$$\cancel{mgl + Ka^2 - ml^2\omega^2 = \mp Ka^2}$$

either $mgl + \cancel{Ka^2} - ml^2\omega^2 = \cancel{Ka^2}$

$$\cancel{m}g\cancel{l} = \cancel{m}l^{\cancel{2}}\omega^2 \longrightarrow \omega^2 = \frac{g}{l} \longrightarrow \omega = \sqrt{\frac{g}{l}}$$

or

$$mgl + Ka^2 - ml^2\omega^2 = -Ka^2$$

$$ml^2\omega^2 = mgl + 2Ka^2 \longrightarrow \omega^2 = \frac{\cancel{m}g\cancel{l}}{\cancel{m}l^{\cancel{2}}} + \frac{2Ka^2}{ml^2}$$

$$\therefore \omega = \sqrt{\frac{g}{l} + \frac{2K}{m}\left(\frac{a}{l}\right)^2}$$

$$\therefore \omega_1 = \sqrt{\frac{g}{l}} \qquad \omega_2 = \sqrt{\frac{g}{l} + \frac{2K}{m}\left(\frac{a}{l}\right)^2}$$

Substituting $\omega_1^2 = \frac{g}{l}$ in Eq. ③

$$\begin{bmatrix} mgl + Ka^2 - ml^{\cancel{2}}\frac{g}{\cancel{l}} & -Ka^2 \\ -Ka^2 & mgl + Ka^2 - ml^{\cancel{2}}\frac{g}{\cancel{l}} \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} Ka^2 & -Ka^2 \\ -Ka^2 & Ka^2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$Ka^2\Theta_1 - Ka^2\Theta_2 = 0 \longrightarrow \frac{\Theta_1}{\Theta_2} = \frac{Ka^2}{Ka^2} = \frac{1}{1}$$

$$-Ka^2\Theta_1 + Ka^2\Theta_2 = 0 \longrightarrow \frac{\Theta_1}{\Theta_2} = \frac{Ka^2}{Ka^2} = \frac{1}{1}$$

$\therefore$ The first mode shape $\left(\frac{\Theta_1}{\Theta_2}\right)^{(1)} = 1.0$ corresponding to the first (fundamental) natural frequency

$$\omega_1 = \sqrt{\frac{g}{l}}$$

Substituting $\omega_2^2 = \frac{g}{l} + \frac{2k}{m}\frac{a^2}{l^2}$ into Eq. (5)

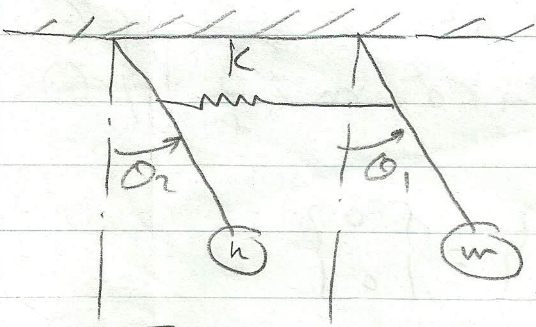
$$\begin{bmatrix} mgl + ka^2 - ml^2 * \left(\frac{g}{l} + \frac{2k}{m}\frac{a^2}{l^2}\right) & -ka^2 \\ -ka^2 & mgl + ka^2 - ml^2\left(\frac{g}{l} + \frac{2k}{m}\frac{a^2}{l^2}\right) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} \cancel{mgl} + ka^2 - \cancel{mgl} - 2ka^2 & -ka^2 \\ -ka^2 & \cancel{mgl} + ka^2 - \cancel{mgl} - 2ka^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

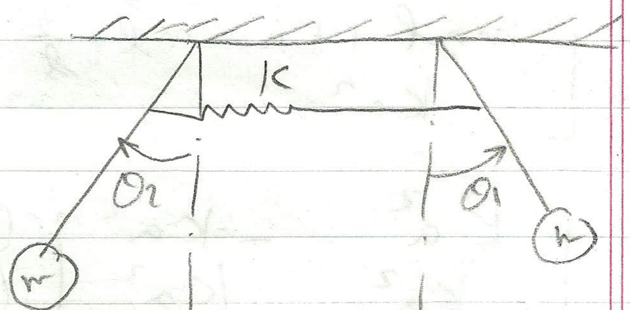
$$\begin{bmatrix} -ka^2 & -ka^2 \\ -ka^2 & -ka^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\therefore -ka^2\theta_1 - ka^2\theta_2 = 0 \longrightarrow \frac{\theta_1}{\theta_2} = -\frac{ka^2}{ka^2} = -1$

The second mode shape $\left(\frac{\theta_1}{\theta_2}\right)^{(2)} = -1$



First mode shape
unstreched spring



Second mode shape
stretched spring