Infinite Series

Infinite series are sequences of a special kind: those in which the ***n***th-term is the sum of the first ***n*** terms of a related sequence.

Example

Suppose that we start with the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$$

If we denote the above sequence as a_n, and the resultant sequence of the series as s_n, then

$$s_1 = a_1 = 1,$$

$$s_2 = a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2},$$

$$s_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4},$$

as the first three terms of the sequence $\{s_n\}$.

When the sequence $\{s_n\}$ is formed in this way from a given sequence $\{a_n\}$ by the rule

$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

the result is called an ***Infinite Series.***

- ❖ The number $s_n = \sum_{k=1}^{n} a_k$ is called the ***n***th ***partial sum*** of the series.
- ❖ Instead of $\{s_n\}$, we usually write $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$.
- ❖ The series $\sum a_n$ is said to ***converge*** to a number L if and only if

$$L = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

in which case we call L the sum of the series and write

$$\sum_{n=1}^{\infty} a_n = L \qquad \text{or} \qquad a_1 + a_2 + \ldots + a_n + \ldots = L$$

If no such limit exists, the series is said to ***diverge***.

Geometric Series

A series of the form

$$a + ar + ar^2 + ar^3 + \ldots + +ar^{n-1} + \ldots$$

is called a ***Geometric Series***. The ratio of any term to the one before it is r. If $|r| < 1$, the geometric series converges to $a/(1-r)$. If $|r| \geq 1$, the series diverges unless $a = 0$. If $a = 0$, the series converges to 0.

Example

Geometric series with $a = \frac{1}{9}$ and $r = \frac{1}{3}$.

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots = \frac{1}{9}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\right) = \frac{1/9}{1-(1/3)} = \frac{1}{6}$$

Geometric series with $a = 4$ and $r = -\frac{1}{2}$.

$$4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \ldots = 4\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \ldots\right)$$

$$= \frac{4}{1+(1/2)} = \frac{8}{3}$$

Example

Determine whether each series converges or diverges. If it converges, find its sum.

(a) $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$, (b) $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$, (c) $\sum_{n=1}^{\infty}2\left(\cos\frac{\pi}{3}\right)^n$, (d) $\sum_{n=0}^{\infty}\left(\tan\frac{\pi}{4}\right)^n$, (e) $\sum_{n=1}^{\infty}\frac{5(-1)^n}{4^n}$

Solution

(a) Since the series is a geometric series with $r=\frac{2}{3}<1$, so the series is convergent with a sum of $\frac{1}{1-(2/3)}=3$

(b) Since the series is a geometric series with $r=\frac{3}{2}>1$, so the series is divergent.

(c) $\cos\pi/3=1/2$. This is a geometric series with first term $a_1=1$ and the ratio $r=1/2$; so the series converges and its sum is $1/(1-\frac{1}{2})=2$.

(d) $\tan\pi/4=1$. This is a geometric series with $r=1$, so the series diverges.

(e) This is a geometric series with first term $a_1=-5/4$ and ratio $r=-1/4$. So the series converges and its sum is $\frac{-5/4}{1+(1/4)}=-1$.

Test Convergence of Series with Non-negative Terms

1) The n^{th}- Term Test

- If $\lim_{n\to\infty}a_n\neq 0$, or if $\lim_{n\to\infty}a_n$ fails to exist, then $\sum_{n=1}^{\infty}a_n$ diverges.
- If $\sum_{n=1}^{\infty}a_n$ converges, then $a_n\to 0$.
- If $\lim_{n\to\infty}a_n=0$, then the test fails.

From the above, it can not be concluded that if $a_n \to 0$ then $\sum_{n=1}^{\infty} a_n$ converges. The series $\sum_{n=1}^{\infty} a_n$ may diverge even though $a_n \to 0$. Thus $\lim_{n\to\infty} a_n = 0$ is a necessary but not a sufficient condition for the series $\sum_{n=1}^{\infty} a_n$ to converge.

Examples

$\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$,

$\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1 \neq 0$,

$\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty}(-1)^{n+1}$ does not exist,

$\sum_{n=1}^{\infty} \frac{n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{n}{2n+5} = \frac{1}{2} \neq 0$,

$\sum_{n=1}^{\infty} \frac{1}{n}$ can not be tested by the nth-term test for divergence because $\frac{1}{n} \to 0$.

2) The Integral Test

Let the function $y = f(x)$, obtained by introducing the continuous variable x in place of the discrete variable n in the nth-term of the positive series $\sum_{n=1}^{\infty} a_n$, then

$$\int_1^{\infty} f(x)dx = \begin{cases} +\infty & Div. \\ -\infty & Div. \\ -\infty < c < \infty & Conv. \end{cases}$$

Example

Prove that, for the p-series, if p is a real constant, the series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots$$

converges if $p>1$ and diverges if $p\le 1$.

Solution

To prove this, let

$$f(x)=\frac{1}{x^p}$$

Then, if $p>1$, we have

$$\int_1^{\infty}x^{-p}dx=\lim_{b\to\infty}\left.\frac{x^{-p+1}}{-p+1}\right|_1^b=\frac{1}{p-1}$$

which is finite. Hence, the p-series converges if $p>1$.

If $p=1$, which is called a harmonic series, we have

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots,$$

and the integral test is

$$\int_1^{\infty}x^{-1}dx=\lim_{b\to\infty}\ \ln x\Big|_1^b=+\infty$$

which diverges.

Finally, for $p<1$, then the terms of the series are greater than the corresponding terms of the divergent harmonic series. Hence the p-series diverges for $p<1$.

Thus, we have a convergence for $p>1$, but divergence for $p\le 1$.

Example

Test the convergence of

(a) $\sum_{n=1}^{\infty} \frac{1}{e^n}$, (b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Solution

(a) $\int_1^{\infty} e^{-x} dx = -e^{-x}\Big|_1^{\infty} = -(e^{-\infty} - e^{-1}) = \frac{1}{e}$ ***(Conv.)***

(b) $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \int_2^{\infty} \frac{1/x}{(\ln x)^2} dx = \frac{-1}{\ln x}\Big|_2^{\infty} = \frac{-1}{\infty} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$ ***(Conv.)***

3) The Ratio Test

Let $\sum a_n$ be a series with positive terms, and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

Then

- ❖ The series converges if $\rho < 1$,
- ❖ The series diverges if $\rho > 1$,
- ❖ The series may converge or it may diverge if $\rho = 1$. (Test fails)

The Ratio Test is often effective when the terms of the series contain factorials of expressions involving n or expressions raised to a power involving n.

Example

Test the following series for convergence or divergence , using the Ratio Test.

(a) $\sum_{n=1}^{\infty}\frac{n!n!}{(2n)!}$, (b) $\sum_{n=1}^{\infty}\frac{4^n n!n!}{(2n)!}$, (c) $\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$, (d) $\sum_{n=1}^{\infty}\frac{n!}{3^n}$, (e) $\sum_{n=1}^{\infty}\frac{n^n}{n!}$

Solution

(a) If $a_n=\frac{n!n!}{(2n)!}$, then $a_{n+1}=\frac{(n+1)!(n+1)!}{(2n+2)!}$ and

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)(2n+1)(2n)!}=\frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$=\frac{n+1}{4n+2}\to\frac{1}{4}<1 \qquad \textit{(Conv.)}$$

(b) If $a_n=\frac{4^n n!n!}{(2n)!}$, then $a_{n+1}=\frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)!}$ and

$$\frac{a_{n+1}}{a_n}=\frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)(2n+1)(2n)!}\times\frac{(2n)!}{4^n n!n!}=\frac{4(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$=\frac{2(n+1)}{2n+1}\to 1 \qquad \textit{(Test fails)}$$

(c) If $a_n=\frac{2^n+5}{3^n}$, then $a_{n+1}=\frac{2^{n+1}+5}{3^{n+1}}$ and

$$\frac{a_{n+1}}{a_n}=\frac{(2^{n+1}+5)/3^{n+1}}{(2^n+5)/3^n}=\frac{1}{3}\times\frac{2^{n+1}+5}{2^n+5}$$

$$=\frac{1}{3}\times\left(\frac{2+5\times 2^{-n}}{1+5\times 2^{-n}}\right)\to\frac{1}{3}\times\frac{2}{1}=\frac{2}{3}<1 \qquad \textit{(Conv.)}$$

(d) If $a_n = \dfrac{n!}{3^n}$, then $a_{n+1} = \dfrac{(n+1)!}{3^{n+1}}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3^{n+1}} \times \frac{3^n}{n!} = \frac{n+1}{3} \to \infty > 1 \qquad \textit{(Div.)}$$

(e) If $a_n = \dfrac{n^n}{n!}$, then $a_{n+1} = \dfrac{(n+1)^{n+1}}{(n+1)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n} = \frac{(n+1)^n (n+1) n!}{(n+1) n! n^n}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e^1 = 2.7 > 1 \qquad \textit{(Div.)}$$

4) The n^th Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n > n_0$ and suppose that

$$\sqrt[n]{a_n} \to \rho$$

Then

- ❖ The series converges if $\rho < 1$.
- ❖ The series diverges if $\rho > 1$.
- ❖ The test is not conclusive if $\rho = 1$.

Example

Test the convergence of the following series using the n^th Root Test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$, (c) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$, (d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$, (e) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

Solution

(a) $\sqrt[n]{\dfrac{1}{n^n}}=\dfrac{1}{n}\to 0<1$ ***(Conv.)***

(b) $\sqrt[n]{\dfrac{2^n}{n^2}}=\dfrac{2}{\sqrt[n]{n^2}}=\dfrac{2}{\left(\sqrt[n]{n}\right)^2}\to\dfrac{2}{1^2}=2>1$ ***(Div.)***

(c) $\sqrt[n]{\left(1-\dfrac{1}{n}\right)^n}=\left(1-\dfrac{1}{n}\right)\to 1$ ***(Test fails)***

(d) $\sqrt[n]{\left(\dfrac{n}{n+1}\right)^{n^2}}=\left(\dfrac{n}{n+1}\right)^{\frac{n^2}{n}}=\left(\dfrac{n}{n+1}\right)^n=\left(\dfrac{1}{1+1/n}\right)^n\to\dfrac{1}{e}=\dfrac{1}{2.7}<1$ ***(Conv.)***

(e) $\sqrt[n]{\left(\dfrac{2n}{n+1}\right)^n}=\dfrac{2n}{n+1}\to 2>1$ ***(Div.)***

Exercises on Series

Find the sum of the following series

1) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}$ ***Ans.*** $\dfrac{4}{5}$

2) $\displaystyle\sum_{n=1}^{\infty}\frac{7}{4^n}$ ***Ans.*** $\dfrac{7}{3}$

3) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)$ ***Ans.*** $\dfrac{23}{2}$

4) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2^n}+\frac{(-1)^n}{5^n}\right)$ ***Ans.*** $\dfrac{17}{6}$

5) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$ ***Ans.*** 1

6) $\displaystyle\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2}$ ***Ans.*** 5

7) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$ ***Ans.*** 1

8) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)} \right)$ ***Ans.*** $-\dfrac{1}{\ln 2}$

Which of the following series converges and which diverges? Find the sum of the convergent series.

1) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}} \right)^n$ ***Ans.*** *Converges,* $2+\sqrt{2}$

2) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n}$ ***Ans.*** *Converges,* 1

3) $\displaystyle\sum_{n=0}^{\infty} \cos(n\pi)$ ***Ans.*** *Diverges*

4) $\displaystyle\sum_{n=0}^{\infty} e^{-2n}$ ***Ans.*** *Converges,* $\dfrac{e^2}{e^2-1}$

5) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{10^n}$ ***Ans.*** *Converges,* $\dfrac{2}{9}$

6) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n-1}{3^n}$ ***Ans.*** *Converges,* $\dfrac{3}{2}$

7) $\sum_{n=0}^{\infty} \frac{n!}{1000^n}$ ***Ans.*** *Diverges*

8) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ ***Ans.*** *Diverges*

9) $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$ ***Ans.*** *Converges,* $\frac{\pi}{\pi - e}$

Which of the following series converges and which diverges?

1) $\sum_{n=1}^{\infty} \frac{1}{10^n}$ ***Ans. Converges (Geometric)***

2) $\sum_{n=1}^{\infty} \frac{n}{n+1}$ ***Ans. Diverges (*** n^{th} ***-term test)***

3) $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ ***Ans. Diverges (p-series)***

4) $\sum_{n=1}^{\infty} \frac{-1}{8^n}$ ***Ans. Converges (Geometric)***

5) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ ***Ans. Diverges (Integral Test)***

6) $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ ***Ans. Converges (Geometric)***

7) $\sum_{n=0}^{\infty} \frac{-2}{n+1}$ ***Ans. Diverges (Integral Test)***

8) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n+1}$ ***Ans.*** ***Diverges (n^{th}-term test)***

9) $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$ ***Ans.*** ***Diverges (n^{th}-term test)***

10) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$ ***Ans.*** ***Diverges (Geometric)***

11) $\displaystyle\sum_{n=3}^{\infty} \frac{(1/n)}{(\ln n)\sqrt{\ln^2 n - 1}}$ ***Ans.*** ***Converges (Integral Test)***

12) $\displaystyle\sum_{n=1}^{\infty} n \sin \frac{1}{n}$ ***Ans.*** ***Diverges (n^{th}-term test)***

13) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$ ***Ans.*** ***Converges (Integral Test)***

14) $\displaystyle\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}$ ***Ans.*** ***Converges (Integral Test)***

15) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n-1}$ ***Ans.*** ***Diverges (n^{th}-term test)***

16) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$ ***Ans.*** ***Converges (Ratio Test)***

17) $\displaystyle\sum_{n=1}^{\infty} n!\,e^{-n}$ ***Ans.*** ***Diverges (Ratio Test)***

18) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ ***Ans.*** ***Converges (Ratio Test)***

19) $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{3}{n}\right)^n$ ***Ans.*** ***Diverges (n^{th}-term test)***

20) $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$ ***Ans.*** ***Converges (Ratio Test)***

21) $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\,n!\,3^n}$ ***Ans.*** ***Converges (Ratio Test)***

22) $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$ ***Ans.*** ***Converges (Ratio Test)***

23) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$ ***Ans.*** ***Converges (Root Test)***

24) $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$ ***Ans.*** ***Diverges (Root Test)***

25) $\sum_{n=1}^{\infty} \frac{n^n}{2^{(n^2)}}$ ***Ans.*** ***Converges (Root Test)***

Alternating Series

A series in which the terms are alternately positive and negative.

Example

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots+\frac{(-1)^{n+1}}{n}+\ldots$$

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\frac{(-1)^{n}4}{2^{n}}+\ldots$$

$$1-2+3-4+5-6+\ldots+(-1)^{n+1}n+\ldots$$

The Convergence Test of Alternating Series

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1-u_2+u_3-u_4+\ldots$$

converges if all three of the following conditions are satisfied:

1) The u_n's are all positive.
2) $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3) $u_n \to 0$.

Example

The alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$$

satisfies the three requirements of convergence; it therefore converges.

Absolute Convergence

A series $\sum a_n$ ***converges absolutely*** (is ***absolutely convergent***) if the corresponding series of absolute values, $\sum |a_n|$, converges, i.e.,

$$\text{If } \sum_{n=1}^{\infty} |a_n| \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Example

The geometric series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$ converges absolutely because the corresponding series of absolute values $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ converges .

Conditional Convergence

A series that converges but does not converge absolutely ***converges conditionally***.

Example

The alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ does not converge absolutely. The corresponding series of absolute values $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is the divergent harmonic series.

Power Series

- A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

- A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n + \dots$$

in which the center a and the coefficients $c_0, c_1, c_2, \dots, c_n, \dots$ are constants.

Example

The series $\sum_{n=0}^{\infty} x^n$ is a geometric series with first term 1 and ratio x. It converges to

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots \qquad \text{for } |x| < 1$$

Convergence of Power Series

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ converges for $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.

The test of power series is done using the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \rho \begin{cases} < 1 & Conv. \\ > 1 & Div. \\ = 1 & Fails \end{cases}$$

Notes:

- Use the Ratio Test to find the interval where the series converges absolutely.
- If the interval of absolute convergence is finite, test the convergence or divergence at each endpoint. Use the integral test or the Alternating Series Test for endpoints.

- If the interval of absolute convergence is $|x-a|<R$, the series diverges for $|x-a|>R$ (it does not even converge conditionally), because the n^th^-term does not approach zero for those values of x.

Example

For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots,$ (b) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$

(c) $\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots,$ (d) $\sum_{n=0}^{\infty}n!x^n=1+x+2!x^2+3!x^3+\ldots$

Solution

(a) $\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{x^{n+1}}{n+1}\times\frac{n}{x^n}\right|=\frac{n}{n+1}|x|\rightarrow|x|.$

The series converges absolutely for $|x|<1$. It diverges if $|x|>1$ because the n^th^-term does not converge to zero. At $x=1$, we get the alternating harmonic series $1-1/2+1/3-1/4+\ldots$, which converges. At $x=-1$, we get $-1-1/2-1/3-1/4-\ldots$, the negative of the harmonic series; it diverges. So, the series converges for $-1<x\le 1$ and diverges elsewhere.

(b) $\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{x^{2n+1}}{2n+1}\times\frac{2n-1}{x^{2n-1}}\right|=\frac{2n-1}{2n+1}x^2\rightarrow x^2.$

The series converges absolutely for $x^2<1$. It diverges for $x^2>1$ because the n^th^-term does not converge to zero. At $x=1$, the series becomes $1-1/3+1/5-1/7+\ldots$, which converges because it satisfies the three conditions of convergence of alternating series. It also converges at $x=-1$ because it is again an alternating series

that satisfies the conditions for convergence. The value at $x=-1$ is the negative of the value at $x=1$. So, the series converges for $-1 \le x \le 1$ and diverges elsewhere.

(c) $\left|\dfrac{u_{n+1}}{u_n}\right| = \left|\dfrac{x^{n+1}}{(n+1)!}\cdot\dfrac{n!}{x^n}\right| = \dfrac{|x|}{n+1} \to 0$ for every x.

The series converges absolutely for all x.

(d) $\left|\dfrac{u_{n+1}}{u_n}\right| = \left|\dfrac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = (n+1)|x| \to \infty$ unless $x=0$.

The series diverges for all values of x except $x=0$.

Exercises on Alternating & Power Series

Which of the following series converges and which diverges?

1) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ ***Ans.*** *Converges*

2) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$ ***Ans.*** *Diverges,* $a_n \to \infty$

3) $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\ln n}$ ***Ans.*** *Converges*

4) $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\ln n}{\ln\left(n^2\right)}$ ***Ans.*** *Diverges,* $a_n \to \dfrac{1}{2}$

5) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$ ***Ans.*** *Converges*

Which of the following series converges absolutely, conditionally, and which diverges?

1) $\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$ ***Ans.*** *Converges absolutely*

2) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}}$ ***Ans.*** *Converges conditionally*

3) $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{3+n}{5+n}$ ***Ans.*** *Diverges,* $a_n \to 1$

4) $\sum_{n=1}^{\infty}(-1)^n n^2 \left(\frac{2}{3}\right)^n$ ***Ans.*** *Converges absolutely*

5) $\sum_{n=1}^{\infty}(-1)^n \frac{\tan^{-1} n}{n^2+1}$ ***Ans.*** *Converges absolutely*

6) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1}$ ***Ans.*** *Diverges,* $a_n \to 1$

7) $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ ***Ans.*** *Converges absolutely*

8) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n\sqrt{n}}$ ***Ans.*** *Converges absolutely*

9) $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)^n}{(2n)^n}$ ***Ans.*** *Converges absolutely*

10) $\sum_{n=1}^{\infty}(-1)^n \frac{(2n)!}{2^n n! n}$ ***Ans.*** *Diverges,* $a_n \to \infty$

Find the interval of convergence for the following series

1) $\sum_{n=0}^{\infty} x^n$ ***Ans.*** $-1 < x < 1$

2) $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$ ***Ans.*** $-\frac{1}{2} < x < 0$

3) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$ ***Ans.*** $-8 < x < 12$

4) $\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$ ***Ans.*** $-1 < x < 1$

5) $\sum_{n=1}^{\infty} \frac{x^n}{3^n n\sqrt{n}}$ ***Ans.*** $-3 \le x \le 3$

6) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ ***Ans.*** *For all* x

7) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$ ***Ans.*** *For all* x

8) $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$ ***Ans.*** $-1 \le x < 1$

9) $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$ ***Ans.*** $-8 < x < 2$

10) $\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{3^n}$ ***Ans.*** $-3 < x < 3$

11) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$ ***Ans.*** $-1 < x < 1$

12) $\sum_{n=1}^{\infty} n^n x^n$ ***Ans.*** $x = 0$

13) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{2^n n}$ ***Ans.*** $-4 < x \le 0$

Find the interval of convergence and the sum within this interval for the following series

1) $\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4n}$ ***Ans.*** $-1 < x < 3, \frac{4}{3+2x-x^2}$

2) $\sum_{n=0}^{\infty} \left(\frac{\sqrt{x}}{2} - 1\right)^n$ ***Ans.*** $0 < x < 16, \frac{2}{4-\sqrt{x}}$

3) $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$ ***Ans.*** $-\sqrt{2} < x < \sqrt{2}, \frac{3}{2-x^2}$