Forced vibration

The excitation in vibration can be classified as follows

- Excitation
  - initial conditions
    - initial disp.
    - initial vel.
  - harmonic excitation
  - periodic excitation
  - non-periodic excitation

  (harmonic, periodic and non-periodic excitation: external forces)

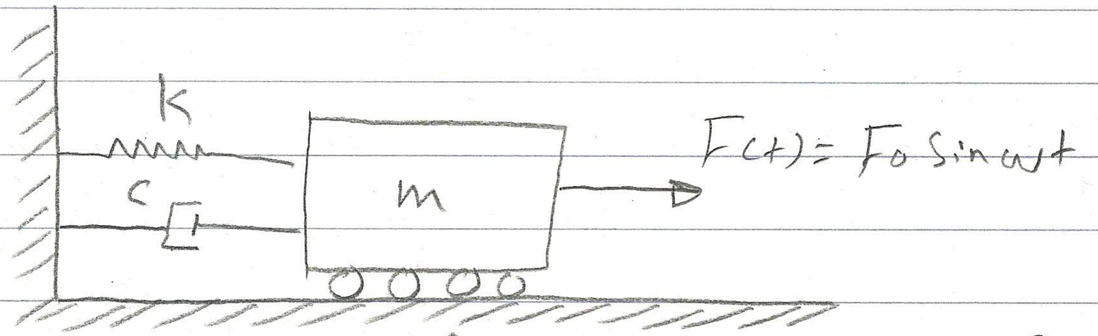
Response of the system = Response to the initial excitation + response to the external forces.

which is called the principle of Superposition.

Note that the periodic excitation can be reduced to harmonic excitation by regarding the periodic forcing function as a superposition of harmonic functions through the use of standard Fourier Series.

Response to harmonic excitation (Freauency Response)

Consider damped second order linear system.



The governing differential equation GDE of the system is $\boxed{m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_o \sin\omega t}$
where ω - driving frequency or the excitation frequency in rad/sec

General solution = Complementary function CF
damped free vibration

+ Particular Integral PI

The particular integral part of solution is assumed $x = X \sin(\omega t - \psi)$
Where X is the amplitude of oscillation and ψ is the phase angle of the displacement with respect to the exciting force.

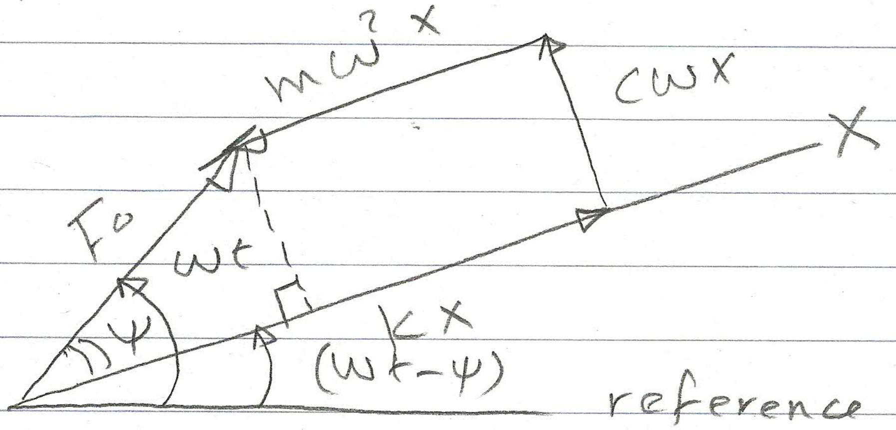
$\dot{x} = X\omega \cos(\omega t - \psi)$
$\ddot{x} = -X\omega^2 \sin(\omega t - \psi)$
$F_s = kx = kX \sin(\omega t - \psi)$ if $\sin(\omega t - \psi) = 1$
$F_s = kX$
$F_d = c\dot{x} = c\omega X \cos(\omega t - \psi)$ if $\cos(\omega t - \psi) = 1$
$F_d = cX\omega$

F_i - inertia force $\quad F_i = m\ddot{x}$

$F_i = -mX\omega^2 \sin(\omega t - \psi)$ if $\sin(\omega t - \psi) = 1$

$F_i = -mX\omega^2$

Now the governing differential equation GDE can be represented in a force vector as shown



$$F_o^2 = [kX - m\omega^2 X]^2 + (c\omega X)^2$$

$$F_o^2 = X^2 \left\{ (k - m\omega^2)^2 + (c\omega)^2 \right\}$$

$$X^2 = \frac{F_o^2}{\left\{ (k - m\omega^2)^2 + (c\omega)^2 \right\}}$$

$$\therefore X = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

and

$$\psi = \tan^{-1} \frac{c\omega X}{kX - m\omega^2 X}$$

$$\therefore \psi = \tan^{-1} \frac{c\omega}{k - m\omega^2}$$

$$X = \frac{F_o}{k\sqrt{(1-\frac{m}{k}\omega^2)^2 + (\frac{c\omega}{k})^2}}$$

but $\omega_n^2 = \frac{k}{m}$

$\frac{c}{m} = 2\zeta\omega_n \longrightarrow c = 2\zeta m\omega_n$

$$\frac{c\omega}{k} = \frac{2\zeta m\omega_n\omega}{k} = 2\zeta\frac{\omega_n}{\omega_n^2}\omega = 2\zeta\frac{\omega}{\omega_n}$$

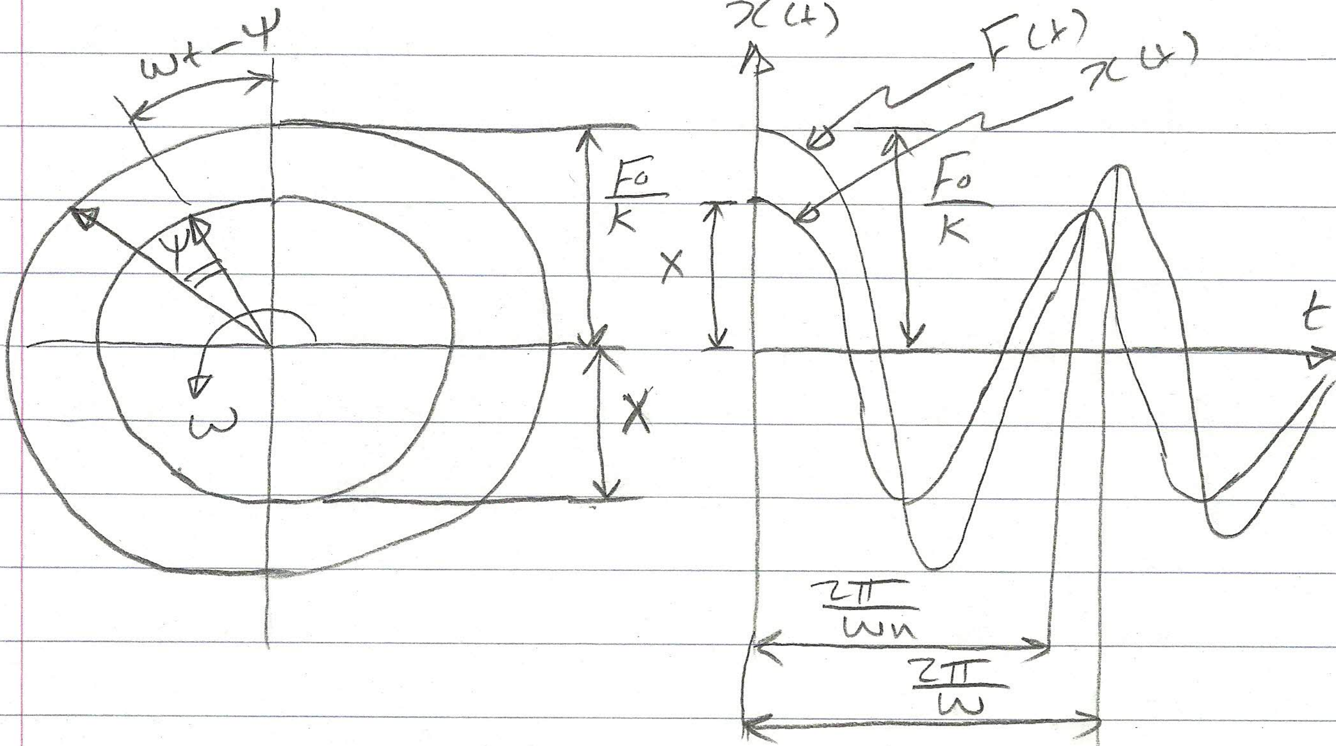
$\therefore \frac{m}{k}\omega^2 = (\frac{\omega}{\omega_n})^2$ and $(\frac{c\omega}{k})^2 = (2\zeta\frac{\omega}{\omega_n})^2$

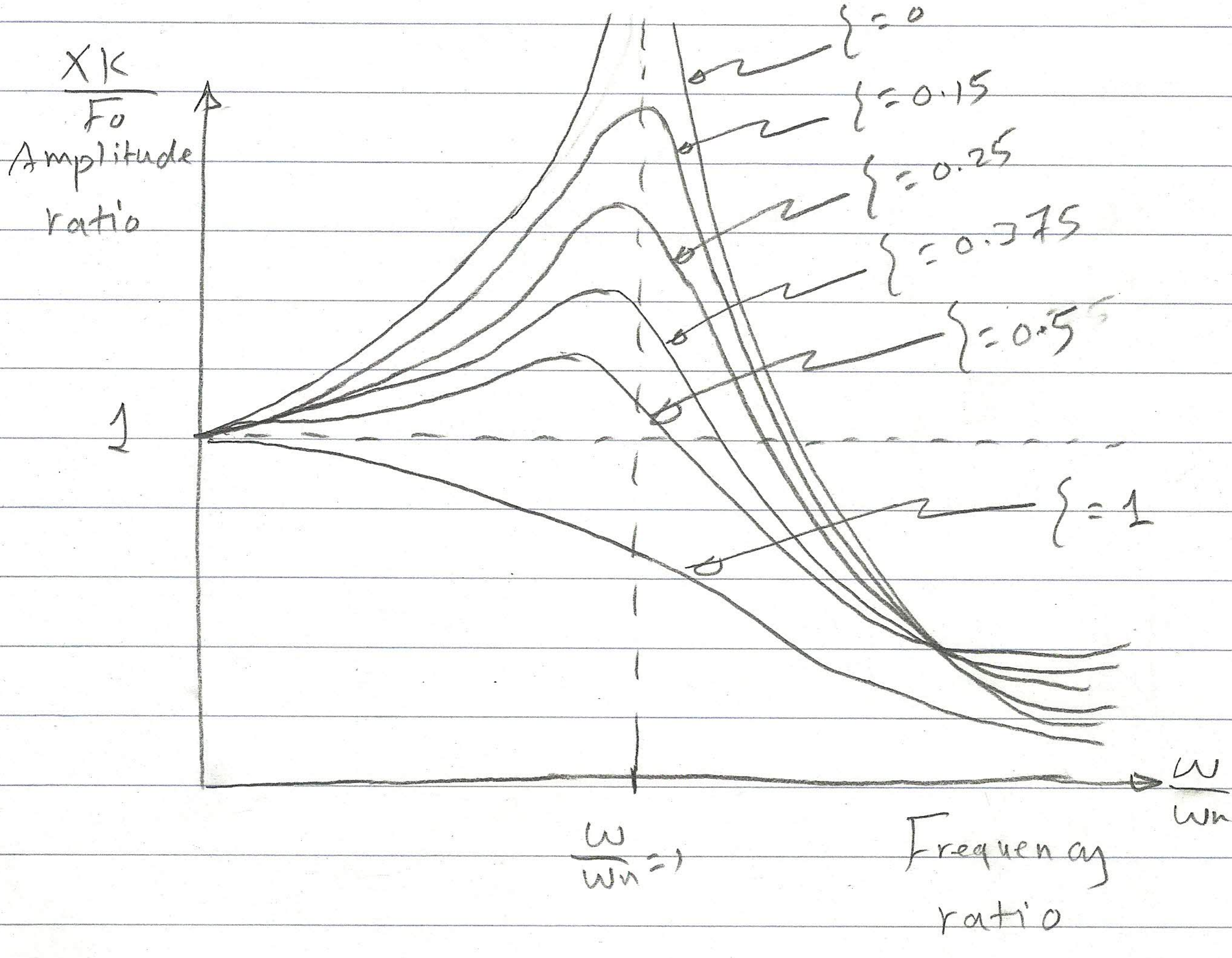
$$\therefore \boxed{X = \frac{F_o}{k\sqrt{[1-(\frac{\omega}{\omega_n})^2]^2 + [2\zeta\frac{\omega}{\omega_n}]^2}}}$$

Also $\psi = \tan^{-1}\frac{c\omega}{k(1-\frac{m}{k}\omega^2)}$

$$\therefore \boxed{\psi = \tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-(\frac{\omega}{\omega_n})^2}}$$

$\frac{Xk}{F_o}$ is called non-dimensional amplitude (amplitude ratio).


ωt-ψ
ψ
ω
Fo/k
X
x(t)
F(t)
x(t)
Fo/k
X
t
2π/ωn
2π/ω


XK/Fo
Amplitude ratio
1
ξ=0
ξ=0.15
ξ=0.25
ξ=0.375
ξ=0.5
ξ=1
ω/ωn=1
ω/ωn
Frequency ratio

(5)

From the above figure, it is observed that the damping tends to diminish amplitudes and to shift peaks to the left of the vertical through $\frac{\omega}{\omega_n}=1$

To find the values at which the peaks of the curves occur for different values of ζ

$$\frac{d}{d\omega}\left(\frac{KX}{F_o}\right)=0$$

$$\frac{XK}{F_o}=\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{-\frac{1}{2}} \quad \text{--- (1)}$$

$$\frac{dxk}{F_o}=-\frac{1}{2}\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{-\frac{3}{2}}\left\{2\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]\right.$$

$$\left.*\left(\frac{-2\omega}{\omega_n^2}\right)+8\zeta^2\frac{\omega}{\omega_n^2}\right\}=0$$

$$\cancel{2}\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]\left(\frac{-\cancel{2}\omega}{\omega_n^2}\right)+\frac{\cancel{8}^{2}\zeta^2\omega}{\omega_n^2}=0$$

$$\frac{-\omega}{\cancel{\omega_n^2}}\left\{1-\left(\frac{\omega}{\omega_n}\right)^2-2\zeta^2\right\}=0$$

$$1-\left(\frac{\omega}{\omega_n}\right)^2-2\zeta^2=0 \longrightarrow \left(\frac{\omega}{\omega_n}\right)^2=1-2\zeta^2$$

$$\frac{\omega}{\omega_n} = \sqrt{1-2\zeta^2} \quad \text{--- (2)}, \quad \omega = \omega_n\sqrt{1-2\zeta^2}$$

Substitute Eq. (2) into Eq. (1) to find the maximum amplitude ratio, this leads

$$\left(\frac{Xk}{F_o}\right)_{max} = \frac{1}{2\zeta\sqrt{1-2\zeta^2}}$$

Referring to Eq. (2)

Now if $\zeta \geq \frac{1}{\sqrt{2}} \longrightarrow \frac{\omega}{\omega_n} \longrightarrow$ imaginary

that means no peak at $\zeta \geq \frac{1}{\sqrt{2}} = 0.707$

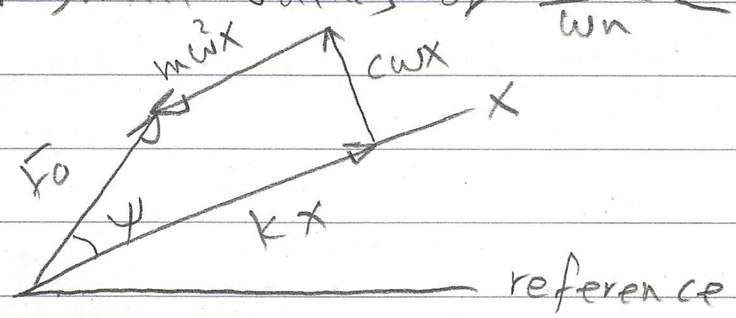
If $\zeta = 0 \longrightarrow$ undamped case at $\frac{\omega}{\omega_n} = 1$

$\frac{Xk}{F_o} \longrightarrow \infty$ discontinuity at $\frac{\omega}{\omega_n} = 1$

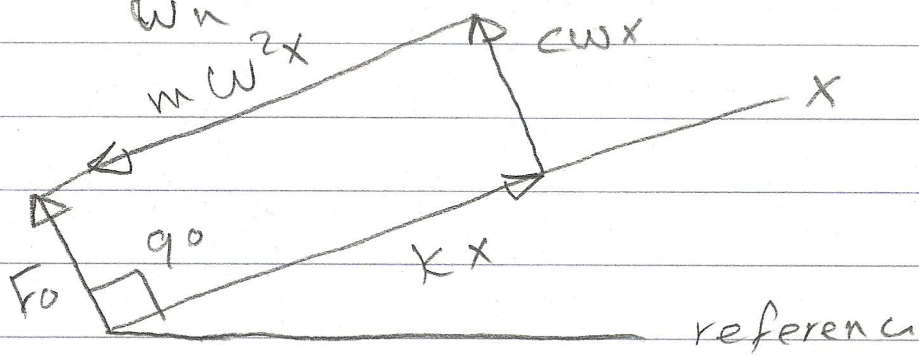
In this case a resonance condition characterized by violent vibration occurs.

Now discuss the effect of the frequency ratio $\frac{\omega}{\omega_n}$ on the force vector

For small values of $\frac{\omega}{\omega_n} << 1$, $\psi < 90$



For $\frac{\omega}{\omega_n} = 1$, $\tan\psi = \infty$, $\psi = 90°$

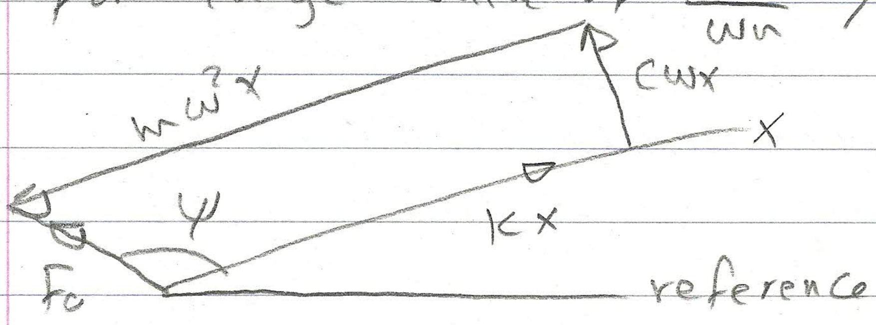


$$F_0 = c\omega X \longrightarrow X = \frac{F_0}{c\omega}$$

$$\omega = \omega_n \longrightarrow c = 2\zeta m\omega_n$$

$$X = \frac{F_0}{c\omega_n} = \frac{F_0}{2\zeta m\omega_n^2} = \frac{F_0}{2\zeta \cancel{m} \frac{k}{\cancel{m}}} = \frac{F_0}{2\zeta k}$$

For large value of $\frac{\omega}{\omega_n} \gg 1$, $\psi \to 180°$



The general solution (response), or the equation of motion for the GDE

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\sin\omega t \quad \text{is}$$

General solution = Complementary function + Particular integral

(5)

$$x(t) = A e^{-\zeta \omega_n t} \cos\left[\left(\omega_n t \sqrt{1-\zeta^2}\right) - \phi\right]$$

complementary function

$$+ \frac{F_0}{K} \frac{\sin(\omega t - \psi)}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta \frac{\omega}{\omega_n}\right]^2}}$$

Particular integral

where A, ϕ can be determined by applying the initial condition, $x(0)$ and $\dot{x}(0)$ and ψ can be determined by the equation

$$\psi = \tan^{-1} \frac{2\zeta \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$