(6)

$$\frac{\partial KE}{\partial \theta_2} = \text{zero}$$

$$\frac{\partial PE}{\partial \theta_2} = \frac{1}{2} k_2 \times 2 (\theta_1 - \theta_2) \times -1 + \frac{1}{2} k_3 \times 2 \theta_2$$

$$= K_2 \theta_2 - K_2 \theta_1 + K_3 \theta_2 = -K_2 \theta_1 + \theta_2 (K_2 + K_3)$$

$\therefore J_2 \ddot{\theta}_2 - K_2 \theta_1 + \theta_2 (K_2 + K_3) = 0 \longrightarrow$ The same as in Eq. (2)

Arranging Eqs. (1) and (2) in matrix form

$$\underbrace{\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}}_{\text{mass matrix}} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \underbrace{\begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 + K_3 \end{bmatrix}}_{\text{stiffness matrix}} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$J \rightarrow m$ $\quad K_1$ – rotational stiffness N.m/rad

Mass matrix is diagonal

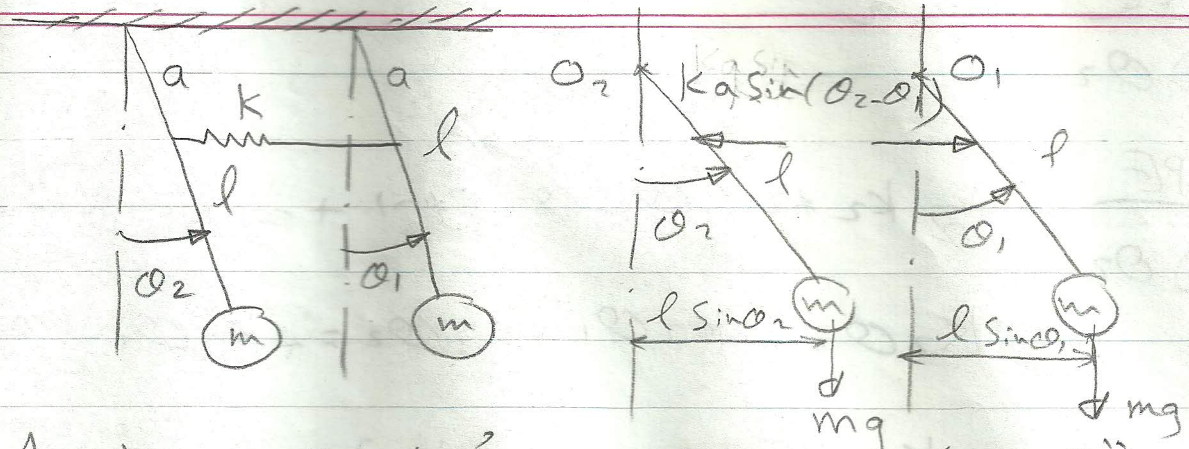
Stiffness matrix is symmetric about the diagonal

$$\therefore [K]^T = [K] \quad \underline{[M]^T = [M]} \quad [M]^{-1} = \left[\frac{1}{M}\right]$$

Coupled Pendulum

Ex. 5.1.3 page (130) (Thomson)

In the Fig. below, the two pendulums are coupled by means of a weak spring K, which is unstrained when the two pendulum rods are in the vertical position. Determine the normal mode vibrations




Applying Newton's second law $\sum M_{O_1} = J_1\ddot{\theta}_1$

$$J_1 = m_1 k^2 = m(l\cos\theta_1)^2$$

$$-mgl\sin\theta_1 + Ka\sin(\theta_2-\theta_1) \times a\cos(\theta_2-\theta_1) = ml^2\cos^2\theta_1\,\ddot{\theta}_1$$

For small oscillation $\sin\theta_1 = \theta_1$ $\cos\theta_1 = 1$

$\sin(\theta_2-\theta_1) = \theta_2-\theta_1$ $\cos\theta_2-\theta_1 = 1$

$$-mgl\theta_1 + Ka^2(\theta_2-\theta_1) = ml^2\ddot{\theta}_1$$

$$ml^2\ddot{\theta}_1 + mgl\theta_1 - Ka^2(\theta_2-\theta_1) = 0$$

$$ml^2\ddot{\theta}_1 + mgl\theta_1 - Ka^2\theta_2 + ka^2\theta_1 = 0$$

$$ml^2\ddot{\theta}_1 + \theta_1(mgl + Ka^2) - Ka^2\theta_2 = 0 \quad \text{--- (1)}$$

$\sum M_{O2} = J_2\ddot{\theta}_2$ $J_2 = m(l\cos\theta_2)^2$

$$-mgl\sin\theta_2 - Ka\sin(\theta_2-\theta_1) \times a\cos(\theta_2-\theta_1) = ml^2\cos^2\theta_2\,\ddot{\theta}_2$$

For small oscillation $\sin\theta_1 = \theta_1$ $\cos\theta_1 = 1$

$\sin(\theta_2-\theta_1) = \theta_2-\theta_1$ $\cos(\theta_2-\theta_1) = 1$

$$-mgl\theta_2 - Ka^2(\theta_2-\theta_1) = ml^2\ddot{\theta}_2$$

$$ml^2\ddot{\theta}_2 + mgl\theta_2 + Ka^2\theta_2 - Ka^2\theta_1 = 0$$

$$ml^2\ddot{\theta}_2 - Ka^2\theta_1 + \theta_2(mgl + Ka^2) = 0 \quad \text{--- (2)}$$