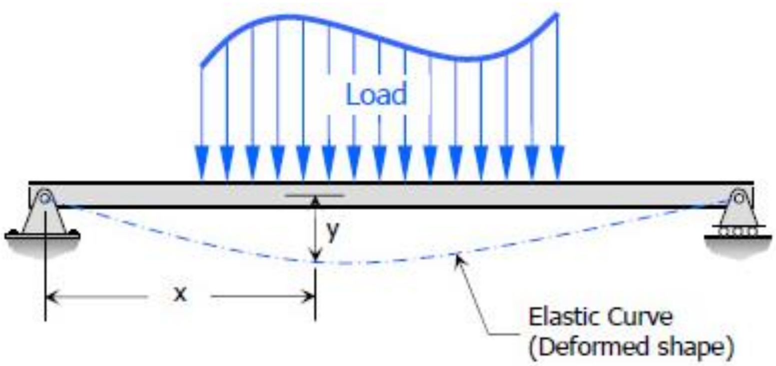
Deflection of Beams

The deformation of a beam is usually expressed in terms of its deflection from its original unloaded position. The deflection is measured from the original neutral surface of the beam to the neutral surface of the deformed beam. The configuration assumed by the deformed neutral surface is known as the elastic curve of the beam.



Numerous methods are available for the determination of beam deflections. These methods include:

1. Double-integration method
2. Area-moment method
3. Strain-energy method (Castigliano's Theorem)
4. Conjugate-beam method
5. Method of superposition

Of these methods, the first two are the ones that are commonly used.

double integration method

The double integration method is a powerful tool in solving deflection and slope of a beam at any point because we will be able to get the equation of the elastic curve.

In calculus, the radius of curvature of a curve y = f(x) is given by

$$\rho = \frac{[\,1 + (dy/dx)^2\,]^{3/2}}{|\,d^2y/dx^2\,|}$$

In the derivation of flexure formula, the radius of curvature of a beam is given as:

$$\rho = \frac{EI}{M}$$

Deflection of beams is so small, such that the slope of the elastic curve dy/dx is very small, and squaring this expression the value becomes practically negligible, hence:

$$\rho = \frac{1}{d^2y/dx^2} = \frac{1}{y''}$$

Thus, EI / M = 1 / y''

$$y'' = \frac{M}{EI} = \frac{1}{EI}M$$

If EI is constant, the equation may be written as:

$$EI\,y'' = M$$

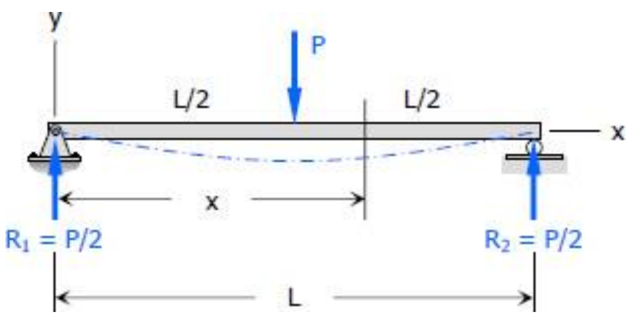
where x and y are the coordinates shown in the figure of the elastic curve of the beam under load, y is the deflection of the beam at any distance x. E is the modulus of elasticity of the beam, I represent the moment of inertia about the neutral axis, and M represents the bending moment at a distance x from the end of the beam. The product EI is called the flexural rigidity of the beam.

The first integration y' yields the slope of the elastic curve and the second integration y gives the deflection of the beam at any distance x. The resulting solution must contain two constants of integration since EI y'' = M is of second order. These two constants must be evaluated from known conditions concerning the slope deflection at certain points of the beam. For instance, in the case of a simply supported beam with rigid supports, at x = 0 and x = L, the deflection y = 0, and in locating the point of maximum deflection, we simply set the slope of the elastic curve y' to zero.

Example 1

Determine the maximum deflection δ in a simply supported beam of length L carrying a concentrated load P at midspan.

Solution



$$EI\,y'' = \tfrac{1}{2}Px - P\langle x - \tfrac{1}{2}L\rangle$$

$$EI\,y' = \tfrac{1}{4}Px^2 - \tfrac{1}{2}P\langle x - \tfrac{1}{2}L\rangle^2 + C_1$$

$$EI\,y = \tfrac{1}{12}Px^3 - \tfrac{1}{6}P\langle x - \tfrac{1}{2}L\rangle^3 + C_1x + C_2$$

At x = 0, y = 0, therefore, C2 = 0

At x = L, y = 0

$$0 = \tfrac{1}{12}PL^3 - \tfrac{1}{6}P\langle L - \tfrac{1}{2}L\rangle^3 + C_1L$$

$$0 = \tfrac{1}{12}PL^3 - \tfrac{1}{48}PL^3 + C_1L$$

$$C_1 = -\tfrac{1}{16}PL^2$$

Thus,

$$EI\,y = \tfrac{1}{12}Px^3 - \tfrac{1}{6}P\langle x - \tfrac{1}{2}L\rangle^3 - \tfrac{1}{16}PL^2x$$

Maximum deflection will occur at x = ½ L (midspan)

$$EI\,y_{max} = \frac{1}{12}P(\frac{1}{2}L)^3 - \frac{1}{6}P(\frac{1}{2}L - \frac{1}{2}L)^3 - \frac{1}{16}PL^2(\frac{1}{2}L)$$

$$EI\,y_{max} = \frac{1}{96}PL^3 - 0 - \frac{1}{32}PL^3$$

$$y_{max} = -\frac{PL^3}{48EI}$$

The negative sign indicates that the deflection is below the undeformed neutral axis.

Therefore,

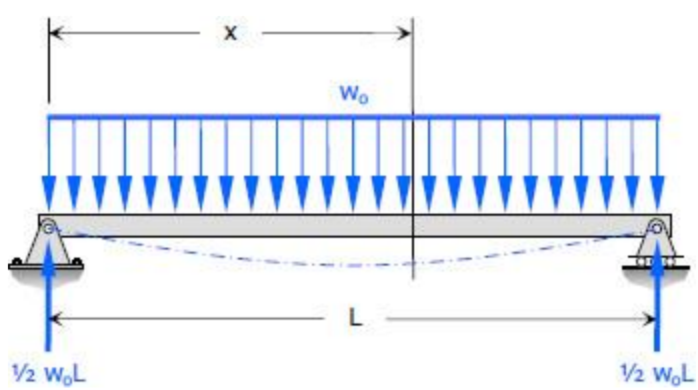
$$\delta_{max} = \frac{PL^3}{48EI}$$ answer

Example 2

Determine the maximum deflection δ in a simply supported beam of length L carrying a uniformly distributed load of intensity w_o applied over its entire length.

Solution

From the figure



$$EI\,y'' = \frac{1}{2}w_oLx - w_ox(\frac{1}{2}x)$$

$$EI\,y'' = \frac{1}{2}w_oLx - \frac{1}{2}w_ox^2$$

$$EI\,y' = \frac{1}{4}w_oLx^2 - \frac{1}{6}w_ox^3 + C_1$$

$$EI\,y = \frac{1}{12}w_oLx^3 - \frac{1}{24}w_ox^4 + C_1x + C_2$$

At x = 0, y = 0, therefore C2 = 0

At x = L, y = 0

$$0 = \frac{1}{12}w_o L^4 - \frac{1}{24}w_o L^4 + C_1 L$$

$$C_1 = -\frac{1}{24}w_o L^3$$

Therefore,

$$EI\,y = \frac{1}{12}w_o L x^3 - \frac{1}{24}w_o x^4 - \frac{1}{24}w_o L^3 x$$

Maximum deflection will occur at x = ½ L (midspan)

$$EI\,y_{max} = \frac{1}{12}w_o L(\frac{1}{2}L)^3 - \frac{1}{24}w_o(\frac{1}{2}L)^4 - \frac{1}{24}w_o L^3(\frac{1}{2}L)$$

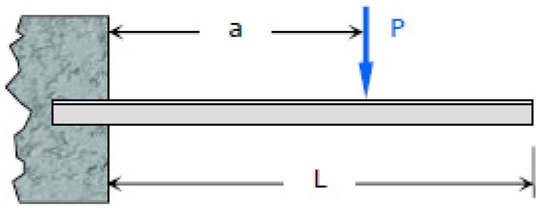
$$EI\,y_{max} = \frac{1}{96}w_o L^4 - \frac{1}{384}w_o L^4 - \frac{1}{48}w_o L^4$$

$$EI\,y_{max} = -\frac{5}{384}w_o L^4$$

$\delta_{max} = \dfrac{5w_o L^4}{384EI}$ answer

Example 3

Determine the maximum value of EIy for the cantilever beam loaded as shown in the figure. Take the origin at the wall.

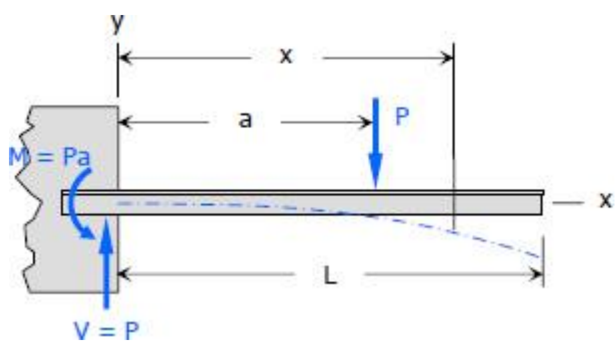


Solution

$$EI\,y'' = -Pa + Px - P\langle x - a\rangle$$

$$EI\,y' = -Pax + \tfrac{1}{2}Px^2 - \tfrac{1}{2}P\langle x - a\rangle^2 + C_1$$

$$EI\,y = -\tfrac{1}{2}Pax^2 + \tfrac{1}{6}Px^3 - \tfrac{1}{6}P\langle x - a\rangle^3 + C_1x + C_2$$



At x = 0, y' = 0, therefore C1 = 0

At x = 0, y = 0, therefore C2 = 0

Therefore,

$$EI\,y = -\tfrac{1}{2}Pax^2 + \tfrac{1}{6}Px^3 - \tfrac{1}{6}P\langle x-a\rangle^3$$

The maximum value of EI y is at x = L (free end)

$$EI\,y_{max} = -\tfrac{1}{2}PaL^2 + \tfrac{1}{6}PL^3 - \tfrac{1}{6}P(L-a)^3$$

$$EI\,y_{max} = -\tfrac{1}{2}PaL^2 + \tfrac{1}{6}PL^3 - \tfrac{1}{6}P(L^3 - 3L^2a + 3La^2 - a^3)$$

$$EI\,y_{max} = -\tfrac{1}{2}PaL^2 + \tfrac{1}{6}PL^3 - \tfrac{1}{6}PL^3 + \tfrac{1}{2}PL^2a - \tfrac{1}{2}PLa^2 + \tfrac{1}{6}Pa^3$$

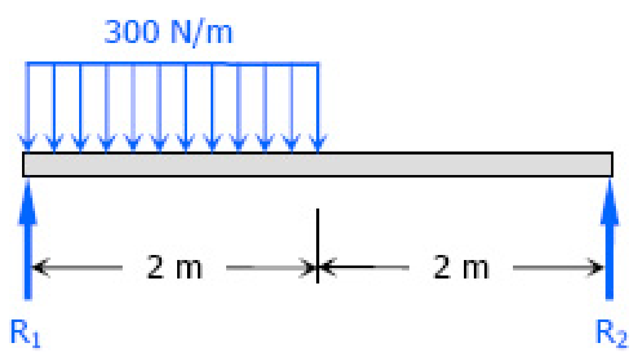
$$EI\,y_{max} = -\tfrac{1}{2}PLa^2 + \tfrac{1}{6}Pa^3$$

$$EI\,y_{max} = -\tfrac{1}{2}PLa^2 + \tfrac{1}{6}Pa^3$$

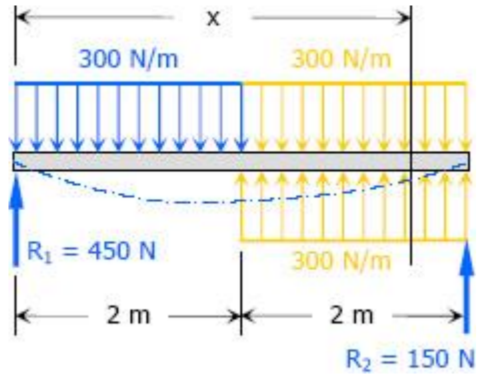
$EI\,y_{max} = -\tfrac{1}{6}Pa^2(3L-a)$ answer

Example 4

Compute the value of EI δ at midspan for the beam loaded as shown. If E = 10 GPa, what value of (I) is required to limit the midspan deflection to 1/360 of the span?



Solution



$\Sigma M_{R2} = 0$

$4R_1 = 300(2)(3)$

$R_1 = 450\,\text{N}$

$\Sigma M_{R1} = 0$

$4R_2 = 300(2)(1)$

$R_2 = 150\,\text{N}$

$EI\,y'' = 450x - \frac{1}{2}(300)x^2 + \frac{1}{2}(300)\langle\, x - 2\,\rangle^2$

$EI\,y'' = 450x - 150x^2 + 150\langle\, x - 2\,\rangle^2$

$EI\,y' = 225x^2 - 50x^3 + 50\langle\, x - 2\,\rangle^3 + C_1$

$EI\,y = 75x^3 - 12.5x^4 + 12.5\langle\, x - 2\,\rangle^4 + C_1x + C_2$

At x = 0, y = 0, therefore C2 = 0

At x = 4 m, y = 0

$0 = 75(4^3) - 12.5(4^4) + 12.5(4 - 2)^4 + 4C_1$

$C_1 = -450\,\text{N}\cdot\text{m}^2$

Therefore,

$EI\,y = 75x^3 - 12.5x^4 + 12.5\langle\, x - 2\,\rangle^4 - 450x$

At x = 2 m (midspan)

$$EI\,y_{midspan} = 75(2^3) - 12.5(2^4) + 12.5(2-2)^4 - 450(2)$$

$$EI\,y_{midspan} = -500\,\mathrm{N\cdot m^3}$$

$$EI\,\delta_{midspan} = 500\,\mathrm{N\cdot m^3}$$

Maximum midspan deflection

$$\delta_{midspan} = \frac{1}{360}L = \frac{1}{360}(4) = \frac{1}{90}\,\mathrm{m}$$

$$\delta_{midspan} = \frac{100}{9}\,\mathrm{mm}$$

Thus,

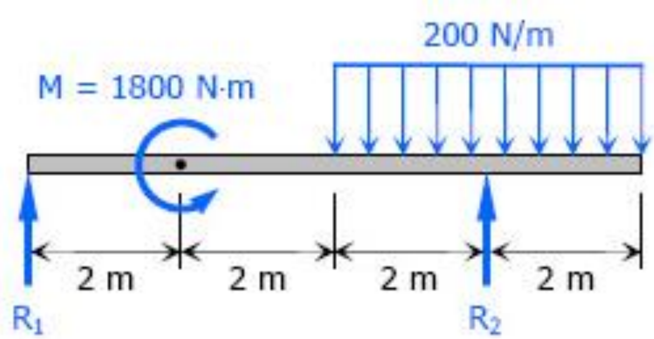
$$10\,000I\left(\frac{100}{9}\right) = 500(1000^3)$$

$$I = 4\,500\,000\,\mathrm{mm^4}$$

$I = 4.5 \times 10^6\,\mathrm{mm^4}$ answer

Example 5

Determine the value of EIy midway between the supports for the beam loaded as shown.

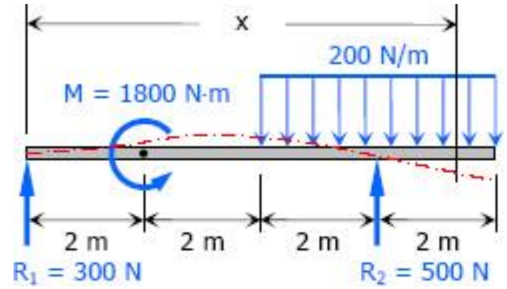


Solution

$\Sigma M_{R2} = 0$

$6R_1 + 200(4)(0) = 1800$

$R_1 = 300\,\mathrm{N}$



$\Sigma M_{R1} = 0$

$6R_2 + 1800 = 200(4)(6)$

$R_2 = 500\,\mathrm{N}$

$$EI\,y'' = 300x - 1800\,\langle\, x - 2\,\rangle^0 + 500\,\langle\, x - 6\,\rangle - \tfrac{1}{2}(200)\,\langle\, x - 4\rangle^2$$

$$EI\,y'' = 300x - 1800\,\langle\, x - 2\,\rangle^0 + 500\,\langle\, x - 6\,\rangle - 100\,\langle\, x - 4\rangle^2$$

$$EI\,y' = 150x^2 - 1800\,\langle\, x - 2\,\rangle + 250\,\langle\, x - 6\,\rangle^2 - \tfrac{100}{3}\,\langle\, x - 4\rangle^3 + C_1$$

$$EI\,y = 50x^3 - 900\,\langle\, x - 2\rangle^2 + \tfrac{250}{3}\,\langle\, x - 6\,\rangle^3 - \tfrac{25}{3}\,\langle\, x - 4\rangle^4 + C_1x + C_2$$

At x = 0, y = 0, therefore C2 = 0

At x = 6 m, y = 0

0 = 50(63) - 900(42) - (25/3)(24) + 6C1

C1 = 5600/9 N·m3

Therefore,

$$EI\,y = 50x^3 - 900\,\langle\, x - 2\rangle^2 + \tfrac{250}{3}\,\langle\, x - 6\,\rangle^3 - \tfrac{25}{3}\,\langle\, x - 4\rangle^4 + \tfrac{5600}{9}x$$

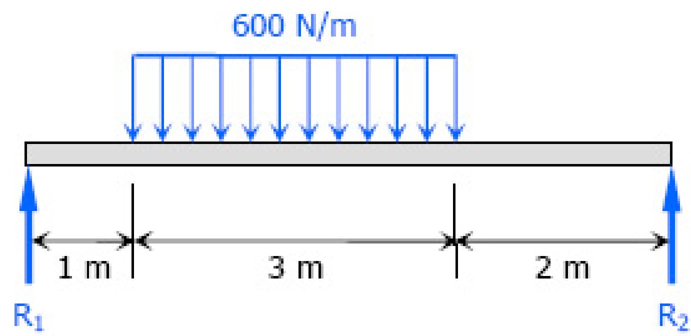
At x = 3 m

$$EI\,y = 50(3^3) - 900(1^2) + \tfrac{5600}{9}(3)$$

$EI\,y = \frac{6950}{3}\,\mathrm{N}\cdot\mathrm{m}^3$ answer

Example 6

Compute the midspan value of EI δ for the beam loaded shown.



Example 7

Compute the value of EI y at the right end of the overhanging beam shown.