Single Degree of freedom System
Free Vibration of undamped system

In case of undamped system $C=0$ and for free vibration $F(t)=0$ in Eq. 1

So Eq. 1 will be reduced to

$$m\ddot{x}(t)+Kx(t)=0$$

or $$\ddot{x}(t)+\frac{K}{m}x(t)=0$$ let $\omega_n^2=\frac{K}{m}$

where ω_n - natural frequency of vibration of undamped system in rad/sec

$$\therefore \ddot{x}(t)+\omega_n^2 x(t)=0$$

but $K\delta_{st}=mg \longrightarrow m=\frac{K\delta_{st}}{g}$

$$\omega_n=\sqrt{\frac{K}{m}}=\sqrt{\frac{K}{\frac{K\delta_{st}}{g}}}=\sqrt{\frac{g}{\delta_{st}}}$$

$$f_n=\frac{\omega_n}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}} \quad \text{in (Hz)}$$

That means that the natural frequency of a single degree of freedom system is uniquely determined by the static deflection of spring.

The general solution is

$$x(t)=A_1\cos\omega_n t+A_2\sin\omega_n t$$

where A_1, A_2 are constants depending on the initial displacement $x(0)$ and initial velocity $\dot{x}(0)$.

$x(0) = A_1$

$\dot{x}(t) = -A_1 \omega_n \sin \omega_n t + A_2 \omega_n \cos \omega_n t$

$\dot{x}(0) = A_2 \omega_n \longrightarrow A_2 = \dfrac{\dot{x}(0)}{\omega_n}$

$\therefore x(t) = x(0) \cos \omega_n t + \dfrac{\dot{x}(0)}{\omega_n} \sin \omega_n t$

Introducing the notations A, ϕ such that $A_1 = A \cos \phi$ and $A_2 = A \sin \phi$

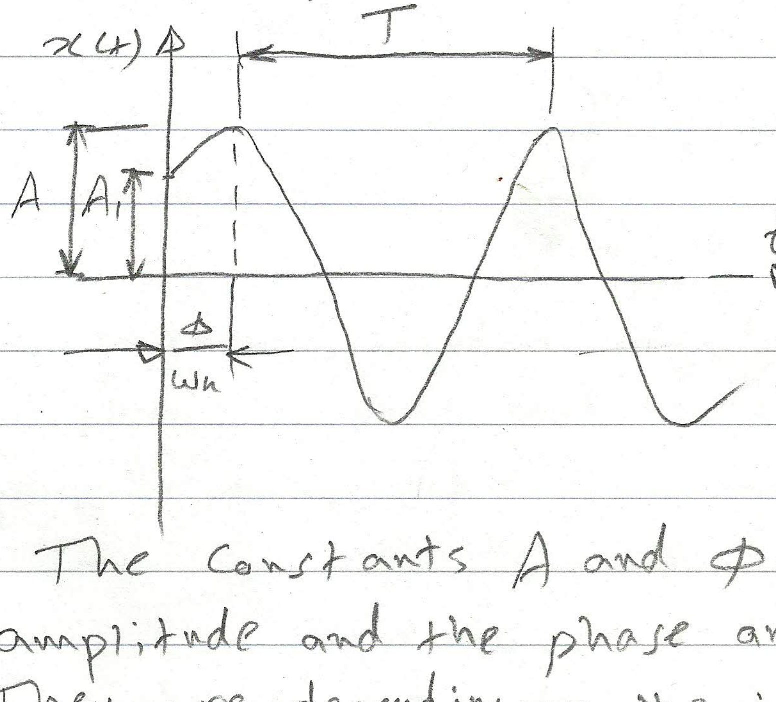
$A = \sqrt{A_1^2 + A_2^2} \qquad \phi = \tan^{-1} \dfrac{A_2}{A_1}$

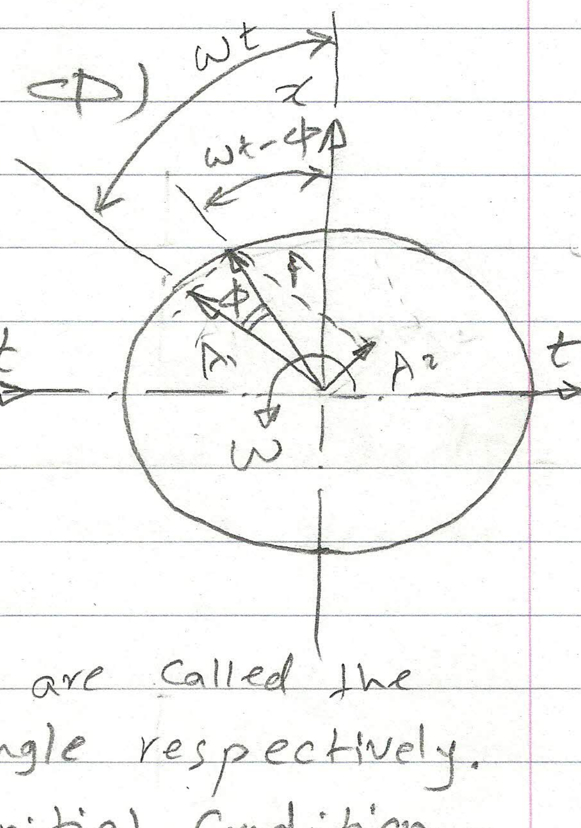
$\therefore x(t) = A \cos \phi \cos \omega_n t + A \sin \phi \sin \omega t$

But $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

$\therefore$ The solution is

$\therefore x(t) = A \cos(\omega t - \phi)$





The constants A and ϕ are called the amplitude and the phase angle respectively. They are depending on the initial condition, $x(0)$ and $\dot{x}(0)$.

Now $\dot{x}(t) = -A \omega_n \sin(\omega_n t - \phi)$

(2)

$$\dot{x}(0) = -A\,\omega_n \sin(-\phi)$$

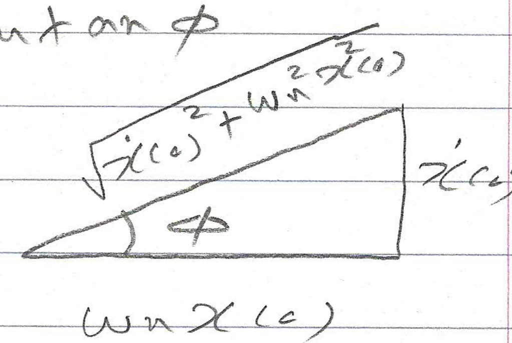
$$x(0) = A\cos(-\phi) \quad --- \quad (*)$$

But $\cos(-\phi) = \cos(\phi)$ and $\sin(-\phi) = -\sin\phi$

$\therefore x(0) = A\cos\phi$ and $\dot{x}(0) = A\,\omega_n \sin\phi$

$$\frac{\dot{x}(0)}{x(0)} = \frac{A\,\omega_n \sin\phi}{A\cos\phi} = \omega_n \tan\phi$$

$$\therefore \tan\phi = \frac{\dot{x}(0)}{x(0)\,\omega_n}$$



$$\therefore \cos\phi = \frac{\omega_n x(0)}{\sqrt{\dot{x}(0)^2 + \omega_n^2 x^2(0)}} = \frac{x(0)\,\omega_n}{x(0)\sqrt{\omega_n^2 + \left(\frac{\dot{x}(0)}{x(0)}\right)^2}}$$

$$\therefore \cos\phi = \frac{\omega_n}{\sqrt{\omega_n^2 + \left(\frac{\dot{x}(0)}{x(0)}\right)^2}}$$

From Eq. (*) $\quad A = \dfrac{x(0)}{\cos\phi} = \dfrac{x_0}{\omega_n}\sqrt{\omega_n^2 + \left(\dfrac{\dot{x}(0)}{x(0)}\right)^2}$

$$\therefore A = \sqrt{\frac{x_0^2}{\omega_n^2}\left[\omega_n^2 + \left(\frac{\dot{x}(0)}{x_0}\right)^2\right]} = \sqrt{x^2(0) + \left(\frac{\dot{x}(0)}{\omega_n}\right)^2}$$

$$\therefore A = \sqrt{x^2(0) + \left(\frac{\dot{x}(0)}{\omega_n}\right)^2}$$

Let $\dot{x}(0) = V_0$ and $x(0) = x_0$

$$\therefore A = \sqrt{x_0^2 + \left(\frac{V_0}{\omega_n}\right)^2}, \quad \phi = \tan^{-1}\frac{V_0}{x_0\,\omega_n}$$

and $T = \frac{2\pi}{\omega_n}$

Example: A spring-mass system have $m = 2\,lb.sec^2/in$, $K = 600\,lb/in$ with initial conditions $x_o = 1$ in and $V_o = 10$ in/sec, find the equation of the resulting motion $x(t)$.

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{600}{2}} = \sqrt{300} = 10\sqrt{3}\ \text{rad/sec}$$

$$A = \sqrt{x_o^2 + \left(\frac{V_o}{\omega_n}\right)^2} = \sqrt{1^2 + \left(\frac{10}{10\sqrt{3}}\right)^2} = \sqrt{1 + \frac{1}{3}}$$

$$A = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$ inch (the amplitude)

$$\phi = \tan^{-1}\frac{V_o}{\omega_n x_o} = \tan^{-1}\frac{10}{10\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}}$$

$\therefore \phi = 30^\circ$ $\phi = \frac{\pi}{6}$ rad (the phase angle)

$\therefore$ The equation of motion is

$$x(t) = \frac{2}{\sqrt{3}}\cos\left(10\sqrt{3}\,t - \frac{\pi}{6}\right) \text{ inch}$$

The energy method:

"In conservative system, the total energy is constant". The undamped system is conservative. So the kinetic energy stored in the mass + Potential energy stored in the

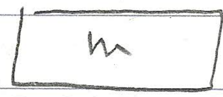
form of strain energy in elastic deformation of the spring = Constant

or $\frac{d}{dt}(KE + PE) = \text{Zero}$

where $PE = \frac{1}{2}kx^2$ and $KE = \frac{1}{2}mv^2$

Another form of the energy method can be obtained as follows

Maximum displacement position ——————— 2



Static equilibrium position ——————— 1

$(PE)_1 = \text{Zero}$ because it is a reference position

$(KE)_1 = (KE)_{max}$

$(PE)_2 = (PE)_{max}$

$(KE)_2 = 0$

Since $(PE)_1 + (KE)_1 = (PE)_2 + (KE)_2$

$(KE)_{max} = (PE)_{max}$.

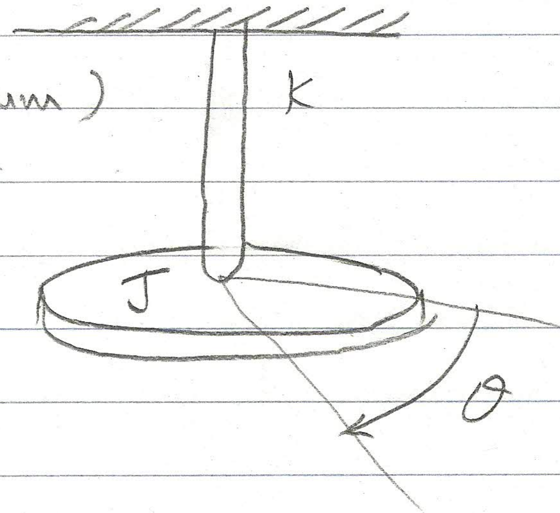
The above equation leads directly to the natural frequency.

Ex. 1:

In torsional vibration, find the natural frequency of a system resulted from

5

a disc of mass moment of inertia J attached to the end of a shaft of stiffness k fixed at the other end



(Torsional Pendulum)

let $\theta = A \sin \omega_n t$

For mass

$$(KE)_{max} = \frac{1}{2} J \dot{\theta}^2_{max}$$

$$\dot{\theta} = A\omega_n \cos \omega_n t$$

$$\dot{\theta}_{max} = A\omega_n$$

$$\therefore (KE)_{max} = \frac{1}{2} J A^2 \omega_n^2 \quad \text{--- (1)}$$

For spring

$$(PE)_{max} = \frac{1}{2} k \theta^2_{max}$$

$$\theta_{max} = A$$

$$(PE)_{max} = \frac{1}{2} K A^2 \quad \text{--- (2)}$$

Applying the energy method

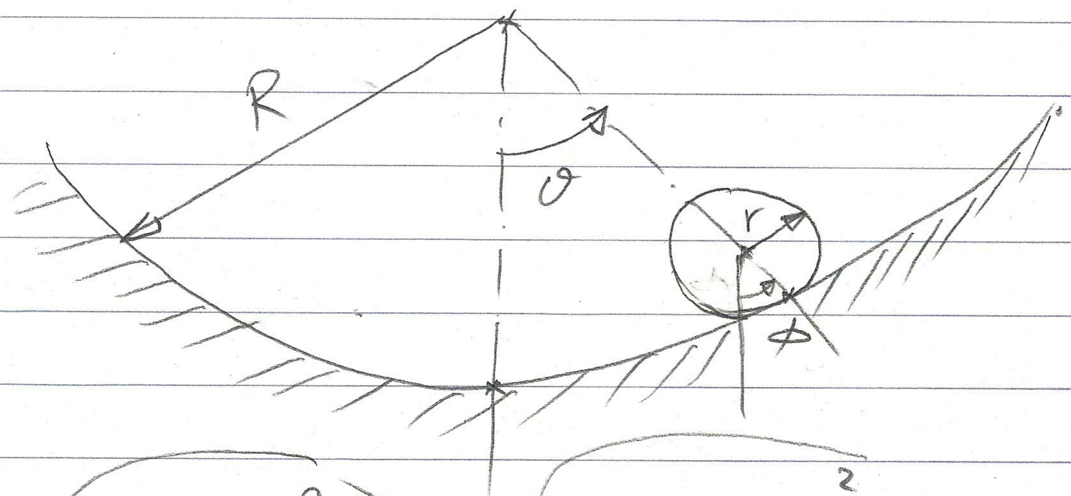
$$(KE)_{max} = (PE)_{max}$$

$$\cancel{\tfrac{1}{2}} J \cancel{A^2} \omega_n^2 = \cancel{\tfrac{1}{2}} K \cancel{A^2} \rightarrow \omega_n^2 = \frac{k}{J}$$

$$\therefore \omega_n = \sqrt{\frac{k}{J}}$$

Ex2: A Cylinder of weight W and radius r rolls without slipping on a Cylindrical surface of radius R. Determine its differential equation of motion for small oscillations

about the lowest point. For no slipping condition $r\phi = R\theta$



$$(KE) = \frac{1}{2} m v^2 + \frac{1}{2} J \omega^2$$

translation　　rotation

$$v = (R-r)\dot{\theta} \longrightarrow \omega = \dot{\phi} - \dot{\theta}$$

but $\phi = \frac{R}{r}\theta \longrightarrow \dot{\phi} = \frac{R}{r}\dot{\theta}$

$$\therefore \omega = \frac{R}{r}\dot{\theta} - \dot{\theta} = \dot{\theta}\left(\frac{R}{r} - 1\right)$$

J for a disc $J = \frac{1}{2} m r^2 = \frac{1}{2}\frac{W}{g} r^2$

$$\therefore KE = \frac{1}{2}\frac{W}{g}\left[(R-r)\dot{\theta}\right]^2 + \frac{1}{2}\left(\frac{1}{2}\frac{W}{g} r^2\right)\left(\frac{R}{r}-1\right)^2 \dot{\theta}^2$$

$$KE = \frac{1}{2}\frac{W}{g}(R-r)^2\dot{\theta}^2 + \frac{1}{4}\frac{W}{g}\cancel{r^2}\,\dot{\theta}^2\,\frac{(R-r)^2}{\cancel{r^2}}$$

$$KE = \frac{3}{4}\frac{W}{g}(R-r)^2\dot{\theta}^2$$

PE = weight of the disc * rise of the disc mass center.

$$PE = W\left[(R-r) - (R-r)\cos\theta\right]$$

$$PE = W(R-r)(1-\cos\theta)$$

Applying energy method

$$\frac{d}{dt}(PE + KE) = 0$$

$$\frac{d}{dt}\left[W(R-r)(1-\cos\theta) + \frac{3}{4}\frac{W}{g}(R-r)^2\dot{\theta}^2\right] = 0$$

$$W(R-r)\sin\theta\frac{d\theta}{dt} + \frac{3}{4}\frac{W}{g}(R-r)^2 \times 2\dot{\theta}\ddot{\theta} = 0$$

but for small oscillations

$\sin\theta = \theta$, $\cos\theta = 1$ and $\tan\theta = 1$

$$\frac{d\theta}{dt} = \dot{\theta}$$

$$\cancel{W}\cancel{(R-r)}\,\theta\,\cancel{\dot{\theta}} + \frac{3}{\cancel{4}_{2}}\frac{\cancel{W}}{g}(R-r)^{\cancel{2}} \times \cancel{2}\,\cancel{\dot{\theta}}\,\ddot{\theta} = 0$$

$$\frac{3}{2g}(R-r)\ddot{\theta} + \theta = 0$$

$$\ddot{\theta} + \frac{2g}{3(R-r)}\theta = 0 \rightarrow \text{Similar to the form } \ddot{\theta} + w_n^2\theta = 0$$

$$\therefore w_n^2 = \frac{2g}{3(R-r)} \rightarrow w_n = \sqrt{\frac{2g}{3(R-r)}}$$

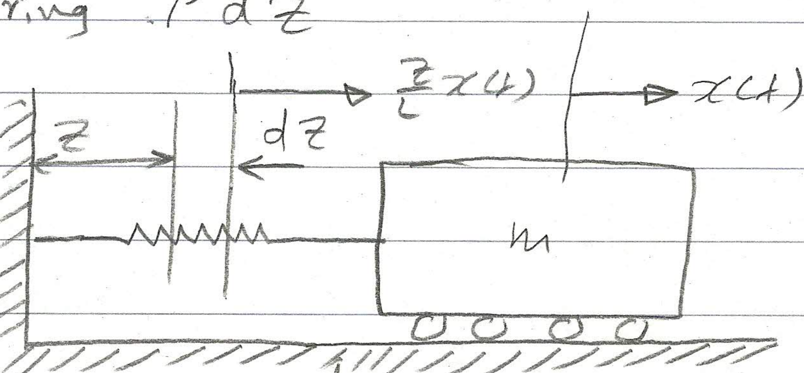
Effective mass:

let ρ _ mass per unit length of the spring

L _ total length of the spring

$x(t)$ _ displacement of mass

$\frac{z}{L}x(t)$ _ displacement of an element of mass spring $\rho\, dz$



$$KE = \frac{1}{2} m \dot{x}^2(t) + \frac{1}{2}\int_0^L \left(\frac{z}{2}\right)^2 \dot{x}^2(t)\, \rho\, dz$$

$$KE = \frac{1}{2} m \dot{x}^2(t) + \frac{1}{2L^2} \rho\, \dot{x}^2(t) \int_0^L z^2\, dz$$

$$KE = \frac{1}{2} m \dot{x}^2(t) + \frac{\rho}{2L^2} \dot{x}^2(t) \left[\frac{z^3}{3}\right]_0^L$$

$$KE = \frac{1}{2} m \dot{x}^2(t) + \frac{\rho \dot{x}^2(t)}{2\cancel{L^2}} * \frac{L^{\cancel{3}}}{3}$$

$$KE = \frac{1}{2} m \dot{x}^2(t) + \frac{1}{2} \rho \dot{x}^2(t) \frac{L}{3}$$

$$KE = \frac{1}{2}\left(m + \frac{\rho L}{3}\right)\dot{x}^2(t)$$

$$PE = \frac{1}{2} K x^2(t)$$

$$KE + PE = \frac{1}{2}\left(m + \frac{\rho L}{3}\right)\dot{x}^2(t) + \frac{1}{2} k x^2(t)$$

Applying energy method

$$\frac{d}{dt}(KE+PE)=0$$

$$\frac{d}{dt}\left[\frac{1}{2}\left(m+\frac{\rho L}{3}\right)\dot{x}^2(t)+\frac{1}{2}kx^2\right]=0$$

$$\frac{1}{\cancel{2}}\left(m+\frac{\rho L}{3}\right)\cancel{2}\cancel{\dot{x}}(t)\ddot{x}+\frac{1}{\cancel{2}}k(\cancel{2}x)\cancel{\dot{x}}=0$$

$$\left(m+\frac{\rho L}{3}\right)\ddot{x}+kx=0$$ which is equivalent

to $m_{eff}\ddot{x}+kx=0$ where $m_{eff}=m+\frac{\rho L}{3}$
which is called effective mass

$$\ddot{x}(t)+\omega_n^2x(t)=0$$

where $\omega_n^2=\dfrac{k}{m_{eff}} \rightarrow \omega_n=\sqrt{\dfrac{k}{m+\frac{\rho L}{3}}}$

That means:-

① The effect of the spring mass is to Lower the system natural frequency

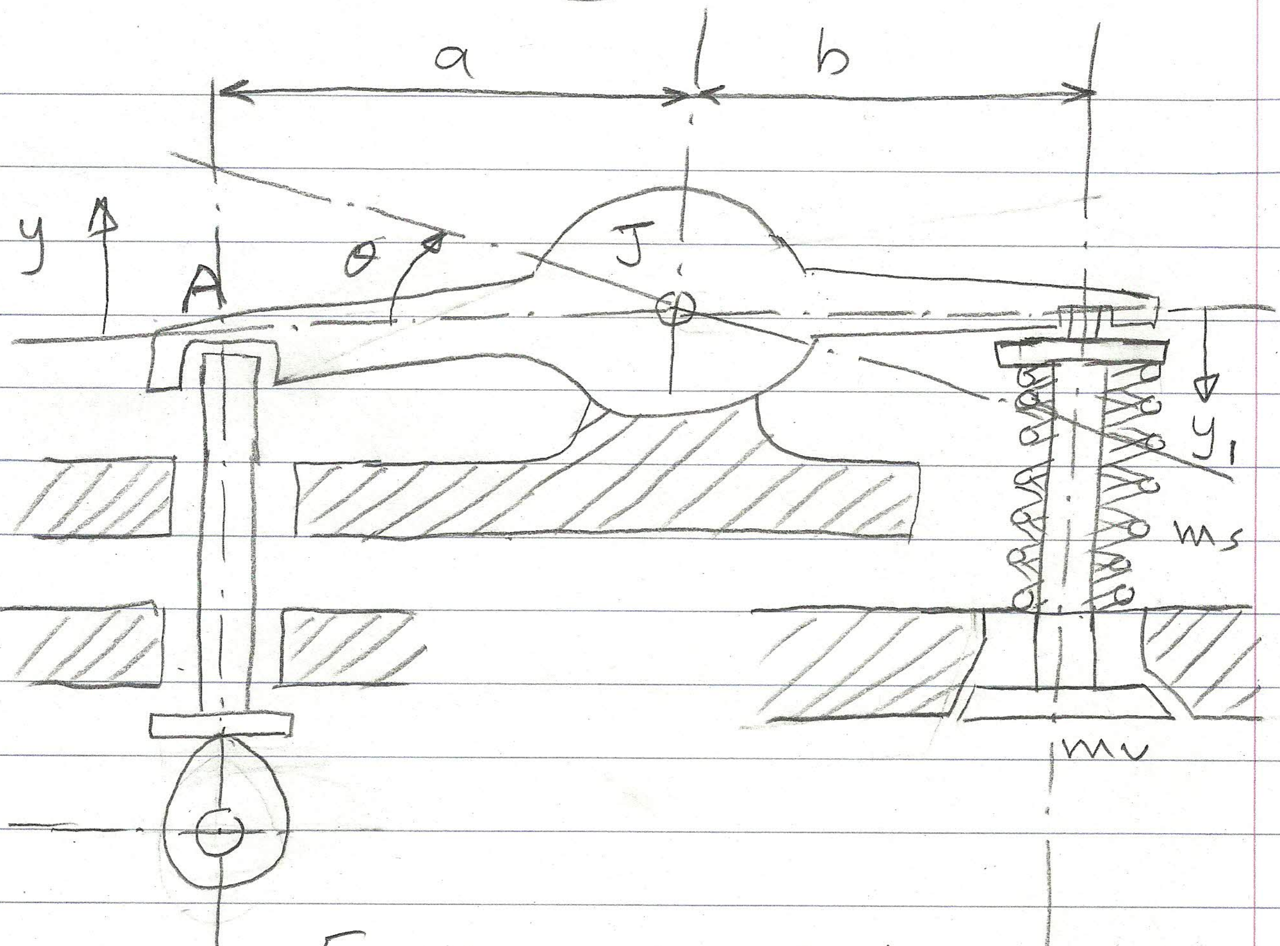
② The spring contributes one third of its mass to the inertia of the system.

Ex:- Find the effective mass at A for the engine valve system shown in the figure below, where

m_v - mass of the valve

m_s - mass of the spring

J - mass moment of inertia of the rocker arm



Engine valve system

$$KE = \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} m_v \dot{y}_1^2 + \frac{1}{2} \frac{m_s}{3} \dot{y}_1^2$$

$$\theta = \frac{y_1}{b} = \frac{y}{a} \qquad y_1 = b\theta \quad \dot{y}_1 = b\dot{\theta} \quad \dot{y} = a\dot{\theta}$$

$$KE = \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} m_v b^2 \dot{\theta}^2 + \frac{1}{2} \frac{m_s}{3} b^2 \dot{\theta}^2$$

$$KE = \frac{1}{2} \dot{\theta}^2 \left[J + b^2 m_v + b^2 \frac{m_s}{3} \right]$$

KE at A $\quad (KE)_A = \frac{1}{2} m_A \dot{y}^2$

$$(KE)_A = \frac{1}{2} m_A a^2 \dot{\theta}^2$$

$$\cancel{\frac{1}{2}} m_A a^2 \cancel{\dot{\theta}^2} = \cancel{\frac{1}{2}} \cancel{\dot{\theta}^2} \left[J + b^2 m_v + \frac{m_s}{3} b^2 \right]$$

$$\therefore -mA = \boxed{\frac{J + b^2 m u + \frac{ms}{3} b^2}{a^2}}$$

6