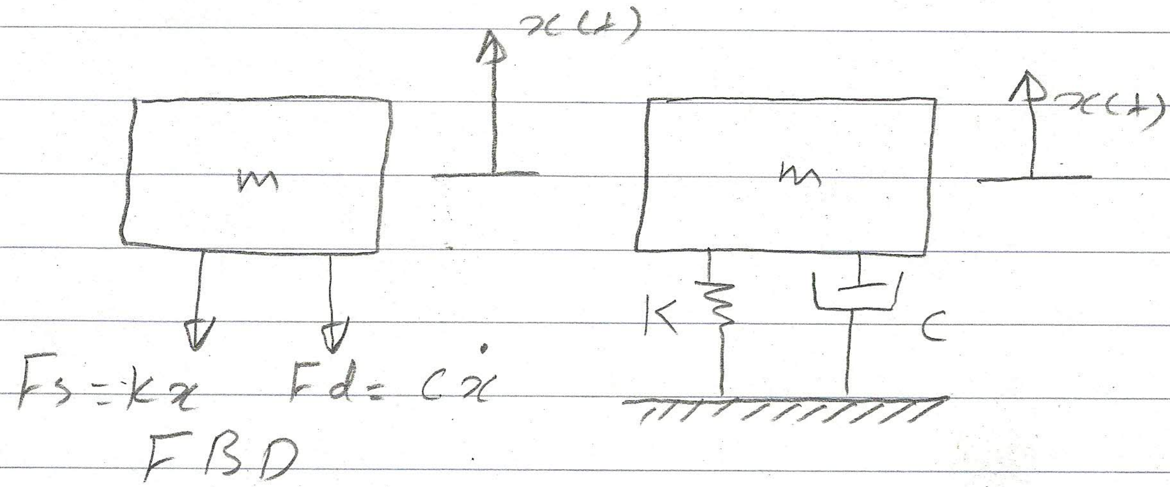
Free vibration of damped system

For simple mass-spring-damper system



Applying Newton's second law $\sum F = m\ddot{x}$

$-c\dot{x} - kx = m\ddot{x} \rightarrow m\ddot{x} + c\dot{x} + kx = 0$

$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$

let $\frac{c}{m} = 2\zeta\omega_n \quad \frac{k}{m} = \omega_n^2$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \quad \text{--- } (1)$$

Where $\zeta = \frac{c}{2m\omega_n}$ is called viscous damping factor

ζ is non-dimensional parameter

Solution $x(t) = Ae^{st}$ where A, s are constants to be determined

$\dot{x}(t) = Ase^{st} = s\,x(t)$ and $\ddot{x} = As^2e^{st}$

$\ddot{x} = s^2 x(t)$

Substitute in Eq. (1)

$s^2 x(t) + 2\zeta\omega_n s\,\cancel{x(t)} + \omega_n^2 \cancel{x(t)} = \text{zero}$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad \text{-- } (2)$$

Equation (2) is called characteristic

equation of the system and it is a quadratic equation in S

$$S_{1,2} = \frac{-2\zeta\omega_n \mp \sqrt{4\omega_n^2\zeta^2 - 4\omega_n^2}}{2}$$

$$S_{1,2} = \omega_n\left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]$$

$$S_{1,2} = \omega_n\left[-\zeta \pm i\sqrt{1-\zeta^2}\right]$$

The nature of the roots $S_{1,2}$ depends on the value of ζ

Discuss four values of ζ namely $\zeta = 0$, $\zeta = 1$, $0 < \zeta < 1$ and $\zeta > 1$

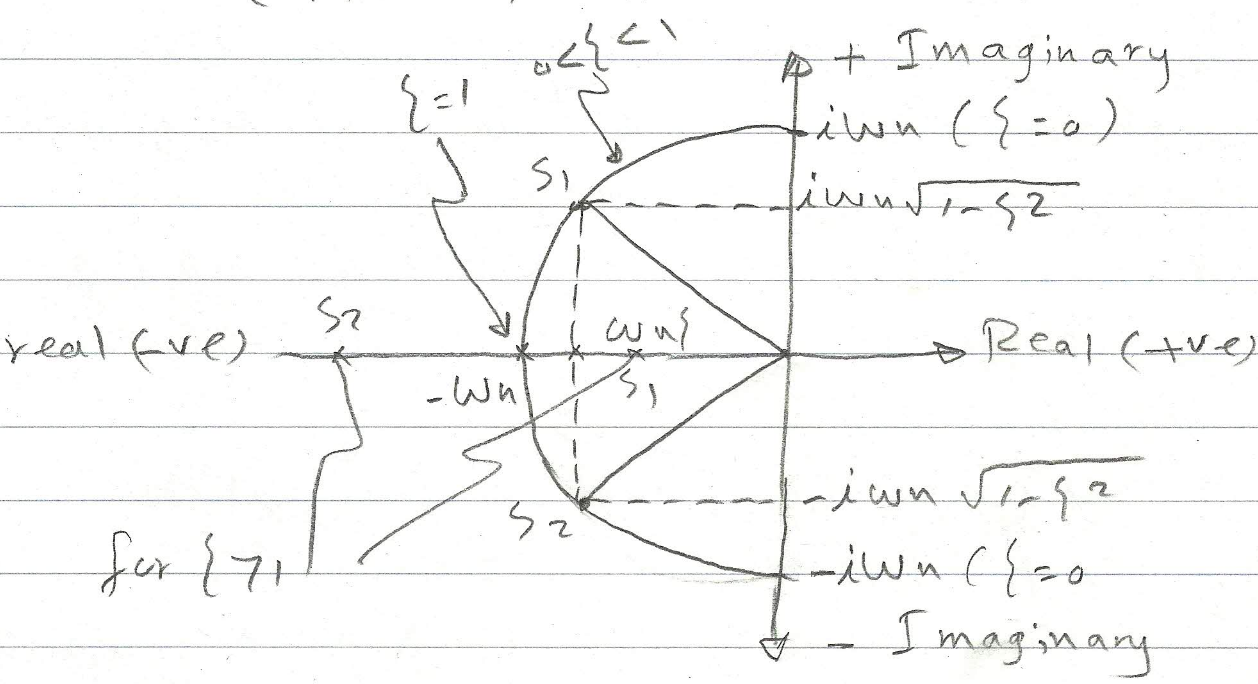
when $\zeta = 0$ $\quad S_{1,2} = \pm i\omega_n$

when $0 < \zeta < 1$

$S_{1,2}$ are complex conjugates with respect to the real axis on a circle of radius ω_n

when $\zeta = 1$ $\quad S_{1,2} = -\omega_n$

when $\zeta > 1$ $\quad S_1 \to 0$ and $S_2 \to -\infty$



(2)

The solution will be

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Case (a) underdamped case $0 < \zeta < 1$

$$x(t) = A_1 e^{-\zeta \omega_n t + i\omega_n t\sqrt{1-\zeta^2}} + A_2 e^{-\zeta\omega_n t - i\omega_n t\sqrt{1-\zeta^2}}$$

$$x(t) = e^{-\zeta\omega_n t}\left[A_1 e^{i\omega_n t\sqrt{1-\zeta^2}} + A_2 e^{-i\omega_n t\sqrt{1-\zeta^2}}\right]$$

let $\omega_d = \omega_n\sqrt{1-\zeta^2}$, ω_d is called frequency of damped free vibration

$$x(t) = e^{-\zeta\omega_n t}\left[A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}\right]$$

but $e^{\pm i\omega_d t} = \cos\omega_d t \pm i\sin\omega_d t$

$$\therefore x(t) = e^{-\zeta\omega_n t}\left[A_1\cos\omega_d t + iA_1\sin\omega_d t + A_2\cos\omega_d t - iA_2\sin\omega_d t\right]$$

$$x(t) = e^{-\zeta\omega_n t}\left[\cos\omega_d t\,(A_1 + A_2) + i\sin\omega_d t\,(A_1 - A_2)\right]$$

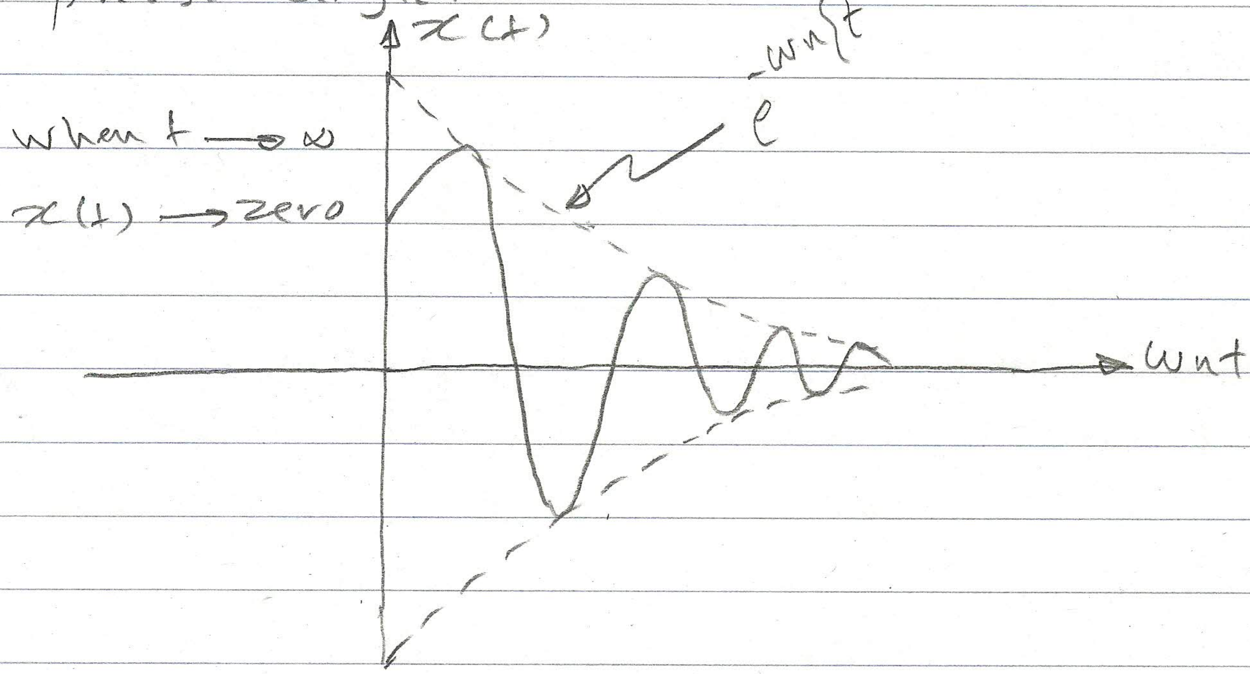
let $A_1 + A_2 = A\cos\phi$

and $i(A_1 - A_2) = A\sin\phi$

$$\therefore x(t) = e^{-\zeta\omega_n t}\left[A\cos\phi\cos\omega_d t + A\sin\phi\sin\omega_d t\right]$$

$$x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

where A, ϕ are constant depending on the initial condition $x(0)$, $\dot{x}(0)$ and they are called amplitude and phase angle.



Thus this case is called Oscillatory motion case.

Case (b) Overdamped case $\zeta > 1$

The roots

$$S_{1,2} = \omega_n \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

are always real roots

$S_1 \to$ zero and $S_2 \to -\infty$

The general solution is

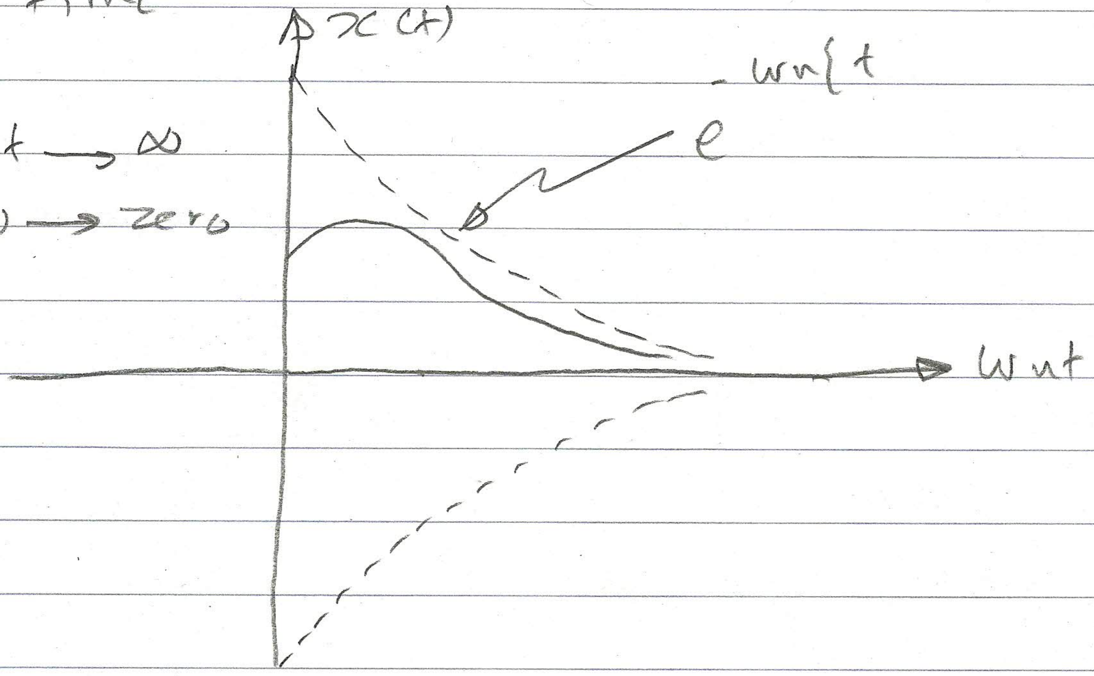
$$x(t) = e^{-\zeta \omega_n t} \left[A_1 e^{(\sqrt{\zeta^2-1})\omega_n t} + A_2 e^{-\omega_n t \sqrt{\zeta^2 - 1}} \right]$$

So there is no imaginary roots, hence no harmonic motion and the motion

is an exponentially decreasing function of time

as $t \to \infty$

$x(t) \to$ zero



So this case is called non-oscillatory motion case.

Case (c) Critical damping $\xi = 1$

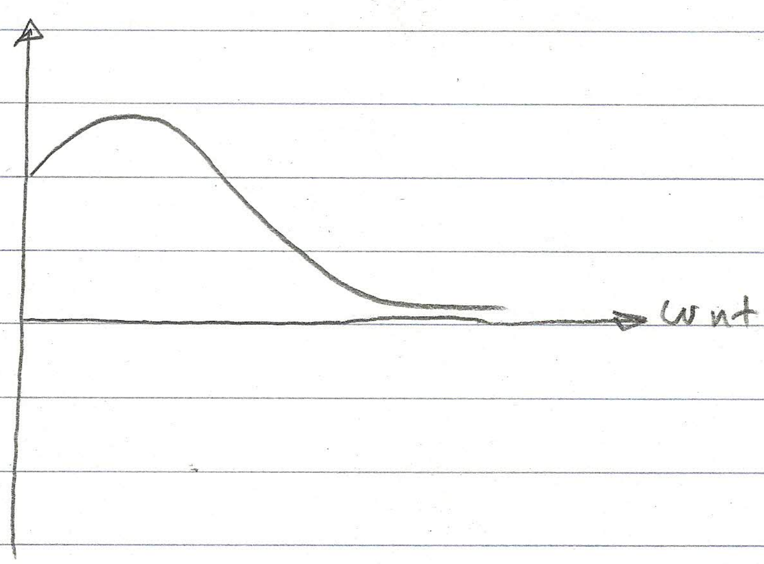
$\frac{c}{m} = 2\xi\omega_n$ when $\xi = 1$ $C = C_{cr}$

$C_{cr} = 2m\omega_n$ but $\omega_n = \sqrt{\frac{k}{m}}$

$C_{cr} = 2m\sqrt{\frac{k}{m}} = 2\sqrt{\frac{m^2 k}{m}} = 2\sqrt{mk}$

It is not important case because it is the borderline between the oscillatory and non-oscillatory cases.

The roots $S_{1,2} = -\omega_n$ and the solution is $x(t) = (A_1 + t A_2) e^{-\omega_n t}$



let $x(0) = x_0$

and $\dot{x}(0) = \dot{x}_0$

$A_1 = x_0$ and $A_2 = \dot{x}_0 + x_0 \omega_n$

$\therefore$ The solution will be

$$x(t) = \left[x_0 + (\dot{x}_0 + x_0 \omega_n) t \right] e^{-\omega_n t}$$

Case (d) undamped case $\zeta = 0$

which is discussed previously in vibration of undamped system

Solution

$$x(t) = A \cos(\omega_n t - \phi)$$

Example: For a simple spring-mass-damper system, the initial conditions are $x(0) = 0$ and $\dot{x}(0) = v_0$. Calculate the response for $\zeta > 1$, $\zeta = 1$ and $0 < \zeta < 1$

(1) for $\zeta > 1$

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$$x(t) = e^{-\zeta\omega_n t}\left[A_1 e^{\omega_n t\sqrt{\zeta^2-1}} + A_2 e^{-\omega_n t\sqrt{\zeta^2-1}}\right]$$

at $t=0$ $\quad x(0) = A_1 + A_2 = 0 \longrightarrow A_1 = -A_2$

$$x(t) = e^{-\zeta\omega_n t}\left[A_1 e^{\omega_n t\sqrt{\zeta^2-1}} - A_1 e^{-\omega_n t\sqrt{\zeta^2-1}}\right] \times \frac{2}{2}$$

$$\therefore x(t) = 2A_1 e^{-\zeta\omega_n t} \sinh \omega_n t\sqrt{\zeta^2-1}$$

$$\dot{x}(t) = 2A_1(-\zeta\omega_n) e^{-\zeta\omega_n t} \sinh \omega_n t\sqrt{\zeta^2-1}$$

$$+ 2A_1 e^{-\zeta\omega_n t}\left(\omega_n\sqrt{\zeta^2-1}\right) \cosh \omega_n t\sqrt{\zeta^2-1}$$

$$\dot{x}(t) = 2A_1 e^{-\zeta\omega_n t}\,\omega_n\left[\sqrt{\zeta^2-1}\cosh \omega_n t\sqrt{\zeta^2-1} - \zeta \sinh \omega_n t\sqrt{\zeta^2-1}\right]$$

at $t=0$ $\quad \dot{x}(0) = 2A_1\omega_n\sqrt{\zeta^2-1} = V_0$

$$\therefore A_1 = \frac{V_0}{2\omega_n\sqrt{\zeta^2-1}} = -A_2$$

$\therefore$ The solution (Response) will be

$$x(t) = \frac{V_0}{\omega_n\sqrt{\zeta^2-1}}\, e^{-\zeta\omega_n t} \sinh \omega_n t\sqrt{\zeta^2-1}$$

for $\zeta = 1$ $\quad x(t) = (A_1 + tA_2)e^{-\omega_n t}$

at $t=0$ $\quad x(0) = 0 \longrightarrow A_1 = 0$

$\dot{x}(t) = tA_2(-\omega_n)e^{-\omega_n t} + A_2 e^{-\omega_n t}$

$\dot{x}(t) = A_2\left[e^{-\omega_n t} - t\omega_n e^{-\omega_n t}\right]$

$\dot{x}(t) = e^{-\omega_n t} A_2(1 - t\omega_n)$

at $t=0$ $\quad \dot{x}(0) = V_0 \longrightarrow V_0 = A_2$

$\therefore$ The solution (Response) will be

$x(t) = V_0 t e^{-\omega_n t}$

for $0 < \zeta < 1$ $\quad x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$

at $t=0$ $\quad x(0) = 0 \longrightarrow 0 = A\cos(-\phi)$

$A \neq 0 \quad \therefore \cos -\phi = 0 \longrightarrow \phi = \frac{\pi}{2}$

$x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \frac{\pi}{2}) = Ae^{-\zeta\omega_n t}\sin\omega_d t$

$\dot{x}(t) = A\left[\omega_d e^{-\zeta\omega_n t}\cos\omega_d t - \omega_n\zeta e^{-\zeta\omega_n t}\sin\omega_d t\right]$

at $t=0$ $\quad \dot{x}(0) = V_0 \longrightarrow V_0 = A\omega_d \longrightarrow A = \frac{V_0}{\omega_d}$

$\therefore$ The solution (Response) will be

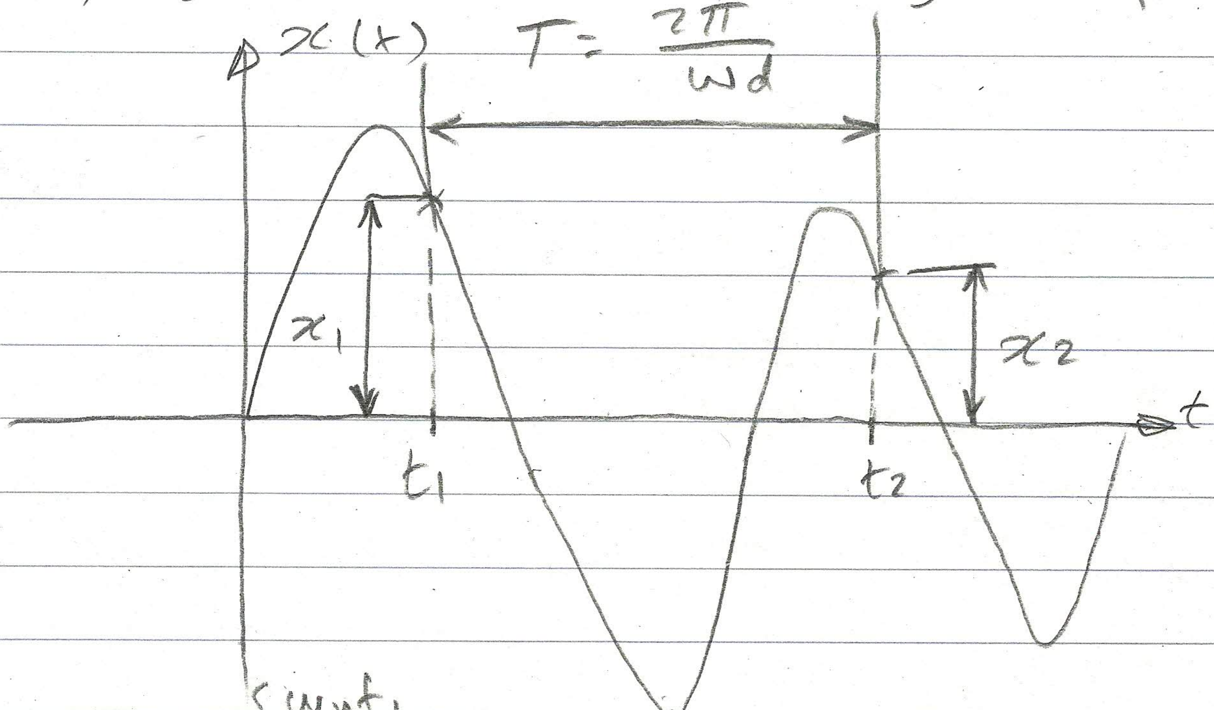
$x(t) = \frac{V_0}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Logarithmic Decrement

In case of underdamped free vibration, viscous damping causes the vibration to decay exponentially, so ζ can be determined from the observation this decay.

let t_1, t_2 the time corresponding to two consecutive displacements x_1, x_2 measured one cycle apart.



$$x_1 = A e^{-\zeta \omega_n t_1} \cos(\omega_d t_1 - \phi)$$

$$x_2 = A e^{-\zeta \omega_n t_2} \cos(\omega_d t_2 - \phi)$$

$$\frac{x_1}{x_2} = \frac{\cancel{A} e^{-\zeta \omega_n t_1} \cos(\omega_d t_1 - \phi)}{\cancel{A} e^{-\zeta \omega_n t_2} \cos(\omega_d t_2 - \phi)}$$

but $t_2 = t_1 + T$, $T = \frac{2\pi}{\omega_d}$

$\therefore \cos(\omega_d t_1 - \phi) = \cos(\omega_d t_2 - \phi)$

$$\therefore \frac{x_1}{x_2} = \frac{e^{-\zeta \omega_n t_1}}{e^{-\zeta \omega_n (t_1 + T)}} = e^{\zeta \omega_n T}$$

let δ - logarithmic decrement

$$\delta = \ln \frac{x_1}{x_2}$$

So $\ln \frac{x_1}{x_2} = \delta = \zeta \omega_n T = \zeta \omega_n \times \frac{2\pi}{\omega_d}$

$\omega_d = \omega_n \sqrt{1-\zeta^2}$

$$\boxed{\delta = \zeta \times \frac{2\pi}{\sqrt{1-\zeta^2}}}$$

$\delta \sqrt{1-\zeta^2} = 2\pi \zeta \longrightarrow \delta^2 (1-\zeta^2) = (2\pi)^2 \zeta^2$

$\zeta^2 \left((2\pi)^2 + \delta^2 \right) = \delta^2 \longrightarrow \boxed{\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}}$

for small damping δ is small quantity

$$\therefore \boxed{\zeta = \frac{\delta}{2\pi}}$$

The viscous damping factor ζ can also be determined by measuring two displacements separated by any number of complete cycles (j).
let x_1 and x_{j+1} be the amplitudes corresponding to times t_1 and t_{j+1} and $t_{j+1} = t_1 + jT$, one can coclude that

$$\frac{x_1}{x_{j+1}} = \frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdot \frac{x_3}{x_4} \cdots \frac{x_j}{x_{j+1}}$$

Because the ratio between any two consecutive displacements, one cycle apart is equal to $e^{\zeta \omega_n T}$

$$\therefore \ln \frac{x_1}{x_{j+1}} = j \zeta \omega_n T \quad \text{but} \quad \zeta \omega_n T = \delta$$

$$\boxed{\ln \frac{x_1}{x_{j+1}} = j\delta} \longrightarrow \boxed{\delta = \frac{1}{j} \ln \frac{x_1}{x_{j+1}}}$$

Ex1: It was observed that the vibration amplitude of a damped single degree of freedom system has fallen by 50 per cent after five complete cycles. Calculate the viscous damping factor.

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$$\delta = \frac{1}{j} \ln \frac{x_1}{x_{j+1}} \qquad j = 5 \quad \delta = \frac{1}{5} \ln \frac{x_1}{x_6}$$

$$\frac{x_1}{x_6} = \frac{1}{0.5} \longrightarrow x_6 = 0.5 x_1$$

$$\delta = \frac{1}{5} \ln \frac{\cancel{x_1}}{0.5 \cancel{x_1}} = \frac{1}{5} \ln 2 = 0.13863$$

(Small damping)

$$\therefore \zeta \simeq \frac{\delta}{2\pi} \longrightarrow \zeta \simeq \frac{0.13863}{2\pi}$$

$$\zeta = 0.02206$$

<u>EX2</u>: For small damping show that the Logarithmic decrement can be expressed in terms of the vibrational energy U and the energy dissipated per cycle DU.

$$\delta = \ln \frac{x_1}{x_2} \longrightarrow \frac{x_2}{x_1} = e^{-\delta} = 1 - \delta + \frac{\delta^2}{2!} - \frac{\delta^3}{3!}$$

$$+ \frac{\delta^4}{4!} - \frac{\delta^5}{5!} + \cdots$$

The vibrational energy of a system is that stored in the spring at maximum displacement.

$$U_1 = \frac{1}{2} k x_1^2 = U \qquad U_2 = \frac{1}{2} k x_2$$

$$\frac{DU}{U} = \frac{U_1 - U_2}{U_1} = 1 - \frac{U_2}{U_1} = 1 - \left(\frac{x_2}{x_1}\right)^2 = 1 - e^{-2\delta}$$

$$\frac{DU}{U} = 1 - \left(1 - 2\delta + \frac{(2\delta)^2}{2!} - \frac{(3\delta)^3}{3!} + \cdots\right)$$

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$$\frac{\Delta U}{U} = \cancel{1} - \cancel{1} + 2\delta - \frac{(2\delta)^2}{2!} + \frac{(2\delta)^3}{3!} - \dots$$

For small damping $\delta^2 \simeq 0$

$$\therefore \boxed{\frac{\Delta U}{U} = 2\delta}$$