 *)
Apply [Clear,Names["Global`*"]];

Off[General::spell1];
Off[General::spell];

(*position analysis*)

m1=m2=m;

L1=L2=L;

rA={L1*Sin[q1[t]],L1*Cos[q1[t]],0};

rC1={L1*Sin[q1[t]]/2.,L1*Cos[q1[t]]/2.,0};

rB=rA+{L2*Sin[q2[t]],L2*Cos[q2[t]],0};

rC2=rA+{L2*Sin[q2[t]]/2.,L2*Cos[q2[t]]/2.,0};

(*velocity analysis*)

vA=D[rA,t];

vC1=D[rC1,t];

vB=D[rB,t];

vC2=D[rC2,t];

w1={0,0,-q1'[t]};

w2={0,0,-q2'[t]};

JC1=m1*L1^2/12.;

JC2=m2*L2^2/12.;
T1=m1 vC1.vC1/2.+JC1 w1.w1/2.;
T2=m2 vC2.vC2/2.+JC2 w2.w2/2.;

(*T1 can be calculated as T1=J0 w1.w1/2.
where  J0=m1*L1^2/3 *)
Simplify[T1];
Simplify[T2];

T=T1+T2;

Simplify[T];
(*external forces*)

FC1={0,m1 g,0};

FC2={0,m2 g,0};
(*generalized forces*)

Q1=FC1.D[rC1,q1[t]]+FC2.D[rC2,q1[t]];

Q2=FC1.D[rC1,q2[t]]+FC2.D[rC2,q2[t]];
(*Lagrange's equations*)

eq1=D[D[T,q1'[t],t]]-D[T,q1[t]]-Q1;

eq2=D[D[T,q2'[t],t]]-D[T,q2[t]]-Q2;
```

```
(*input data*)

inp={m1->1.,L1->1.,m2->1.,L2->1.,g->10.};

equation1=eq1/.inp;

equation2=eq2/.inp;

sol=NDSolve[{equation1==0,equation2==0,
q1[0]==.1,q2[0]==.1,q1'[0]==0.,q2'[0]==0.},
{q1,q2},{t,0.,40.},MaxSteps->2000];

Plot[Evaluate[q1[t]]/.sol,{t,0.,40.}]
Plot[Evaluate[q2[t]]/.sol,{t,0.,40.}]
```



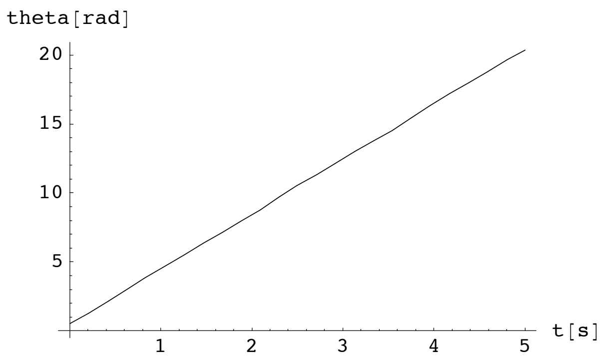
-Graphics-

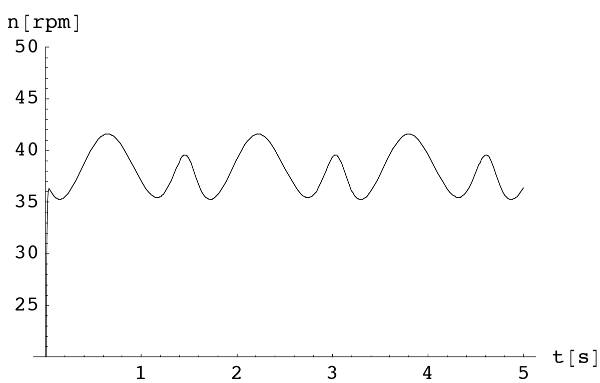


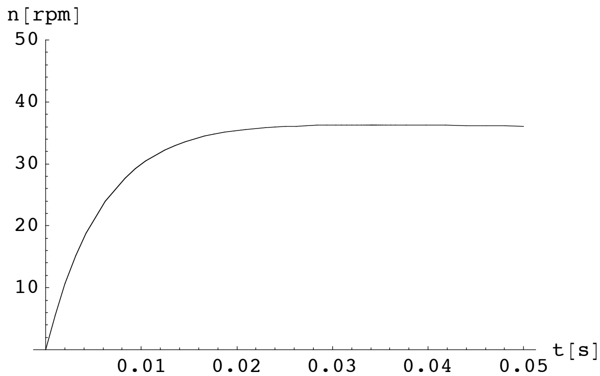
-Graphics-

```
(*Lagrange' s equations of motion  - example 3 *)
Apply[Clear, Names["Global`*"]];
Off[General::spell];

(*Input data*)

data = {L1 → .100, L2 → .470, L3 → .047, AC → .280, h → .01,
   h3 → .025, d → .005, d3 → .008, ro → 7850, g → 9.807, M0 → 1., w0 → 4.};

m1 = ro * L1 * h * d;
m2 = ro * L2 * h * d;
m3a = ro * L3 * h3 * d3;
m3b = ro * L3 * h * d;
m3 = m3a - m3b;
IC1 = m1 / 12 * (L1 ^ 2 + h ^ 2);
IC2 = m2 / 12 * (L2 ^ 2 + h ^ 2);
IC3 = m3a / 12 * (L3 ^ 2 + h3 ^ 2) - m3b / 12 * (L3 ^ 2 + h ^ 2);

(*Position,velocity and acceleration vectors*)

xB = L1 * Cos[theta[t]];
yB = L1 * Sin[theta[t]];
rB = {xB, yB, 0};
rC1 = rB / 2.;
vC1 = D[rC1, t];
xC = AC;
yC = 0;
rC = {xC, yC, 0};
theta2 = ArcTan[(yB - yC) / (xB - xC)];
rC2 = {xB + L2 * Cos[theta2] / 2., yB + L2 * Sin[theta2] / 2., 0};
vC2 = D[rC2, t];
rC3 = rC;
vC3 = {0, 0, 0};

(*Angular velocities*)

omega = {0, 0, theta'[t]};
omega2 = Simplify[{0, 0, D[theta2, t]}];
omega3 = omega2;

(*Kinetic energy*)

T1 = m1 * vC1.vC1 / 2. + IC1 * omega.omega / 2.;
T2 = m2 * vC2.vC2 / 2. + IC2 * omega2.omega2 / 2.;
T3 = m3 * vC3.vC3 / 2. + IC3 * omega3.omega3 / 2.;
T = Expand[T1 + T2 + T3];

(*Left hand side of the Lagrange equation*)

LHS = D[D[T, theta'[t]], t] - D[T, theta[t]];

(*Right hand side of the Lagrange equation (generalized forces)*)

G1 = {0, -m1 g, 0};
G2 = {0, -m2 g, 0};
G3 = {0, -m3 g, 0};
```

```
Mm = {0, 0, M0 (1. - theta'[t] / w0)};
RHS =
  D[rC1, theta[t]].G1 + D[rC2, theta[t]].G2 + D[rC3, theta[t]].G3 + D[omega, theta'[t]].Mm;


(*Solution of the Lagrange equation*)

eqnLHS = LHS /. data /. {theta'[t] → w[t], theta''[t] → w'[t]};

eqnRHS = RHS /. data /. {theta'[t] → w[t], theta''[t] → w'[t]};

solution = NDSolve[{eqnLHS == eqnRHS, theta'[t] == w[t], theta[0] == N[Pi] / 6, w[0] == 0},
   {theta[t], w[t]}, {t, 0, 5}];

Plot[Evaluate[theta[t] /. solution], {t, 0, 5}, AxesLabel → {"t[s]", "theta[rad]"}];
Plot[Evaluate[w[t] /. solution] * 30 / N[Pi], {t, 0, 5},
  AxesLabel → {"t[s]", "n[rpm]"}, PlotRange → {All, {20, 50}}];
Plot[Evaluate[w[t] /. solution] * 30 / N[Pi], {t, 0, .05},
  AxesLabel → {"t[s]", "n[rpm]"}, PlotRange → {All, {0, 50}}];
```






n[rpm]
50
40
30
20
10
0.01
0.02
0.03
0.04
0.05
t[s]

```
"Lagrange's equations of motion  - example 4 "
Apply [Clear,Names["Global`*"]];
Off[General::spell]
Off[General::spell1]

"kinematics"
"transformation matrix from RF1 to RF0"
R10 = {{1,0,0},
      {0,Cos[q1[t]],Sin[q1[t]]},
      {0,-Sin[q1[t]],Cos[q1[t]]}};
MatrixForm[R10]
"transformation matrix from RF2 to RF1"
R21={{Cos[q2[t]],0,-Sin[q2[t]]},
     {0,1,0},
     {Sin[q2[t]],0,Cos[q2[t]]}};
MatrixForm[R21]
"angular velocity of link 1 in RF0
expressed in terms of RF1 {i1,j1,k1}"
w10 = {D[q1[t],t],0,0}

"angular velocity of link 2 in RF0
expressed  in terms of RF1 {i1,j1,k1}"
w201 = {D[q1[t],t],D[q2[t],t],0}

"angular velocity of link 2 in RF0
expressed in terms of RF2 {i2,j2,k2}"
w202=w201.Transpose[R21]

"angular velocity of link 2 in RF0
expressed in terms of RF0 {i0,j0,k0}"
w200=w201.R10

"position vector of mass center C1 of link 1
in RF0 expressed in terms of RF1 {i1,j1,k1}"
rC1={L1,0,0}

"linear velocity of mass center C1 of link 1
in RF0 expressed in terms of RF1 {i1,j1,k1}"
vC1 =D[rC1,t]+Cross[w10,rC1]

"position vector of mass center C2 of link 2
in RF0 expressed in terms of RF2 {i2,j2,k2}"
rC2={L2,0,0}.Transpose[R21]

"position vector of mass center C2 of link 2
in RF0 expressed in terms of RF0 {i0,j0,k0}"
rC20={L2,0,0}.R10

"linear velocity of mass center C2 of link 2 in RF0
expressed in terms of RF2 {i2,j2,k2}"
vC2 =D[rC2,t]+Cross[w202,rC2]

"position vector of mass center C3 of link 3 in RF0
expressed in terms of RF2 {i2,j2,k2}"
rC3=rC2+{0,0,q3[t]}

"linear velocity of mass center C3 of link 3 in RF0
expressed in terms of RF2 {i2,j2,k2}"
vC3 =Expand[D[rC3,t]+Cross[w202,rC3]];
Simplify[vC3]

"position vector of mass center C3 of link 3 in RF0
expressed in terms of RF0 {i0,j0,k0}"
rC30=((rC2+{0,0,q3[t]}).R21).R10
```

```
vC30=D[rC30,t];
Expand[vC3.vC3,Trig->True]==Simplify[vC30.vC30];

"position vector of mass center of rigid body RB
in RF0 expressed in terms of RF2 {i2,j2,k2}"
rCR=rC3+{0,0,L/2}


"linear velocity of mass center of rigid body RB
in RF0 expressed in terms of RF2 {i2,j2,k2}"
vCR =Expand[D[rCR,t]+Cross[w202,rCR]];

Simplify[rCR];
Simplify[vCR]

" position vector of mass center CR of rigid body RB in RF0
expressed in terms of RF0 {i0,j0,k0} "
rCR0=(rCR.R21).R10
vCR0=D[rCR0,t];
Expand[vCR.vCR,Trig->True]==Simplify[vCR0.vCR0];

"inertia matrix associated to link 1
expressed in terms of RF1 {i1,j1,k1} "
I1={{I1x,0,0},{0,I1y,0},{0,0,I1z}};
MatrixForm[I1]

"inertia matrix associated to bar 2
expressed in terms of RF2 {i2,j2,k2} "
I2x=I2z=m2 l^2/12;
I2y=0;
I2={{I2x,0,0},{0,I2y,0},{0,0,I2z}};
MatrixForm[I2]

"inertia matrix associated to slider 2'
expressed in terms of RF2 {i2,j2,k2}"
I2S={{I2Sx,0,0},{0,I2Sy,0},{0,0,I2Sz}};
MatrixForm[I2S]

"inertia matrix associated to bar 3
expressed in terms of RF2 {i2,j2,k2} "
I3x=I3y=m3 L^2/12;
I3z=0;
I3={{I3x,0,0},{0,I3y,0},{0,0,I3z}};
MatrixForm[I3]


" kinetic energy "

" kinetic energy of link 1 "
T1=m1 vC1.vC1/2+ w10.I1.w10/2

" kinetic energy of link 2 "
T2=m2 vC2.vC2/2+ w202.I2.w202/2;
Simplify[T2]

" kinetic energy of slider 2' "
T2S=m2 vC2.vC2/2+ w202.I2S.w202/2

" kinetic energy of link 3 "
T3=m3 vC3.vC3/2+ w202.I3.w202/2;
Simplify[T3]

" kinetic energy of RB "
TR=Simplify[mR vCR.vCR/2,Trig->True]
```

```
" total kinetic energy "
Simplify[T1+T2+T2S+T3+TR]
T=Expand[T1+T2+T2S+T3+TR];

" gravitational force that acts on link 1 at C1
in RF0 expressed in terms of RF0 {i0,j0,k0} or in
in terms of RF1 {i1,j1,k1} "
G1={ -m1 g, 0, 0 }

" gravitational force that acts on bar 2 and
slider 2' at C2 in RF0 expressed in terms of
RF0 {i0,j0,k0} or RF1 {i1,j1,k1} "
G2={ -(m2+m2S) g, 0, 0 }

(* gravitational force that acts on bar 2 and
slider 2' at C2 in RF0 expressed in terms of
RF2 {i2,j2,k2} *)
G22={ -(m2+m2S) g, 0, 0 }.Transpose[R21];


" gravitational force that acts on link 3 at C3
in RF0 expressed in terms of RF0 {i0,j0,k0} "
G3={ -m3 g, 0 ,0 }

(* gravitational force that acts on link 3 at C3
in RF0 expressed in terms of RF2 {i2,j2,k2} *)
G32={ -m3 g, 0 ,0 }.Transpose[R21];

" gravitational force that acts on link 3 at CR
in RF0 expressed in terms of RF0 {i0,j0,k0} "
GR={ -mR g, 0, 0 }

(* gravitational force that acts on link 3 at C3
in RF0 expressed in terms of RF2 {i2,j2,k2} *)
GR2={ -mR g, 0, 0 }.Transpose[R21];

"generalized active force Qi=∑Fj.∂(rj)/∂(qi): Q1, Q2, Q3"

Q1=D[rC1,q1[t]].G1+D[rC20,q1[t]].G2+
   D[rC30,q1[t]].G3+D[rCR0,q1[t]].GR;

Q2=D[rC1,q2[t]].G1+D[rC20,q2[t]].G2+
   D[rC30,q2[t]].G3+D[rCR0,q2[t]].GR;

Q3=D[rC1,q3[t]].G1+D[rC20,q3[t]].G2+
   D[rC30,q3[t]].G3+D[rCR0,q3[t]].GR;


Simplify[Q1]
Simplify[Q2]
Simplify[Q3]


(*generalized active force*)


"generalized active force Qi=∑Fj.∂(vj)/∂(qi'): Q1, Q2, Q3"

F1=D[vC1,q1'[t]].G1+
   D[vC2,q1'[t]].G22+
   D[vC3,q1'[t]].G32+
   D[vCR,q1'[t]].GR2;

F2=D[vC1,q2'[t]].G1 +
   D[vC2,q2'[t]].G22+
```

```
   D[vC3,q2'[t]].G32+
   D[vCR,q2'[t]].GR2;

F3=D[vC1,q3'[t]].G1 +
   D[vC2,q3'[t]].G22+
   D[vC3,q3'[t]].G32+
   D[vCR,q3'[t]].GR2;

Expand[Q1]==Expand[F1];
Expand[Q2]==Expand[F2];
Expand[Q3]==Expand[F3];

Simplify[F1]
Simplify[F2]
Simplify[F3]



" Lagrange's eom "

Leq1=D[D[T,q1'[t]],t]-D[T,q1[t]]-Q1==0;
Leq2=D[D[T,q2'[t]],t]-D[T,q2[t]]-Q2==0;
Leq3=D[D[T,q3'[t]],t]-D[T,q3[t]]-Q2==0;

Simplify[Leq1]
Simplify[Leq2]
Simplify[Leq3]
```

Lagrange's equations of motion - example 4

kinematics

transformation matrix from RF1 to RF0

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}[q1[t]] & \text{Sin}[q1[t]] \\ 0 & -\text{Sin}[q1[t]] & \text{Cos}[q1[t]] \end{pmatrix}$$

transformation matrix from RF2 to RF1

$$\begin{pmatrix} \text{Cos}[q2[t]] & 0 & -\text{Sin}[q2[t]] \\ 0 & 1 & 0 \\ \text{Sin}[q2[t]] & 0 & \text{Cos}[q2[t]] \end{pmatrix}$$

angular velocity of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}

{q1′[t], 0, 0}

angular velocity of link 2 in RF0 expressed in terms of RF1 {i1,j1,k1}

{q1′[t], q2′[t], 0}

angular velocity of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}

{Cos[q2[t]] q1′[t], q2′[t], Sin[q2[t]] q1′[t]}

angular velocity of link 2 in RF0 expressed in terms of RF0 {i0,j0,k0}

{q1′[t], Cos[q1[t]] q2′[t], Sin[q1[t]] q2′[t]}

```
position vector of mass center C1
  of link 1  in RF0 expressed in terms of RF1 {i1,j1,k1}

{L1, 0, 0}

linear velocity of mass center C1
  of link 1  in RF0 expressed in terms of RF1 {i1,j1,k1}

{0, 0, 0}

position vector of mass center C2
  of link 2  in RF0 expressed in terms of RF2 {i2,j2,k2}

{L2 Cos[q2[t]], 0, L2 Sin[q2[t]]}

position vector of mass center C2
  of link 2  in RF0 expressed in terms of RF0 {i0,j0,k0}

{L2, 0, 0}

linear velocity of mass center C2 of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}

{0, 0, 0}

position vector of mass center C3 of link 3 in RF0 expressed in terms of RF2 {i2,j2,k2}

{L2 Cos[q2[t]], 0, q3[t] + L2 Sin[q2[t]]}

linear velocity of mass center C3 of link 3 in RF0 expressed in terms of RF2 {i2,j2,k2}

{q3[t] q2'[t], -Cos[q2[t]] q3[t] q1'[t], q3'[t]}

position vector of mass center C3 of link 3 in RF0 expressed in terms of RF0 {i0,j0,k0}

{L2 Cos[q2[t]]^2 + Sin[q2[t]] (q3[t] + L2 Sin[q2[t]]),
 -Sin[q1[t]] (-L2 Cos[q2[t]] Sin[q2[t]] + Cos[q2[t]] (q3[t] + L2 Sin[q2[t]])),
 Cos[q1[t]] (-L2 Cos[q2[t]] Sin[q2[t]] + Cos[q2[t]] (q3[t] + L2 Sin[q2[t]]))}

position vector of mass center of
  rigid body RB in RF0 expressed in terms of RF2 {i2,j2,k2}

{L2 Cos[q2[t]], 0, L/2 + q3[t] + L2 Sin[q2[t]]}

linear velocity of mass center of
  rigid body RB in RF0 expressed in terms of RF2 {i2,j2,k2}

{1/2 (L + 2 q3[t]) q2'[t], -1/2 Cos[q2[t]] (L + 2 q3[t]) q1'[t], q3'[t]}

 position vector of mass center CR of
  rigid body RB in RF0 expressed in terms of RF0 {i0,j0,k0}

{L2 Cos[q2[t]]^2 + Sin[q2[t]] (L/2 + q3[t] + L2 Sin[q2[t]]),
 -Sin[q1[t]] (-L2 Cos[q2[t]] Sin[q2[t]] + Cos[q2[t]] (L/2 + q3[t] + L2 Sin[q2[t]])),
 Cos[q1[t]] (-L2 Cos[q2[t]] Sin[q2[t]] + Cos[q2[t]] (L/2 + q3[t] + L2 Sin[q2[t]]))}
```

inertia matrix associated to link 1 expressed in terms of RF1 {i1,j1,k1}

$$\begin{pmatrix} I1x & 0 & 0 \\ 0 & I1y & 0 \\ 0 & 0 & I1z \end{pmatrix}$$

inertia matrix associated to bar 2 expressed in terms of RF2 {i2,j2,k2}

$$\begin{pmatrix} \frac{l^2\, m2}{12} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{l^2\, m2}{12} \end{pmatrix}$$

inertia matrix associated to slider 2' expressed in terms of RF2 {i2,j2,k2}

$$\begin{pmatrix} I2Sx & 0 & 0 \\ 0 & I2Sy & 0 \\ 0 & 0 & I2Sz \end{pmatrix}$$

inertia matrix associated to bar 3 expressed in terms of RF2 {i2,j2,k2}

$$\begin{pmatrix} \frac{L^2\, m3}{12} & 0 & 0 \\ 0 & \frac{L^2\, m3}{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

kinetic energy

kinetic energy of link 1

$$\frac{1}{2}\, I1x\, q1'[t]^2$$

kinetic energy of link 2

$$\frac{1}{24}\, l^2\, m2\, q1'[t]^2$$

kinetic energy of slider 2'

$$\frac{1}{2}\left(I2Sx\, Cos[q2[t]]^2\, q1'[t]^2 + I2Sz\, Sin[q2[t]]^2\, q1'[t]^2 + I2Sy\, q2'[t]^2\right)$$

kinetic energy of link 3

$$\frac{1}{24}\, m3\, \left(Cos[q2[t]]^2\, \left(L^2 + 12\, q3[t]^2\right) q1'[t]^2 + \left(L^2 + 12\, q3[t]^2\right) q2'[t]^2 + 12\, q3'[t]^2\right)$$

kinetic energy of RB

$$\frac{1}{8}\, mR\, \left(Cos[q2[t]]^2\, (L + 2\, q3[t])^2\, q1'[t]^2 + (L + 2\, q3[t])^2\, q2'[t]^2 + 4\, q3'[t]^2\right)$$

total kinetic energy

$$\frac{1}{24}\Big(\big(12\, I1x + \left(12\, I2Sx + l^2\, m2 + L^2\, (m3 + 3\, mR)\right) Cos[q2[t]]^2 + 12\, L\, mR\, Cos[q2[t]]^2\, q3[t] + 12\, (m3 + mR)\, Cos[q2[t]]^2\, q3[t]^2 + 12\, I2Sz\, Sin[q2[t]]^2 + l^2\, m2\, Sin[q2[t]]^2\big)\, q1'[t]^2 + \left(12\, I2Sy + L^2\, (m3 + 3\, mR) + 12\, L\, mR\, q3[t] + 12\, (m3 + mR)\, q3[t]^2\right) q2'[t]^2 + 12\, (m3 + mR)\, q3'[t]^2\Big)$$

gravitational force that acts on link 1 at C1 in RF0
expressed in terms of RF0 {i0,j0,k0} or in in terms of RF1 {i1,j1,k1}

$$\{-g\,m1,\ 0,\ 0\}$$

gravitational force that acts on bar 2 and slider 2' at
C2 in RF0 expressed in terms of RF0 {i0,j0,k0} or RF1 {i1,j1,k1}

$$\{-g\,(m2+m2S),\ 0,\ 0\}$$

gravitational force that acts on
link 3 at C3 in RF0 expressed in terms of RF0 {i0,j0,k0}

$$\{-g\,m3,\ 0,\ 0\}$$

gravitational force that acts on
link 3 at CR in RF0 expressed in terms of RF0 {i0,j0,k0}

$$\{-g\,mR,\ 0,\ 0\}$$

generalized active force Qi=∑Fj.∂(rj)/∂(qi): Q1, Q2, Q3

$$0$$

$$-\frac{1}{2}\,g\,\mathrm{Cos}[q2[t]]\,(L\,mR+2\,(m3+mR)\,q3[t])$$

$$-g\,(m3+mR)\,\mathrm{Sin}[q2[t]]$$

generalized active force Qi=∑Fj.∂(vj)/∂(qi'): Q1, Q2, Q3

$$0$$

$$-\frac{1}{2}\,g\,\mathrm{Cos}[q2[t]]\,(L\,mR+2\,(m3+mR)\,q3[t])$$

$$-g\,(m3+mR)\,\mathrm{Sin}[q2[t]]$$

Lagrange's eom

$$\begin{aligned}&(12\,I1x+(12\,I2Sx+l^2\,m2+L^2\,(m3+3\,mR))\,\mathrm{Cos}[q2[t]]^2+12\,L\,mR\,\mathrm{Cos}[q2[t]]^2\,q3[t]+\\&\quad 12\,(m3+mR)\,\mathrm{Cos}[q2[t]]^2\,q3[t]^2+12\,I2Sz\,\mathrm{Sin}[q2[t]]^2+l^2\,m2\,\mathrm{Sin}[q2[t]]^2)\,q1''[t]==\\&2\,\mathrm{Cos}[q2[t]]\,q1'[t]\,((12\,I2Sx-12\,I2Sz+L^2\,m3+3\,L^2\,mR+12\,L\,mR\,q3[t]+12\,(m3+mR)\,q3[t]^2)\\&\quad \mathrm{Sin}[q2[t]]\,q2'[t]-6\,\mathrm{Cos}[q2[t]]\,(L\,mR+2\,(m3+mR)\,q3[t])\,q3'[t])\end{aligned}$$

$$\begin{aligned}&(12\,I2Sx-12\,I2Sz+L^2\,(m3+3\,mR))\,\mathrm{Sin}[2\,q2[t]]\,q1'[t]^2+\\&\quad 24\,(m3+mR)\,q3[t]^2\,(\mathrm{Cos}[q2[t]]\,\mathrm{Sin}[q2[t]]\,q1'[t]^2+q2''[t])+\\&\quad 24\,q3[t]\,(g\,m3\,\mathrm{Cos}[q2[t]]+g\,mR\,\mathrm{Cos}[q2[t]]+\\&\qquad L\,mR\,\mathrm{Cos}[q2[t]]\,\mathrm{Sin}[q2[t]]\,q1'[t]^2+2\,(m3+mR)\,q2'[t]\,q3'[t]+L\,mR\,q2''[t])+\\&\quad 2\,(6\,g\,L\,mR\,\mathrm{Cos}[q2[t]]+12\,L\,mR\,q2'[t]\,q3'[t]+(12\,I2Sy+L^2\,(m3+3\,mR))\,q2''[t])==0\end{aligned}$$

$$\begin{aligned}&g\,L\,mR\,\mathrm{Cos}[q2[t]]+2\,(m3+mR)\,q3''[t]==L\,mR\,(\mathrm{Cos}[q2[t]]^2\,q1'[t]^2+q2'[t]^2)+\\&\quad 2\,(m3+mR)\,q3[t]\,(-g\,\mathrm{Cos}[q2[t]]+\mathrm{Cos}[q2[t]]^2\,q1'[t]^2+q2'[t]^2)\end{aligned}$$

```
(* Example 5*)
"LAGRANGE's equations of motion  - 3 DOF Robot "
Apply [Clear,Names["Global`*"]];
Off[General::spell];
Off[General::spell1];

"transformation matrix from RF1 to RF0: R10="
R10 = {{1,0,0},
      {0,Cos[q1[t]],Sin[q1[t]]},
      {0,-Sin[q1[t]],Cos[q1[t]]}};
MatrixForm[R10]

"transformation matrix from RF2 to RF1: R21="
R21={{Cos[q2[t]],0,-Sin[q2[t]]},
     {0,1,0},
     {Sin[q2[t]],0,Cos[q2[t]]}};
MatrixForm[R21]

"angular velocity of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}: w10="
w10 = {D[q1[t],t],0,0}

"angular velocity of link 2 in RF0
expressed  in terms of RF1 {i1,j1,k1}: w201="
w201 = {D[q1[t],t],D[q2[t],t],0}

"angular velocity of link 2 in RF0
expressed in terms of RF2 {i2,j2,k2}: w20="
w20=w201.Transpose[R21]

"angular acceleration of link 1 in RF0
expressed in terms of RF1 {i1,j1,k1}: α10="
α10=D[w10,t]

"angular acceleration of link 2 in RF0
expressed in terms of RF2 {i2,j2,k2}: α20="
α20=D[w20,t]

"position vector of mass center C1 of link 1
in RF0 expressed in terms of RF1 {i1,j1,k1}: rC1="
rC1={0,0,L1}

"linear velocity of mass center C1 of link 1
in RF0 expressed in terms of RF1 {i1,j1,k1}: vC1="
vC1 =D[rC1,t]+Cross[w10,rC1]

"linear velocity of joint B in RF0
expressed in terms of RF1 {i1,j1,k1}: vB="
vB =D[{0,0, 2 L1},t]+Cross[w10,{0,0,2 L1}]

"position vector of mass center C2 of link 2
in RF0 expressed in terms of RF2 {i2,j2,k2}: rC2="
rC2={0,0,2 L1}.Transpose[R21]+{0,0,L2}

"linear velocity of mass center C2 of link 2 in RF0
expressed in terms of RF2 {i2,j2,k2}: vC2="
vC2 =D[rC2,t]+Cross[w20,rC2]

"position vector of mass center C3 of link 3 in RF0
expressed in terms of RF2 {i2,j2,k2}: rC3="
rC3=rC2+{0,0,q3[t]}

"linear velocity of  C32 of link 2
expressed in terms of RF2 {i2,j2,k2}
C32 of link 2 is superposed with C3 of link 3: vC32="
```

```
vC32 =vC2+Cross[w20,{0,0,q3[t]}]

"linear velocity of mass center C3 of link 3 in RF0
expressed in terms of RF2 {i2,j2,k2}: vC3="
vC3 =D[rC3,t]+Cross[w20,rC3]

(* another way of computing vC3 is: *)
vC3'=vC32+D[{0,0,q3[t]},t];
(*vC3-vC3'={0,0,0};*)

"linear acceleration of mass center C1 of link 1
in RF0 expressed in terms of RF1 {i1,j1,k1}: aC1="
aC1 =D[vC1,t]+Cross[w10,vC1]

"linear acceleration of mass center C2 of link 2
in RF0 expressed in terms of RF2 {i2,j2,k2}: aC2="
aC2 =D[vC2,t]+Cross[w20,vC2]

"linear acceleration of mass center C3 of link 3
in RF0 expressed in terms of RF2 {i2,j2,k2}: aC3="
aC3 =D[vC3,t]+Cross[w20,vC3]

"gravitational force that acts on link 1 at C1
in RF0 expressed in terms of RF1 {i1,j1,k1}: G1="
G1={ -m1 g ,0 , 0 }
"gravitational force that acts on link 2 at C2
in RF0 expressed in terms of RF2 {i2,j2,k2}: G2="
G2={ -m2 g ,0 , 0 }.Transpose[R21]
"gravitational force that acts on link 3 at C3
in RF0 expressed in terms of RF2 {i2,j2,k2}: G3="
G3={ -m3 g ,0 , 0 }.Transpose[R21]

"contact torque of 0 that acts on link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}:
T01="
T01={T01x,T01y,T01z}

"contact torque of link 1 that acts on link 2 in RF0 expressed in \
terms of RF2 {i2,j2,k2}: T12="
T12={T12x,T12y,T12x}

"contact force of link 2 that acts on link 3 at C3 in RF0 expressed \
in terms of RF2 {i2,j2,k2}: F23="
F23={F23x,F23y,F23z}

"generalized active force Q1="
Q1=D[w10,q1'[t]].T01+
    D[vC1,q1'[t]].G1+
    D[w10,q1'[t]].Transpose[R21].(-T12)+
    D[w20,q1'[t]].T12+
    D[vC2,q1'[t]].G2+
    D[vC32,q1'[t]].(-F23)+
    D[vC3,q1'[t]].G3+
    D[vC3,q1'[t]].F23

"generalized active force Q2="
Q2=D[w10,q2'[t]].T01+
    D[vC1,q2'[t]].G1+
    D[w10,q2'[t]].Transpose[R21].(-T12)+
    D[w20,q2'[t]].T12+
    D[vC2,q2'[t]].G2+
    D[vC32,q2'[t]].(-F23)+
    D[vC3,q2'[t]].G3+
    D[vC3,q2'[t]].F23

"generalized active force Q3="
```

```
Q3=D[w10,q3'[t]].T01+
    D[vC1,q3'[t]].G1+
    D[w10,q3'[t]].Transpose[R21].(-T12)+
    D[w20,q3'[t]].T12+
    D[vC2,q3'[t]].G2+
    D[vC32,q3'[t]].(-F23)+
    D[vC3,q3'[t]].G3+
    D[vC3,q3'[t]].F23


(* inertia dyadics *)

"central inertia dyadic for link 1
expressed in terms of RF1 {i1,j1,k1}: I1="
I1={{m1(2 L1)^2/12,0,0},{0,m1(2 L1)^2/12,0},{0,0,0}}

"central inertia dyadic for link 2
expressed in terms of RF2 {i2,j2,k2}: I2="
I2={{m2(2 L2)^2/12,0,0},{0,m2(2 L2)^2/12,0},{0,0,0}}

"central inertia torque for link 3
expressed in terms of RF2 {i2,j2,k2}: I3="
I3={{I3x,0,0},{0,I3y,0},{0,0,I3z}}

"kinetic energy of link 1: T1="
T1=m1 vC1.vC1/2+w10.I1.w10/2
"kinetic energy of link 2: T2="
T2=m2 vC2.vC2/2+w20.I2.w20/2;
T2=Simplify[T2]
"kinetic energy of link 3: T3="
T3=m3 vC3.vC3/2+w20.I3.w20/2;
T2=Simplify[T2]
"total kinetic energy: T="
T=Expand[T1+T2+T3];
Simplify[T]


LHS1=D[D[T,q1'[t]],t]-D[T,q1[t]];

LHS2=D[D[T,q2'[t]],t]-D[T,q2[t]];

LHS3=D[D[T,q3'[t]],t]-D[T,q3[t]];

Lagr1=LHS1-Q1;
Lagr2=LHS2-Q2;
Lagr3=LHS3-Q3;

"First Lagrange's equation of motion"
"D[D[T,q1'[t]],t]-D[T,q1[t]]=Q1"
Simplify[Lagr1]
"Second Lagrange's equation of motion"
"D[D[T,q2'[t]],t]-D[T,q2[t]]=Q2"
Simplify[Lagr2]
"Third Lagrange's equation of motion"
"D[D[T,q3'[t]],t]-D[T,q3[t]]=Q3"
Simplify[Lagr3]


"numerical data"
indata={L1→0.4,L2→0.7,LB→0.8,
        I3x→5,I3y→4,I3z→1,
        m1→90,m2→60,m3→40,g→9.81,
    b01→450,g01→300,
    b12→200,g12→300,
    b23→150,g23→50,
```

```
    q1ref→N[Pi/3],q2ref→N[Pi/3],q3ref→0.25}

"control"
control={
T01x→-b01 q1'[t]-g01(q1[t]-q1ref),
T12y→-b12 q2'[t]-g12(q2[t]-q2ref)+g (m2 L2+m3 (L2+q3[t])) Cos[q2[t]],
F23z→-b23 q3'[t]-g23(q3[t]-q3ref)+g m3 Sin[q2[t]]
}/.indata

(*Lagrange's equations of motion*)

lageq={
(Lagr1/.indata)==0,(Lagr2/.indata)==0,(Lagr3/.indata)==0,
q1'[0]==0.,q2'[0]==0.,q3'[0]==0.,
q1[0]==N[Pi/18],q2[0]==N[Pi/6],q3[0]==0.1};

(*numerical simulation of Lagrange's eom*)

lagrange=NDSolve[lageq/.control,{q1,q2,q3},{t,0,15}]

Plot[Evaluate[q1[t]/.lagrange],{t,0,15},PlotRange→{All,All},
  AxesLabel→{"t[s]","q1[rad]"}]

Plot[Evaluate[q2[t]/.lagrange],{t,0,15},PlotRange→{All,All},
  AxesLabel→{"t[s]","q2[rad]"}]

Plot[Evaluate[q3[t]/.lagrange],{t,0,15},PlotRange→{All,All},
  AxesLabel→{"t[s]","q3[m]"}]
```

LAGRANGE's equations of motion - 3 DOF Robot

transformation matrix from RF1 to RF0: R10=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}[\text{q1}[t]] & \text{Sin}[\text{q1}[t]] \\ 0 & -\text{Sin}[\text{q1}[t]] & \text{Cos}[\text{q1}[t]] \end{pmatrix}$$

transformation matrix from RF2 to RF1: R21=

$$\begin{pmatrix} \text{Cos}[\text{q2}[t]] & 0 & -\text{Sin}[\text{q2}[t]] \\ 0 & 1 & 0 \\ \text{Sin}[\text{q2}[t]] & 0 & \text{Cos}[\text{q2}[t]] \end{pmatrix}$$

angular velocity of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}: w10=

{q1′[t], 0, 0}

angular velocity of link 2 in RF0 expressed in terms of RF1 {i1,j1,k1}: w201=

{q1′[t], q2′[t], 0}

angular velocity of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}: w20=

{Cos[q2[t]] q1′[t], q2′[t], Sin[q2[t]] q1′[t]}

angular acceleration of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}: α10=

{q1″[t], 0, 0}

angular acceleration of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}: α20=

$\{-\mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] + \mathrm{Cos}[q2[t]]\, q1''[t],$
$q2''[t], \mathrm{Cos}[q2[t]]\, q1'[t]\, q2'[t] + \mathrm{Sin}[q2[t]]\, q1''[t]\}$

```
position vector of mass center C1 of
  link 1  in RF0 expressed in terms of RF1 {i1,j1,k1}: rC1=
```

$\{0, 0, L1\}$

```
linear velocity of mass center C1 of
  link 1  in RF0 expressed in terms of RF1 {i1,j1,k1}: vC1=
```

$\{0, -L1\, q1'[t], 0\}$

```
linear velocity of joint B in RF0 expressed in terms of RF1 {i1,j1,k1}: vB=
```

$\{0, -2\, L1\, q1'[t], 0\}$

```
position vector of mass center C2 of
  link 2  in RF0 expressed in terms of RF2 {i2,j2,k2}: rC2=
```

$\{-2\, L1\, \mathrm{Sin}[q2[t]], 0, L2 + 2\, L1\, \mathrm{Cos}[q2[t]]\}$

```
linear velocity of mass center C2 of
  link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}: vC2=
```

$\{L2\, q2'[t], -L2\, \mathrm{Cos}[q2[t]]\, q1'[t] - 2\, L1\, \mathrm{Cos}[q2[t]]^2\, q1'[t] - 2\, L1\, \mathrm{Sin}[q2[t]]^2\, q1'[t], 0\}$

```
position vector of mass center C3 of
  link 3 in RF0 expressed in terms of RF2 {i2,j2,k2}: rC3=
```

$\{-2\, L1\, \mathrm{Sin}[q2[t]], 0, L2 + 2\, L1\, \mathrm{Cos}[q2[t]] + q3[t]\}$

```
linear velocity of  C32 of link 2 expressed in terms of
  RF2 {i2,j2,k2}  C32 of link 2 is superposed with C3 of link 3: vC32=
```

$\{L2\, q2'[t] + q3[t]\, q2'[t], -L2\, \mathrm{Cos}[q2[t]]\, q1'[t] -$
$2\, L1\, \mathrm{Cos}[q2[t]]^2\, q1'[t] - \mathrm{Cos}[q2[t]]\, q3[t]\, q1'[t] - 2\, L1\, \mathrm{Sin}[q2[t]]^2\, q1'[t], 0\}$

```
linear velocity of mass center C3 of
  link 3 in RF0 expressed in terms of RF2 {i2,j2,k2}: vC3=
```

$\{L2\, q2'[t] + q3[t]\, q2'[t], -L2\, \mathrm{Cos}[q2[t]]\, q1'[t] -$
$2\, L1\, \mathrm{Cos}[q2[t]]^2\, q1'[t] - \mathrm{Cos}[q2[t]]\, q3[t]\, q1'[t] - 2\, L1\, \mathrm{Sin}[q2[t]]^2\, q1'[t], q3'[t]\}$

```
linear acceleration of mass center C1 of
  link 1  in RF0 expressed in terms of RF1 {i1,j1,k1}: aC1=
```

$\{0, -L1\, q1''[t], -L1\, q1'[t]^2\}$

```
linear acceleration of mass center C2 of
  link 2  in RF0 expressed in terms of RF2 {i2,j2,k2}: aC2=
```

$\{L2\, \mathrm{Cos}[q2[t]]\, \mathrm{Sin}[q2[t]]\, q1'[t]^2 + 2\, L1\, \mathrm{Cos}[q2[t]]^2\, \mathrm{Sin}[q2[t]]\, q1'[t]^2 +$
$2\, L1\, \mathrm{Sin}[q2[t]]^3\, q1'[t]^2 + L2\, q2''[t], 2\, L2\, \mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] - L2\, \mathrm{Cos}[q2[t]]\, q1''[t] -$
$2\, L1\, \mathrm{Cos}[q2[t]]^2\, q1''[t] - 2\, L1\, \mathrm{Sin}[q2[t]]^2\, q1''[t], -L2\, \mathrm{Cos}[q2[t]]^2\, q1'[t]^2 -$
$2\, L1\, \mathrm{Cos}[q2[t]]^3\, q1'[t]^2 - 2\, L1\, \mathrm{Cos}[q2[t]]\, \mathrm{Sin}[q2[t]]^2\, q1'[t]^2 - L2\, q2'[t]^2\}$

```
linear acceleration of mass center C3 of
  link 3  in RF0 expressed in terms of RF2 {i2,j2,k2}: aC3=
```

$$\{L2\,Cos[q2[t]]\,Sin[q2[t]]\,q1'[t]^2 + 2\,L1\,Cos[q2[t]]^2\,Sin[q2[t]]\,q1'[t]^2 + Cos[q2[t]]\,q3[t]\,Sin[q2[t]]\,q1'[t]^2 + 2\,L1\,Sin[q2[t]]^3\,q1'[t]^2 + 2\,q2'[t]\,q3'[t] + L2\,q2''[t] + q3[t]\,q2''[t],\ 2\,L2\,Sin[q2[t]]\,q1'[t]\,q2'[t] + 2\,q3[t]\,Sin[q2[t]]\,q1'[t]\,q2'[t] - 2\,Cos[q2[t]]\,q1'[t]\,q3'[t] - L2\,Cos[q2[t]]\,q1''[t] - 2\,L1\,Cos[q2[t]]^2\,q1''[t] - Cos[q2[t]]\,q3[t]\,q1''[t] - 2\,L1\,Sin[q2[t]]^2\,q1''[t],\ -L2\,Cos[q2[t]]^2\,q1'[t]^2 - 2\,L1\,Cos[q2[t]]^3\,q1'[t]^2 - Cos[q2[t]]^2\,q3[t]\,q1'[t]^2 - 2\,L1\,Cos[q2[t]]\,Sin[q2[t]]^2\,q1'[t]^2 - L2\,q2'[t]^2 - q3[t]\,q2'[t]^2 + q3''[t]\}$$

```
gravitational force that acts on link
  1 at C1 in RF0 expressed in terms of RF1 {i1,j1,k1}: G1=
```

$\{-g\,m1,\ 0,\ 0\}$

```
gravitational force that acts on link
  2 at C2 in RF0 expressed in terms of RF2 {i2,j2,k2}: G2=
```

$\{-g\,m2\,Cos[q2[t]],\ 0,\ -g\,m2\,Sin[q2[t]]\}$

```
gravitational force that acts on link
  3 at C3 in RF0 expressed in terms of RF2 {i2,j2,k2}: G3=
```

$\{-g\,m3\,Cos[q2[t]],\ 0,\ -g\,m3\,Sin[q2[t]]\}$

```
contact torque of 0 that acts on link
  1 in RF0 expressed in terms of RF1 {i1,j1,k1}: T01=
```

$\{T01x,\ T01y,\ T01z\}$

```
contact torque of link 1 that acts on
  link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}: T12=
```

$\{T12x,\ T12y,\ T12x\}$

```
contact force of link 2 that acts on link
  3 at C3 in RF0 expressed in terms of RF2 {i2,j2,k2}: F23=
```

$\{F23x,\ F23y,\ F23z\}$

```
generalized active force Q1=
```

$T01x$

```
generalized active force Q2=
```

$T12y - g\,L2\,m2\,Cos[q2[t]] - g\,m3\,Cos[q2[t]]\,(L2 + q3[t])$

```
generalized active force Q3=
```

$F23z - g\,m3\,Sin[q2[t]]$

```
central inertia dyadic for link 1  expressed in terms of RF1 {i1,j1,k1}: I1=
```

$$\left\{\left\{\frac{L1^2\,m1}{3},\ 0,\ 0\right\},\ \left\{0,\ \frac{L1^2\,m1}{3},\ 0\right\},\ \{0,\ 0,\ 0\}\right\}$$

central inertia dyadic for link 2 expressed in terms of RF2 {i2,j2,k2}: I2=

$$\left\{\left\{\frac{L2^2\,m2}{3},0,0\right\},\left\{0,\frac{L2^2\,m2}{3},0\right\},\{0,0,0\}\right\}$$

central inertia torque for link 3 expressed in terms of RF2 {i2,j2,k2}: I3=

{{I3x, 0, 0}, {0, I3y, 0}, {0, 0, I3z}}

kinetic energy of link 1: T1=

$$\frac{2}{3}\,L1^2\,m1\,q1'[t]^2$$

kinetic energy of link 2: T2=

$$\frac{1}{3}\,m2\,\left(\left(6\,L1^2+L2^2+6\,L1\,L2\,Cos[q2[t]]+L2^2\,Cos[2\,q2[t]]\right)q1'[t]^2+2\,L2^2\,q2'[t]^2\right)$$

kinetic energy of link 3: T3=

$$\frac{1}{3}\,m2\,\left(\left(6\,L1^2+L2^2+6\,L1\,L2\,Cos[q2[t]]+L2^2\,Cos[2\,q2[t]]\right)q1'[t]^2+2\,L2^2\,q2'[t]^2\right)$$

total kinetic energy: T=

$$\begin{aligned}&\frac{1}{12}\\&\Big(\big(3\,I3x+3\,I3z+8\,L1^2\,m1+24\,L1^2\,m2+4\,L2^2\,m2+24\,L1^2\,m3+3\,L2^2\,m3+24\,L1\,L2\,(m2+m3)\,Cos[q2[t]]+\\&\quad 3\,I3x\,Cos[2\,q2[t]]-3\,I3z\,Cos[2\,q2[t]]+4\,L2^2\,m2\,Cos[2\,q2[t]]+3\,L2^2\,m3\,Cos[2\,q2[t]]+\\&\quad 12\,m3\,Cos[q2[t]]\,(2\,L1+L2\,Cos[q2[t]])\,q3[t]+6\,m3\,Cos[q2[t]]^2\,q3[t]^2\big)\,q1'[t]^2+\\&\quad 2\,\left(3\,I3y+L2^2\,(4\,m2+3\,m3)+6\,L2\,m3\,q3[t]+3\,m3\,q3[t]^2\right)q2'[t]^2+6\,m3\,q3'[t]^2\Big)\end{aligned}$$

First Lagrange's equation of motion

D[D[T,q1'[t]],t]-D[T,q1[t]]=Q1

$$\begin{aligned}&\frac{1}{6}\,\Big(-6\,T01x-2\,q1'[t]\,\Big(2\,\big(6\,L1\,L2\,(m2+m3)+(3\,I3x-3\,I3z+L2^2\,(4\,m2+3\,m3))\,Cos[q2[t]]+\\&\qquad 6\,m3\,(L1+L2\,Cos[q2[t]])\,q3[t]+3\,m3\,Cos[q2[t]]\,q3[t]^2\big)\,Sin[q2[t]]\,q2'[t]-\\&\qquad 6\,m3\,Cos[q2[t]]\,(2\,L1+L2\,Cos[q2[t]]+Cos[q2[t]]\,q3[t])\,q3'[t]\Big)+\\&\quad (3\,I3x+3\,I3z+8\,L1^2\,m1+24\,L1^2\,m2+4\,L2^2\,m2+24\,L1^2\,m3+3\,L2^2\,m3+\\&\qquad 24\,L1\,L2\,(m2+m3)\,Cos[q2[t]]+3\,I3x\,Cos[2\,q2[t]]-\\&\qquad 3\,I3z\,Cos[2\,q2[t]]+4\,L2^2\,m2\,Cos[2\,q2[t]]+3\,L2^2\,m3\,Cos[2\,q2[t]]+\\&\qquad 12\,m3\,Cos[q2[t]]\,(2\,L1+L2\,Cos[q2[t]])\,q3[t]+6\,m3\,Cos[q2[t]]^2\,q3[t]^2)\,q1''[t]\Big)\end{aligned}$$

Second Lagrange's equation of motion

D[D[T,q2'[t]],t]-D[T,q2[t]]=Q2

$$\begin{aligned}&-T12y+g\,L2\,m2\,Cos[q2[t]]+g\,L2\,m3\,Cos[q2[t]]+\\&\frac{1}{3}\,\left(6\,L1\,L2\,(m2+m3)+(3\,I3x-3\,I3z+L2^2\,(4\,m2+3\,m3))\,Cos[q2[t]]\right)Sin[q2[t]]\,q1'[t]^2+\\&2\,L2\,m3\,q2'[t]\,q3'[t]+I3y\,q2''[t]+\frac{4}{3}\,L2^2\,m2\,q2''[t]+L2^2\,m3\,q2''[t]+\\&m3\,q3[t]^2\,\left(Cos[q2[t]]\,Sin[q2[t]]\,q1'[t]^2+q2''[t]\right)+\\&m3\,q3[t]\,\left(g\,Cos[q2[t]]+2\,(L1+L2\,Cos[q2[t]])\,Sin[q2[t]]\,q1'[t]^2+2\,q2'[t]\,q3'[t]+2\,L2\,q2''[t]\right)\end{aligned}$$

Third Lagrange's equation of motion

```
D[D[T,q3'[t]],t]-D[T,q3[t]]=Q3
```

$-F23z + g\,m3\,Sin[q2[t]] - m3\,Cos[q2[t]]\,(2\,L1 + L2\,Cos[q2[t]] + Cos[q2[t]]\,q3[t])\,q1'[t]^2 - m3\,(L2 + q3[t])\,q2'[t]^2 + m3\,q3''[t]$

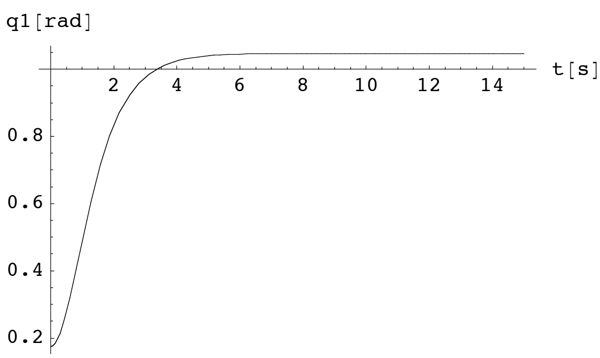
numerical data

{L1 → 0.4, L2 → 0.7, LB → 0.8, I3x → 5, I3y → 4, I3z → 1, m1 → 90,
m2 → 60, m3 → 40, g → 9.81, b01 → 450, g01 → 300, b12 → 200, g12 → 300,
b23 → 150, g23 → 50, q1ref → 1.0472, q2ref → 1.0472, q3ref → 0.25}

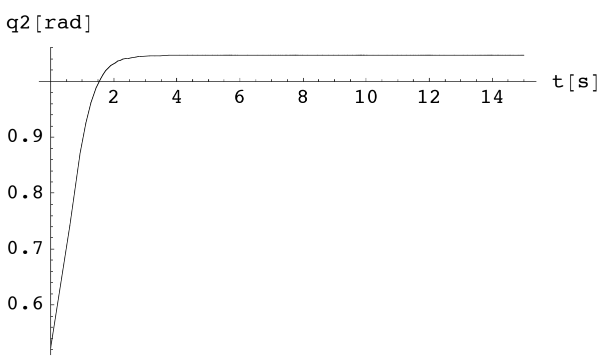
control

{T01x → -300 (-1.0472 + q1[t]) - 450 q1′[t],
T12y → -300 (-1.0472 + q2[t]) + 9.81 Cos[q2[t]] (42. + 40 (0.7 + q3[t])) - 200 q2′[t],
F23z → -50 (-0.25 + q3[t]) + 392.4 Sin[q2[t]] - 150 q3′[t]}

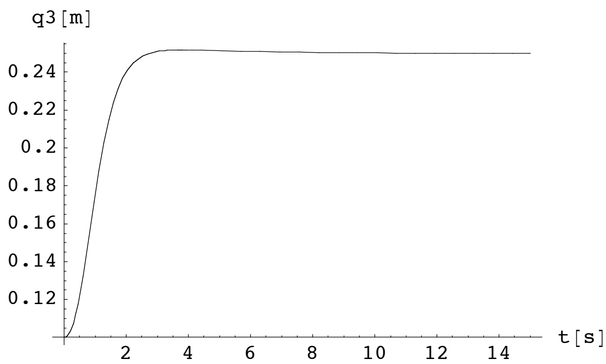
{{q1 → InterpolatingFunction[{{0., 15.}}, <>],
q2 → InterpolatingFunction[{{0., 15.}}, <>],
q3 → InterpolatingFunction[{{0., 15.}}, <>]}}



- Graphics -



- Graphics -



- Graphics -