Sequences and Series

Sequences of Numbers

A ***sequence*** of numbers is a function whose domain is the set of positive integers.

Example

$0, \quad 1, \quad 2, \quad . \quad . \quad . \quad n-1, \quad . \quad . \quad .$ for a sequence whose defining rule is $a_n = n-1$

$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad . \quad . \quad . \quad \frac{1}{n}, \quad . \quad . \quad .$ for a sequence whose defining rule is $a_n = \frac{1}{n}$

The index n is the ***domain*** of the sequence. While the numbers in the ***range*** of the sequence are called the ***terms*** of the sequence, and the number a_n being called the ***nth*-term**, or ***the term with index*** n.

Example $\quad a_n = \dfrac{n+1}{n}$ then the terms are

$$\begin{array}{ccccc} 1^{st}\ term & 2^{nd}\ term & 3^{rd}\ term & & n^{th}\ term \\ a_1 = 2, & a_2 = \frac{3}{2}, & a_3 = \frac{4}{3}, & . \quad . \quad . & a_n = \frac{n+1}{n}, \quad . \quad . \quad . \end{array}$$

and we use the notation $\{a_n\}$ as the sequence a_n.

Example

Find the first five terms of the following:

(a) $\left\{\dfrac{2n-1}{3n+2}\right\}$, (b) $\left\{\dfrac{1-(-1)^n}{n^3}\right\}$, (c) $\left\{(-1)^{n+1}\dfrac{x^{2n-1}}{(2n-1)!}\right\}$

Solution

(a) $\frac{1}{5}, \quad \frac{3}{8}, \quad \frac{5}{11}, \quad \frac{7}{14}, \quad \frac{9}{17}$ **(b)** $2, \quad 0, \quad \frac{2}{27}, \quad 0, \quad \frac{2}{125}$

(c) $x, \quad \frac{-x^3}{3!}, \quad \frac{x^5}{5!}, \quad \frac{-x^7}{7!}, \quad \frac{x^9}{9!}$

Example

Find the n^th-term of the following:

(a) $0, \quad \frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{4},$ (b) $0, \quad \frac{\ln 2}{2}, \quad \frac{\ln 3}{3}, \quad \frac{\ln 4}{4},$ (c) $0, \quad \frac{1}{4}, \quad \frac{2}{9}, \quad \frac{3}{16},$

(d) $2, \quad 1, \quad \frac{2^3}{3^2}, \quad \frac{2^4}{4^2}, \quad \frac{2^5}{5^2}$

Solution

(a) $a_n = \frac{n-1}{n},$ **(b)** $a_n = \frac{\ln n}{n},$ **(c)** $a_n = \frac{n-1}{n^2},$ **(d)** $a_n = \frac{2^n}{n^2}$

Convergence of Sequences

The fact that $\{a_n\}$ converges to L is written as

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty$$

and we call the limit of the sequence $\{a_n\}$. If no such limit exists, we say that $\{a_n\}$ diverges.

From that we can say that

1) $\lim_{n\to\infty} a_n = L$ (Conv.)

2) $\lim_{n\to\infty} a_n = \infty$ (Div.)

3) $\lim_{n\to\infty} a_n = \begin{cases} L_1 \\ L_2 \end{cases}$ (Div.)

Also, if $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$ both exist and are finite, then

i) $\lim_{n\to\infty} \{a_n + b_n\} = A + B$

ii) $\lim_{n\to\infty} \{ka_n\} = kA$

iii) $\lim_{n\to\infty}\{a_n \cdot b_n\} = A \cdot B$

iv) $\lim_{n\to\infty}\left\{\frac{a_n}{b_n}\right\} = \frac{A}{B}$, provided $B \neq 0$ and b_n is never 0

Example

Test the convergence of the following:

(a) $\left\{\frac{1}{n}\right\}$, (b) $\{1+(-1)^n\}$, (c) $\{n^2\}$, (d) $\{\sqrt{n+1}-\sqrt{n}\}$,

(e) $\left\{\frac{3n^2-5n}{5n^2+2n+6}\right\}$, (f) $\left\{\frac{3n^2-4n}{2n-1}\right\}$, (g) $\left\{\left(\frac{2n-3}{3n-7}\right)^4\right\}$, (h) $\left\{\frac{2n^5-4n^2}{3n^7+n^2-10}\right\}$,

(i) $\left\{\frac{2^n}{5n}\right\}$, (j) $\left\{\frac{\ln n}{e^n}\right\}$

Solution

(a) $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$ ***(Conv.)***

(b) $\lim_{n\to\infty}\left(1+(-1)^n\right) = 1 + \lim_{n\to\infty}(-1)^n = \begin{cases} 0 & n \;\; odd \\ 2 & n \;\; even \end{cases}$ ***(Div.)***

(c) $\lim_{n\to\infty}\left(n^2\right) = \infty$ ***(Div.)***

(d) $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = \lim_{n\to\infty}\left(\left(\sqrt{n+1}-\sqrt{n}\right)\times\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right) = \lim_{n\to\infty}\left(\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right)$

$$= \lim_{n\to\infty}\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) = \frac{1}{\infty+\infty} = 0$$ ***(Conv.)***

(e) $\lim_{n\to\infty}\left(\dfrac{3n^2-5n}{5n^2+2n+6}\right)=\lim_{n\to\infty}\left(\dfrac{\dfrac{3n^2}{n^2}-\dfrac{5n}{n^2}}{\dfrac{5n^2}{n^2}+\dfrac{2n}{n^2}+\dfrac{6}{n^2}}\right)=\dfrac{3}{5}$ ***(Conv.)***

(f) $\lim_{n\to\infty}\left(\dfrac{3n^2-4n}{2n-1}\right)=\lim_{n\to\infty}\left(\dfrac{\dfrac{3n^2}{n^2}-\dfrac{4n}{n^2}}{\dfrac{2n}{n^2}-\dfrac{1}{n^2}}\right)=\dfrac{3}{0}=\infty$ ***(Div.)***

(g) $\lim_{n\to\infty}\left(\dfrac{2n-3}{3n-7}\right)^4=\left(\dfrac{2}{3}\right)^4=\dfrac{16}{81}$ ***(Conv.)***

(h) $\lim_{n\to\infty}\left(\dfrac{2n^5-4n^2}{3n^7+n^2-10}\right)=\lim_{n\to\infty}\left(\dfrac{\dfrac{2}{n^2}-\dfrac{4}{n^5}}{3+\dfrac{1}{n^5}-\dfrac{10}{n^7}}\right)=0$ ***(Conv.)***

(i) $\lim_{n\to\infty}\left(\dfrac{2^n}{5n}\right)=\lim_{n\to\infty}\left(\dfrac{2^n.\ln 2}{5}\right)=\infty$ ***(Div.)***

(j) $\lim_{n\to\infty}\left(\dfrac{\ln n}{e^n}\right)=\lim_{n\to\infty}\left(\dfrac{1/n}{e^n}\right)=\lim_{n\to\infty}\left(\dfrac{1}{n.e^n}\right)=\dfrac{1}{\infty}=0$ ***(Conv.)***

<u>Example</u>

Prove the following limits

(a) $\lim_{n\to\infty}\left(\dfrac{\ln n}{n}\right)=0$, (b) $\lim_{n\to\infty}\left(\sqrt[n]{n}\right)=1$, (c) $\lim_{n\to\infty}\left(x^{1/n}\right)=1$ $(x>0)$,

(d) $\lim_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n=e^x$ (any x), (e) $\lim_{n\to\infty}\left(\dfrac{x^n}{n!}\right)=0$ (any x)

Solution

(a) $\lim_{n\to\infty}\left(\dfrac{\ln n}{n}\right)=\lim_{n\to\infty}\left(\dfrac{1/n}{1}\right)=\dfrac{0}{1}=0$

(b) Let $a_n=n^{1/n}$, then $\ln a_n=\ln n^{1/n}=\dfrac{1}{n}\ln n\to 0$,

So, $\lim_{n\to\infty} n^{1/n}=e^{\ln a_n}\to e^0=1$

(c) Let $a_n=x^{1/n}$, then $\ln a_n=\ln x^{1/n}=\dfrac{1}{n}\ln x\to 0$,

So, $\lim_{n\to\infty} x^{1/n}=e^{\ln a_n}\to e^0=1$

(d) Let $a_n=\left(1+\dfrac{x}{n}\right)^n$, then

$$\ln a_n=\ln\left(1+\frac{x}{n}\right)^n=n.\ln\left(1+\frac{x}{n}\right)$$

So, $$\lim_{n\to\infty} n.\ln\left(1+\frac{x}{n}\right)=\lim_{n\to\infty}\frac{\ln(1+x/n)}{1/n}=\lim_{n\to\infty}\frac{\left(\dfrac{1}{1+x/n}\right)\cdot\left(-\dfrac{x}{n^2}\right)}{-1/n^2}$$

$$=\lim_{n\to\infty}\frac{x}{1+x/n}=x,$$

Thus, $\left(1+\dfrac{x}{n}\right)^n=a_n=e^{\ln a_n}\to e^x$

(e) $\lim_{n\to\infty}\left(\dfrac{x^n}{n!}\right)=\lim_{n\to\infty}\left(\dfrac{x}{1}\right)\left(\dfrac{x}{2}\right)\left(\dfrac{x}{3}\right)\ldots\left(\dfrac{x}{n}\right)=0$

Exercises on Sequences

Find the values of a_1, a_2, a_3 and a_4 for the following sequences

1) $a_n = \dfrac{1-n}{n^2}$ **2)** $a_n = \dfrac{1}{n!}$ **3)** $a_n = \dfrac{(-1)^{n+1}}{2n-1}$

4) $a_n = 2 + (-1)^n$ **5)** $a_n = \dfrac{2^n}{2^{n+1}}$ **6)** $a_n = \dfrac{2^n - 1}{2^n}$

Find a formula for the n^{th} term of the following sequences

1) $1,-1,1,-1,1,....$ **2)** $-1,1,-1,1,-1,....$ **3)** $1,-4,9,-16,25,...$

4) $1,-\dfrac{1}{4},\dfrac{1}{9},-\dfrac{1}{16},\dfrac{1}{25},...$ **5)** $0,3,8,15,24,...$ **6)** $-3,-2,-1,0,1,...$

7) $1,5,9,13,17,...$ **8)** $2,6,10,14,18,...$ **9)** $1,0,1,0,1,...$

Which of the following sequences converge and which diverge?

1) $a_n = 2 + (0.1)^n$ ***Ans.*** *Converges,* 2

2) $a_n = \dfrac{1-2n}{1+2n}$ ***Ans.*** *Converges,* -1

3) $a_n = \dfrac{1-5n^4}{n^4+8n^3}$ ***Ans.*** *Converges,* -5

4) $a_n = \dfrac{n^2-2n+1}{n-1}$ ***Ans.*** *Diverges*

5) $a_n = 1 + (-1)^n$ ***Ans.*** *Diverges*

6) $a_n = \left(\frac{n+1}{2n}\right)\left(1-\frac{1}{n}\right)$ **Ans.** *Converges,* $\frac{1}{2}$

7) $a_n = \frac{(-1)^{n+1}}{2n-1}$ **Ans.** *Converges,* 0

8) $a_n = \sqrt{\frac{2n}{n+1}}$ **Ans.** *Converges,* $\sqrt{2}$

9) $a_n = \sin\left(\frac{\pi}{2}+\frac{1}{n}\right)$ **Ans.** *Converges,* 1

10) $a_n = \frac{\sin n}{n}$ **Ans.** *Converges,* 0

11) $a_n = \frac{n}{2^n}$ **Ans.** *Converges,* 0

12) $a_n = \frac{\ln(n+1)}{n}$ **Ans.** *Converges,* 0

13) $a_n = 8^{1/n}$ **Ans.** *Converges,* 1

14) $a_n = \left(1+\frac{7}{n}\right)^n$ **Ans.** *Converges,* e^7

15) $a_n = \sqrt[n]{10n}$ **Ans.** *Converges,* 1

16) $a_n = \left(\frac{3}{n}\right)^{1/n}$ **Ans.** *Converges,* 1

17) $a_n = \frac{\ln n}{n^{1/n}}$ **Ans.** *Diverges*

18) $a_n = \sqrt[n]{4^n n}$ **Ans.** *Converges,* 4

20) $a_n = \dfrac{n!}{10^{6n}}$ ***Ans.*** *Diverges*

21) $a_n = \left(\dfrac{1}{n}\right)^{1/(\ln n)}$ ***Ans.*** *Converges,* e^{-1}

22) $a_n = \left(\dfrac{3n+1}{3n-1}\right)^{n}$ ***Ans.*** *Converges,* $e^{2/3}$

23) $a_n = \left(\dfrac{x^n}{2n+1}\right)^{1/n}, \ x > 0$ ***Ans.*** *Converges,* $x \ (x > 0)$

24) $a_n = \dfrac{3^n \times 6^n}{2^{-n} \times n!}$ ***Ans.*** *Converges,* 0

25) $a_n = \tanh(n)$ ***Ans.*** *Converges,* 1

26) $a_n = \dfrac{n^2}{2n-1}\sin\dfrac{1}{n}$ ***Ans.*** *Converges,* $\dfrac{1}{2}$

27) $a_n = \tan^{-1}(n)$ ***Ans.*** *Converges,* $\dfrac{\pi}{2}$

28) $a_n = \left(\dfrac{1}{3}\right)^{n} + \dfrac{1}{\sqrt{2^n}}$ ***Ans.*** *Converges,* 0

29) $a_n = \dfrac{(\ln n)^{200}}{n}$ ***Ans.*** *Converges,* 0

30) $a_n = n - \sqrt{n^2 - n}$ ***Ans.*** *Converges,* $\dfrac{1}{2}$

31) $a_n = \dfrac{1}{n}\displaystyle\int_1^n \dfrac{1}{x}\,dx$ ***Ans.*** *Converges,* 0