❖ Note that gears 2, 3 & 5 are drivers while gears 3, 4 & 6 are driven

We define the *train value*, "*e* " as:

$$\boxed{e = \frac{Product\ of\ driving\ tooth\ numbers}{Product\ of\ driven\ tooth\ numbers}}$$

The sign of "*e* " is positive if the last gear rotates in the same direction as the first, and negative if the direction is reserved.

Thus we can write:

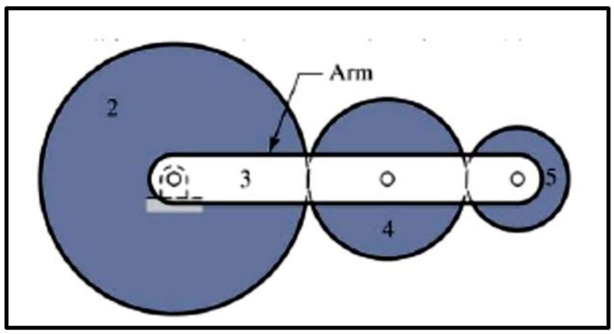
$$\boxed{n_l = e\, n_f}$$

Where: n_l Speed of last gear
n_f Speed of first gear

Planetary gear trains

In this type of trains the axis of some gears rotates about other gears (see fig. 13-28).

- ❖ Planetary trains include: a sun gear, a planetary carrier or arm, and one or more planet gears.
- ❖ Planetary trains have two or more degrees of freedom and thus have two or more inputs.



For the gear train shown

Angular velocity of gear 2 relative to the arm

$$\boxed{n_{23} = n_2 - n_3}$$

Also, angular velocity of gear 5 relative to the arm

$$\boxed{n_{53} = n_5 - n_3}$$

Dividing we obtain the ratio of gear 5 to that of 2

$$\boxed{\frac{n_{53}}{n_{23}} = \frac{n_5 - n_3}{n_2 - n_3} = e}$$

In general we write:

$$e = \frac{n_l - n_A}{n_f - n_A}$$

Ratio of last to first

n_A: Arm velocity

Example 13–3

A gearbox is needed to provide a 30:1 ($\pm$ 1 percent) increase in speed, while minimizing the overall gearbox size. Specify appropriate teeth numbers.

Solution

Since the ratio is greater than 10:1, but less than 100:1, a two-stage compound gear train, such as in Figure 13–28, is needed. The portion to be accomplished in each stage is $\sqrt{30} = 5.4772$. For this ratio, assuming a typical 20° pressure angle, the minimum number of teeth to avoid interference is 16, according to Eq. (13–11). The number of teeth necessary for the mating gears is

$$16\sqrt{30} = 87.64 \doteq 88$$

From Eq. (13–30), the overall train value is

$$e = (88/16)(88/16) = 30.25$$

This is within the 1 percent tolerance. If a closer tolerance is desired, then increase the pinion size to the next integer and try again.

Example 13–4

A gearbox is needed to provide an *exact* 30:1 increase in speed, while minimizing the overall gearbox size. Specify appropriate teeth numbers.

Solution

The previous example demonstrated the difficulty with finding integer numbers of teeth to provide an exact ratio. In order to obtain integers, factor the overall ratio into two integer stages.

$$e = 30 = (6)(5)$$

$$N_2/N_3 = 6 \quad \text{and} \quad N_4/N_5 = 5$$

With two equations and four unknown numbers of teeth, two free choices are available. Choose N_3 and N_5 to be as small as possible without interference. Assuming a 20° pressure angle, Eq. (13–11) gives the minimum as 16.

Then

$$N_2 = 6\,N_3 = 6\,(16) = 96$$
$$N_4 = 5\,N_5 = 5\,(16) = 80$$

The overall train value is then exact.

$$e = (96/16)(80/16) = (6)(5) = 30$$

Example 13–5

A gearbox is needed to provide an exact 30:1 increase in speed, while minimizing the overall gearbox size. The input and output shafts should be in-line. Specify appropriate teeth numbers.

Solution

The governing equations are

$$N_2/N_3 = 6$$
$$N_4/N_5 = 5$$
$$N_2 + N_3 = N_4 + N_5$$

With three equations and four unknown numbers of teeth, only one free choice is available. Of the two smaller gears, N_3 and N_5, the free choice should be used to minimize N_3 since a greater gear ratio is to be achieved in this stage. To avoid interference, the minimum for N_3 is 16.

Applying the governing equations yields

$$N_2 = 6N_3 = 6(16) = 96$$
$$N_2 + N_3 = 96 + 16 = 112 = N_4 + N_5$$

Substituting $N_4 = 5N_5$ gives

$$112 = 5N_5 + N_5 = 6N_5$$
$$N_5 = 112/6 = 18.67$$

If the train value need only be approximated, then this can be rounded to the nearest integer. But for an exact solution, it is necessary to choose the initial free choice for N_3 such that solution of the rest of the teeth numbers results exactly in integers. This can be done by trial and error, letting $N_3 = 17$, then 18, etc., until it works. Or, the problem can be normalized to quickly determine the minimum free choice. Beginning again, let the free choice be $N_3 = 1$. Applying the governing equations gives

$$N_2 = 6N_3 = 6(1) = 6$$

$$N_2 + N_3 = 6 + 1 = 7 = N_4 + N_5$$

Substituting $N_4 = 5N_5$, we find

$$7 = 5N_5 + N_5 = 6N_5$$

$$N_5 = 7/6$$

This fraction could be eliminated if it were multiplied by a multiple of 6. The free choice for the smallest gear N_3 should be selected as a multiple of 6 that is greater than the minimum allowed to avoid interference. This would indicate that $N_3 = 18$.

Repeating the application of the governing equations for the final time yields

$$N_2 = 6N_3 = 6(18) = 108$$

$$N_2 + N_3 = 108 + 18 = 126 = N_4 + N_5$$

$$126 = 5N_5 + N_5 = 6N_5$$

$$N_5 = 126/6 = 21$$

$$N_4 = 5N_5 = 5(21) = 105$$

Thus,

$$N_2 = 108$$

$$N_3 = 18$$

$$N_4 = 105$$

$$N_5 = 21$$

Checking, we calculate $e = (108/18)(105/21) = (6)(5) = 30$.

And checking the geometry constraint for the in-line requirement, we calculate

$$N_2 + N_3 = N_4 + N_5$$

$$108 + 18 = 105 + 21$$

$$126 = 126$$

Example 13–6

In Fig. 13–30 the sun gear is the input, and it is driven clockwise at 100 rev/min. The ring gear is held stationary by being fastened to the frame. Find the rev/min and direction of rotation of the arm and gear 4.

Designate $n_F = n_2 = -100$ rev/min, and $n_L = n_5 = 0$. Unlocking gear 5 and holding the arm stationary, in our imagination, we find

$$e = -\left(\frac{20}{30}\right)\left(\frac{30}{80}\right) = -0.25$$

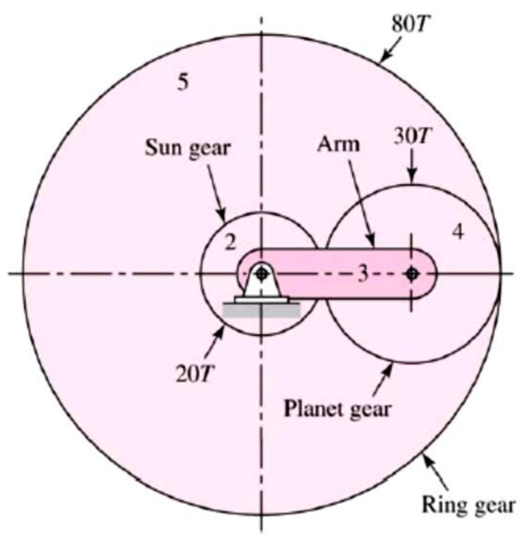


Fig. 13–30

Substituting this value in Eq. (13–32) gives

$$-0.25 = \frac{0 - n_A}{(-100) - n_A}$$

or

$$n_A = -20 \text{ rev/min}$$

To obtain the speed of gear 4, we follow the procedure outlined by Eqs. (*b*), (*c*), and (*d*). Thus

$$n_{43} = n_4 - n_3 \qquad n_{23} = n_2 - n_3$$

and so

$$\frac{n_{43}}{n_{23}} = \frac{n_4 - n_3}{n_2 - n_3} \qquad (1)$$

But

$$\frac{n_{43}}{n_{23}} = -\frac{20}{30} = -\frac{2}{3} \qquad (2)$$

Substituting the known values in Eq. (1) gives

$$-\frac{2}{3} = \frac{n_4 - (-20)}{(-100) - (-20)}$$

Solving gives

$$n_4 = 33\tfrac{1}{3} \text{ rev/min}$$

Interference with Helical Gears

- On spur and gear with one-to-one gear ratio, smallest number of teeth which will not have interference is

$$N_P = \frac{2k\cos\psi}{3\sin^2\phi_t}\left(1 + \sqrt{1 + 3\sin^2\phi_t}\right) \qquad (13\text{–}21)$$

- k =1 for full depth teeth. k = 0.8 for stub teeth
- On spur meshed with larger gear with gear ratio $m_G = N_G/N_P = m$, the smallest number of teeth which will not have interference is

$$N_P = \frac{2k\cos\psi}{(1+2m)\sin^2\phi_t}\left[m + \sqrt{m^2 + (1+2m)\sin^2\phi_t}\right] \qquad (13\text{–}22)$$

Interference with Helical Gears

- Largest gear with a specified pinion that is interference-free is

$$N_G = \frac{N_P^2\sin^2\phi_t - 4k^2\cos^2\psi}{4k\cos\psi - 2N_P\sin^2\phi_t} \qquad (13\text{–}23)$$

- Smallest spur pinion that is interference-free with a rack is

$$N_P = \frac{2k\cos\psi}{\sin^2\phi_t} \qquad (13\text{–}24)$$

Force Analysis- Spur Gears

Notation:

- Number "1" corresponds to the frame of machine.
- The input gear is given the number "2" and then gears are numbered successively 3, 4, etc.
- Shafts are designated using lowercase letters; *a, b, c*, etc.
- Force exerted by gear "2" on gear "3" will be named F_{23}.
- Similarly, F_{2a} is the force exerted by gear "2" on shaft "*a* ".
- Superscripts indicate the direction, directions are indicated using coordinates x, y and z for the shafts, radial and tangential, r and t for the gear teeth reactions.

Example: F^t_{34} is the "tangential" component of the force exerted by gear "3" on gear "4".

- Consider gear "2" mounted on shaft "*a*" which rotates at velocity "n_2" and it drives gear "3" which is mounted on shaft "*b*" causing it to rotate at velocity "n_3".

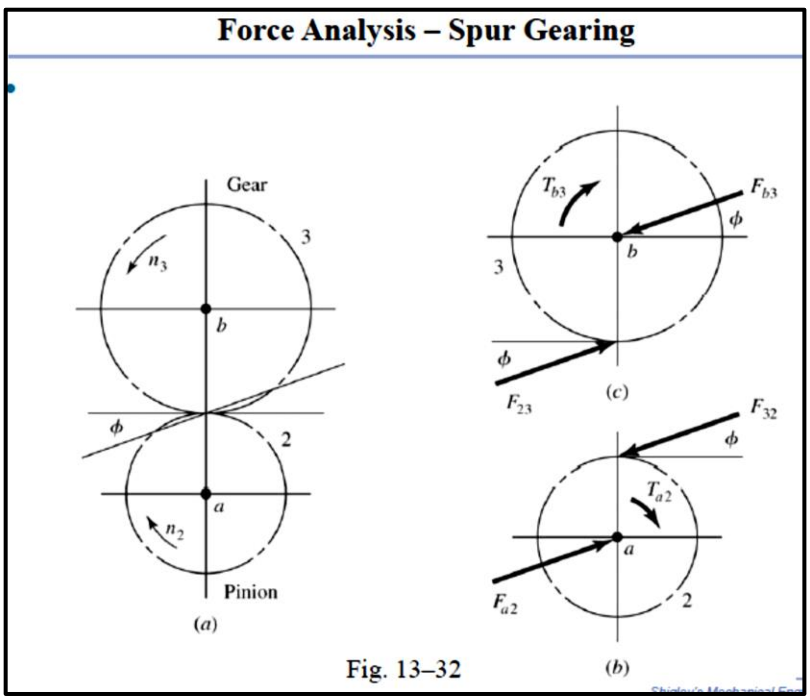


Fig. 13–32

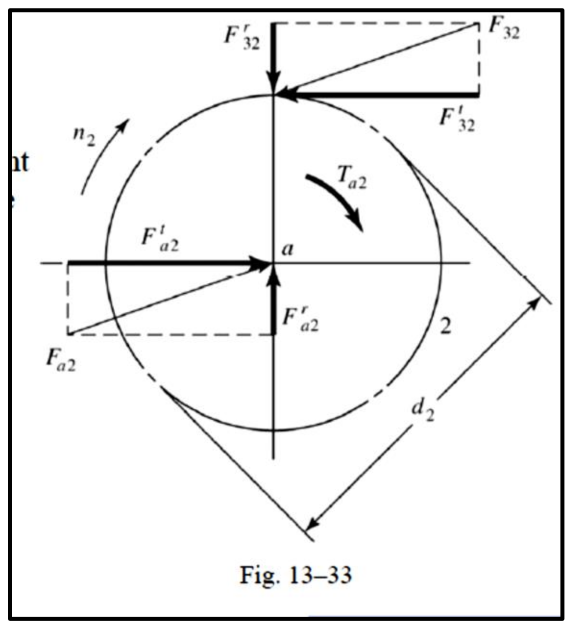


Fig. 13–33

- ❖ A free body diagram of each gear will show the forces acting on it.
- ❖ For "driver gears" the torque direction is same as the direction of rotation.
- ❖ The reactions between mating gears will be directed along the pressure line, and its direction will be opposite to the direction of rotation of "*driver gears*".

- The reaction forces can be further split into their components.
- The only useful component of $\boldsymbol{F_{32}}$ is the tangential component, and it is defined as the " *transmitted load* " $W_t = F_{32}^t$
- The applied torque can be related to the transmitted load as $T = \frac{d}{2} W_t$

If we designate the pitch line velocity as "***V***" where; $V = {}^{\pi dn}/_{12}$ *(feet/min)*

where V = pitch-line velocity, ft/min
d = gear diameter, in
n = gear speed, rev/min

The power "*H*" can be obtained as:

$$H = T\omega = (W_t d/2)\,\omega \qquad (13\text{–}33)$$

- Useful power relation in customary units,

$$W_t = 33\,000 \frac{H}{V} \qquad (13\text{–}35)$$

where W_t = transmitted load, lbf
H = power, hp
V = pitch-line velocity, ft/min

- Or in *SI units*

$$W_t = \frac{60\,000 H}{\pi d n} \qquad (13\text{–}36)$$

where W_t = transmitted load, kN
H = power, kW
d = gear diameter, mm
n = speed, rev/min

Example 13–7

Pinion 2 in Fig. 13–34*a* runs at 1750 rev/min and transmits 2.5 kW to idler gear 3. Th teeth are cut on the 20° full-depth system and have a module of m = 2.5 mm. Draw free-body diagram of gear 3 and show all the forces that act upon it.

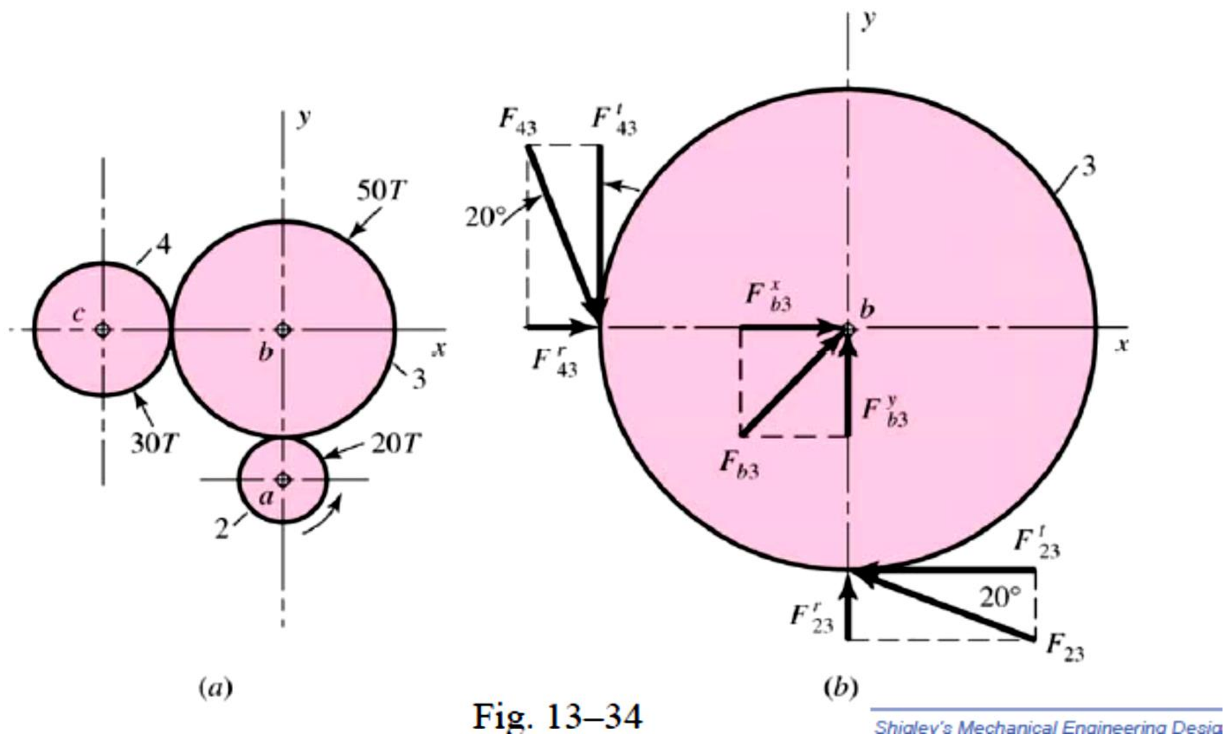


Fig. 13–34

The pitch diameters of gears 2 and 3 are

$$d_2 = N_2 m = 20(2.5) = 50 \text{ mm}$$

$$d_3 = N_3 m = 50(2.5) = 125 \text{ mm}$$

From Eq. (13–36) we find the transmitted load to be

$$W_t = \frac{60\,000H}{\pi d_2 n} = \frac{60\,000(2.5)}{\pi(50)(1750)} = 0.546 \text{ kN}$$

Thus, the tangential force of gear 2 on gear 3 is $F^t_{23} = 0.546$ kN, as shown in Fig. 13–34*b*. Therefore

$$F^r_{23} = F^t_{23} \tan 20° = (0.546) \tan 20° = 0.199 \text{ kN}$$

and so

$$F_{23} = \frac{F^t_{23}}{\cos 20°} = \frac{0.546}{\cos 20°} = 0.581 \text{ kN}$$

Since gear 3 is an idler, it transmits no power (torque) to its shaft, and so the tangential reaction of gear 4 on gear 3 is also equal to W_t. Therefore

$$F^t_{43} = 0.546 \text{ kN} \qquad F^r_{43} = 0.199 \text{ kN} \qquad F_{43} = 0.581 \text{ kN}$$

and the directions are shown in Fig. 13–34*b*.

The shaft reactions in the x and y directions are

$$F_{b3}^{x} = -(F_{23}^{t} + F_{43}^{r}) = -(-0.546 + 0.199) = 0.347 \text{ kN}$$

$$F_{b3}^{y} = -(F_{23}^{r} + F_{43}^{t}) = -(0.199 - 0.546) = 0.347 \text{ kN}$$

The resultant shaft reaction is

$$F_{b3} = \sqrt{(0.347)^2 + (0.347)^2} = 0.491 \text{ kN}$$

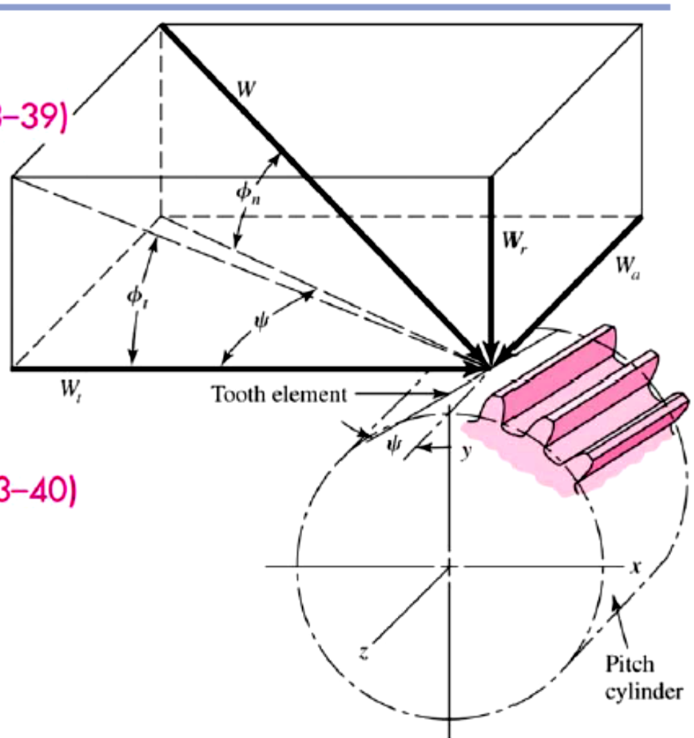
These are shown on the figure.

Force Analysis- Helical Gears

The point of action of forces is assumed to be at the middle of the tooth.
Forces acting on the tooth are shown in the figure.
The transmitted load is found from the input torque same as in spur gears.
From trigonometry, the radial force W_r and axial force W_a can be found from the transmitted load W_t (tangential force) as: (W is the total force)

Force Analysis – Helical Gearing

$$W_r = W \sin\phi_n$$
$$W_t = W \cos\phi_n \cos\psi \qquad (13\text{–}39)$$
$$W_a = W \cos\phi_n \sin\psi$$



$$W_r = W_t \tan\phi_t$$
$$W_a = W_t \tan\psi \qquad (13\text{–}40)$$
$$W = \frac{W_t}{\cos\phi_n \cos\psi}$$

Fig. 13–37

Where, ψ **: Helix angle**

ϕ_t**: Transverse pressure angle**

ϕ_n**: Normal pressure angle**

Example 13–9

In Fig. 13–38 a 1-hp electric motor runs at 1800 rev/min in the clockwise direction, as viewed from the positive x axis. Keyed to the motor shaft is an 18-tooth helical pinion having a normal pressure angle of 20°, a helix angle of 30°, and a normal diametral pitch of 12 teeth/in. The hand of the helix is shown in the figure. Make a three-dimensional sketch of the motor shaft and pinion, and show the forces acting on the pinion and the bearing reactions at A and B. The thrust should be taken out at A.

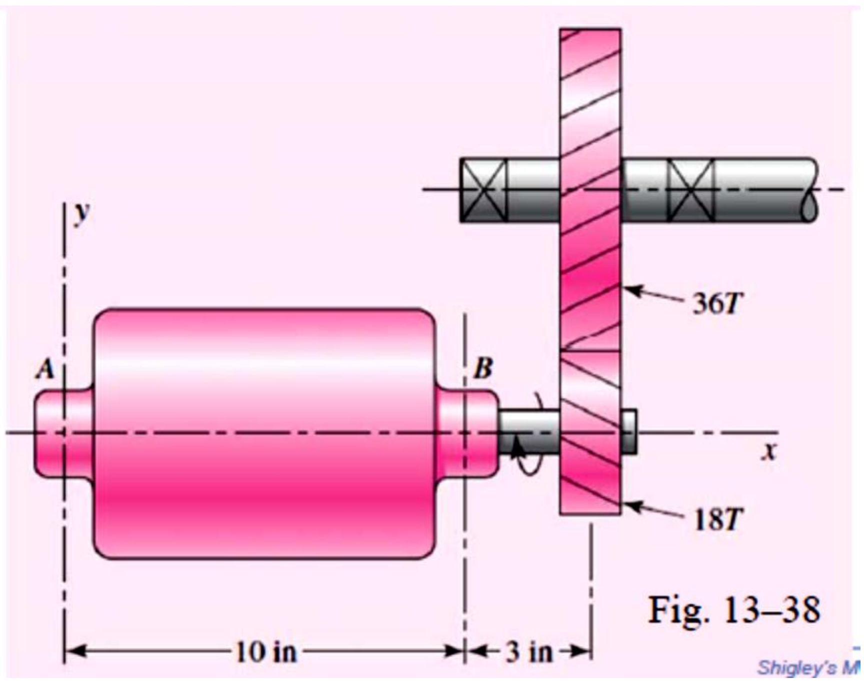


Fig. 13–38

From Eq. (13–19) we find

$$\phi_t = \tan^{-1}\frac{\tan\phi_n}{\cos\psi} = \tan^{-1}\frac{\tan 20^\circ}{\cos 30^\circ} = 22.8^\circ$$

Also, $P_t = P_n\cos\psi = 12\cos 30^\circ = 10.39$ teeth/in. Therefore the pitch diameter of the pinion is $d_p = 18/10.39 = 1.732$ in. The pitch-line velocity is

$$V = \frac{\pi d n}{12} = \frac{\pi(1.732)(1800)}{12} = 816 \text{ ft/min}$$

The transmitted load is

$$W_t = \frac{33\,000H}{V} = \frac{(33\,000)(1)}{816} = 40.4 \text{ lbf}$$

From Eq. (13–40) we find

$$W_r = W_t \tan\phi_t = (40.4)\tan 22.8^\circ = 17.0 \text{ lbf}$$

$$W_a = W_t \tan\psi = (40.4)\tan 30^\circ = 23.3 \text{ lbf}$$

$$W = \frac{W_t}{\cos\phi_n \cos\psi} = \frac{40.4}{\cos 20^\circ \cos 30^\circ} = 49.6 \text{ lbf}$$

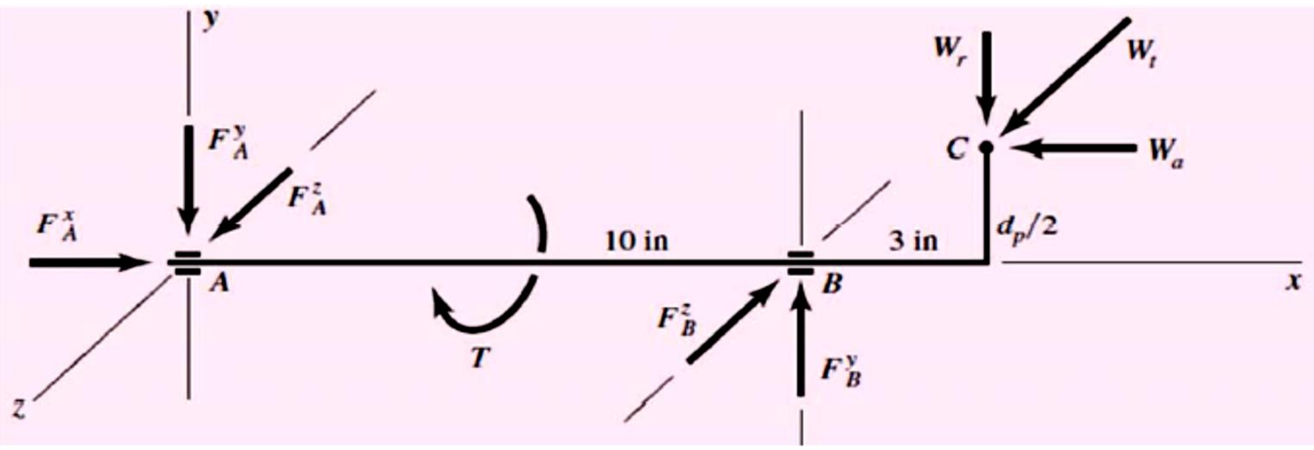


Fig. 13–39

These three forces, W_r in the $-y$ direction, W_a in the $-x$ direction, and W_t in the $+z$ direction, are shown acting at point C in Fig. 13–39. We assume bearing reactions at A and B as shown. Then $F_A^x = W_a = 23.3$ lbf. Taking moments about the z axis,

$$-(17.0)(13) + (23.3)\left(\frac{1.732}{2}\right) + 10F_B^y = 0$$

or $F_B^y = 20.1$ lbf. Summing forces in the y direction then gives $F_A^y = 3.1$ lbf. Taking moments about the y axis, next

$$10F_B^z - (40.4)(13) = 0$$

or $F_B^z = 52.5$ lbf. Summing forces in the z direction and solving gives $F_A^z = 12.1$ lbf. Also, the torque is $T = W_t d_p/2 = (40.4)(1.732/2) = 35 \text{ lbf} \cdot \text{in}$.

For comparison, solve the problem again using vectors. The force at C is

$$\mathbf{W} = -23.3\mathbf{i} - 17.0\mathbf{j} + 40.4\mathbf{k}\ \text{lbf}$$

Position vectors to B and C from origin A are

$$\mathbf{R}_B = 10\mathbf{i} \qquad \mathbf{R}_C = 13\mathbf{i} + 0.866\mathbf{j}$$

Taking moments about A, we have

$$\mathbf{R}_B \times \mathbf{F}_B + \mathbf{T} + \mathbf{R}_C \times \mathbf{W} = \mathbf{0}$$

Using the directions assumed in Fig. 13–39 and substituting values gives

$$10\mathbf{i} \times \left(F_B^y\mathbf{j} - F_B^z\mathbf{k}\right) - T\mathbf{i} + (13\mathbf{i} + 0.866\mathbf{j}) \times (-23.3\mathbf{i} - 17.0\mathbf{j} + 40.4\mathbf{k}) = \mathbf{0}$$

When the cross products are formed, we get

$$\left(10F_B^y\mathbf{k} + 10F_B^z\mathbf{j}\right) - T\mathbf{i} + (35\mathbf{i} - 525\mathbf{j} - 201\mathbf{k}) = \mathbf{0}$$

whence $T = 35\ \text{lbf}\cdot\text{in}$, $F_B^y = 20.1\ \text{lbf}$, and $F_B^z = 52.5\ \text{lbf}$.

Next,

$$\mathbf{F}_A = -\mathbf{F}_B - \mathbf{W}, \text{ and so } \mathbf{F}_A = 23.3\mathbf{i} - 3.1\mathbf{j} + 12.1\mathbf{k}\ \text{lbf}.$$