Post-Newtonian theory for the common reader

Lecture notes (July 2007)

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Contents

Chapter 1
Preliminaries

We collect in this chapter a number of results and techniques that will be required in the following chapters. The formulation of the Einstein field equations that is best suited to a post-Newtonian expansion is due to Landau and Lifshitz, and this is reviewed in Secs. 1.1 and 1.2. In Secs. 1.3 and 1.4 we refine this formulation by imposing the harmonic coordinate conditions, and we show that the exact field equations can be expressed as a set of ten wave equations in Minkowski spacetime (with complicated and highly nonlinear source terms). We illustrate the formalism in Sec. 1.5, by showing how the Schwarzschild metric can be cast in harmonic coordinates. The post-Newtonian method builds on approximate solutions to the wave equations, and in Sec. 1.6 we show how the metric can be systematically expanded in powers of the gravitational constant G and inserted within the field equations; these are iterated a number of times, and each iteration of the field equations increases the accuracy of the solution by one power of G. In Sec. 1.7 we construct the energy-momentum tensor of a point mass, and in Sec. 1.8 we summarize the elegant theory of symmetric-tracefree (STF) angular tensors and their relations with the spherical-harmonic functions.

1.1 Landau-Lifshitz formulation of the Einstein field equations

The post-Newtonian approach to integrate the Einstein field equations is based on the Landau and Lifshitz formulation of these equations. In this formulation the main variables are not the components of the metric tensor $g_{\alpha\beta}$ but those of the "gothic inverse metric"

$$\mathfrak{g}^{\alpha\beta} := \sqrt{-g}\, g^{\alpha\beta}, \tag{1.1.1}$$

where $g^{\alpha\beta}$ is the inverse metric and g the metric determinant. Knowledge of the gothic metric is sufficient to determine the metric itself: Note first that $\det[\mathfrak{g}^{\alpha\beta}] = g$,

so that g can be directly obtained from the gothic metric; then Eq. (1.1.1) gives $g^{\alpha\beta}$, which can be inverted to give $g_{\alpha\beta}$.

In the Landau-Lifshitz formulation, the left-hand side of the field equations is built from

$$H^{\alpha\mu\beta\nu} := \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu}. \tag{1.1.2}$$

This tensor density is readily seen to possess the same symmetries as the Riemann tensor, namely,

$$H^{\mu\alpha\beta\nu} = -H^{\alpha\mu\beta\nu}, \qquad H^{\alpha\mu\nu\beta} = -H^{\alpha\mu\beta\nu}, \qquad H^{\beta\nu\alpha\mu} = H^{\alpha\mu\beta\nu}. \tag{1.1.3}$$

The Einstein field equations take the form

$$\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = \frac{16\pi G}{c^4}(-g)\bigl(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}\bigr), \tag{1.1.4}$$

where $T^{\alpha\beta}$ is the energy-momentum tensor of the matter distribution, and

$$\begin{aligned}
(-g)t^{\alpha\beta}_{\rm LL} \quad := \quad & \frac{c^4}{16\pi G}\biggl\{\partial_\lambda \mathfrak{g}^{\alpha\beta}\partial_\mu \mathfrak{g}^{\lambda\mu} - \partial_\lambda \mathfrak{g}^{\alpha\lambda}\partial_\mu \mathfrak{g}^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\lambda\mu}\partial_\rho \mathfrak{g}^{\lambda\nu}\partial_\nu \mathfrak{g}^{\mu\rho} \\
& - g^{\alpha\lambda}g_{\mu\nu}\partial_\rho \mathfrak{g}^{\beta\nu}\partial_\lambda \mathfrak{g}^{\mu\rho} - g^{\beta\lambda}g_{\mu\nu}\partial_\rho \mathfrak{g}^{\alpha\nu}\partial_\lambda \mathfrak{g}^{\mu\rho} + g_{\lambda\mu}g^{\nu\rho}\partial_\nu \mathfrak{g}^{\alpha\lambda}\partial_\rho \mathfrak{g}^{\beta\mu} \\
& + \frac{1}{8}\bigl(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu}\bigr)\bigl(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau}\bigr)\partial_\lambda \mathfrak{g}^{\nu\tau}\partial_\mu \mathfrak{g}^{\rho\sigma}\biggr\}
\end{aligned} \tag{1.1.5}$$

is the Landau-Lifshitz pseudotensor, which (very loosely speaking) represents the distribution of gravitational-field energy. We use the notation $\partial_\mu f := \partial f/\partial x^\mu$ and $\partial_{\mu\nu} f := \partial^2 f/\partial x^\mu \partial x^\nu$ for any field $f(x^\mu)$ in spacetime.

By virtue of the antisymmetry of $H^{\alpha\mu\beta\nu}$ in the last pair of indices, we have that the equation

$$\partial_{\beta\mu\nu} H^{\alpha\mu\beta\nu} = 0 \tag{1.1.6}$$

holds as a trivial identity. This, together with Eq. (1.1.4), imply that

$$\partial_\beta\Bigl[(-g)\bigl(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}\bigr)\Bigr] = 0. \tag{1.1.7}$$

These are conservation equations for the total energy-momentum tensor (which includes a contribution from the matter and another contribution from the gravitational field), expressed in terms of a partial-derivative operator. These equations are strictly equivalent to the usual expression of energy-momentum conservation, $\nabla_\beta T^{\alpha\beta} = 0$, which involves only the matter's energy-momentum tensor and a covariant-derivative operator.

Equations (1.1.2)–(1.1.7) form the core of the Landau-Lifshitz framework. It is out of the question to provide here a derivation of these equations (the calculations are straightforward but extremely tedious), but the following considerations will provide a partial understanding of where they come from.

Let us write down the Einstein field equations, in their usual tensorial form

$$G^{\alpha\beta} = \frac{8\pi G}{c^4}T^{\alpha\beta},$$

at an event P in spacetime, in a local coordinate system such that $\partial_\gamma g_{\alpha\beta}(P) \stackrel{*}{=} 0$. (The special equality sign $\stackrel{*}{=}$ means "equals in the selected coordinate system.") In these coordinates the Riemann tensor at P involves only second derivatives of the metric, and a short computation reveals that the Einstein tensor is given by

$$\begin{aligned}
G^{\alpha\beta} \quad \stackrel{*}{=} \quad & \frac{1}{2}\bigl(g^{\alpha\lambda}g^{\beta\mu}g^{\nu\rho} + g^{\beta\lambda}g^{\alpha\mu}g^{\nu\rho} - g^{\alpha\lambda}g^{\beta\rho}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu}g^{\lambda\rho} \\
& - g^{\alpha\beta}g^{\mu\lambda}g^{\nu\rho} + g^{\alpha\beta}g^{\mu\nu}g^{\lambda\rho}\bigr)\partial_{\mu\nu}g_{\lambda\rho}.
\end{aligned}$$

If we now compute $\partial_{\mu\nu} H^{\alpha\mu\beta\nu}$, at the same point and in the same coordinate system, we find after straightforward manipulations that it is given by

$$\begin{aligned}\partial_{\mu\nu} H^{\alpha\mu\beta\nu} \overset{*}{=} \;& (-g)\big(g^{\alpha\lambda}g^{\beta\mu}g^{\nu\rho} + g^{\beta\lambda}g^{\alpha\mu}g^{\nu\rho} - g^{\alpha\lambda}g^{\beta\rho}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu}g^{\lambda\rho} \\ & - g^{\alpha\beta}g^{\mu\lambda}g^{\nu\rho} + g^{\alpha\beta}g^{\mu\nu}g^{\lambda\rho}\big)\partial_{\mu\nu} g_{\lambda\rho}.\end{aligned}$$

To arrive at this result we had to differentiate $(-g)$ using the rule $\partial_\mu(-g) = (-g)g^{\alpha\beta}\partial_\mu g_{\alpha\beta}$, which leads to $\partial_{\mu\nu}(-g) \overset{*}{=} (-g)g^{\alpha\beta}\partial_{\mu\nu} g_{\alpha\beta}$. We also had to relate derivatives of the inverse metric to derivatives of the metric itself; here we used the rule $\partial_\mu g^{\alpha\beta} = -g^{\alpha\lambda}g^{\beta\rho}\partial_\mu g_{\lambda\rho}$, which leads to $\partial_{\mu\nu} g^{\alpha\beta} \overset{*}{=} -g^{\alpha\lambda}g^{\beta\rho}\partial_{\mu\nu} g_{\lambda\rho}$.

Comparing the last two displayed equations reveals that

$$G^{\alpha\beta} \overset{*}{=} \frac{1}{2(-g)} \partial_{\mu\nu} H^{\alpha\mu\beta\nu},$$

and we conclude that at P, the Einstein field equations take the form of

$$\partial_{\mu\nu} H^{\alpha\mu\beta\nu} \overset{*}{=} \frac{16\pi G}{c^4}(-g)T^{\alpha\beta}.$$

This is the same as Eq. (1.1.4), because $t^{\alpha\beta}_{\mathrm{LL}} \overset{*}{=} 0$ at P, by virtue of the fact that each term in the Landau-Lifshitz pseudotensor is quadratic in $\partial_\mu \mathfrak{g}^{\alpha\beta}$, which vanishes at P in the selected coordinate system. It is therefore plausible that at any other event in spacetime, and in an arbitrary coordinate system, the Einstein field equations should take the form of Eq. (1.1.4), with a pseudotensor $t^{\alpha\beta}_{\mathrm{LL}}$ that restores all first-derivative terms that were made to vanish at P in the special coordinate system. To show that this pseudotensor takes the specific form of Eq. (1.1.5) requires a long computation.

1.2 Momentum and flux: Integral identities

Because they involve a partial-derivative operator, the differential identities of Eq. (1.1.7) can immediately be turned into integral identities. Consider a three-dimensional volume V, a fixed (time-independent) domain of the spatial coordinates x^a, bounded by a two-dimensional surface S. We assume that V contains at least some of the matter (so that $T^{\alpha\beta}$ is nonzero somewhere within V), but that S does not intersect any of the matter (so that $T^{\alpha\beta} = 0$ everywhere on S). We formally define a momentum vector $P^\alpha[V]$ associated with the volume V by the three-dimensional integral

$$P^\alpha[V] := \frac{1}{c}\int_V (-g)\big(T^{\alpha 0} + t^{\alpha 0}_{\mathrm{LL}}\big)\, d^3x. \tag{1.2.1}$$

We assume that the coordinate x^0 has a dimension of length, and the factor of c^{-1} on the right-hand side ensures that $P^\alpha[V]$ has the dimension (mass) × (velocity) of a momentum vector; it follows that $cP^0[V]$ has the dimension of an energy. In flat spacetime, and in Cartesian coordinates, $P^\alpha[V]$ would have the interpretation of being the total momentum vector associated with the energy-momentum tensor $T^{\alpha\beta}$. In curved spacetime, and in a coordinate system that cannot be assumed to be Cartesian, the quantity defined by Eq. (1.2.1) does not have any physical meaning. It is, nevertheless, a useful quantity to introduce, as we shall see in Chapter 5. In the limit in which V includes all of three-dimensional space, $P^\alpha[V]$ is known to coincide with the ADM four-momentum of an asymptotically-flat spacetime; in this limit, therefore, the physical interpretation of the momentum vector is robust.

Substituting Eq. (1.1.4) into Eq. (1.2.1) gives

$$P^\alpha[V] = \frac{c^3}{16\pi G} \int_V \partial_{\mu\nu} H^{\alpha\mu 0\nu}\, d^3x.$$

Summation over ν must exclude $\nu = 0$, because $H^{\alpha\mu 00} \equiv 0$. We therefore have

$$P^\alpha[V] = \frac{c^3}{16\pi G} \int_V \partial_c \bigl(\partial_\mu H^{\alpha\mu 0c}\bigr)\, d^3x,$$

and this can be written as a surface integral by invoking Gauss's theorem. We now have

$$P^\alpha[V] = \frac{c^3}{16\pi G} \oint_S \partial_\mu H^{\alpha\mu 0c}\, dS_c, \tag{1.2.2}$$

where dS_c is an outward-directed surface element on the two-dimensional surface S. Equation (1.2.2) can be adopted as an alternative definition for the momentum enclosed by S. This is advantageous when the volume integral of Eq. (1.2.1) is ill-defined or difficult to compute.

Assuming (as we have done) that the surface S does not move on the coordinate grid, the rate of change of the momentum vector is given by

$$\frac{d}{dx^0} P^\alpha[V] = \frac{c^3}{16\pi G} \oint_S \partial_{\mu 0} H^{\alpha\mu 0c}\, dS_c.$$

We have $\partial_{\mu 0} H^{\alpha\mu 0c} = -\partial_{\mu 0} H^{\alpha\mu c0} = -\partial_{\mu\nu} H^{\alpha\mu c\nu} + \partial_{\mu d} H^{\alpha\mu cd}$. The first term on the right-hand side can be related to the total energy-momentum tensor on S, which is equal to $(-g)t^{\alpha c}_{\rm LL}$ because the matter contribution vanishes on the surface. The second term is the spatial divergence of an antisymmetric tensor field, and its integral vanishes (by virtue of Stokes's theorem) because S does not have a boundary. Collecting results, we find that

$$\frac{d}{dx^0} P^\alpha[V] = -\frac{1}{c} \oint_S (-g) t^{\alpha c}_{\rm LL}\, dS_c. \tag{1.2.3}$$

The rate of change of $P^\alpha[V]$ is therefore expressed as a flux integral over S; and the flux is given by the Landau-Lifshitz pseudotensor. The integral identity of Eq. (1.2.3), and others similar to it, will be put to good use in Chapter 5.

1.3 Harmonic coordinates and wave equation

It is advantageous at this stage to impose the four coordinate conditions

$$\partial_\beta \mathfrak{g}^{\alpha\beta} = 0 \tag{1.3.1}$$

on the gothic metric. These are known as the *harmonic coordinate conditions*, and they play a helpful role in post-Newtonian theory. It is also useful to introduce the potentials

$$h^{\alpha\beta} := \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta}, \tag{1.3.2}$$

where $\eta^{\alpha\beta} := \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric expressed in Cartesian coordinates ($x^0 := ct, x^a$). In terms of the potentials the harmonic coordinate conditions read

$$\partial_\beta h^{\alpha\beta} = 0, \tag{1.3.3}$$

and in this context they are usually referred to as the *harmonic gauge conditions*.

The introduction of the potentials $h^{\alpha\beta}$ and the imposition of the harmonic gauge conditions simplify the appearance of the Einstein field equations. It is easy to verify that the left-hand side becomes

$$\partial_{\mu\nu} H^{\alpha\mu\beta\nu} = -\Box h^{\alpha\beta} + h^{\mu\nu}\partial_{\mu\nu} h^{\alpha\beta} - \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu},$$

where $\Box = \eta^{\mu\nu}\partial_{\mu\nu}$ is the flat-spacetime wave operator. The right-hand side of the field equations stays essentially unchanged, but the harmonic conditions do slightly simplify the form of the Landau-Lifshitz pseudotensor, as can be seen in Eq. (1.1.5). Isolating the wave operator on the left-hand side, and putting everything else on the right-hand side, gives us the formal wave equation

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta} \tag{1.3.4}$$

for the potentials $h^{\alpha\beta}$, where

$$\tau^{\alpha\beta} := (-g)\bigl(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL} + t^{\alpha\beta}_{\rm H}\bigr) \tag{1.3.5}$$

is the *effective energy-momentum pseudotensor* for the wave equation. We have introduced

$$(-g)t^{\alpha\beta}_{\rm H} := \frac{c^4}{16\pi G}\Bigl\{\partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} - h^{\mu\nu}\partial_{\mu\nu} h^{\alpha\beta}\Bigr\} \tag{1.3.6}$$

as an additional (harmonic-gauge) contribution to the effective energy-momentum pseudotensor. The wave equation of Eq. (1.3.4) is the main starting point of post-Newtonian theory. It is worth emphasizing that Eq. (1.3.4), together with Eq. (1.3.5), are an *exact formulation* of the Einstein field equations; no approximations have been introduced at this stage.

It is easy to verify that $(-g)t^{\alpha\beta}_{\rm H}$ is separately conserved, in the sense that it satisfies the equation $\partial_\beta[(-g)t^{\alpha\beta}_{\rm H}] = 0$. This, together with Eq. (1.1.7), imply that

$$\partial_\beta \tau^{\alpha\beta} = 0. \tag{1.3.7}$$

The effective energy-momentum pseudotensor is conserved.

Because it involves second derivatives of the potentials, the term $h^{\mu\nu}\partial_{\mu\nu}h^{\alpha\beta}$ on the right-hand side of the field equations might have been more appropriately placed on the left-hand side, and joined with the wave-operator term. In fact, there is a way of combining all second-order derivatives into a *curved-spacetime wave operator*. For this purpose we treat $h^{\alpha\beta}$ as a collection of ten scalar fields instead of as a tensor field. The scalar wave operator associated with the metric $g_{\alpha\beta}$ (which is to be constructed from the potentials) is denoted $\Box_g$, and it has the following action on the potentials:

$$\begin{aligned}
\Box_g h^{\alpha\beta} &:= \frac{1}{\sqrt{-g}}\partial_\mu\bigl(\sqrt{-g}g^{\mu\nu}\partial_\nu h^{\alpha\beta}\bigr) \\
&= \frac{1}{\sqrt{-g}}\partial_\mu\Bigl[\bigl(\eta^{\mu\nu} - h^{\mu\nu}\bigr)\partial_\nu h^{\alpha\beta}\Bigr] \\
&= \frac{1}{\sqrt{-g}}\Bigl[\Box h^{\alpha\beta} - h^{\mu\nu}\partial_{\mu\nu} h^{\alpha\beta}\Bigr],
\end{aligned}$$

where we have used the harmonic gauge conditions in the last step. This does indeed involve all second-derivative terms that appear in Eq. (1.3.4). The field equations could then be formulated in terms of $\Box_g$, and this was, in fact, the approach adopted by Kovacs and Thorne in their series of papers on the generation of gravitational waves. This approach, while conceptually compelling, is not as immediately useful for post-Newtonian theory as the approach adopted here, which is based on the

Minkowski wave operator. It is indeed much simpler to solve the wave equation in flat spacetime than it is to solve it in a curved spacetime with an unknown metric.

The wave equation of Eq. (1.3.4) admits the formal solution

$$h^{\alpha\beta}(x) = \frac{4G}{c^4}\int G(x,x')\tau^{\alpha\beta}(x')\,d^4x', \tag{1.3.8}$$

where $x = (ct, \boldsymbol{x})$ is a field point and $x' = (ct', \boldsymbol{x'})$ a source point. The two-point function $G(x, x')$ is the *retarded Green's function* of the Minkowski wave operator, which satisfies

$$\Box G(x,x') = -4\pi\delta(x-x'), \tag{1.3.9}$$

and which is known to be a function of $x - x'$ only. (An explicit expression will be presented in Chapter 2.) This property is sufficient to prove that if the effective energy-momentum pseudotensor $\tau^{\alpha\beta}$ satisfies the conservation identities of Eq. (1.3.7), then the potentials $h^{\alpha\beta}$ will satisfy the harmonic gauge conditions of Eq. (1.3.3). The proof involves simple manipulations and integration by parts.

1.4 Conservation identities

The conservation identities of Eq. (1.3.7) can be expressed as

$$\partial_0\tau^{00} + \partial_a\tau^{0a} = 0, \qquad \partial_0\tau^{0a} + \partial_b\tau^{ab} = 0, \tag{1.4.1}$$

in which we have separated the time derivatives from the spatial derivatives. From these we can easily derive the useful consequences

$$\tau^{0a} = \partial_0\big(\tau^{00}x^a\big) + \partial_b\big(\tau^{0b}x^a\big), \tag{1.4.2}$$

$$\tau^{ab} = \frac{1}{2}\partial_{00}\big(\tau^{00}x^ax^b\big) + \frac{1}{2}\partial_c\big(\tau^{ac}x^b + \tau^{bc}x^a - \partial_d\tau^{cd}x^ax^b\big), \tag{1.4.3}$$

and

$$\tau^{ab}x^c = \frac{1}{2}\partial_0\big(\tau^{0a}x^bx^c + \tau^{0b}x^ax^c - \tau^{0c}x^ax^b\big) + \frac{1}{2}\partial_d\big(\tau^{ad}x^bx^c + \tau^{bd}x^ax^c - \tau^{cd}x^ax^b\big). \tag{1.4.4}$$

As we shall see in Chapter 6, these conservation identities play an important role in the theory of gravitational-wave generation.

1.5 Schwarzschild metric in harmonic coordinates

The usual form of the Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2GM}{c^2\rho}\right)d(ct)^2 + \left(1 - \frac{2GM}{c^2\rho}\right)^{-1}d\rho^2 + \rho^2\big(d\theta^2 + \sin^2\theta\,d\phi^2\big), \tag{1.5.1}$$

where (t, ρ, θ, ϕ) are the usual Schwarzschild coordinates. To help us gain experience with the harmonic coordinates of post-Newtonian theory, we wish here to transform the Schwarzschild metric to a new form that is compatible with the harmonic conditions of Eq. (1.3.1).

We motivate the transformation with the observation that each one of the four scalar fields (cT, X, Y, Z), defined by

$$cT := ct, \tag{1.5.2}$$
$$X := (\rho - GM/c^2)\sin\theta\cos\phi, \tag{1.5.3}$$
$$Y := (\rho - GM/c^2)\sin\theta\sin\phi, \tag{1.5.4}$$
$$Z := (\rho - GM/c^2)\cos\theta, \tag{1.5.5}$$

and collectively denoted $X^{(\mu)}$, satisfies the scalar wave equation in the Schwarzschild metric:

$$\Box_g X^{(\mu)} := g^{\alpha\beta}\nabla_\alpha\nabla_\beta X^{(\mu)} = \frac{1}{\sqrt{-g}}\partial_\alpha\bigl(\sqrt{-g}g^{\alpha\beta}\partial_\beta X^{(\mu)}\bigr) = 0. \tag{1.5.6}$$

This statement, which is invariant under coordinate transformations, is easily verified by a quick computation. Suppose now that (cT, X, Y, Z) are adopted as coordinates, and that the Schwarzschild metric is expressed in terms of these coordinates. In these circumstances we would have $\partial_\beta X^{(\mu)} \stackrel{*}{=} \delta^\mu_\beta$ and Eq. (1.5.6) would become

$$\partial_\beta\bigl(\sqrt{-g}g^{\alpha\beta}\bigr) \stackrel{*}{=} 0.$$

This, in view of the definition $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$, is the harmonic coordinate condition of Eq. (1.3.1). The conclusion, therefore, is that (cT, X, Y, Z) is a set of harmonic coordinates for the Schwarzschild spacetime.

The transformation from the Schwarzschild coordinates (t, ρ, θ, ϕ) to the harmonic coordinates $(x^0 \equiv ct \equiv cT, x^1 \equiv x \equiv X, x^2 \equiv y \equiv Y, x^3 \equiv z \equiv Z)$ is effected by the relations

$$x^0 = ct, \qquad x^a = r\Omega^a, \qquad r := \rho - GM/c^2, \tag{1.5.7}$$

where

$$\Omega^1 := \sin\theta\cos\phi, \qquad \Omega^2 := \sin\theta\sin\phi, \qquad \Omega^3 := \cos\theta. \tag{1.5.8}$$

These definitions imply $\delta_{ab}\Omega^a\Omega^b = 1$, and we have the usual relation

$$r^2 = \delta_{ab}x^a x^b = x^2 + y^2 + z^2 \tag{1.5.9}$$

between r and the Cartesian-like coordinates (x, y, z).

The differential form of $x^a = r\Omega^a$ is

$$dx^a = \Omega^a\, d\rho + r\Omega^a_A\, d\theta^A, \tag{1.5.10}$$

where $\Omega^a_A = \partial\Omega^a/\partial\theta^A$ and $\theta^A = (\theta, \phi)$. This allows us to transform the inverse metric $g^{\alpha\beta}$ from its original Schwarzschild form to its new harmonic form. The computation involves the identity $\Omega^{AB}\Omega^a_A\Omega^b_B = \delta^{ab} - \Omega^a\Omega^b$, where Ω^{AB} is the inverse of $\Omega_{AB} := \mathrm{diag}(1, \sin^2\theta)$, the metric on a unit two-sphere. It gives

$$g^{00} = -\frac{r + GM/c^2}{r - GM/c^2}, \tag{1.5.11}$$

$$g^{ab} = \frac{r - GM/c^2}{r + GM/c^2}\Omega^a\Omega^b + \frac{r^2}{(r + GM/c^2)^2}\bigl(\delta^{ab} - \Omega^a\Omega^b\bigr). \tag{1.5.12}$$

In these expressions, r is defined by Eq. (1.5.9) and $\Omega^a := x^a/r$ forms the components of a unit vector. In Eq. (1.5.12) the spatial components of the inverse metric are decomposed into a longitudinal part proportional to $\Omega^a\Omega^b$ and a transverse part proportional to $\delta^{ab} - \Omega^a\Omega^b$; notice that this last tensor is orthogonal to Ω^a and Ω^b.

The metric is next obtained by inverting Eqs. (1.5.11) and (1.5.12). We obtain

$$g_{00} = -\frac{r - GM/c^2}{r + GM/c^2}, \tag{1.5.13}$$

$$g_{ab} = \frac{r + GM/c^2}{r - GM/c^2}\Omega_a\Omega_b + \frac{(r + GM/c^2)^2}{r^2}\bigl(\delta_{ab} - \Omega_a\Omega_b\bigr). \tag{1.5.14}$$

It is understood that $\Omega_a := \delta_{ab}\Omega^b$. It is worth noticing that in harmonic coordinates, the event horizon is located at $r = GM/c^2$; the familiar factor of 2 is missing.

The metric determinant is easily calculated to be $\sqrt{-g} = (r + GM/c^2)^2/r^2$, and the gothic inverse metric is then

$$\mathfrak{g}^{00} = -\frac{(r + GM/c^2)^3}{r^2(r - GM/c^2)}, \tag{1.5.15}$$

$$\mathfrak{g}^{ab} = \delta^{ab} - \left(\frac{GM/c^2}{r}\right)^2 \Omega^a\Omega^b. \tag{1.5.16}$$

The potentials $h^{\alpha\beta} = \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta}$ are

$$h^{00} = -1 + \frac{(r + GM/c^2)^3}{r^2(r - GM/c^2)} = 4\frac{GM/c^2}{r} + 7\left(\frac{GM/c^2}{r}\right)^2 + \cdots, \tag{1.5.17}$$

$$h^{ab} = \left(\frac{GM/c^2}{r}\right)^2 \Omega^a\Omega^b. \tag{1.5.18}$$

In Eq. (1.5.17) the exact expression for h^{00} is expanded in powers of $GM/(c^2r)$. This is an example of a post-Newtonian expansion; the leading term in h^{00} is said to be of Newtonian order, while the next term is of first post-Newtonian order.

It is easy to substitute Eqs. (1.5.16) and (1.5.17) into Eq. (1.1.2) to calculate $H^{\alpha\mu0c}$. This can then be substituted into Eq. (1.2.2) to calculate $P^\alpha[r]$, the momentum vector associated with a surface S described by r = constant. The computations are simple, they involve the surface element $dS_c = r^2\Omega_c\, d\Omega$ (where $d\Omega = \sin\theta\, d\theta d\phi$ is an element of solid angle), and they lead to $P^a[r] = 0$ and

$$P^0[r] = \frac{1}{2}Mc\frac{(2r - GM/c^2)(r + GM/c^2)}{r(r - GM/c^2)}. \tag{1.5.19}$$

The spatial momentum vanishes (as expected, since the coordinates are centered on the black hole), and in the limit $r \to \infty$ Eq. (1.5.20) reduces to

$$P^0[\infty] = Mc. \tag{1.5.20}$$

The total energy is $cP^0[\infty] = Mc^2$, and M is the total gravitational mass of the spacetime.

1.6 Iteration of the Einstein field equations

A practical way of integrating the Einstein field equations, in the form of the wave equation of Eq. (1.3.4),

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}, \qquad \tau^{\alpha\beta} = (-g)\big(T^{\alpha\beta}[g] + t^{\alpha\beta}_{\rm LL} + t^{\alpha\beta}_{\rm H}\big), \tag{1.6.1}$$

is to involve a *post-Minkowskian expansion* of the form

$$h^{\alpha\beta} = Gk_1^{\alpha\beta} + G^2k_2^{\alpha\beta} + G^3k_3^{\alpha\beta} + \cdots. \tag{1.6.2}$$

The strategy consists of integrating the wave equation order-by-order in G. This method gives rise to an adequate asymptotic expansion of the metric when the spacetime does not deviate too strongly from Minkowski spacetime. Notice that as was indicated in Eq. (1.6.1), the matter's energy-momentum tensor is actually a functional of the metric $g_{\alpha\beta}$, and this dependence comes in addition to its dependence on the purely material variables. Part of the challenge of finding a solution to the wave equation resides in this implicit dependence on the metric.

A zeroth-order approximation for the potentials is $h_0^{\alpha\beta} = 0$, which implies that $\mathfrak{g}_0^{\alpha\beta} = \eta^{\alpha\beta}$ and $g^0_{\alpha\beta} = \eta_{\alpha\beta}$. If one substitutes this into the right-hand of the

wave equation, one obtains $\Box h^{\alpha\beta} = -(16\pi G/c^4)\tau_0^{\alpha\beta}$, where $\tau_0^{\alpha\beta}$ is the zeroth-order approximation to the effective energy-momentum pseudotensor. This is known, because at this order of approximation it is equal to $T^{\alpha\beta}[\eta]$, the material energy-momentum expressed as a functional of the Minkowski metric. The solution to the wave equation is $h_1^{\alpha\beta} \equiv Gk_1^{\alpha\beta}$, and the Einstein field equations have been integrated to first order in G.

The next iteration begins by substituting $h_1^{\alpha\beta}$ into the right-hand side of the wave equation to form $\tau_1^{\alpha\beta}$, the first-order approximation to the effective energy-momentum pseudotensor. This is known, because it is constructed from $t_{\rm LL}^{\alpha\beta}$ and $t_{\rm H}^{\alpha\beta}$, which can both be computed from $h_1^{\alpha\beta}$, and also from $T^{\alpha\beta}[g]$, which is now expressed as a functional of the first-order approximation to the metric, $g_{\alpha\beta} = \eta_{\alpha\beta} + O(G)$. The new solution to the wave equation is $h_2^{\alpha\beta} \equiv Gk_1^{\alpha\beta} + G^2k_2^{\alpha\beta}$, and the Einstein field equations have been integrated to second order in G.

The iterations are continued until a desired degree of accuracy has been achieved. At this stage we have nth-iterated potentials $h_n^{\alpha\beta}$ that depend on the position in spacetime, and that depend also on the matter variables contained in $T^{\alpha\beta}$. These must be determined as well, and this is done by imposing the conservation identities of Eq. (1.3.7), $\partial_\beta \tau^{\alpha\beta} = 0$. Or equivalently [see the discussion following Eq. (1.3.9)], the matter variables are determined by imposing the harmonic gauge conditions, $\partial_\beta h^{\alpha\beta} = 0$. After this final procedure the potentials, and the associated metric, become proper tensor fields in spacetime. The point, of course, is that solving the wave equation order-by-order in G amounts to integrating only *a subset* of the Einstein field equations; to get a solution to the complete set of equations it is necessary also to impose the coordinate conditions. And since doing this is equivalent to enforcing energy-momentum conservation, the motion of the matter is determined, along with the metric, by a complete integration of the Einstein field equations.

The post-Minkowskian method requires an efficient way of computing the metric and various associated quantities from the potentials. The following approximate relations are easy to verify:

$$\begin{aligned} g_{\alpha\beta} &= \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta} + h_{\alpha\mu}h^\mu{}_\beta - \frac{1}{2}hh_{\alpha\beta} \\ &\quad + \left(\frac{1}{8}h^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\right)\eta_{\alpha\beta} + O(G^3), \qquad (1.6.3) \\ g^{\alpha\beta} &= \eta^{\alpha\beta} - h^{\alpha\beta} + \frac{1}{2}h\eta^{\alpha\beta} - \frac{1}{2}hh^{\alpha\beta} \\ &\quad + \left(\frac{1}{8}h^2 + \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\right)\eta^{\alpha\beta} + O(G^3), \qquad (1.6.4) \\ (-g) &= 1 - h + \frac{1}{2}h^2 - \frac{1}{2}h^{\mu\nu}h_{\mu\nu} + O(G^3), \qquad (1.6.5) \\ \sqrt{-g} &= 1 - \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu} + O(G^3). \qquad (1.6.6) \end{aligned}$$

It is understood that here, indices on $h^{\alpha\beta}$ are lowered with the Minkowski metric. Thus, $h_{\alpha\beta} := \eta_{\alpha\mu}\eta_{\beta\nu}h^{\mu\nu}$ and $h := \eta_{\mu\nu}h^{\mu\nu}$.

1.7 Energy-momentum tensor of a point mass

Let a particle of mass m follow a world line described by the equations $x^\alpha = z^\alpha(\lambda)$, with λ denoting proper time. Its energy-momentum tensor is given by

$$T^{\alpha\beta}(x) = mc \int \frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\frac{\delta(x^0 - z^0)\delta(\boldsymbol{x} - \boldsymbol{z})}{\sqrt{-g}}\, d\lambda, \qquad (1.7.1)$$

in which $\boldsymbol{x} = (x^1, x^2, x^3)$ denotes a purely spatial vector with components x^a. The energy-momentum tensor depends on the metric explicitly through the factor $\sqrt{-g}$, and also implicitly through the calibration of the proper time λ.

It is useful to change the parameterization of the world line from λ to z^0. We write

$$T^{\alpha\beta} = mc \int \frac{dz^\alpha}{dz^0} \frac{dz^\beta}{dz^0} \frac{dz^0}{d\lambda} \frac{\delta(x^0 - z^0)\delta(\boldsymbol{x} - \boldsymbol{z})}{\sqrt{-g}}\, dz^0,$$

and we carry out the integration with respect to z^0. This eliminates one of the δ-functions, and we obtain

$$T^{\alpha\beta} = mc \frac{dz^\alpha}{dz^0} \frac{dz^\beta}{dz^0} \frac{dz^0}{d\lambda} \frac{\delta(\boldsymbol{x} - \boldsymbol{z})}{\sqrt{-g}}.$$

Letting $z^0 \equiv x^0 = ct$, we write this as

$$T^{\alpha\beta} = \frac{m}{c} v^\alpha v^\beta \frac{dz^0}{d\lambda} \frac{\delta(\boldsymbol{x} - \boldsymbol{z})}{\sqrt{-g}},$$

where $v^\alpha = dz^\alpha/dt$ is the velocity four-vector. We next write $d\lambda^2 = -g_{\mu\nu} dz^\mu dz^\nu / c^2$ for proper time, and deduce that

$$\frac{d\lambda}{dz^0} = \frac{1}{c}\sqrt{-g_{\mu\nu} v^\mu v^\nu / c^2}.$$

Inserting this into our previous expression returns

$$T^{\alpha\beta}(t, \boldsymbol{x}) = \frac{m v^\alpha v^\beta}{\sqrt{-g}\sqrt{-g_{\mu\nu} v^\mu v^\nu / c^2}} \delta(\boldsymbol{x} - \boldsymbol{z}), \qquad v^\alpha := \frac{dz^\alpha}{dt} = (c, \boldsymbol{v}), \tag{1.7.2}$$

our final expression for the energy-momentum tensor. This is expressed in terms of $\boldsymbol{z}(t)$, the spatial position of the particle as a function of time, and in terms of $\boldsymbol{v}(t) = d\boldsymbol{z}/dt$, the spatial velocity vector. The dependence on the metric is now fully explicit.

The wave equation of Eq. (1.6.1) involves $(-g)T^{\alpha\beta}$ instead of just $T^{\alpha\beta}$, and we shall be interested in a situation in which there is an arbitrary number of point particles in the spacetime. Assigning a label A to each particle, and denoting their masses by m_A, their position vectors by $\boldsymbol{z}_A$, and their velocity vectors by $\boldsymbol{v}_A$, we find that Eq. (1.7.2) generalizes to

$$\begin{aligned} (-g)T^{\alpha\beta}(t, \boldsymbol{x}) &= \sum_A m_A v_A^\alpha v_A^\beta \frac{\sqrt{-g}}{\sqrt{-g_{\mu\nu} v_A^\mu v_A^\nu / c^2}} \delta\bigl(\boldsymbol{x} - \boldsymbol{z}_A\bigr), && (1.7.3) \\ v_A^\alpha &= \frac{dz_A^\alpha}{dt} = \bigl(c, \boldsymbol{v}_A\bigr), && (1.7.4) \end{aligned}$$

where the sum extends over each particle. It is understood that the metric $g_{\mu\nu}$, and its determinant g, are to be evaluated at the position $\boldsymbol{x} = \boldsymbol{z}_A$ of each particle.

1.8 Angular STF tensors and spherical harmonics

1.8.1 Angular STF tensors

The angular vector

$$\boldsymbol{\Omega} := \frac{\boldsymbol{x}}{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{1.8.1}$$

will appear frequently throughout these notes, together with various products $\Omega^a\Omega^b\Omega^c\cdots$, and together with *tracefree versions* $\Omega^{\langle abc\cdots\rangle}$ of such products. These tracefree tensors, distinguished by the angular bracket notation, are constructed by removing all traces from the "raw" products $\Omega^a\Omega^b\Omega^c\cdots$. Explicit examples are

$$\Omega^{\langle ab\rangle} := \Omega^a\Omega^b - \frac{1}{3}\delta^{ab}, \tag{1.8.2}$$

$$\Omega^{\langle abc\rangle} := \Omega^a\Omega^b\Omega^c - \frac{1}{5}\bigl(\delta^{ab}\Omega^c + \delta^{ac}\Omega^b + \delta^{bc}\Omega^a\bigr), \tag{1.8.3}$$

$$\begin{aligned}\Omega^{\langle abcd\rangle} := \Omega^a\Omega^b\Omega^c\Omega^d &- \frac{1}{7}\bigl(\delta^{ab}\Omega^c\Omega^d + \delta^{ac}\Omega^b\Omega^d + \delta^{ad}\Omega^b\Omega^c + \delta^{bc}\Omega^a\Omega^d \\ &+ \delta^{bd}\Omega^a\Omega^c + \delta^{cd}\Omega^a\Omega^b\bigr) + \frac{1}{35}\bigl(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\bigr).\end{aligned} \tag{1.8.4}$$

For example, $\Omega^{\langle abc\rangle}$ is tracefree because $\delta_{ab}\Omega^{\langle abc\rangle} = \Omega^c - \frac{1}{5}(3\Omega^c + \Omega^c + \Omega^c) = 0$ and similarly, $\delta_{ac}\Omega^{\langle abc\rangle} = \delta_{bd}\Omega^{\langle abc\rangle} = 0$. Because these tensors are also symmetric with respect to all pairs of indices, they are called *symmetric-tracefree tensors*, or STF tensors.

The angular STF tensors $\Omega^{\langle abc\cdots\rangle}$ play a useful role in the construction of irreducible solutions to Laplace's equation, $\nabla^2\psi = 0$. In preparation for this discussion we record the useful identities

$$\partial_a r = \Omega_a, \qquad \partial_a\Omega_b = \frac{1}{r}\bigl(\delta_{ab} - \Omega_a\Omega_b\bigr) \tag{1.8.5}$$

and introduce a *multi-index* $L := a_1a_2\cdots a_\ell$ as well as the notation

$$\Omega^L := \Omega^{a_1}\Omega^{a_2}\cdots\Omega^{a_\ell} \qquad (\ell \text{ factors}), \tag{1.8.6}$$

in which the number of factors matches the number of indices contained in the multi-index. The STF version of this product is denoted $\Omega^{\langle L\rangle}$. A tensor such as A_L is assumed to be completely symmetric, and a tensor such as $A_{\langle L\rangle}$ is completely tracefree. It is understood that summation over a repeated multi-index involves summation over each individual index contained in the multi-index.

1.8.2 Solutions to Laplace's equation

Let us first consider the growing solutions to Laplace's equation,

$$\nabla^2\psi = 0.$$

The simplest solution is the monopole, $\psi = A =$ constant, and next in order of complexity is the dipole $\psi = A_a x^a = rA_a\Omega^a$, where A_a is a constant vector (3 independent components). For a quadrupole solution we might try $\psi = r^2 A_{ab}\Omega^a\Omega^b$, but this is a solution if and only if $\delta^{ab}A_{ab} = 0$; the constant tensor must be tracefree, and we find that our quadrupole solution can be expressed as $\psi = r^2 A_{\langle ab\rangle}\Omega^a\Omega^b$, or as $\psi = r^2 A_{\langle ab\rangle}\Omega^{\langle ab\rangle}$, because the difference between $\Omega^a\Omega^b$ and $\Omega^{\langle ab\rangle}$ is $\frac{1}{3}\delta^{ab}$, and this vanishes after multiplication by a STF tensor. Notice that the number of independent components contained in $A_{\langle ab\rangle}$ is equal to 5. Continuing along these lines would eventually reveal that a general ℓ-pole solution to Laplace's equation can be expressed as

$$\psi = r^\ell A_{\langle L\rangle}\Omega^{\langle L\rangle} \qquad (\text{growing solution}), \tag{1.8.7}$$

in which the constant STF tensor $A_{\langle L\rangle}$ contains $2\ell + 1$ independent components.

Let us consider next the decaying solutions. The simplest is the monopole $\psi = Ar^{-1}$, which involves a single constant A. To generate a dipole solution we

simply differentiate r^{-1} and multiply this by a constant vector. The result is $\psi = A^a\partial_a r^{-1}$, and this clearly is a solution to Laplace's equation; the vector A^a contains 3 independent components. To generate a quadrupole solution we differentiate once more, and write $\psi = A^{ab}\partial_{ab}r^{-1}$. The solution is currently expressed in terms of a symmetric tensor that contains 6 independent components. The trace part of A^{ab}, however, provides only irrelevant information, because $\delta^{ab}\partial_{ab}r^{-1} = \nabla^2 r^{-1} = 0$ (away from $r = 0$). We may therefore remove the trace part of A^{ab} without sacrificing the generality of the solution, which we now write as $\psi = A^{\langle ab\rangle}\partial_{ab}r^{-1}$, or as $\psi = A^{\langle ab\rangle}\partial_{\langle ab\rangle}r^{-1}$; the STF tensor $A^{\langle ab\rangle}$ contains 5 independent components. Continuing along these lines would eventually reveal that a general ℓ-pole solution to Laplace's equation can be expressed as

$$\psi = A^{\langle L\rangle}\partial_{\langle L\rangle}r^{-1} \qquad \text{(decaying solution)}, \tag{1.8.8}$$

in which the constant STF tensor $A^{\langle L\rangle}$ contains $2\ell + 1$ independent components.

The decaying solutions of Eq. (1.8.8) are not yet expressed in terms of $\Omega^{\langle L\rangle}$. This is easily remedied. Involving Eq. (1.8.5) we note first that $\partial_a r^{-1} = -r^{-2}\Omega_a$, that $\partial_{ab}r^{-1} = r^{-3}(3\Omega_a\Omega_b - \delta_{ab})$, and that $\partial_{abc}r^{-1} = 3r^{-4}(-5\Omega_a\Omega_b\Omega_c + \delta_{ab}\Omega_c + \delta_{ac}\Omega_b + \delta_{bc}\Omega_a)$. Because r^{-1} is a solution to Laplace's equation, its derivatives form the components of a STF tensor, and the preceding results can be expressed as $\partial_{\langle ab\rangle}r^{-1} = 3r^{-3}\Omega_{\langle ab\rangle}$ and $\partial_{\langle abc\rangle}r^{-1} = -15r^{-4}\Omega_{\langle abc\rangle}$. To derive the general statement we assume that there is an ℓ for which we know that

$$\partial_L r^{-1} = M_\ell\, r^{-(\ell+1)}\Omega_L + \text{trace terms},$$

and we proceed by induction. (Here M_ℓ is a constant that will be determined for all values of ℓ.) An additional differentiation yields

$$\partial_{aL}r^{-1} = M_\ell\big[-(\ell+1)r^{-(\ell+2)}\Omega_a\Omega_L + r^{-(\ell+1)}\partial_a\Omega_L\big] + \text{trace terms},$$

and we compute

$$\begin{aligned}
\partial_a\Omega_L &= \partial_a\big(\Omega_{b_1}\Omega_{b_2}\cdots\Omega_{b_\ell}\big) \\
&= \big(\partial_a\Omega_{b_1}\big)\Omega_{b_2}\cdots\Omega_{b_\ell} + \cdots + \Omega_{b_1}\Omega_{b_2}\cdots\big(\partial_a\Omega_{b_\ell}\big) \\
&= r^{-1}\big(\delta_{ab_1} - \Omega_a\Omega_{b_1}\big)\Omega_{b_2}\cdots\Omega_{b_\ell} + \cdots + r^{-1}\Omega_{b_1}\Omega_{b_2}\cdots(\delta_{ab_\ell} - \Omega_a\Omega_{b_\ell}) \\
&= -\ell r^{-1}\Omega_a\Omega_L + \text{trace terms}.
\end{aligned}$$

Incorporating this into our previous result gives

$$\partial_{aL}r^{-1} = -(2\ell+1)M_\ell\, r^{-(\ell+2)}\Omega_a\Omega_L + \text{trace terms},$$

and this allows us to conclude that the expression for $\partial_{aL}r^{-1}$ is of the same general form as the expression for $\partial_L r^{-1}$, and that $M_{\ell+1} = -(2\ell+1)M_\ell$. The solution to this recurrence relation is $M_\ell = (-1)^\ell(2\ell-1)!!M_0$, and using our previous special cases we can verify that $M_0 \equiv 1$. What we have, at this stage, is a proof that $\partial_L r^{-1} = (-1)^\ell(2\ell-1)!!\, r^{-(\ell+1)}\Omega_L + \text{trace terms}$. Because $\partial_L r^{-1}$ is a STF tensor, we may express this in its final form as

$$\partial_L r^{-1} = \partial_{\langle L\rangle}r^{-1} = (-1)^\ell(2\ell-1)!!\, r^{-(\ell+1)}\Omega_{\langle L\rangle}. \tag{1.8.9}$$

It follows that Eq. (1.8.8) can be written as

$$\psi = r^{-(\ell+1)}A^{\langle L\rangle}\Omega_{\langle L\rangle} \qquad \text{(decaying solution)}, \tag{1.8.10}$$

after absorbing the factor $(-1)^\ell(2\ell-1)!!$ into a re-definition of $A^{\langle L\rangle}$.

1.8.3 Spherical harmonics

It is well known that irreducible solutions to Laplace's equations can also be expressed as

$$\psi = r^\ell \sum_{m=-\ell}^{\ell} A_{\ell m} Y_{\ell m}(\theta,\phi) \qquad \text{(growing solution)} \tag{1.8.11}$$

and

$$\psi = r^{-(\ell+1)} \sum_{m=-\ell}^{\ell} A_{\ell m} Y_{\ell m}(\theta,\phi) \qquad \text{(decaying solution)}, \tag{1.8.12}$$

where $Y_{\ell m}(\theta,\phi)$ are the usual spherical-harmonic functions, which satisfy $Y_{\ell,-m} = (-1)^m \bar{Y}_{\ell m}$, with the overbar indicating complex conjugation. The sums over m contain $2\ell+1$ terms, and for a real ψ the number of independent real constants contained in $A_{\ell m}$ is also $2\ell+1$. This number matches the number of independent components contained in the STF tensor $A^{\langle L\rangle}$.

Comparing Eq. (1.8.7) to Eq. (1.8.11), and also Eq. (1.8.10) to Eq. (1.8.12), it is clear that there must exist a strict correspondence between the angular STF tensors $\Omega^{\langle L\rangle}$ on the one hand, and the spherical harmonics $Y_{\ell m}(\theta,\phi)$ on the other hand. We can, in fact, express this correspondence as

$$Y_{\ell m}(\theta,\phi) = \mathscr{Y}^{\langle L\rangle}_{\ell m} \Omega_{\langle L\rangle}, \tag{1.8.13}$$

where $\mathscr{Y}^{\langle L\rangle}_{\ell m}$ is a constant STF tensor that satisfies $\mathscr{Y}^{\langle L\rangle}_{\ell,-m} = (-1)^m \bar{\mathscr{Y}}^{\langle L\rangle}_{\ell m}$. As specific examples, it is easy to check that

$$\mathscr{Y}^{\langle xx\rangle}_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}, \qquad \mathscr{Y}^{\langle xy\rangle}_{22} = \mathscr{Y}^{\langle yx\rangle}_{22} = \frac{i}{4}\sqrt{\frac{15}{2\pi}}, \qquad \mathscr{Y}^{\langle yy\rangle}_{22} = -\frac{1}{4}\sqrt{\frac{15}{2\pi}},$$

with all other components of $\mathscr{Y}^{\langle ab\rangle}_{22}$ vanishing, that

$$\mathscr{Y}^{\langle xz\rangle}_{21} = \mathscr{Y}^{\langle zx\rangle}_{21} = -\frac{1}{4}\sqrt{\frac{15}{2\pi}}, \qquad \mathscr{Y}^{\langle yz\rangle}_{21} = \mathscr{Y}^{\langle zy\rangle}_{21} = -\frac{i}{4}\sqrt{\frac{15}{2\pi}},$$

with all other components of $\mathscr{Y}^{\langle ab\rangle}_{21}$ vanishing, and that

$$\mathscr{Y}^{\langle xx\rangle}_{20} = -\frac{1}{4}\sqrt{\frac{5}{\pi}}, \qquad \mathscr{Y}^{\langle yy\rangle}_{20} = -\frac{1}{4}\sqrt{\frac{5}{\pi}}, \qquad \mathscr{Y}^{\langle zz\rangle}_{20} = \frac{1}{2}\sqrt{\frac{5}{\pi}},$$

with all other components of $\mathscr{Y}^{\langle ab\rangle}_{20}$ vanishing. It is also easy to verify that the functions of θ and ϕ defined by Eq. (1.8.13) satisfy the familiar eigenvalue equation for spherical harmonics.

The inverted form of Eq. (1.8.13) is

$$\Omega^{\langle L\rangle} = N_\ell \sum_{m=-\ell}^{\ell} \bar{\mathscr{Y}}^{\langle L\rangle}_{\ell m} Y_{\ell m}(\theta,\phi), \qquad N_\ell = \frac{4\pi \ell!}{(2\ell+1)!!}. \tag{1.8.14}$$

This can easily be checked for specific cases, such as $\ell = 2$. To illustrate the truth of this statement we introduce another angular vector $\boldsymbol{\Omega'} := (\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$, defined in terms of a distinct set of angles (θ',ϕ'), and we multiply each side of Eq. (1.8.14) by $\Omega'_{\langle L\rangle}$. We get

$$\Omega^{\langle L\rangle}\Omega'_{\langle L\rangle} = N_\ell \sum_{m=-\ell}^{\ell} \big(\bar{\mathscr{Y}}^{\langle L\rangle}_{\ell m}\Omega'_{\langle L\rangle}\big) Y_{\ell m}(\theta,\phi),$$

and substituting Eq. (1.8.14) gives

$$\Omega^{\langle L\rangle}\Omega'_{\langle L\rangle} = N_\ell \sum_{m=-\ell}^{\ell} \bar{Y}_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi) = \frac{2\ell+1}{4\pi} N_\ell P_\ell(\mathbf{\Omega}\cdot\mathbf{\Omega'}),$$

where we have invoked the addition theorem for spherical harmonics. Let us examine a few special cases. When $\ell = 1$ a direct computation reveals that $\Omega^a\Omega'_a = \mathbf{\Omega}\cdot\mathbf{\Omega'} =: \cos\gamma \equiv P_1(\cos\gamma)$, where γ is the angle between the vectors $\mathbf{\Omega}$ and $\mathbf{\Omega'}$. When $\ell = 2$ we have $\Omega^{\langle ab\rangle}\Omega'_{\langle ab\rangle} = \cos^2\gamma - \frac{1}{3} \equiv \frac{2}{3}P_2(\cos\gamma)$. And when $\ell = 3$ we have $\Omega^{\langle abc\rangle}\Omega'_{\langle abc\rangle} = \cos^3\gamma - \frac{3}{5}\cos\gamma \equiv \frac{2}{5}P_3(\cos\gamma)$. All these results are compatible with the general statement, which follows from Eq. (1.8.14). Now, the number that multiplies $P_\ell(\cos\gamma)$ in the general expression is $(2\ell+1)N_\ell/(4\pi)$. It is also, as we can see from the special cases, the reciprocal of the coefficient multiplying $\cos^\ell\gamma$ in an expansion of $P_\ell(\cos\gamma)$ in powers of $\cos\gamma$. This coefficient is equal to $(2\ell-1)!!/\ell!$ and we conclude that $(2\ell+1)N_\ell/(4\pi) = \ell!/(2\ell-1)!!$, so that N_ℓ is indeed given by the expression displayed in Eq. (1.8.14). We conclude also that as a consequence of Eqs. (1.8.13) and (1.8.14), we have

$$\Omega^{\langle L\rangle}\Omega'_{\langle L\rangle} = \frac{\ell!}{(2\ell-1)!!} P_\ell(\mathbf{\Omega}\cdot\mathbf{\Omega'}), \tag{1.8.15}$$

a useful identity involving the contraction of angular STF tensors that refer to two distinct directions.

The foregoing results give rise to another useful identity. We rewrite Eq. (1.8.14) as

$$\Omega'_{\langle L'\rangle} = N_{\ell'} \sum_{m'=-\ell'}^{\ell'} \mathscr{Y}^{\ell' m'}_{\langle L'\rangle} \bar{Y}_{\ell' m'}(\theta',\phi')$$

and insert it into the integral $\int Y_{\ell m}(\theta',\phi')\Omega'_{\langle L'\rangle}\, d\Omega'$, where $d\Omega' = \sin\theta'\, d\theta' d\phi'$. This gives

$$\int Y_{\ell m}(\theta',\phi')\Omega'_{\langle L'\rangle}\, d\Omega' = N_{\ell'} \sum_{m'=-\ell'}^{\ell'} \mathscr{Y}^{\ell' m'}_{\langle L'\rangle} \int \bar{Y}_{\ell' m'}(\theta',\phi') Y_{\ell m}(\theta',\phi')\, d\Omega',$$

and the orthonormality of the spherical harmonics allows us to simplify this as

$$\int Y_{\ell m}(\theta',\phi')\Omega'_{\langle L'\rangle}\, d\Omega' = \delta_{\ell\ell'} N_\ell\, \mathscr{Y}^{\ell m}_{\langle L\rangle}$$

If we now multiply each side by $\bar{Y}_{\ell m}(\theta,\phi)$ and sum over m, we obtain

$$\sum_{m=-\ell}^{\ell} \bar{Y}_{\ell m}(\theta,\phi) \int Y_{\ell m}(\theta',\phi')\Omega'_{\langle L'\rangle}\, d\Omega' = \delta_{\ell\ell'} N_\ell \sum_{m=-\ell}^{\ell} \mathscr{Y}^{\ell m}_{\langle L\rangle} \bar{Y}_{\ell m}(\theta,\phi).$$

In view of Eq. (1.8.14), this is

$$\sum_{m=-\ell}^{\ell} \bar{Y}_{\ell m}(\theta,\phi) \int Y_{\ell m}(\theta',\phi')\Omega'_{\langle L'\rangle}\, d\Omega' = \delta_{\ell\ell'}\, \Omega_{\langle L\rangle}. \tag{1.8.16}$$

This identity will be put to good use in Chapters 2 and 6.

1.8.4 Spherical averages

We denote by $\langle\langle\psi\rangle\rangle$ the average of a quantity $\psi(\theta,\phi)$ over the surface of a unit two-sphere:

$$\langle\!\langle \psi \rangle\!\rangle := \frac{1}{4\pi}\int \psi(\theta,\phi)\, d\Omega, \tag{1.8.17}$$

where $d\Omega = \sin\theta\, d\theta d\phi$. Of particular interest are the spherical average of products $\Omega^a\Omega^b\Omega^c\cdots$ of angular vectors. These are easily computed using Eqs. (1.8.2) and the fact that the average of an angular STF tensor $\Omega^{\langle abc\cdots\rangle}$ must be zero; this property follows directly from Eq. (1.8.14) and the identity $\int Y_{\ell m}(\theta,\phi)\, d\Omega = \delta_{\ell,0}\delta_{m,0}$. We obtain

$$\langle\!\langle \Omega^a \rangle\!\rangle = 0, \tag{1.8.18}$$

$$\langle\!\langle \Omega^a\Omega^b \rangle\!\rangle = \frac{1}{3}\delta^{ab}, \tag{1.8.19}$$

$$\langle\!\langle \Omega^a\Omega^b\Omega^c \rangle\!\rangle = 0, \tag{1.8.20}$$

$$\langle\!\langle \Omega^a\Omega^b\Omega^c\Omega^d \rangle\!\rangle = \frac{1}{15}\big(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\big). \tag{1.8.21}$$

These results can also be established directly, by recognizing that the tensorial structure on the right-hand side is uniquely determined by the complete symmetry of the left-hand side and the fact that δ^{ab} is the only available geometrical object. The numerical coefficient can then be determined by taking traces; for example, $1 = \delta_{ab}\delta_{cd}\langle\langle\Omega^a\Omega^b\Omega^c\Omega^d\rangle\rangle = \frac{1}{15}(9+3+3)$, and this confirms that the numerical coefficient must indeed be $\frac{1}{15}$.

Chapter 2
Integration techniques

We saw in Chapter 1 that in their Landau-Lifshitz formulation, the Einstein field equations take the form of wave equations in Minkowski spacetime. Integrating the field equations requires finding solutions to the wave equation, and in this chapter we introduce the relevant techniques. We begin by giving a precise formulation of the problem in Sec. 2.1, and after introducing the retarded Green's function for the wave equation, we describe how its solution can be expressed as an integral over the past light cone of the spacetime point at which the field is evaluated. In Sec. 2.2 we partition three-dimensional space into near-zone and wave-zone regions, and in Sec. 2.3 we follow Will and Wiseman (1996) and explain how the light-cone integral can be decomposed into near-zone and wave-zone contributions. Techniques to evaluate near-zone integrals are introduced in Sec. 2.4, and techniques to evaluate wave-zone integrals are developed in Sec. 2.5.

2.1 Formulation of the mathematical problem

We wish to integrate the wave equation

$$\Box\psi = -4\pi\mu \tag{2.1.1}$$

for a potential $\psi(x)$ generated by a source $\mu(x)$. Here $x = (ct, \boldsymbol{x})$ labels a spacetime event, and

$$\Box := \eta^{\alpha\beta}\partial_{\alpha\beta} = -\frac{\partial^2}{\partial(ct)^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.1.2}$$

is the wave operator of Minkowski spacetime. In this chapter ψ plays the role of the gravitational potentials $h^{\alpha\beta}$, and the source function μ plays the role of the effective energy-momentum pseudotensor $\tau^{\alpha\beta}$. The source function is assumed to be known, as $\tau^{\alpha\beta}$ would be in the post-Minkowskian formulation of the Einstein field equations — see Sec. 1.6. But unlike the typical situation encountered in electrodynamics, it is not assumed to be bounded. Instead, the source is assumed to be distributed over all of Minkowski spacetime, because $\tau^{\alpha\beta}$ is constructed in part from $h^{\alpha\beta}$, which does extend over all of spacetime. The source does not have compact support, but it is assumed to fall off sufficiently fast to ensure that the solution to the wave equation decays at least as fast as $|\boldsymbol{x}|^{-1}$.

The central tool to integrate Eq. (2.1.1) is the *retarded Green's function* $G(x, x')$, a solution to

$$\Box G(x, x') = -4\pi\delta(x - x') = -4\pi\delta(ct - ct')\delta(\boldsymbol{x} - \boldsymbol{x'}) \tag{2.1.3}$$

with the property that $G(x, x')$ vanishes if x is in the past of x'. The Green's function is given explicitly by

$$G(x, x') = \frac{\delta(ct - ct' - |\boldsymbol{x} - \boldsymbol{x'}|)}{|\boldsymbol{x} - \boldsymbol{x'}|}, \tag{2.1.4}$$

where $|\boldsymbol{x} - \boldsymbol{x'}| := \sqrt{(\boldsymbol{x} - \boldsymbol{x'}) \cdot (\boldsymbol{x} - \boldsymbol{x'})} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$ is the length of the three-dimensional vector $\boldsymbol{x} - \boldsymbol{x'}$ and the spatial distance between x and x'. Alternatively, the Green's function can be expressed as

$$G(x, x') = 2\Theta(ct - ct')\,\delta\big[(ct - ct')^2 - |\boldsymbol{x} - \boldsymbol{x'}|^2\big], \tag{2.1.5}$$

in terms of the spacetime interval between x and x'; here $\Theta(ct - ct')$ is the Heaviside step function, which is equal to one when $ct > ct'$ and to zero when $ct < ct'$.

In terms of the Green's function, the solution to Eq. (2.1.1) is

$$\psi(x) = \int G(x, x')\mu(x')\,d^4x', \tag{2.1.6}$$

where $d^4x' = d(ct')d^3x'$. After substitution of Eq. (2.1.4) and integration over $d(ct')$, this becomes

$$\psi(ct, \boldsymbol{x}) = \int \frac{\mu(ct - |\boldsymbol{x} - \boldsymbol{x'}|, \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|} d^3x'. \tag{2.1.7}$$

This is the *retarded solution* to the wave equation, and the domain of integration extends over $\mathscr{C}(x)$, the *past light cone* of the field point $x = (ct, \boldsymbol{x})$.

2.2 Near zone and wave zone

The domain $\mathscr{C}(x)$ will be partitioned into a *near-zone domain* $\mathscr{N}$ and a *wave-zone domain* $\mathscr{W}$. Before we formally introduce these notions, let us examine the solution to a specific version of Eq. (2.1.1),

$$\psi = \boldsymbol{p} \cdot \boldsymbol{\Omega} \left[\frac{\cos\omega(t - r/c)}{r^2} - \frac{\omega}{c} \frac{\sin\omega(t - r/c)}{r} \right],$$

which corresponds to $\mu = -\boldsymbol{p} \cdot \boldsymbol{\nabla}\delta(\boldsymbol{x})\cos\omega t$. Here $\boldsymbol{p}$ is a constant vector, $r := |\boldsymbol{x}|$, $\boldsymbol{\Omega} := \boldsymbol{x}/r$ is the angular vector of Eq. (1.8.1), and ω is an angular frequency. Physically, this solution represents the scalar potential of a dipole of constant direction $\boldsymbol{p}$, oscillating in strength with a frequency $f = \omega/(2\pi)$; the wavelength of the radiation produced by the oscillating dipole is $\lambda = c/f = 2\pi c/\omega$.

Our first observation is that ψ behaves very differently depending on whether r is small or large compared with λ. When $r \ll \lambda = 2\pi c/\omega$, the trigonometric functions can be expanded in powers of $\omega r/c$, and the result is

$$\psi = \boldsymbol{p} \cdot \boldsymbol{\Omega} \frac{\cos\omega t}{r^2} \left[1 + O\Big(\frac{\omega^2 r^2}{c^2}\Big) \right] \qquad \text{(near zone)},$$

with a correction term that is quadratic in $r/\lambda \ll 1$. We observe also that in the *near zone* — the region $r \ll \lambda$ — the derivatives of ψ are related by

$$\frac{\partial_t \psi}{c|\boldsymbol{\nabla}\psi|} = O\Big(\frac{\omega r}{c}\Big) \qquad \text{(near zone)}.$$

In the near zone, therefore, a time derivative is smaller than a spatial derivative (multiplied by c) by a factor of order $r/\lambda \ll 1$.

When, on the other hand, $r \gg \lambda = 2\pi c/\omega$, it is no longer appropriate to expand the trigonometric functions, and the potential must be expressed as

$$\psi = -\boldsymbol{p} \cdot \boldsymbol{\Omega} \frac{\omega}{c} \frac{\sin \omega\tau}{r} \biggl[1 + O\Bigl(\frac{c}{\omega r}\Bigr) \biggr] \qquad \text{(wave zone)},$$

in terms of the *retarded-time* variable $\tau := t - r/c$; here the difference between τ and t is not small, and the correction term is linear in $\lambda/r \ll 1$. We observe also that in the *wave zone* — the region $r \gg \lambda$ — the derivatives of ψ are related by

$$\frac{\partial_t \psi}{c|\boldsymbol{\nabla}\psi|} = O(1) \qquad \text{(wave zone)}.$$

To obtain this result we use the fact that the spatial dependence contained in $\boldsymbol{\Omega}$ and r^{-1} produces a spatial derivative of fractional order λ/r, while the spatial dependence contained in $\tau = t - r/c$ produces a spatial derivative of order unity. In the wave zone, therefore, a time derivative has the same order of magnitude as a spatial derivative (multiplied by c).

To define the notions of near zone and wave zone in the general context of the wave equation of Eq. (2.1.1), we introduce the following scaling quantities:

$$t_c := \text{characteristic time scale of the source}, \tag{2.2.1}$$

$$\omega_c := \frac{2\pi}{t_c} = \text{characteristic frequency of the source}, \tag{2.2.2}$$

$$\lambda_c := \frac{2\pi c}{\omega_c} = ct_c = \text{characteristic wavelength of the radiation}. \tag{2.2.3}$$

The characteristic time scale t_c is the time required for noticeable changes to occur within the source; it is defined such that $\partial_t \mu$ is typically of order μ/t_c over the support of the source function. If, as in the previous example, μ oscillates with a frequency ω, then $t_c \sim 1/\omega$ and $\omega_c \sim \omega$.

The near zone and the wave zone are defined as

$$\text{near zone:} \qquad r \text{ or } r' \ll \lambda_c = \frac{2\pi c}{\omega_c} = ct_c, \tag{2.2.4}$$

$$\text{wave zone:} \qquad r \text{ or } r' \gg \lambda_c = \frac{2\pi c}{\omega_c} = ct_c. \tag{2.2.5}$$

Thus, the near zone is the region of space in which $r := |\boldsymbol{x}|$ or $r' := |\boldsymbol{x'}|$ is small compared with a characteristic wavelength λ_c, while the wave zone is the region of space in which r or r' is large compared with this length scale. As we have seen in the dipole example, the potential behaves very differently in the two zones: In the near zone the difference between $\tau = t - r/c$ and t is small (the field retardation is unimportant), and time derivatives are small compared with spatial derivatives; in the wave zone the difference between $\tau = t - r/c$ and t is large, and time derivatives are comparable to spatial derivatives. These properties are shared by all generic solutions to the wave equation.

Another important feature of the near zone concerns the quantity $(r'/c)\partial_t \mu$, in which μ is understood to be a function of time and the spatial variables $\boldsymbol{x'}$. This quantity is of order $(r'/c)(\mu/t_c)$, or $(r'/\lambda_c)\mu$, which is much smaller than μ. In the near zone, therefore,

$$\frac{r'}{c} \frac{\partial \mu}{\partial t} = O\Bigl(\frac{r'}{\lambda_c}\mu\Bigr) \ll \mu. \tag{2.2.6}$$

This states, simply, that the source retardation is unimportant within the near zone. This was to be expected, because the field retardation itself was seen to be unimportant in the near zone.

2.3 Integration domains

The integral of Eq. (2.1.7) extends over the past light cone $\mathscr{C}(x)$ of the field point x. Following Will and Wiseman, we partition the integration domain into two pieces, the *near-zone domain* $\mathscr{N}(x)$ and the *wave-zone domain* $\mathscr{W}(x)$. We place the boundary of the near/wave zones at an arbitrarily selected radius $\mathcal{R}$, with $\mathcal{R}$ imagined to be of the same order of magnitude as λ_c, the characteristic wavelength of the radiation associated with ψ. We let $\mathscr{N}(x)$ be the intersection between $\mathscr{C}(x)$ and the near zone, formally defined as the spatial region such that $r' := |\boldsymbol{x'}| < \mathcal{R}$. Similarly, we let $\mathscr{W}(x)$ be the intersection between $\mathscr{C}(x)$ and the wave zone, formally defined as the spatial region such that $r' > \mathcal{R}$. The near-zone and wave-zone domains join together to form the complete light cone of the field point x: $\mathscr{N}(x) + \mathscr{W}(x) = \mathscr{C}(x)$.

We write Eq. (2.1.6) as

$$\psi(x) = \psi_{\mathscr{N}}(x) + \psi_{\mathscr{W}}(x), \tag{2.3.1}$$

where

$$\psi_{\mathscr{N}}(x) = \int_{\mathscr{N}} G(x,x')\mu(x')\, d^4x' \tag{2.3.2}$$

is the near-zone portion of the light-cone integral, while

$$\psi_{\mathscr{W}}(x) = \int_{\mathscr{W}} G(x,x')\mu(x')\, d^4x' \tag{2.3.3}$$

is its wave-zone portion. We recall that the boundary between the near and wave zones is placed at $r' = \mathcal{R} = O(\lambda_c)$, where λ_c is defined by Eq. (2.2.3). Methods to evaluate $\psi_{\mathscr{N}}$ and $\psi_{\mathscr{W}}$ will be devised in the following sections. It is an important fact that while $\psi_{\mathscr{N}}$ and $\psi_{\mathscr{W}}$ will individually depend on the cutoff parameter $\mathcal{R}$, their sum $\psi = \psi_{\mathscr{N}} + \psi_{\mathscr{W}}$ will necessarily be independent of this parameter. The $\mathcal{R}$-dependence of $\psi_{\mathscr{N}}$ and $\psi_{\mathscr{W}}$ is therefore unimportant, and it can freely be ignored. This observation will serve as a helpful simplifying tool in many subsequent computations.

2.4 Near-zone integration

2.4.1 Wave-zone field point

To begin, we evaluate

$$\psi_{\mathscr{N}}(x) = \int_{\mathscr{N}} \frac{\mu(ct - |\boldsymbol{x} - \boldsymbol{x'}|, \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|} d^3x' \tag{2.4.1}$$

when x is situated in the wave zone, that is, when $r = |\boldsymbol{x}| > \mathcal{R}$. We recall that the domain of integration $\mathscr{N}$ is the intersection between $\mathscr{C}(x)$, the past light cone of the field point x, and the near zone, defined as the spatial region such that $r' := |\boldsymbol{x'}| < \mathcal{R}$.

For this purpose we introduce a modified integrand,

$$\frac{\mu(ct - |\boldsymbol{x} - \boldsymbol{x'}|, \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{x'}|} =: f(|\boldsymbol{x} - \boldsymbol{x'}|) =: g(\boldsymbol{x'}),$$

in which the spatial dependence of the source function on $\boldsymbol{x'}$ has been replaced by a dependence on arbitrary parameters $\boldsymbol{y}$. We have indicated that if t and $\boldsymbol{y}$ are kept fixed, then the modified integrand can be viewed as a function f of argument

$|\boldsymbol{x}-\boldsymbol{x'}|$. If, in addition, $\boldsymbol{x}$ is kept fixed, then we have a function g that depends only on the vector $\boldsymbol{x'}$.

Knowing that $\boldsymbol{x'}$ lies within the near zone, we treat it as a small vector, and we Taylor-expand g about $\boldsymbol{x'}=\boldsymbol{0}$. Keeping just a few terms in this expansion, we obtain

$$g(\boldsymbol{x'}) = g(\boldsymbol{0}) + \frac{\partial g}{\partial x'^a}x'^a + \frac{1}{2}\frac{\partial^2 g}{\partial x'^a x'^b}x'^a x'^b + \cdots,$$

in which all derivatives are evaluated at $\boldsymbol{x'}=\boldsymbol{0}$. But

$$\frac{\partial g}{\partial x'^a} = \frac{\partial f}{\partial x'^a} = -\frac{\partial f}{\partial x^a},$$

because f depends on $\boldsymbol{x'}$ only through the combination $|\boldsymbol{x}-\boldsymbol{x'}|$. Our Taylor expansion can therefore be expressed as

$$g(\boldsymbol{x'}) = f(|\boldsymbol{x}|) - \frac{\partial f}{\partial x^a}x'^a + \frac{1}{2}\frac{\partial^2 f}{\partial x^a x^b}x'^a x'^b + \cdots.$$

The derivatives of f are still evaluated at $\boldsymbol{x'}=\boldsymbol{0}$. But because the differentiation is now carried out with respect to $\boldsymbol{x}$, we can set $\boldsymbol{x'}=\boldsymbol{0}$ in f *before* taking the derivatives. Observing that f then becomes a function of $r=|\boldsymbol{x}|$ only, we have

$$g(\boldsymbol{x'}) = f(r) - \frac{\partial f(r)}{\partial x^a}x'^a + \frac{1}{2}\frac{\partial^2 f(r)}{\partial x^a x^b}x'^a x'^b + \cdots.$$

Keeping all terms of the Taylor expansion, this is

$$g(\boldsymbol{x'}) = \sum_{q=0}^{\infty}\frac{(-1)^q}{q!}x'^Q\partial_Q f(r),$$

where $Q := a_1 a_2 \cdots a_q$ is a multi-index of the sort introduced in Sec. 1.8.1. More explicitly, we have established the identity

$$\frac{\mu(ct-|\boldsymbol{x}-\boldsymbol{x'}|,\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{x'}|} = \sum_{q=0}^{\infty}\frac{(-1)^q}{q!}x'^Q\partial_Q\frac{\mu(ct-r,\boldsymbol{y})}{r}. \tag{2.4.2}$$

The dependence of μ/r on the variables x^a is contained entirely within r.

We may now set $\boldsymbol{y}$ equal to $\boldsymbol{x'}$ and substitute Eq. (2.4.2) into Eq. (2.4.1). This gives

$$\psi_{\mathscr{N}}(ct,\boldsymbol{x}) = \sum_{q=0}^{\infty}\frac{(-1)^q}{q!}\partial_Q\left[\frac{1}{r}\int_{\mathscr{M}}\mu(u,\boldsymbol{x'})x'^Q\,d^3x'\right], \tag{2.4.3}$$

where

$$u := ct - r = c(t-r/c) =: c\tau \tag{2.4.4}$$

is a retarded-time variable. Notice that the temporal dependence of the source function no longer involves $\boldsymbol{x'}$, the variable of integration. The integration domain has therefore become a surface of constant time (the constant being equal to $\tau = t - r/c$) bounded externally by the sphere $r' = \mathcal{R}$. This domain is denoted $\mathscr{M}$ in Eq. (2.4.3).

Equation (2.4.3) is valid everywhere within the wave zone. It simplifies when $r\to\infty$, that is, when $\psi_{\mathscr{N}}$ is evaluated in the *far-away wave zone*, a neighbourhood of future null infinity. In this limit we retain only the dominant, r^{-1} term in $\psi_{\mathscr{N}}$, and we approximate Eq. (2.4.3) by

$$\psi_{\mathscr{N}} = \frac{1}{r}\sum_{q=0}^{\infty}\frac{(-1)^q}{q!}\int_{\mathscr{M}}\partial_Q\mu(u,\boldsymbol{x'})x'^Q\,d^3x' + O(r^{-2}).$$

The dependence of μ on x^a is contained in $u = ct - r$, so that $\partial_a \mu = -\mu^{(1)}\partial_a r = -\mu^{(1)}\Omega_a$, in which $\mu^{(1)}$ denotes the first derivative of μ with respect to u, and where we used Eq. (1.8.5). Involving this equation once more, we find that $\partial_{ab}\mu = \mu^{(2)}\Omega_a\Omega_b + O(r^{-1})$, and continuing along these lines reveals that in general, $\partial_Q \mu = (-1)^q \mu^{(q)}\Omega_Q + O(r^{-1})$. Inserting this into the previously displayed equation, we find that Eq. (2.4.3) becomes

$$\psi_{\mathscr{N}}(t,\boldsymbol{x}) = \frac{1}{r}\sum_{q=0}^{\infty}\frac{1}{q!}\Omega_Q\left(\frac{\partial}{\partial u}\right)^q \int_{\mathscr{M}} \mu(u,\boldsymbol{x}')x'^Q\, d^3x' + O(r^{-2}) \tag{2.4.5}$$

in the far-away wave zone. This is a *multipole expansion* for the potential $\psi_{\mathscr{N}}$. Notice that $\Omega_Q x'^Q = \Omega_{a_1}\Omega_{a_2}\cdots\Omega_{a_q}x'^{a_1}x'^{a_2}\cdots x'^{a_q} = (\boldsymbol{\Omega}\cdot\boldsymbol{x}')^q$.

2.4.2 Near-zone field point

We next evaluate

$$\psi_{\mathscr{N}}(x) = \int_{\mathscr{N}} \frac{\mu(ct - |\boldsymbol{x}-\boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x' \tag{2.4.6}$$

when x is situated in the near zone, that is, when $r = |\boldsymbol{x}| < \mathcal{R}$.

In this situation, both $\boldsymbol{x}$ and $\boldsymbol{x}'$ lie within the near zone, and $|\boldsymbol{x}-\boldsymbol{x}'|$ can be treated as a small quantity. To evaluate the integral we simply Taylor-expand the time-dependence of the source function, as in

$$\mu(ct - |\boldsymbol{x}-\boldsymbol{x}'|) = \mu(ct) - \frac{\partial\mu}{\partial(ct)}|\boldsymbol{x}-\boldsymbol{x}'| + \frac{1}{2}\frac{\partial^2\mu}{\partial(ct)^2}|\boldsymbol{x}-\boldsymbol{x}'|^2 + \cdots,$$

in which all derivatives are evaluated at ct. Substituting this expansion into Eq. (2.4.6) produces

$$\psi_{\mathscr{N}}(t,\boldsymbol{x}) = \sum_{q=0}^{\infty}\frac{(-1)^q}{q!}\left(\frac{\partial}{\partial(ct)}\right)^q \int_{\mathscr{M}} \mu(ct,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|^{q-1}\, d^3x', \tag{2.4.7}$$

which is valid everywhere within the near zone. Notice that once more, the domain of integration is $\mathscr{M}$, a surface of constant time bounded externally by the sphere $r' = \mathcal{R}$.

2.5 Wave-zone integration

In this section we develop a method to evaluate

$$\psi_{\mathscr{W}}(x) = \int_{\mathscr{W}} G(x,x')\mu(x')\, d^4x', \tag{2.5.1}$$

the wave-zone portion of the complete solution ψ to the wave equation. We recall that the domain of integration $\mathscr{W}$ is the intersection between $\mathscr{C}(x)$, the past light cone of the field point x, and the wave zone, defined as the spatial region such that $r' := |\boldsymbol{x}'| > \mathcal{R}$. The wave-zone integral of Eq. (2.5.1) is much more difficult to evaluate than the near-zone integral encountered in Sec. 2.4. To proceed it will be necessary to restrict our attention to source functions of the form

$$\mu(x') = \frac{1}{4\pi}\frac{f(u')}{r'^n}\Omega'^{\langle L\rangle}, \tag{2.5.2}$$

where f is an arbitrary function of argument $u' = ct' - r'$ (it is unrelated to the function f introduced in Sec. 2.4.1), n is an arbitrary integer, and $\Omega'^{\langle L\rangle}$ is an

angular STF tensor of degree ℓ, of the sort introduced in Sec. 1.8.1. Fortunately, this restriction is not too important from a practical point of view: All sources functions that will be involved in wave-zone integrals in the remaining chapters of these notes will be seen to be superpositions of the irreducible forms displayed in Eq. (2.5.2).

2.5.1 Reduced Green's function

The source function of Eq. (2.5.2) can be neatly expressed in terms of spherical harmonics $Y_{\ell m}(\theta', \phi')$ — see Sec. 1.8.3 — and for this reason it is advantageous to express $G(x, x')$, the retarded Green's function of Eqs. (2.1.4) and (2.1.5), as a spherical-harmonic decomposition. We therefore write

$$G(x, x') = \sum_{\ell m} g_\ell(ct, r; ct', r') \bar{Y}_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi), \tag{2.5.3}$$

where $g_\ell(ct, r; ct', r')$ is a *reduced Green's function* for each multipole order ℓ. Substitution of Eq. (2.5.3) into Eq. (2.1.3) reveals that each g_ℓ satisfies the reduced wave equation

$$\left[-\frac{\partial^2}{\partial (ct)^2} + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} \right] g_\ell = -\frac{4\pi}{r^2} \delta(ct - ct') \delta(r - r'). \tag{2.5.4}$$

It follows from this equation that (as was already indicated) the reduced Green's function depends on ℓ but is independent of m.

We shall not attempt to integrate Eq. (2.5.4) directly. It is simpler to take the known expression for $G(x, x')$, as it appears in Eq. (2.1.5), and to extract its multipole components using Eq. (2.5.3) and the orthonormality of the spherical harmonics. Introducing the notation

$$\Delta := c(t - t'), \qquad R := |\boldsymbol{x} - \boldsymbol{x'}|, \tag{2.5.5}$$

Eq. (2.1.5) can be expressed as

$$G(x, x') = 2\Theta(\Delta) \delta(\Delta^2 - R^2).$$

We substitute this on the left-hand side of Eq. (2.5.1), multiply each side by $Y_{\ell' m'}(\theta', \phi')$, and integrate over $d\Omega' = \sin\theta' \, d\theta' d\phi'$. The result is

$$2\Theta(\Delta) \int \delta(\Delta^2 - R^2) Y_{\ell m}(\theta', \phi') \, d\Omega' = g_\ell Y_{\ell m}(\theta, \phi).$$

We next set $m = 0$ and use the fact that $Y_{\ell 0}(\theta, \phi) \propto P_\ell(\cos\theta)$. The previous equation reduces to

$$2\Theta(\Delta) \int \delta(\Delta^2 - R^2) P_\ell(\cos\theta') \, d\cos\theta' \, d\phi' = g_\ell P_\ell(\cos\theta).$$

Finally, we set $\cos\theta = 1$ and use the fact that $P_\ell(1) = 1$. This gives

$$g_\ell = 2\Theta(\Delta) \int \delta(\Delta^2 - R^2) \Big|_{\cos\theta = 1} P_\ell(\cos\theta') \, d\cos\theta' \, d\phi',$$

and since $\Delta^2 - R^2$ evaluated at $\cos\theta = 1$ is independent of ϕ' (as we shall see), we have

$$g_\ell(ct, r; ct', r') = 4\pi\Theta(\Delta) \int \delta(\Delta^2 - R^2) \Big|_{\cos\theta = 1} P_\ell(\cos\theta') \, d\cos\theta'. \tag{2.5.6}$$

To evaluate the remaining integral we must compute $R^2 = (\boldsymbol{x} - \boldsymbol{x'}) \cdot (\boldsymbol{x} - \boldsymbol{x'})$ and evaluate it at $\cos\theta = 1$. This gives $R^2 = r^2 - 2rr'\cos\theta' + r'^2$, and substitution inside the δ-function produces

$$g_\ell = \frac{2\pi\Theta(\Delta)}{rr'} \int \delta(\cos\theta' - \xi) P_\ell(\cos\theta')\, d\cos\theta',$$

where $\xi := (r^2 + r'^2 - \Delta^2)/(2rr')$. The integral is nonzero whenever ξ lies in the interval between -1 and $+1$; when this condition is satisfied it evaluates to

$$g_\ell = \frac{2\pi\Theta(\Delta)}{rr'} P_\ell(\xi).$$

The condition $-1 < \xi$ implies $-2rr' < r^2 + r'^2 - \Delta^2$, so that $\Delta < r + r'$. The condition $\xi < 1$ implies $2rr' > r^2 + r'^2 - \Delta^2$, so that $\Delta > |r - r'|$. This last condition supersedes the requirement $\Delta > 0$, which comes from the step function appearing in $G(x, x')$. Altogether, we find that the reduced Green's function is given by

$$g_\ell(ct, r; ct', r') = \frac{2\pi}{rr'} \Theta(\Delta - |r - r'|) \Theta(r + r' - \Delta) P_\ell(\xi), \tag{2.5.7}$$

where

$$\xi := \frac{r^2 + r'^2 - \Delta^2}{2rr'}. \tag{2.5.8}$$

The temporal support of the reduced Green's function is the interval $|r - r'| < \Delta < r + r'$.

For later convenience we wish to express g_ℓ in terms of the retarded-time variables

$$u := ct - r, \qquad u' := ct' - r'. \tag{2.5.9}$$

We have $\Delta = u - u' + r - r'$, and for $r > r'$ the condition $\Delta > |r - r'|$ translates to $u - u' > 0$, while for $r < r'$ it translates to $u - u' > 2(r' - r)$. In both cases the condition $\Delta < r + r'$ translates to $u - u' < 2r'$. Finally, rewriting ξ in terms of $u - u'$ reveals that the reduced Green's function of Eq. (2.5.7) can be expressed as

$$g_\ell(u, r; u', r') = \frac{2\pi H}{rr'} P_\ell(\xi), \tag{2.5.10}$$

where

$$H := \begin{cases} \Theta\bigl[u - u'\bigr]\Theta\bigl[2r' - (u - u')\bigr] & r > r' \\ \Theta\bigl[u - u' - 2(r' - r)\bigr]\Theta\bigl[2r' - (u - u')\bigr] & r < r' \end{cases} \tag{2.5.11}$$

and

$$\xi = 1 - \frac{r - r'}{rr'}(u - u') - \frac{1}{2rr'}(u - u')^2. \tag{2.5.12}$$

The simplicity of the reduced Green's function comes as a great help in the evaluation of wave-zone integrals.

2.5.2 Wave-zone field point

To begin, we evaluate

$$\psi_{\mathscr{W}}(x) = \int_{\mathscr{W}} G(x, x') \mu(x')\, d^4x'$$

when x is situated in the wave zone, that is, when $r = |\boldsymbol{x}| > \mathcal{R}$. We do this for the specific source term displayed in Eq. (2.5.2), using the representation of the retarded Green's function given in Eq. (2.5.3).

The first step is to change the variables of integration from (ct', x', y', z') to (u', r', θ', ϕ'), using $u' = ct' - r'$ and the usual relation between Cartesian and spherical coordinates. The new volume element is $d^4x' = r'^2 du' dr' d\Omega'$, where $d\Omega' = \sin\theta'\, d\theta' d\phi'$. After inserting this, together with Eqs. (2.5.2) and (2.5.3), inside the integral, we obtain

$$\begin{aligned}\psi_{\mathscr{W}} &= \frac{1}{4\pi}\sum_{\ell'}\int du' dr' \frac{f(u')}{r'^{(n-2)}} g_{\ell'}(u, r; u', r') \\ &\quad \times \sum_{m'=-\ell'}^{\ell'} Y_{\ell' m'}(\theta, \phi) \int \bar{Y}_{\ell' m'}(\theta', \phi') \Omega'^{\langle L\rangle}\, d\Omega'.\end{aligned}$$

The angular integration is carried out with the help of Eq. (1.8.16), and we arrive at

$$\psi_{\mathscr{W}} = \frac{\Omega^{\langle L\rangle}}{4\pi}\int du' dr' \frac{f(u')}{r'^{(n-2)}} g_\ell(u, r; u', r').$$

To evaluate this we partition the spatial domain of integration into the two intervals $\mathcal{R} < r' < r$ and $r < r' < \infty$. (Because x is chosen to be within the wave zone, it is automatic that $r > \mathcal{R}$.) We next refer to Eq. (2.5.11) and use the step functions to define the temporal domain of integration. After also involving Eq. (2.5.10) and changing the integration variable from u' to $s := \frac{1}{2}(u - u')$, we obtain

$$\psi_{\mathscr{W}} = \frac{\Omega^{\langle L\rangle}}{r}\biggl\{\int_{\mathcal{R}}^{r} dr' \int_0^{r'} ds \frac{f(u-2s)}{r'^{(n-1)}} P_\ell(\xi) + \int_r^\infty dr' \int_{r'-r}^{r'} ds \frac{f(u-2s)}{r'^{(n-1)}} P_\ell(\xi)\biggr\},$$

where ξ is now given by $\xi = 1 - 2(r'-r)s/(rr') - 2s^2/(rr')$.

We can make additional progress if we change the order of integration and rewrite the preceding expression as

$$\begin{aligned}\psi_{\mathscr{W}} &= \frac{\Omega^{\langle L\rangle}}{r}\biggl\{\int_0^{\mathcal{R}} ds f(u-2s)\int_{\mathcal{R}}^{r} dr' \frac{P_\ell(\xi)}{r'^{(n-1)}} + \int_{\mathcal{R}}^{r} ds f(u-2s)\int_s^r dr' \frac{P_\ell(\xi)}{r'^{(n-1)}} \\ &\quad + \int_0^r ds f(u-2s)\int_r^{r+s} dr' \frac{P_\ell(\xi)}{r'^{(n-1)}} + \int_r^\infty ds f(u-2s)\int_s^{r+s} dr' \frac{P_\ell(\xi)}{r'^{(n-1)}}\biggr\}.\end{aligned}$$

The integrals over dr' can now be evaluated. Let

$$G(k) := \int^k dr' \frac{P_\ell(\xi)}{r'^{(n-1)}}$$

be a function of the parameter k, in addition to the dependence on r and s contained in ξ. In terms of this function we have

$$\begin{aligned}\psi_{\mathscr{W}} &= \frac{\Omega^{\langle L\rangle}}{r}\biggl\{\int_0^{\mathcal{R}} ds f(u-2s)\bigl[G(r) - G(\mathcal{R})\bigr] \\ &\quad + \int_{\mathcal{R}}^{r} ds f(u-2s)\bigl[G(r) - G(s)\bigr] \\ &\quad + \int_0^{r} ds f(u-2s)\bigl[G(r+s) - G(r)\bigr] \\ &\quad + \int_r^{\infty} ds f(u-2s)\bigl[G(r+s) - G(s)\bigr]\biggr\},\end{aligned}$$

which can be rewritten as

$$\psi_{\mathscr{W}} = \frac{\Omega^{\langle L\rangle}}{r}\biggl\{-\int_0^{\mathcal{R}} ds f(u-2s) G(\mathcal{R})$$

$$- \int_{\mathcal{R}}^{\infty} ds f(u-2s)G(s)$$
$$+ \int_0^{\infty} ds f(u-2s)G(r+s)\biggr\},$$

or as

$$\begin{aligned}\psi_{\mathscr{W}} &= \frac{\Omega^{\langle L\rangle}}{r}\biggl\{\int_0^{\mathcal{R}} ds f(u-2s)\bigl[G(r+s)-G(\mathcal{R})\bigr] \\ &\quad + \int_{\mathcal{R}}^{\infty} ds f(u-2s)\bigl[G(r+s)-G(s)\bigr]\biggr\}.\end{aligned}$$

To put this in its final form we introduce the functions

$$A(s,r) := \int_{\mathcal{R}}^{r+s} \frac{P_\ell(\xi)}{p^{n-1}}\, dp \tag{2.5.13}$$

and

$$B(s,r) := \int_{s}^{r+s} \frac{P_\ell(\xi)}{p^{n-1}}\, dp, \tag{2.5.14}$$

in which p stands for r', s for $\frac{1}{2}(u-u')$, and

$$\xi = \frac{r+2s}{r} - \frac{2s(r+s)}{rp}. \tag{2.5.15}$$

We next observe that $A = G(r+s) - G(\mathcal{R})$ and $B = G(r+s) - G(s)$, and we write our previous expression as

$$\psi_{\mathscr{W}}(u,r,\theta,\phi) = \frac{\Omega^{\langle L\rangle}}{r}\biggl\{\int_0^{\mathcal{R}} ds f(u-2s)A(s,r) + \int_{\mathcal{R}}^{\infty} ds f(u-2s)B(s,r)\biggr\}. \tag{2.5.16}$$

This is a concrete expression for the $\psi_{\mathscr{W}}$ of Eq. (2.5.1), corresponding to a source function of the form displayed in Eq. (2.5.2), when the field point x is in the wave zone. Here, $f(u')$ describes the temporal behaviour of the source function, which decays spatially as r'^{-n}; the angular dependence is given by $\Omega'^{\langle L\rangle}$, an angular STF tensor of degree ℓ.

2.5.3 Near-zone field point

We next evaluate

$$\psi_{\mathscr{W}}(x) = \int_{\mathscr{W}} G(x,x')\mu(x')\, d^4x'$$

when x is situated in the near zone, that is, when $r = |\boldsymbol{x}| < \mathcal{R}$.

We return to

$$\psi_{\mathscr{W}} = \frac{\Omega^{\langle L\rangle}}{4\pi}\int du' dr' \frac{f(u')}{r'^{(n-2)}} g_\ell(u,r;u',r')$$

and notice that here, r is always smaller than r', so that there is no need to partition the spatial domain of integration. The step functions of Eq. (2.5.11) define the temporal domain of integration, and after substituting Eq. (2.5.10) and changing the integration variable from u' to $s := \frac{1}{2}(u-u')$, we obtain

$$\psi_{\mathscr{W}} = \frac{\Omega^{\langle L\rangle}}{r}\int_{\mathcal{R}}^{\infty} dr' \int_{r'-r}^{r'} ds \frac{f(u-2s)}{r'^{(n-1)}}\, P_\ell(\xi),$$

where ξ is still given by $\xi = 1 - 2(r' - r)s/(rr') - 2s^2/(rr')$. This becomes

$$\begin{aligned}\psi_{\mathscr{W}} &= \frac{\Omega^{\langle L\rangle}}{r}\biggl\{\int_{\mathcal{R}-r}^{\mathcal{R}} ds f(u-2s)\int_{\mathcal{R}}^{s+r} dr'\,\frac{P_\ell(\xi)}{r'^{(n-1)}}\\ &\quad + \int_{\mathcal{R}}^{\infty} ds f(u-2s)\int_{s}^{s+r} dr'\,\frac{P_\ell(\xi)}{r'^{(n-1)}}\biggr\}\end{aligned}$$

after changing the order of integration. Proceeding through the same steps as before, we finally obtain

$$\psi_{\mathscr{W}}(u,r,\theta,\phi) = \frac{\Omega^{\langle L\rangle}}{r}\biggl\{\int_{\mathcal{R}-r}^{\mathcal{R}} ds f(u-2s)A(s,r) + \int_{\mathcal{R}}^{\infty} ds f(u-2s)B(s,r)\biggr\}, \quad (2.5.17)$$

where the functions $A(s,r)$ and $B(s,r)$ are defined by Eqs. (2.5.13) and (2.5.14), respectively. This is a concrete expression for the $\psi_{\mathscr{W}}$ of Eq. (2.5.1), corresponding to a source function of the form displayed in Eq. (2.5.2), when the field point x is in the near zone. Here, $f(u')$ describes the temporal behaviour of the source function, which decays spatially as r'^{-n}; the angular dependence is given by $\Omega'^{\langle L\rangle}$, an angular STF tensor of degree ℓ.

2.5.4 Estimates

It is possible to give crude estimates to the integrals appearing on the right-hand side of Eqs. (2.5.16) and (2.5.17).

Suppose first that we wish to evaluate Eq. (2.5.16) in the far-away wave zone, and keep only its dominant, r^{-1} part. Taking $P_\ell(\xi)$ to be of order unity, we approximate the functions defined by Eqs. (2.5.13) and (2.5.14) as

$$A \sim \int_{\mathcal{R}}^{\infty} \frac{dp}{p^{n-1}} \sim \frac{1}{\mathcal{R}^{n-2}}$$

and

$$B \sim \int_{s}^{\infty} \frac{dp}{p^{n-1}} \sim \frac{1}{s^{n-2}};$$

we ignore all numerical factors and exclude the special case $n = 2$. Inserting A into the first integral of Eq. (2.5.16) yields

$$\frac{1}{\mathcal{R}^{n-2}}\int_0^{\mathcal{R}} f(u-2s)\,ds.$$

Taking $\mathcal{R}$ to be small, we Taylor-expand $f(u - 2s)$ about $s = 0$ and integrate term-by-term. A typical term in the expansion is

$$\frac{1}{\mathcal{R}^{n-2}} f^{(q)}(u)\mathcal{R}^{q+1},$$

where the superscript (q) indicates the number of derivatives with respect to u. As was motivated at the end of Sec. 2.3, we are interested in the $\mathcal{R}$-independent part of $\psi_{\mathscr{W}}$. In order to extract this from your previous expansion, we retain the term $q = n - 3$ and discard all others. An estimate for the first integral is therefore $f^{(n-3)}(u)$. We next substitute B into the second integral of Eq. (2.5.16) and obtain

$$\int_{\mathcal{R}}^{\infty} f(u-2s)\frac{ds}{s^{n-2}}.$$

Assuming that f and all its derivatives vanish in the infinite past, repeated integration by parts returns an expression of the form

$$\frac{f(u-2\mathcal{R})}{\mathcal{R}^{n-3}} + \frac{f^{(1)}(u-2\mathcal{R})}{\mathcal{R}^{n-4}} + \frac{f^{(2)}(u-2\mathcal{R})}{\mathcal{R}^{n-5}} + \cdots.$$

The $\mathcal{R}$-independent part of this is easily seen to be of the form $f^{(n-3)}(u)$, as we had for the first integral. We conclude that a crude estimate for Eq. (2.5.16) is

$$\psi_{\mathscr{W}} \sim \frac{\Omega^{\langle L\rangle}}{r} f^{(n-3)}(u) \qquad \text{(far-away wave zone).} \tag{2.5.18}$$

This estimate ignores numerical factors, $\mathcal{R}$-dependent terms, and terms that decay faster than r^{-1}.

Suppose next that we wish to evaluate Eq. (2.5.17) deep within the near zone, for $r \ll \mathcal{R}$. Here the first integral of Eq. (2.5.17) is approximated as

$$\int_{\mathcal{R}-r}^{\mathcal{R}} ds f(u-2s)A(s,r) \sim r f(u-2\mathcal{R})A(\mathcal{R},r),$$

with

$$A(\mathcal{R},r) \sim \int_{\mathcal{R}}^{r+\mathcal{R}} \frac{dp}{p^{n-1}} \sim \frac{r}{\mathcal{R}^{n-1}}.$$

This produces the estimate

$$\frac{r^2}{\mathcal{R}^{n-1}} f(u-2\mathcal{R})$$

for the first integral, and the $\mathcal{R}$-independent part of this is $r^2 f^{(n-1)}(u)$. The second integral of Eq. (2.5.17) involves the domain of integration $\mathcal{R} < s < \infty$. Because s is large compared with r, we have the estimate

$$B \sim \int_{s}^{r+s} \frac{dp}{p^{n-1}} \sim \frac{r}{s^{n-1}}.$$

Inserting this inside the integral gives

$$r \int_{\mathcal{R}}^{\infty} f(u-2s) \frac{ds}{s^{n-1}},$$

and repeated integration by parts returns an expression of the form

$$\frac{r f(u-2\mathcal{R})}{\mathcal{R}^{n-2}} + \frac{r f^{(1)}(u-2\mathcal{R})}{\mathcal{R}^{n-3}} + \frac{r f^{(2)}(u-2\mathcal{R})}{\mathcal{R}^{n-4}} + \cdots.$$

The $\mathcal{R}$-independent part of this is easily seen to be $r f^{(n-2)}(u)$. Collecting results, we conclude that a crude estimate for Eq. (2.5.17) is

$$\psi_{\mathscr{W}} \sim \Omega^{\langle L\rangle} \Big\{ f^{(n-2)}(u) + r f^{(n-1)}(u) \Big\} \qquad \text{(near zone).} \tag{2.5.19}$$

This estimate ignores numerical factors and all $\mathcal{R}$-dependent terms.

Chapter 3
First post-Minkowskian approximation

Our strategy to integrate the Einstein field equations, in their Landau-Lifshitz formulation introduced in Chapter 1, relies on a post-Minkowskian expansion of the gravitational potentials in powers of G. This strategy leads to an iterative approach to the solution, and each iteration of the field equations increases the order of accuracy by one power of G. In this chapter we carry out the first iteration of this program, and construct the *first post-Minkowskian approximation* to the gravitational potentials. Because the wave equations are linear at this level of approximation, this is the domain of the *linearized theory*, that is, general relativity linearized about Minkowski spacetime. We begin in Sec. 3.1 with a statement of the field equations, for which we can write down immediate integral solutions. In Sec. 3.2 we evaluate the integrals when the field point lies within the near zone, and in Sec. 3.3 we construct expressions that are valid in the wave zone. In each case the gravitational potentials are presented in the form of a post-Newtonian expansion in powers of c^{-2}.

3.1 Field equations

In the first post-Minkowskian approximation, Eq. (1.6.1) reduces to the linear wave equation

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4} T^{\alpha\beta}, \tag{3.1.1}$$

in which the energy-momentum tensor is a functional of $\eta_{\alpha\beta}$, the metric of Minkowski spacetime. The coordinate system is $x^\alpha = (ct, x^a)$, and $\Box := \eta^{\alpha\beta}\partial_{\alpha\beta}$ is the wave operator of Minkowski spacetime. The potentials must also satisfy the harmonic gauge conditions

$$\partial_\beta h^{\alpha\beta} = 0, \tag{3.1.2}$$

which are enforced automatically when the energy-momentum tensor satisfies the conservation identities

$$\partial_\beta T^{\alpha\beta} = 0; \tag{3.1.3}$$

this statement was established at the end of Sec. 1.3. At this order of approximation, the metric and other related quantities are given by

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2} h \eta_{\alpha\beta} + O(G^2), \tag{3.1.4}$$

$$
\begin{aligned}
g^{\alpha\beta} &= \eta^{\alpha\beta} - h^{\alpha\beta} + \frac{1}{2} h \eta^{\alpha\beta} + O(G^2), && (3.1.5)\\
(-g) &= 1 - h + O(G^2), && (3.1.6)\\
\sqrt{-g} &= 1 - \frac{1}{2} h + O(G^2). && (3.1.7)
\end{aligned}
$$

These expressions are a special case of Eqs. (1.6.3)–(1.6.6). Notice that $h_{\alpha\beta}$ is the "trace-reversed" metric perturbation. It is understood that here, indices on $h^{\alpha\beta}$ are lowered with the Minkowski metric. Thus, $h_{\alpha\beta} := \eta_{\alpha\mu}\eta_{\beta\nu}h^{\mu\nu}$ and $h := \eta_{\mu\nu}h^{\mu\nu}$.

For our purposes in this chapter, it will not be necessary to specify the nature of the matter distribution. We shall leave $T^{\alpha\beta}$ arbitrary, except for the conditions of Eq. (3.1.3). We shall assume, however, that the matter distribution is bounded, and that it has enough internal dynamics to produce an interesting gravitational field. (By interesting we mean, for example, a nonspherical and time-dependent gravitational field.) This dynamics must have a nongravitational origin, because the conservation identities of Eq. (3.1.3) forbid the existence of significant gravitational interactions within the matter distribution.

We introduce the field variables Φ, A^a, and B^{ab}, which are related to the gravitational potentials by

$$
h^{00} := \frac{4}{c^2}\Phi, \qquad h^{0a} := \frac{4}{c^3}A^a, \qquad h^{ab} := \frac{4}{c^4}B^{ab}. \tag{3.1.8}
$$

We also introduce the matter variables ρ, j^a, and T^{ab}, which are related to the energy-momentum tensor by

$$
T^{00} := c^2\rho, \qquad T^{0a} := cj^a, \qquad T^{ab} = T^{ab}. \tag{3.1.9}
$$

The quantity ρ has the dimension of a mass density, j^a has the dimension of (mass density)×(velocity), and T^{ab} has the dimension of (mass density)×(velocity)^2.

The field equations are

$$
\begin{aligned}
\Box\Phi &= -4\pi G\rho, && (3.1.10)\\
\Box A^a &= -4\pi G j^a, && (3.1.11)\\
\Box B^{ab} &= -4\pi G T^{ab}, && (3.1.12)
\end{aligned}
$$

and the gauge conditions are

$$
\partial_t\Phi + \partial_a A^a = 0, \qquad \partial_t A^a + \partial_b B^{ab} = 0. \tag{3.1.13}
$$

The conservation identities are

$$
\partial_t\rho + \partial_a j^a = 0, \qquad \partial_t j^a + \partial_b T^{ab} = 0. \tag{3.1.14}
$$

The retarded solutions to the wave equations are

$$
\begin{aligned}
\Phi(ct, \boldsymbol{x}) &= G\int \frac{\rho(ct - |\boldsymbol{x}-\boldsymbol{x'}|, \boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x', && (3.1.15)\\
A^a(ct, \boldsymbol{x}) &= G\int \frac{j^a(ct - |\boldsymbol{x}-\boldsymbol{x'}|, \boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x', && (3.1.16)\\
B^{ab}(ct, \boldsymbol{x}) &= G\int \frac{T^{ab}(ct - |\boldsymbol{x}-\boldsymbol{x'}|, \boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x'. && (3.1.17)
\end{aligned}
$$

We assume that the matter distribution is contained within the near zone (refer back to Secs. 2.2 and 2.3), so that $h^{\alpha\beta}_{\mathscr{W}} = 0$ and $h^{\alpha\beta} = h^{\alpha\beta}_{\mathscr{N}}$.

3.2 Near-zone expressions

To express the potentials in the near zone we rely on the method described in Sec. 2.4.2, which consists of treating $|\boldsymbol{x}-\boldsymbol{x}'|$ as a small quantity and Taylor-expanding the time dependence of each source term. For example, we write

$$\rho(ct-|\boldsymbol{x}-\boldsymbol{x}'|)=\rho-\frac{1}{c}\frac{\partial\rho}{\partial t}|\boldsymbol{x}-\boldsymbol{x}'|+\frac{1}{2c^2}\frac{\partial^2\rho}{\partial t^2}|\boldsymbol{x}-\boldsymbol{x}'|^2+\cdots$$

and insert this into Eq. (3.1.15). We obtain

$$\begin{aligned}\Phi &= G\int\frac{\rho(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x'-\frac{G}{c}\frac{\partial}{\partial t}\int\rho(t,\boldsymbol{x}')\,d^3x'\\ &\quad+\frac{G}{2c^2}\frac{\partial^2}{\partial t^2}\int\rho(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\,d^3x'+\cdots.\end{aligned}$$

The second term vanishes by virtue of the conservation identities:

$$\int\partial_t\rho\,d^3x=-\int\partial_a j^a\,d^3x=-\oint j^a dS_a=0,$$

because there is no flux of matter across the surface bounding the matter distribution. Our final expression for the scalar potential is

$$\Phi=U+\frac{1}{2c^2}\frac{\partial^2X}{\partial t^2}+O(c^{-3}),\tag{3.2.1}$$

where

$$U(t,\boldsymbol{x}):=G\int\frac{\rho(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x'\tag{3.2.2}$$

is the *Newtonian potential* associated with the mass density ρ, while

$$X(t,\boldsymbol{x}):=G\int\rho(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\,d^3x'\tag{3.2.3}$$

is known as the *superpotential*. It is easy to verify that these satisfy the Poisson equations

$$\nabla^2U=-4\pi G\rho,\qquad\nabla^2X=2U.\tag{3.2.4}$$

It is evident that the Newtonian potential ignores all retardation effects within the near-zone, and that these are contained in the superpotential as well as the higher-order corrections discarded in Eq. (3.2.1).

Similar considerations reveal that the vector potential is given by

$$A^a=U^a+O(c^{-2}),\tag{3.2.5}$$

where

$$U^a(t,\boldsymbol{x}):=G\int\frac{j^a(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x'\tag{3.2.6}$$

is another instanteneous potential that satisfies

$$\nabla^2U^a=-4\pi Gj^a.\tag{3.2.7}$$

Notice that by virtue of the conservation identity $\partial_t j^a+\partial_b T^{ab}=0$, a term of order c^{-1} that should be present in Eq. (3.2.5) actually vanishes. The discarded term of order c^{-2} would involve $\partial_t^2 j^a$ and would partially incorporate the retardation effects that do not appear in U^a.

Finally, the near-zone expression for the tensor potential is

$$B^{ab} = P^{ab} + O(c^{-1}), \tag{3.2.8}$$

where

$$P^{ab}(t, \boldsymbol{x}) := G \int \frac{T^{ab}(t, \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|}\, d^3x' \tag{3.2.9}$$

is an instanteneous potential that satisfies

$$\nabla^2 P^{ab} = -4\pi G T^{ab}. \tag{3.2.10}$$

Here it should be noticed that the discarded term of order c^{-1} does not vanish — we have run out of conservation identities.

Substituting Eqs. (3.2.1), (3.2.5), and (3.2.8) into Eqs. (3.1.8) gives

$$\begin{aligned} h^{00} &= \frac{4}{c^2}U + \frac{2}{c^4}\frac{\partial^2 X}{\partial t^2} + O(c^{-5}), && (3.2.11)\\ h^{0a} &= \frac{4}{c^3}U^a + O(c^{-5}), && (3.2.12)\\ h^{ab} &= \frac{4}{c^4}P^{ab} + O(c^{-5}). && (3.2.13) \end{aligned}$$

These expansions in powers of c^{-1} are known as *post-Newtonian expansions*. The leading term in h^{00}, of order c^{-2} and involving U, is said to be of Newtonian order, or 0PN order. The second term in h^{00}, of order c^{-4} and involving X, is said to be of first post-Newtonian order, or 1PN order. The leading term in h^{0a}, of order c^{-3} and involving U^a, is said to be of one-half post-Newtonian order, or $\frac{1}{2}$PN order. And finally, the leading term in h^{ab}, of order c^{-4} and involving P^{ab}, is also of 1PN order. The counting of post-Newtonian order depends on the power of c^{-1}; to an additional power of c^{-1} one assigns a *half* PN order, and to an additional power of c^{-2} one assignes a full post-Newtonian order.

3.3 Wave-zone expressions

3.3.1 Multipole moments and identities

To express the potentials in the wave zone we rely on the method described in Sec. 2.4.1, which led to the multipole expansion displayed in Eq. (2.4.3). Our expressions will involve the *mass multipole moments*

$$\begin{aligned} I(u) &:= \int \rho(u, \boldsymbol{x'})\, d^3x', && (3.3.1)\\ I^a(u) &:= \int \rho(u, \boldsymbol{x'}) x'^a\, d^3x', && (3.3.2)\\ I^{ab}(u) &:= \int \rho(u, \boldsymbol{x'}) x'^a x'^b\, d^3x', && (3.3.3)\\ I^{abc}(u) &:= \int \rho(u, \boldsymbol{x'}) x'^a x'^b x'^c\, d^3x', && (3.3.4) \end{aligned}$$

and so on; these tensors are all completely symmetric. Recall that

$$u := ct - r = c(t - r/c) =: c\tau \tag{3.3.5}$$

is a retarded-time variable, and notice that I is in fact the total mass associated with the mass density ρ. Our expressions will involve also the *current multipole*

moments

$$\begin{aligned} J^{ab}(u) &:= \int \bigl[j^a(u,\boldsymbol{x'})x'^b - j^b(u,\boldsymbol{x'})x'^a \bigr]\, d^3x', && (3.3.6)\\ J^{abc}(u) &:= \int \bigl[j^a(u,\boldsymbol{x'})x'^b - j^b(u,\boldsymbol{x'})x'^a \bigr] x'^c\, d^3x', && (3.3.7) \end{aligned}$$

and so on; these tensors are antisymmetric in the first pair of indices. Notice that J^{ab} is in fact the total angular-momentum tensor associated with the current density j^a; this is related to the angular-momentum vector J^a by $J^a := \frac{1}{2}\varepsilon^a{}_{bc}J^{bc}$, where ε_{abc} is the permutation symbol.

We will use the conservation identities involving ρ, j^a, and T^{ab} to derive the important consequences

$$\begin{aligned} I &= \text{constant}, && (3.3.8)\\ I^a &= 0 \qquad \text{(in centre-of-mass frame)}, && (3.3.9)\\ J^{ab} &= \text{constant}, && (3.3.10)\\ \int j^a\, d^3x' &= \dot{I}^a = 0, && (3.3.11)\\ \int j^a x'^b\, d^3x' &= \frac{1}{2}\dot{I}^{ab} + \frac{1}{2}J^{ab}, && (3.3.12)\\ \int j^a x'^b x'^c\, d^3x' &= \frac{1}{3}\dot{I}^{abc} + \frac{1}{3}\bigl(J^{abc} + J^{acb}\bigr), && (3.3.13)\\ \int T^{ab}\, d^3x' &= \frac{1}{2}\ddot{I}^{ab}, && (3.3.14)\\ \int T^{ab}x'^c\, d^3x' &= \frac{1}{6}\ddot{I}^{abc} + \frac{1}{3}\bigl(\dot{J}^{acb} + \dot{J}^{bca}\bigr), && (3.3.15) \end{aligned}$$

in which an overdot indicates differentiation with respect to τ: $\dot{I}^Q := dI^Q/d\tau = c\,dI^Q/du$.

We begin by differentiating Eq. (3.3.1) with respect to u. We obtain

$$\frac{dI}{du} = \int \partial_u \rho\, d^3x = -c^{-1}\int \partial_a j^a\, d^3x = -c^{-1}\oint j^a\, dS_a = 0,$$

because (as was observed before) there is no flux of matter across the surface bounding the matter distribution. Because $dI/du = 0$, $I(u)$ must be a constant, and we conclude that the total mass I is conserved, as was stated in Eq. (3.3.8). Notice that to simplify the notation we have dropped the primes on the integration variables; we shall continue to do so in the remainder of this subsection.

We proceed by differentiating Eq. (3.3.2) with respect to u, which gives

$$\frac{dI^a}{du} = \int (\partial_u\rho)x^a\, d^3x = -c^{-1}\int \bigl(\partial_b j^b\bigr)x^a\, d^3x.$$

We rewrite the integrand as

$$\bigl(\partial_b j^b\bigr)x^a = \partial_b\bigl(j^b x^a\bigr) - j^a,$$

which is a special case of Eq. (1.4.2) expressed in a different notation. The divergence produces a surface integral that vanishes (because, as usual, $j^b dS_b$ vanishes on the surface bounding the matter distribution), and we obtain

$$\frac{dI^a}{du} = c^{-1}\int j^a\, d^3x,$$

which is just Eq. (3.3.11). Differentiating once more, we get

$$\frac{d^2 I^a}{du^2} = c^{-1} \int \partial_u j^a \, d^3x = -c^{-2} \int \partial_b T^{ab} \, d^3x = -c^{-2} \oint T^{ab} dS_b = 0,$$

because there can be no normal stress on the surface bounding the matter distribution. We conclude that $d^2 I^a/d\tau^2 = 0$, so that the mass dipole moment $I^a(\tau)$ must be at all times of the form $I^a(0) + \dot{I}^a(0)\tau$. We may choose the initial conditions $I^a(0) = 0 = \dot{I}^a(0)$ and set $I^a(\tau) \equiv 0$. This defines the system's centre-of-mass frame, and this is the statement of Eq. (3.3.9).

To establish Eq. (3.3.12) we express $j^a x^b$ in terms of its symmetric and anti-symmetric parts, and we integrate over d^3x. Taking Eq. (3.3.6) into account, we obtain

$$\int j^a x^b \, d^3x = \frac{1}{2} \int \bigl(j^a x^b + j^b x^a\bigr) \, d^3x + \frac{1}{2} J^{ab}.$$

Going back to Eq. (3.3.3), we have

$$\frac{dI^{ab}}{du} = \int (\partial_u \rho) x^a x^b \, d^3x = -c^{-1} \int \bigl(\partial_c j^c\bigr) \, x^a x^b \, d^3x.$$

The integrand can be expressed as

$$\bigl(\partial_c j^c\bigr) x^a x^b = \partial_c\bigl(j^c x^a x^b\bigr) - j^a x^b - j^b x^a,$$

and integration produces

$$\frac{dI^{ab}}{du} = c^{-1} \int \bigl(j^a x^b + j^b x^a\bigr) \, d^3x$$

because, as usual, there is no contribution from the surface integral. Collecting results, we have

$$\int j^a x^b \, d^3x = \frac{c}{2} \frac{dI^{ab}}{du} + \frac{1}{2} J^{ab},$$

and this is just Eq. (3.3.12). The derivation of Eq. (3.3.13) follows a very similar path, and we shall not go through the details here.

To establish Eq. (3.3.14) we invoke the identity

$$T^{ab} = \frac{c^2}{2} \partial_{uu}\bigl(\rho x^a x^b\bigr) + \frac{1}{2} \partial_c\bigl(T^{ac} x^b + T^{bc} x^a - \partial_d T^{cd} x^a x^b\bigr),$$

which is a special case of Eq. (1.4.3). After integrating over d^3x and discarding the surface integral, we obtain Eq. (3.3.14). For Eq. (3.3.15) our starting point is

$$T^{ab} x^c = \frac{c}{2} \partial_u\bigl(j^a x^b x^c + j^b x^a x^c - j^c x^a x^b\bigr) + \frac{1}{2} \partial_d\bigl(T^{ad} x^b x^c + T^{bd} x^a x^c - T^{cd} x^a x^b\bigr),$$

a special case of Eq. (1.4.4). After integration and involvement of Eq. (3.3.13), we obtain

$$\begin{aligned} \int T^{ab} x^c \, d^3x \quad = \quad & \frac{c^2}{6} \frac{d^2}{du^2} I^{abc} + \frac{c}{6} \frac{d}{du}\bigl(J^{abc} + J^{acb}\bigr) + \frac{c^2}{6} \frac{d^2}{du^2} I^{bac} \\ & + \frac{c}{6} \frac{d}{du}\bigl(J^{bac} + J^{bca}\bigr) - \frac{c^2}{6} \frac{d^2}{du^2} I^{cab} - \frac{c}{6} \frac{d}{du}\bigl(J^{cab} + J^{cba}\bigr). \end{aligned}$$

Noting the complete symmetry of I^{abc} and the antisymmetry of J^{abc} in the first pair of indices, this is just Eq. (3.3.15).

3.3.2 Wave-zone potentials

According to Eq. (2.4.3), the scalar potential is given by

$$\begin{aligned}\Phi &= G\biggl[\frac{1}{r}\int \rho\, d^3x' - \partial_a\biggl(\frac{1}{r}\int \rho x'^a\, d^3x'\biggr) + \frac{1}{2}\partial_{ab}\biggl(\frac{1}{r}\int \rho x'^a x'^b\, d^3x'\biggr) \\ &\quad - \frac{1}{6}\partial_{abc}\biggl(\frac{1}{r}\int \rho x'^a x'^b x'^c\, d^3x'\biggr) + \cdots\biggr]\end{aligned}$$

in the wave zone. With the definitions of Eqs. (3.3.1)–(3.3.4), this is

$$\Phi = G\biggl[\frac{I}{r} - \partial_a\biggl(\frac{I^a}{r}\biggr) + \frac{1}{2}\partial_{ab}\biggl(\frac{I^{ab}}{r}\biggr) - \frac{1}{6}\partial_{abc}\biggl(\frac{I^{abc}}{r}\biggr) + \cdots\biggr].$$

Taking into account Eq. (3.3.9) gives us the final expression

$$\Phi = G\biggl[\frac{I}{r} + \frac{1}{2}\partial_{ab}\biggl(\frac{I^{ab}}{r}\biggr) - \frac{1}{6}\partial_{abc}\biggl(\frac{I^{abc}}{r}\biggr) + \cdots\biggr], \qquad (3.3.16)$$

in which each multipole moment I^Q, with the exception of the constant monopole moment I, is a function of the retarded-time variable $\tau = t - r/c$.

For the vector potential we have

$$A^a = G\biggl[\frac{1}{r}\int j^a\, d^3x' - \partial_b\biggl(\frac{1}{r}\int j^a x'^b\, d^3x'\biggr) + \frac{1}{2}\partial_{bc}\biggl(\frac{1}{r}\int j^a x'^b x'^c\, d^3x'\biggr) + \cdots\biggr].$$

Taking into account Eqs. (3.3.11)–(3.3.13), this is

$$A^a = G\biggl[-\frac{1}{2}\partial_b\biggl(\frac{\dot{I}^{ab} + J^{ab}}{r}\biggr) + \frac{1}{6}\partial_{bc}\biggl(\frac{\dot{I}^{abc} + J^{abc} + J^{acb}}{r}\biggr) + \cdots\biggr]. \qquad (3.3.17)$$

We recall that J^{ab} is the constant angular-momentum tensor; all other multipole moments depend on τ. Because J^{ab} is constant, we have that $\partial_b(J^{ab}r^{-1}) = J^{ab}\partial_b r^{-1} = -r^{-2}J^{ab}\Omega_b$, where $\Omega^a = x^a/r$.

Finally, the wave-zone tensor potential is

$$B^{ab} = G\biggl[\frac{1}{r}\int T^{ab}\, d^3x' - \partial_c\biggl(\frac{1}{r}\int T^{ab}x'^c\, d^3x'\biggr) + \cdots\biggr],$$

which becomes

$$B^{ab} = G\biggl[\frac{\ddot{I}^{ab}}{2r} - \frac{1}{6}\partial_c\biggl(\frac{\ddot{I}^{abc} + 2\dot{J}^{acb} + 2\dot{J}^{bca}}{r}\biggr) + \cdots\biggr] \qquad (3.3.18)$$

after taking into account the identities displayed in Eqs. (3.3.14) and (3.3.15).

Substituting Eqs. (3.3.16), (3.3.17), and (3.3.18) into Eqs. (3.1.8) gives

$$h^{00} = \frac{4G}{c^2}\biggl[\frac{I}{r} + \frac{1}{2}\partial_{ab}\biggl(\frac{I^{ab}}{r}\biggr) - \frac{1}{6}\partial_{abc}\biggl(\frac{I^{abc}}{r}\biggr) + \cdots\biggr], \qquad (3.3.19)$$

$$\begin{aligned}h^{0a} &= \frac{4G}{c^3}\biggl[\frac{1}{2}J^{ab}\frac{\Omega_b}{r^2} - \frac{1}{2}\partial_b\biggl(\frac{\dot{I}^{ab}}{r}\biggr) \\ &\quad + \frac{1}{6}\partial_{bc}\biggl(\frac{\dot{I}^{abc} + J^{abc} + J^{acb}}{r}\biggr) + \cdots\biggr],\end{aligned} \qquad (3.3.20)$$

$$h^{ab} = \frac{4G}{c^4}\biggl[\frac{1}{2}\frac{\ddot{I}^{ab}}{r} - \frac{1}{6}\partial_c\biggl(\frac{\ddot{I}^{abc} + 2\dot{J}^{acb} + 2\dot{J}^{bca}}{r}\biggr) + \cdots\biggr]. \qquad (3.3.21)$$

These are the gravitational potentials in the wave zone, expressed as multipole expansions involving the mass multipole moments I^Q and the current multipole moments J^Q. The dependence on x^a of each quantity within round brackets is contained in the factor r^{-1}, and also in the dependence of each moment on $\tau = t - r/c$.

3.3.3 Post-Newtonian counting

Counting post-Newtonian orders is more subtle in the wave zone than in the near zone. Recalling our discussion in Sec. 2.2, let us introduce the scaling quantities

$$m_c := \text{characteristic mass scale of the source,} \tag{3.3.22}$$

$$r_c := \text{characteristic length scale of the source,} \tag{3.3.23}$$

$$t_c := \text{characteristic time scale of the source,} \tag{3.3.24}$$

$$v_c := \frac{r_c}{t_c} = \text{characteristic velocity within the source,} \tag{3.3.25}$$

$$\lambda_c := ct_c = \text{characteristic wavelength of the radition.} \tag{3.3.26}$$

The characteristic radius r_c is defined such that the matter variables vanish outside a sphere of radius r_c; the matter distribution is confined within this sphere. The characteristic time scale t_c, we recall, is the time required for noticeable changes to occur within the matter distribution. We assume that the characteristic velocity v_c is small compared with the speed of light:

$$v_c \ll c. \tag{3.3.27}$$

This, of course, is the standard *slow-motion approximation* of post-Newtonian theory. It follows from Eq. (3.3.27) that

$$r_c \ll \lambda_c; \tag{3.3.28}$$

in the slow-motion approximation, the matter distribution is always situated deep within the near zone.

Let us examine the various terms that make up $h^{\alpha\beta}$, and let us estimate their orders of magnitude in the wave zone, when $r > \lambda_c$. Based on these estimates, we shall assign a post-Newtonian order to each term.

We begin with the first term on the right of Eq. (3.3.19). This is evidently of order $Gm_c/(c^2 r)$, which is what was called *Newtonian order* at the end of Sec. 3.2. We therefore assign a 0PN order to this term.

We continue with the second term on the right of Eq. (3.3.19). Recalling that I^{ab} is a function of $\tau = t - r/c$, we find that after differentiation, the second term is of the schematic form

$$\frac{G}{c^2}\left(\frac{\ddot{I}^{ab}}{c^2 r} + \frac{\dot{I}^{ab}}{cr^2} + \frac{I^{ab}}{r^3}\right),$$

where we ignore the angular dependence and all numerical factors. Noting that I^{ab} is of order $m_c r_c^2$, this term is of order

$$\frac{G}{c^2}\left(\frac{m_c r_c^2}{c^2 t_c^2 r} + \frac{m_c r_c^2}{ct_c r^2} + \frac{m_c r_c^2}{r^3}\right) = \frac{Gm_c}{c^2 r}\frac{r_c^2}{c^2 t_c^2}\left(1 + \frac{ct_c}{r} + \frac{c^2 t_c^2}{r^2}\right).$$

In view of Eqs. (3.3.25) and (3.3.26), this is of order

$$\left(\frac{v_c}{c}\right)^2\left[1 + \frac{\lambda_c}{r} + \left(\frac{\lambda_c}{r}\right)^2\right]$$

relative to $Gm_c/(c^2 r)$. The term within square brackets is of order unity in the wave zone, and we conclude that the second term in Eq. (3.3.19) is smaller than the first term by a factor of order $(v_c/c)^2$. To this term we therefore assign a 1PN order.

Similar considerations reveal that the third term on the right of Eq. (3.3.19) is smaller than the first by a factor of order $(v_c/c)^3$. To this term we therefore assign a $\frac{3}{2}$PN order. The discarded terms in Eq. (3.3.19) are terms of 2PN order and higher.

We next move on to the first term on the right of Eq. (3.3.20). The angular-momentum tensor J^{ab} is of order $m_c v_c r_c$, and it follows that the first term is of order

$$\frac{G}{c^3}\frac{m_c v_c r_c}{r^2} = \frac{Gm_c}{c^2 r}\Big(\frac{v_c}{c}\Big)^2\frac{\lambda_c}{r}.$$

Relative to $Gm_c/(c^2 r)$ this is of order $(v_c/c)^2(\lambda_c/r) < (v_c/c)^2$, and to this term we assign a 1PN order. The same conclusion applies to the second term of Eq. (3.3.20). For the third term we note that both $\dot{I}^{abc}$ and J^{abc} are of order $m_c r_c^3/t_c$, which allows us to focus on only one of these contributions. After differentiation we find that the second term has the schematic form

$$\frac{G}{c^3}\bigg(\frac{\ddot{J}^{abc}}{c^2 r} + \frac{\dot{J}^{abc}}{cr^2} + \frac{J^{abc}}{r^3}\bigg),$$

which leads to an estimate of

$$\frac{G}{c^2}\bigg(\frac{m_c r_c^3}{c^3 t_c^3 r} + \frac{m_c r_c^3}{c^2 t_c^2 r^2} + \frac{m_c r_c^3}{c t_c r^3}\bigg) = \frac{Gm_c}{c^2 r}\Big(\frac{v_c}{c}\Big)^3\bigg[1 + \frac{\lambda_c}{r} + \Big(\frac{\lambda_c}{r}\Big)^2\bigg].$$

Because this is smaller than $Gm_c/(c^2 r)$ by a factor of order $(v_c/c)^3$, we assign a $\frac{3}{2}$PN order to this term. The discarded terms in Eq. (3.3.20) are of 2PN order and higher.

Finally, it is easy to see that the first term on the right of Eq. (3.3.21) is of order $(v_c/c)^2$ relative to $Gm_c/(c^2 r)$, and is therefore of 1PN order. The second term is of $\frac{3}{2}$PN order, and the discarded terms are of 2PN order and higher.

These conclusions are summarized in the following equations:

$$h^{00} = \frac{4G}{c^2}\bigg[\underbrace{\frac{I}{r}}_{0\text{PN}} + \underbrace{\frac{1}{2}\partial_{ab}\Big(\frac{I^{ab}}{r}\Big)}_{1\text{PN}} - \underbrace{\frac{1}{6}\partial_{abc}\Big(\frac{I^{abc}}{r}\Big)}_{\frac{3}{2}\text{PN}} + \cdots\bigg], \tag{3.3.29}$$

$$\begin{aligned} h^{0a} = \frac{4G}{c^2}\bigg[&\underbrace{\frac{1}{2c}J^{ab}\frac{\Omega_b}{r^2}}_{1\text{PN}} - \underbrace{\frac{1}{2c}\partial_b\Big(\frac{\dot{I}^{ab}}{r}\Big)}_{1\text{PN}} \\ &+ \underbrace{\frac{1}{6c}\partial_{bc}\Big(\frac{\dot{I}^{abc} + J^{abc} + J^{acb}}{r}\Big)}_{\frac{3}{2}\text{PN}} + \cdots\bigg], \end{aligned} \tag{3.3.30}$$

$$h^{ab} = \frac{4G}{c^2}\bigg[\underbrace{\frac{1}{2c^2}\frac{\ddot{I}^{ab}}{r}}_{1\text{PN}} - \underbrace{\frac{1}{6c^2}\partial_c\Big(\frac{\ddot{I}^{abc} + 2\dot{J}^{acb} + 2\dot{J}^{bca}}{r}\Big)}_{\frac{3}{2}\text{PN}} + \cdots\bigg]. \tag{3.3.31}$$

This reveals that the gravitational potentials have been calculated consistently through $\frac{3}{2}$PN order. Comparison with Eqs. (3.2.11)–(3.2.13) shows that the counting of post-Newtonian orders is different in the near and wave zones. For example, in the near zone h^{0a} begins at $\frac{1}{2}$PN order, while in the wave zone it begins at 1PN order.

Another difference concerns the post-Newtonian order of a time derivative relative to that of a spatial derivative. As we have seen in Sec. 2.2, in the near zone we have

$$\frac{\partial_t h^{\alpha\beta}}{c|\boldsymbol{\nabla} h^{\alpha\beta}|} = O\Big(\frac{v_c}{c}\Big) \qquad \text{(near zone)},$$

so that a time-differentiated potential is assigned a post-Newtonian order that is *half a unit higher* than the spatially-differentiated potential. In the wave zone, on

the other hand,

$$\frac{\partial_t h^{\alpha\beta}}{c|\boldsymbol{\nabla} h^{\alpha\beta}|} = O(1) \qquad \text{(wave zone)},$$

so that a time-differentiated potential is assigned *the same* post-Newtonian order as the spatially-differentiated potential. There are, however, important exceptions to this rule. For example, it is clear from Eq. (3.3.29) and the constancy of I that $c^{-1}\partial_t h^{00}$ is of 1PN order while $\partial_a h^{00}$ is of 0PN order. Similarly, the constancy of J^{ab} implies that $c^{-1}\partial_t h^{0a}$ is of 2PN order while $\partial_b h^{0a}$ is of 1PN order. These exceptions are consequences of the harmonic gauge conditions,

$$c^{-1}\partial_t h^{00} + \partial_a h^{0a} = 0, \qquad c^{-1}\partial_t h^{0a} + \partial_b h^{ab} = 0,$$

which indeed imply that $c^{-1}\partial_t h^{00}$, for example, must be of the same post-Newtonian order as $\partial_a h^{0a}$ *everywhere in spacetime*, and not just in the near zone.

Chapter 4
Second post-Minkowskian approximation

In this chapter we continue the iterative program initiated in Chapter 3 and construct the *second post-Minkowskian approximation* to the gravitational potentials. These will be accurate to second order in the gravitational constant G, and they will apply specifically to a system of N bodies moving under their mutual gravitational attraction. The chapter is structured much as the preceding one. We begin in Sec. 4.1 with a statement of the field equations and a presentation of their integral solutions. In Sec. 4.2 we construct expressions for the potentials that are valid in the near zone, and in Sec. 4.4 we do the same for the wave zone. In each case the potentials are presented in the form of a post-Newtonian expansion in powers of c^{-2}. In Sec. 4.3 we make a brief excursion off the main path and indicate how the (Newtonian) equations of motion for the N bodies can be obtained from the conservation identities satisfied by the source terms. A more complete derivation of the post-Newtonian equations of motions is postponed until Chapter 5.

4.1 Field equations

4.1.1 Wave equation

As was shown in Sec. 1.3, the Einstein field equations are written in the form of the wave equation

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta} \tag{4.1.1}$$

for the potentials $h^{\alpha\beta}$, where

$$\tau^{\alpha\beta} := (-g)\big(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL} + t^{\alpha\beta}_{\rm H}\big) \tag{4.1.2}$$

is an effective energy-momentum pseudotensor. Here, $T^{\alpha\beta}$ is the energy-momentum tensor of the matter distribution, $t^{\alpha\beta}_{\rm LL}$ is the Landau-Lifshitz pseudotensor of Eq. (1.1.5), and $t^{\alpha\beta}_{\rm H}$ is an additional contribution to $\tau^{\alpha\beta}$, defined by Eq. (1.3.6), and associated with the harmonic-gauge conditions $\partial_\beta h^{\alpha\beta} = 0$.

Equation (4.1.1) is exact, but in this chapter the right-hand side of Eq. (4.1.2) will be approximated to first order in G. This will make the right-hand side of

Eq. (4.1.1) accurate to second order in G, which is the required degree of accuracy in the second post-Minkowskian approximation.

4.1.2 Material energy-momentum tensor

In Chapter 3 the nature of the matter distribution was not specified, and $T^{\alpha\beta}$ was left arbitrary. In this chapter we assume specifically that the matter distribution is a collection of N point particles of masses m_A at positions $\boldsymbol{z}_A(t)$; the index $A = 1, 2, \cdots, N$ labels each particle. The energy-momentum tensor for such a matter distribution was constructed in Sec. 1.7. According to Eq. (1.7.3), it is given by

$$(-g)T^{\alpha\beta}(t,\boldsymbol{x}) = \sum_A m_A v_A^\alpha v_A^\beta \frac{\sqrt{-g}}{\sqrt{-g_{\mu\nu}v_A^\mu v_A^\nu/c^2}}\delta\big(\boldsymbol{x}-\boldsymbol{z}_A\big), \tag{4.1.3}$$

where

$$v_A^\alpha = \big(c, \boldsymbol{v}_A\big) \tag{4.1.4}$$

is the velocity four-vector of each particle, with zeroth component c and spatial components $\boldsymbol{v}_A = d\boldsymbol{z}_A/dt$.

We assume that the particles move slowly, so that

$$v_A \ll c, \tag{4.1.5}$$

with $v_A := \sqrt{\boldsymbol{v}_A \cdot \boldsymbol{v}_A}$ denoting the length of the spatial vector $\boldsymbol{v}_A$. We assume also that the system of particles is gravitationally bound, which implies (as a consequence of the Newtonian virial theorem) that v_A^2 is of the same order of magnitude as $Gm_A/|\boldsymbol{z}_A - \boldsymbol{z}_B|$. This approximate equality,

$$v_A^2 \sim \frac{Gm_A}{|\boldsymbol{z}_A - \boldsymbol{z}_B|}, \tag{4.1.6}$$

implies that an expansion in powers of $(v_A/c)^2$ is intimately linked to an expansion in powers of G. To be consistent in this context of gravitationally-bound systems, a post-Newtonian expansion must keep the order of accuracy in G in step with the order of accuracy in c^{-2}.

The energy-momentum tensor of Eq. (4.1.3) is a functional of the metric $g_{\alpha\beta}$, which must be calculated to first-order in G. This calculation was carried out in Chapter 3, and from Sec. 1.6 (as well as Sec. 3.1) we recall the relations

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta} + O(G^2) \tag{4.1.7}$$

and

$$\sqrt{-g} = 1 - \frac{1}{2}h + O(G^2), \tag{4.1.8}$$

where it is understood that indices on $h^{\alpha\beta}$ are lowered with the Minkowski metric $\eta_{\alpha\beta}$; thus, $h_{\alpha\beta} := \eta_{\alpha\mu}\eta_{\beta\nu}h^{\mu\nu}$ and $h := \eta_{\mu\nu}h^{\mu\nu}$.

To evaluate the potentials we rely on the observation made in Sec. 3.3.3, that in a slow-motion approximation the matter distribution is always situated deep within the near zone. This means that we can rely on the near-zone expressions obtained in Sec. 3.2. Recalling Eqs. (3.2.11)–(3.2.13), we have

$$h^{00} = \frac{4}{c^2}U + O(c^{-4}), \qquad h^{0a} = O(c^{-3}), \qquad h^{ab} = O(c^{-4}), \tag{4.1.9}$$

where the Newtonian potential U is determined by Poisson's equation $\nabla^2 U = -4\pi G\rho$, in which $\rho := T^{00}/c^2$ is the mass density calculated from Eq. (4.1.3) *while*

neglecting all correction terms of order c^{-2} *and* G. Indeed, corrections of fractional order c^{-2} have already been discarded in h^{00}, and a correction of order G in ρ would produce a term of order G^2 in U that must be neglected. In other words, $\rho = \sum_B m_B \delta(\boldsymbol{x} - \boldsymbol{z}_B)$, and the solution to Poisson's equation is

$$U(t, \boldsymbol{x}) = \sum_B \frac{G m_B}{|\boldsymbol{x} - \boldsymbol{z}_B|}. \tag{4.1.10}$$

This is indeed the Newtonian potential for a system of point masses at positions $\boldsymbol{z}_B(t)$.

Combining Eqs. (4.1.4), (4.1.7), and (4.1.9) produces

$$-g_{\mu\nu} v_A^\mu v_A^\nu / c^2 = 1 - v_A^2/c^2 - 2U/c^2 + O(c^{-4}),$$

and we see that this expression was calculated consistently through order c^{-2}; according to Eq. (4.1.6), the terms v_A^2/c^2 and $2U/c^2$ are of the same order of magnitude. We also have

$$\sqrt{-g} = 1 + 2U/c^2 + O(c^{-4}),$$

and inserting these relations into Eq. (4.1.3) gives

$$(-g)T^{\alpha\beta}(t, \boldsymbol{x}) = \sum_A m_A v_A^\alpha v_A^\beta \left[1 + \frac{v_A^2}{2c^2} + \frac{3\lfloor U \rfloor_A}{c^2} + O(c^{-4}) \right] \delta\bigl(\boldsymbol{x} - \boldsymbol{z}_A\bigr), \tag{4.1.11}$$

in which $\lfloor U \rfloor_A$ is formally equal to $U(t, \boldsymbol{z}_A)$, the Newtonian potential evaluated at the position of particle A — the potential must be evaluated there because it multiplies $\delta(\boldsymbol{x} - \boldsymbol{z}_A)$.

Equation (4.1.11) is an explicit expression for the energy-momentum tensor, but it is formally ill-defined because *the Newtonian potential is infinite* at $\boldsymbol{x} = \boldsymbol{z}_A$. As it stands, the energy-momentum tensor cannot be defined as a proper distribution, and there exist no solutions to the wave equation of Eq. (4.1.1). (It is a bad idea to incorporate infinite densities within a nonlinear field theory.) Following Blanchet and his collaborators, we shall step around this problem by postulating a prescription to regularize the expression of Eq. (4.1.11). We shall assert, simply, that the quantity $\delta(\boldsymbol{x} - \boldsymbol{z})/|\boldsymbol{x} - \boldsymbol{z}|$, which is too singular to be a proper distribution, is to be set equal to zero. Thus,

$$\text{regularization prescription:} \qquad \frac{\delta(\boldsymbol{x} - \boldsymbol{z})}{|\boldsymbol{x} - \boldsymbol{z}|} \equiv 0. \tag{4.1.12}$$

This prescription, known as *taking Hadamard's partie finie*, can be loosely interpreted as a renormalization of each mass parameter m_A by the infinite self-energy of the particle: $m_A(1 + 3U^{\text{self}}/c^2) \to m_A$. In the case of extended bodies, the gravitational self-energy would indeed contribute to the total mass-energy of each body.

The prescription of Eq. (4.1.12) allows us to formally define $\lfloor U \rfloor_A$ as

$$\lfloor U \rfloor_A := \sum_{B \neq A} \frac{G m_B}{|\boldsymbol{z}_A - \boldsymbol{z}_B|}. \tag{4.1.13}$$

The sum now excludes body A, and the result can safely be substituted into Eq. (4.1.11).

4.1.3 Landau-Lifshitz pseudotensor

We next compute $(-g)t_{\text{LL}}^{\alpha\beta}$, the Landau-Lifshitz pseudotensor of Eq. (1.1.5), to first order in G. This is a fairly labourious calculation, but an important source of

simplification comes from the fact that we need expressions that are accurate only up to some power of c^{-1}. A source of caution, on the other hand, is that we need expressions that sufficiently accurate both in the near zone and in the wave zone.

To begin, we recall the scalings that are implied by Eq. (3.1.8),

$$h^{00} = O(c^{-2}), \qquad h^{0a} = O(c^{-3}), \qquad h^{ab} = O(c^{-4}); \tag{4.1.14}$$

it is understood that each potential carries also a factor of G. It follows that the spatial derivatives of the potentials scale as

$$\partial_c h^{00} = O(c^{-2}), \qquad \partial_c h^{0a} = O(c^{-3}), \qquad \partial_c h^{ab} = O(c^{-4}). \tag{4.1.15}$$

The temporal derivatives, on the other hand, scale as

$$\partial_0 h^{00} = O(c^{-3}), \qquad \partial_0 h^{0a} = O(c^{-4}), \qquad \partial_0 h^{ab} = O(c^{-4}), \tag{4.1.16}$$

because the gauge conditions imply that $\partial_0 h^{00}$ must be of the same order as $\partial_a h^{0a}$, and $\partial_0 h^{0a}$ must be of the same order as $\partial_b h^{ab}$. In the *near zone* the temporal derivative of h^{ab} would come with an additional factor of c^{-1}, and would therefore scale as $O(c^{-5})$, but this does not happen in the wave zone.

Substituting the potentials $h^{\alpha\beta}$ into Eq. (1.1.5) and keeping careful track of the orders in c^{-1} eventually returns

$$\begin{aligned}
\frac{16\pi G}{c^4}(-g)t^{00}_{\rm LL} &= -\frac{7}{8}\partial_c h^{00}\partial^c h^{00} + O(c^{-6}), && (4.1.17)\\
\frac{16\pi G}{c^4}(-g)t^{0a}_{\rm LL} &= \frac{3}{4}\partial^a h^{00}\partial_0 h^{00} + \bigl(\partial^a h^{0c} - \partial^c h^{0a}\bigr)\partial_c h^{00} + O(c^{-7}), && (4.1.18)\\
\frac{16\pi G}{c^4}(-g)t^{ab}_{\rm LL} &= \frac{1}{4}\partial^a h^{00}\partial^b h^{00} - \frac{1}{8}\delta^{ab}\partial_c h^{00}\partial^c h^{00} + O(c^{-6}). && (4.1.19)
\end{aligned}$$

These results are sufficiently accurate for our immediate purposes. At a later stage, however, we shall need additional accuracy in our expression for $(-g)t^{ab}_{\rm LL}$, and we record this improved expression here:

$$\begin{aligned}
\frac{16\pi G}{c^4}(-g)t^{ab}_{\rm LL} = {}& \frac{1}{4}\bigl(1-2h^{00}\bigr)\partial^a h^{00}\partial^b h^{00} - \frac{1}{8}\delta^{ab}\bigl(1-2h^{00}\bigr)\partial_c h^{00}\partial^c h^{00}\\
&- \partial^a h^{0c}\partial^b h^0{}_c + \partial^a h^{0c}\partial_c h^{0b} + \partial^b h^{0c}\partial_c h^{0a} - \partial_c h^{0a}\partial^c h^{0b}\\
&+ \partial^a h^{00}\partial_0 h^{0b} + \partial^b h^{00}\partial_0 h^{0a} + \frac{1}{4}\partial^a h^{00}\partial^b h^c{}_c + \frac{1}{4}\partial^b h^{00}\partial^a h^c{}_c\\
&+ \delta^{ab}\biggl[-\frac{3}{8}\bigl(\partial_0 h^{00}\bigr)^2 - \partial_c h^{00}\partial_0 h^{0c} - \frac{1}{4}\partial_c h^{00}\partial^c h^d{}_d\\
&+ \frac{1}{2}\partial_c h^0{}_d\bigl(\partial^c h^{0d} - \partial^d h^{0c}\bigr)\biggr] + O(c^{-8}). \qquad (4.1.20)
\end{aligned}$$

It should be noted that this incorporates corrections of fractional order c^{-2} relative to the leading-order expression of Eq. (4.1.19), and that to be consistent, we have terms (such as $h^{00}\partial^a h^{00}\partial^b h^{00}$) which contain an additional power of the gravitational constant G.

4.1.4 Harmonic-gauge pseudotensor

We next compute $(-g)t^{\alpha\beta}_{\rm H}$, the "harmonic-gauge" pseudotensor of Eq. (1.3.6), to first order in G. Here the computations are quite simple, and the scalings of Eqs. (4.1.14)–(4.1.16) imply

$$\frac{16\pi G}{c^4}(-g)t^{00}_{\rm H} = O(c^{-6}), \tag{4.1.21}$$

$$\frac{16\pi G}{c^4}(-g)t_{\rm H}^{0a} = O(c^{-7}), \qquad (4.1.22)$$

$$\frac{16\pi G}{c^4}(-g)t_{\rm H}^{ab} = O(c^{-6}). \qquad (4.1.23)$$

For later reference we record the improved expression

$$\frac{16\pi G}{c^4}(-g)t_{\rm H}^{ab} = -h^{00}\partial_{00}h^{ab} + O(c^{-8}) \qquad (4.1.24)$$

for the spatial components of the pseudotensor.

4.1.5 Explicit form of the wave equations

We may now substitute Eqs. (4.1.11), (4.1.17)–(4.1.19), and (4.1.21)–(4.1.23) into Eq. (4.1.2), and insert this into the right-hand side of Eq. (4.1.1). Before we do this, however, we recall that the h^{00} that appears within the Landau-Lifshitz pseudotensor is the one that was determined during the first iteration of the Einstein field equations. Returning to the notation of Sec. 3.1, we write this as $h^{00} = 4\Phi/c^2$, and we note that Φ satisfies the wave equation

$$\Box\Phi = -4\pi G\sum_A m_A\delta\bigl(\boldsymbol{x}-\boldsymbol{z}_A\bigr); \qquad (4.1.25)$$

to get this we have inserted our previous expression for ρ [see the text preceding Eq. (4.1.10)] into Eq. (3.1.10). The wave equations for the second post-Minkowskian potentials are then

$$\Box h^{00} = -\frac{16\pi G}{c^2}\sum_A m_A\biggl(1+\frac{v_A^2}{2c^2}+\frac{3\lfloor U\rfloor_A}{c^2}\biggr)\delta\bigl(\boldsymbol{x}-\boldsymbol{z}_A\bigr) + \frac{14}{c^4}\partial_c\Phi\partial^c\Phi + O(c^{-6}), \qquad (4.1.26)$$

$$\Box h^{0a} = -\frac{16\pi G}{c^3}\sum_A m_A v_A^a\delta\bigl(\boldsymbol{x}-\boldsymbol{z}_A\bigr) + O(c^{-5}), \qquad (4.1.27)$$

$$\Box h^{ab} = -\frac{16\pi G}{c^4}\sum_A m_A v_A^a v_A^b\delta\bigl(\boldsymbol{x}-\boldsymbol{z}_A\bigr) - \frac{4}{c^4}\biggl(\partial^a\Phi\partial^b\Phi - \frac{1}{2}\delta^{ab}\partial_c\Phi\partial^c\Phi\biggr) + O(c^{-6}). \qquad (4.1.28)$$

We recall that $\lfloor U\rfloor_A$ is given by Eq. (4.1.13).

The structure of each wave equation simplifies if we introduce new potentials V, V^a, and W^{ab}, defined by

$$h^{00} = \frac{4}{c^2}V - \frac{4}{c^4}W + \frac{8}{c^4}\Phi^2, \qquad (4.1.29)$$

$$h^{0a} = \frac{4}{c^3}V^a, \qquad (4.1.30)$$

$$h^{ab} = \frac{4}{c^4}W^{ab}, \qquad (4.1.31)$$

where $W := \delta_{ab}W^{ab}$ is the trace of the tensor potential. The wave equation satisfied by V^a follows immediately from Eq. (4.1.27), and the wave equation satisfied by W^{ab} follows from Eq. (4.1.28); from this we deduce that $\Box W = -4\pi G\sum_A m_A v_A^2\delta(\boldsymbol{x}-\boldsymbol{z}_A) + \frac{1}{2}\partial_c\Phi\partial^c\Phi + O(c^{-2})$. The equation for V follows from Eq. (4.1.26) and our result for $\Box W$, but we need also an expression for $\Box\Phi^2$.

We apply the wave operator on Φ^2 and find that

$$\Box\Phi^2 = 2\Phi\Box\Phi + 2\partial_c\Phi\partial^c\Phi - 2\bigl(\partial_0\Phi\bigr)^2.$$

We next involve Eq. (4.1.25) and write

$$2\Phi\Box\Phi = -4\pi G\sum_A m_A\bigl(2\Phi\bigr)\delta\bigl(\boldsymbol{x}-\boldsymbol{z}_A\bigr)$$

for the first term, which we copy as

$$2\Phi\Box\Phi = -4\pi G\sum_A m_A\bigl[2\lfloor U\rfloor_A + O(c^{-2})\bigr]\delta\bigl(\boldsymbol{x}-\boldsymbol{z}_A\bigr),$$

because Φ is evaluated in the near zone, at $\boldsymbol{x}=\boldsymbol{z}_A$. We further notice that thanks to Eq. (4.1.16), $(\partial_0\Phi)^2$ is of order c^{-2}, and we arrive at

$$\Box\Phi^2 = -4\pi G\sum_A m_A\bigl(2\lfloor U\rfloor_A\bigr)\delta\bigl(\boldsymbol{x}-\boldsymbol{z}_A\bigr) + 2\partial_c\Phi\partial^c\Phi + O(c^{-2}). \tag{4.1.32}$$

With this we find that the wave equations for V, V^a, and W^{ab} are

$$\Box V = -4\pi G\sum_A m_A\biggl(1+\frac{3}{2}\frac{v_A^2}{c^2}-\frac{\lfloor U\rfloor_A}{c^2}\biggr)\delta\bigl(\boldsymbol{x}-\boldsymbol{z}_A\bigr) + O(c^{-4}), \tag{4.1.33}$$

$$\Box V^a = -4\pi G\sum_A m_A v_A^a\delta\bigl(\boldsymbol{x}-\boldsymbol{z}_A\bigr) + O(c^{-2}), \tag{4.1.34}$$

$$\Box W^{ab} = -4\pi G\sum_A m_A v_A^a v_A^b\delta\bigl(\boldsymbol{x}-\boldsymbol{z}_A\bigr) - \biggl(\partial^a\Phi\partial^b\Phi - \frac{1}{2}\delta^{ab}\partial_c\Phi\partial^c\Phi\biggr) + O(c^{-2}). \tag{4.1.35}$$

We recall that $\lfloor U\rfloor_A$ is defined by Eq. (4.1.13), and that Φ is determined by Eq. (4.1.25).

The reason for expressing h^{00} in the form of Eq. (4.1.29) is now clear: By inserting the terms involving W and Φ^2 we were able to make the wave equation for V entirely independent of the field variable Φ. The equation for V^a also is independent of Φ, and the only place in which Φ appears is within the wave equation for W^{ab}.

Equation (4.1.33) allows us to calculate V through order c^{-2}, while Eqs. (4.1.25), (4.1.34), and (4.1.35) allow us to calculate Φ, V^a, and W^{ab} through order c^0. Inserting the results into Eqs. (4.1.29)–(4.1.31) produces gravitational potentials with an order structure given schematically by $h^{00}=c^{-2}+c^{-4}$, $h^{0a}=c^{-3}$, and $h^{ab}=c^{-4}$.

4.2 Near-zone expressions

4.2.1 Computation of V

The wave equation for V has a source term that is confined to the near zone, and recalling the discussion of Sec. 2.3, we observe that as a consequence, $V_{\mathscr{W}}=0$ and $V=V_{\mathscr{N}}$. The near-zone scalar potential is computed with the help of Eq. (2.4.7), and we quickly obtain

$$V = U + \frac{1}{c^2}\psi + \frac{1}{2c^2}\frac{\partial^2 X}{\partial t^2} + O(c^{-3}), \tag{4.2.1}$$

where

$$U(t,\boldsymbol{x}) = \sum_A \frac{Gm_A}{|\boldsymbol{x}-\boldsymbol{z}_A|} \tag{4.2.2}$$

is the Newtonian potential of Eq. (4.1.10),

$$\psi(t,\boldsymbol{x}) = \sum_A \frac{Gm_A\bigl(\frac{3}{2}v_A^2-\lfloor U\rfloor_A\bigr)}{|\boldsymbol{x}-\boldsymbol{z}_A|} \tag{4.2.3}$$

is a post-Newtonian correction to the Newtonian potential, and

$$X(t,\boldsymbol{x}) = \sum_A G m_A |\boldsymbol{x} - \boldsymbol{z}_A| \tag{4.2.4}$$

is the superpotential.

We notice that there is no term of order c^{-1} in Eq. (4.2.1). A contribution at this order would originate from the $q = 1$ term in Eq. (2.4.7), and it would be equal to

$$-\frac{G}{c}\frac{d}{dt}\sum_A m_A\biggl(1 + \frac{3}{2}\frac{v_A^2}{c^2} - \frac{\lfloor U \rfloor_A}{c^2}\biggr).$$

Because $\sum_A m_A$ is a constant, this is actually of order c^{-3}, and is part of the discarded terms in Eq. (4.2.1).

4.2.2 Computation of V^a

The wave equation for V^a has a source term that is also confined to the near zone, and once more we find that $V^a_{\mathscr{W}} = 0$ and $V^a = V^a_{\mathscr{N}}$. The near-zone vector potential also is computed with the help of Eq. (2.4.7), and here we obtain

$$V^a = \sum_A \frac{G m_A v_A^a}{|\boldsymbol{x} - \boldsymbol{z}_A|} - \frac{1}{c}\frac{d}{dt}\sum_A G m_A v_A^a + O(c^{-2}).$$

The second term can be expressed as $-(G/c)dP^a/dt$, in terms of the vector $\boldsymbol{P} = \sum_A m_A \boldsymbol{v}_A$, which is the total (Newtonian) momentum of the N-body system. Anticipating that at leading order the motion of the system is governed by the Newtonian equations of motion (a fact that will be established properly in Sec. 4.3), we declare that the Newtonian momentum is conserved at 0PN order: $d\boldsymbol{P}/dt = O(c^{-2})$. It follows that the second term in V^a is actually of order c^{-3} and part of the discarded terms.

We write our final expression as

$$V^a = U^a + O(c^{-2}), \tag{4.2.5}$$

where

$$U^a(t,\boldsymbol{x}) = \sum_A \frac{G m_A v_A^a}{|\boldsymbol{x} - \boldsymbol{z}_A|} \tag{4.2.6}$$

is the same instantaneous potential that was first introduced in Eq. (3.2.6).

4.2.3 Computation of W^{ab}: Organization

The computation of the tensor potential is much more involved, because its wave equation possesses a source term that contains a field contribution in addition to a material contribution. To distinguish these we shall write

$$W^{ab} = W^{ab}[\mathrm{M}] + W^{ab}[\mathrm{F}], \tag{4.2.7}$$

with $W^{ab}[\mathrm{M}]$ denoting the part of the tensor potential that comes entirely from the material source, while $W^{ab}[\mathrm{F}]$ comes from the field source. The wave equation for $W^{ab}[\mathrm{F}]$ is simplified if we define an auxiliary potential χ^{ab} by

$$W^{ab}[\mathrm{F}] =: \chi^{ab} - \frac{1}{2}\delta^{ab}\chi, \tag{4.2.8}$$

where $\chi := \delta_{ab}\chi^{ab}$. With these definitions, Eq. (4.1.35) becomes the set of equations

$$\Box W^{ab}[\mathrm{M}] = -4\pi G \sum_A m_A v_A^a v_A^b \delta\big(\boldsymbol{x} - \boldsymbol{z}_A\big) + O(c^{-2}), \tag{4.2.9}$$

$$\Box \chi^{ab} = -\partial^a \Phi \partial^b \Phi + O(c^{-2}). \tag{4.2.10}$$

Because the source term of Eq. (4.2.9) is contained within the near zone, we have that $W^{ab}_{\mathscr{W}}[\mathrm{M}] = 0$ and $W^{ab}[\mathrm{M}] = W^{ab}_{\mathscr{N}}[\mathrm{M}]$. The source term of Eq. (4.2.10), on the other hand, is distributed over all space, and

$$\chi^{ab} = \chi^{ab}_{\mathscr{N}} + \chi^{ab}_{\mathscr{W}}. \tag{4.2.11}$$

In the following subsections we will endeavour to compute each one of the quantities introduced here, so as to finally build up a complete expression for W^{ab}.

4.2.4 Computation of W^{ab}: $W^{ab}[\mathrm{M}]$

This is the easiest piece. Following the same steps as in Secs. 4.2.1 and 4.2.2, we arrive at

$$W^{ab}[\mathrm{M}] = \sum_A \frac{G m_A v_A^a v_A^b}{|\boldsymbol{x} - \boldsymbol{z}_A|} + O(c^{-1}). \tag{4.2.12}$$

4.2.5 Computation of W^{ab}: χ

The computation of χ^{ab} is much more involved, and to get us started we first examine its trace χ, which is in fact easy to calculate. From Eq. (4.2.10) we have $\Box\chi = -\partial_c \Phi \partial^c \Phi + O(c^{-2})$. Using Eq. (4.1.32), we write this as

$$\Box\Big(\chi + \frac{1}{2}\Phi^2\Big) = -4\pi G \sum_A m_A \big(\lfloor U \rfloor_A\big) \delta\big(\boldsymbol{x} - \boldsymbol{z}_A\big) + O(c^{-2}). \tag{4.2.13}$$

Because this source term comes entirely from the matter distribution, we follow the familiar steps and obtain

$$\chi = -\frac{1}{2}U^2 + \sum_A \frac{G m_A \lfloor U \rfloor_A}{|\boldsymbol{x} - \boldsymbol{z}_A|} + O(c^{-1}). \tag{4.2.14}$$

Here we used the fact that $\Phi = U + O(c^{-2})$ in the near zone, and the Newtonian potential U is given by Eq. (4.1.10).

4.2.6 Computation of W^{ab}: $\chi^{ab}_{\mathscr{N}}$

Now for a more challenging computation. In this subsection we calculate $\chi^{ab}_{\mathscr{N}}$, the near-zone contribution to the retarded integral associated with Eq. (4.2.10), assuming that the field-point $x = (ct, \boldsymbol{x})$ is within the near zone. The techniques to carry out such a computation were described in Sec. 2.4.2, and according to Eq. (2.4.7) we have

$$\chi^{ab}_{\mathscr{N}}(t, \boldsymbol{x}) = \frac{1}{4\pi} \int_{\mathscr{M}} \frac{\partial^{a'} U \partial^{b'} U}{|\boldsymbol{x} - \boldsymbol{x'}|}\, d^3x' + O(c^{-1}). \tag{4.2.15}$$

Here we have once more substituted U in place of Φ, and inside the integral the Newtonian potential is viewed as a function of t and $\boldsymbol{x'}$; the symbol $\partial^{a'}$ indicates differentiation with respect to x'^a, and the domain of integration $\mathscr{M}$ is a surface of constant time bounded externally by the sphere $r' := |\boldsymbol{x'}| = \mathcal{R}$.

Our starting point is Eq. (4.1.10) for the Newtonian potential, from which we obtain

$$\partial^{a'} U = -\sum_A G m_A \frac{(\boldsymbol{x'} - \boldsymbol{z}_A)^a}{|\boldsymbol{x'} - \boldsymbol{z}_A|^3}$$

and then

$$\begin{aligned} \partial^{a'} U \partial^{b'} U &= \sum_A G^2 m_A^2 \frac{(\boldsymbol{x'} - \boldsymbol{z}_A)^a (\boldsymbol{x'} - \boldsymbol{z}_A)^b}{|\boldsymbol{x'} - \boldsymbol{z}_A|^6} \\ &\quad + \sum_A \sum_{B \neq A} G^2 m_A m_B \frac{(\boldsymbol{x'} - \boldsymbol{z}_A)^a (\boldsymbol{x'} - \boldsymbol{z}_B)^b}{|\boldsymbol{x'} - \boldsymbol{z}_A|^3 |\boldsymbol{x'} - \boldsymbol{z}_B|^3}, \end{aligned} \tag{4.2.16}$$

in which we distinguish "self terms" from "interaction terms".

Following Blanchet, Faye, and Ponsot, we re-express Eq. (4.2.16) in terms of quantities differentiated with respect to $\boldsymbol{z}_A$ and $\boldsymbol{z}_B$. The derivative operators are taken outside the integral of Eq. (4.2.15), and $\chi^{ab}_{\mathscr{N}}$ is written in terms of these operators acting on a generating function $K(\boldsymbol{x}; \boldsymbol{z}_A, \boldsymbol{z}_B)$. This function can be evaluated, and it is straightforward to take the derivatives. The end result is the relatively simple expression for $\chi^{ab}_{\mathscr{N}}$ displayed in Eq. (4.2.23) below.

To proceed we note the identities

$$\frac{\partial}{\partial z_A^a} \frac{1}{|\boldsymbol{x'} - \boldsymbol{z}_A|} = \frac{(\boldsymbol{x'} - \boldsymbol{z}_A)^a}{|\boldsymbol{x'} - \boldsymbol{z}_A|^3}$$

and

$$\frac{\partial^2}{\partial z_A^a \partial z_A^b} \frac{1}{|\boldsymbol{x'} - \boldsymbol{z}_A|} = \frac{3(\boldsymbol{x'} - \boldsymbol{z}_A)^a (\boldsymbol{x'} - \boldsymbol{z}_A)^b}{|\boldsymbol{x'} - \boldsymbol{z}_A|^5} - \frac{\delta^{ab}}{|\boldsymbol{x'} - \boldsymbol{z}_A|^3} - \frac{4\pi}{3} \delta^{ab} \delta\bigl(\boldsymbol{x'} - \boldsymbol{z}_A\bigr);$$

in the last one it is necessary to insert a distributional term proportional to δ^{ab}, to ensure that the Laplacian of $|\boldsymbol{x'} - \boldsymbol{z}_A|^{-1}$ with respect to the variables $\boldsymbol{z}_A$ is properly equal to $-4\pi\delta(\boldsymbol{x'} - \boldsymbol{z}_A)$. More work along these lines produces an additional identity,

$$\begin{aligned} \frac{\partial^2}{\partial z_A^a \partial z_A^b} \frac{1}{|\boldsymbol{x'} - \boldsymbol{z}_A|^2} &= \frac{8(\boldsymbol{x'} - \boldsymbol{z}_A)^a (\boldsymbol{x'} - \boldsymbol{z}_A)^b}{|\boldsymbol{x'} - \boldsymbol{z}_A|^6} - \frac{2\delta^{ab}}{|\boldsymbol{x'} - \boldsymbol{z}_A|^4} \\ &\quad - \frac{8\pi}{3} \frac{\delta^{ab}}{|\boldsymbol{x'} - \boldsymbol{z}_A|} \delta\bigl(\boldsymbol{x'} - \boldsymbol{z}_A\bigr). \end{aligned}$$

The last term is not defined as a distribution, and the identity must be regularized. Following consistently the general prescription of Eq. (4.1.12), we simply drop the last term, and write our last identity as

$$\frac{(\boldsymbol{x'} - \boldsymbol{z}_A)^a (\boldsymbol{x'} - \boldsymbol{z}_A)^b}{|\boldsymbol{x'} - \boldsymbol{z}_A|^6} = \frac{1}{8} \biggl(\frac{\partial^2}{\partial z_A^a \partial z_A^b} + \delta^{ab} \nabla_A^2 \biggr) \frac{1}{|\boldsymbol{x'} - \boldsymbol{z}_A|^2}, \tag{4.2.17}$$

where ∇_A^2 is the Laplacian operator with respect to the variables $\boldsymbol{z}_A$. The last identity we shall need follows directly from the first, and it is

$$\frac{(\boldsymbol{x'} - \boldsymbol{z}_A)^a (\boldsymbol{x'} - \boldsymbol{z}_B)^b}{|\boldsymbol{x'} - \boldsymbol{z}_A|^3 |\boldsymbol{x'} - \boldsymbol{z}_B|^3} = \frac{\partial^2}{\partial z_A^a \partial z_B^b} \frac{1}{|\boldsymbol{x'} - \boldsymbol{z}_A| |\boldsymbol{x'} - \boldsymbol{z}_B|}; \tag{4.2.18}$$

this requires no regularization, and is valid when $\boldsymbol{z}_A \neq \boldsymbol{z}_B$.

We substitute Eqs. (4.2.17) and (4.2.18) into Eq. (4.2.16), and insert the result inside the integral of Eq. (4.2.15). This produces

$$\begin{aligned} \chi^{ab}_{\mathscr{N}} &= \frac{1}{8} \sum_A G^2 m_A^2 \biggl(\frac{\partial^2}{\partial z_A^a \partial z_A^b} + \delta^{ab} \nabla_A^2 \biggr) K(\boldsymbol{x}; \boldsymbol{z}_A, \boldsymbol{z}_A) \\ &\quad + \sum_A \sum_{B \neq A} G^2 m_A m_B \frac{\partial^2}{\partial z_A^a \partial z_B^b} K(\boldsymbol{x}; \boldsymbol{z}_A, \boldsymbol{z}_B) + O(c^{-1}), \end{aligned} \tag{4.2.19}$$

where

$$K(\boldsymbol{x};\boldsymbol{z}_A,\boldsymbol{z}_B) = \frac{1}{4\pi}\int_{\mathscr{M}} \frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x'}||\boldsymbol{x'}-\boldsymbol{z}_A||\boldsymbol{x'}-\boldsymbol{z}_B|} \tag{4.2.20}$$

is the generating function that was mentioned previously.

The generating function will be evaluated in Sec. 4.2.7. The result is

$$K(\boldsymbol{x};\boldsymbol{z}_A,\boldsymbol{z}_B) = 1 - \ln\frac{S}{2\mathcal{R}}, \tag{4.2.21}$$

where

$$S(\boldsymbol{x};\boldsymbol{z}_A,\boldsymbol{z}_B) := |\boldsymbol{x}-\boldsymbol{z}_A| + |\boldsymbol{x}-\boldsymbol{z}_B| + |\boldsymbol{z}_A-\boldsymbol{z}_B|. \tag{4.2.22}$$

The dependence of K on $\mathcal{R}$ comes from the fact that the domain of integration $\mathscr{M}$ is truncated at $r' = \mathcal{R}$. This dependence plays no role, however, because K is differentiated as soon as it is substituted into Eq. (4.2.19).

It is now straightforward to compute derivatives of $K(\boldsymbol{x};\boldsymbol{z}_A,\boldsymbol{z}_B)$ with respect to $\boldsymbol{z}_A$ and $\boldsymbol{z}_B$. We find, for example,

$$\frac{\partial^2}{\partial z_A^a \partial z_A^b} K(\boldsymbol{x};\boldsymbol{z}_A,\boldsymbol{z}_A) = \frac{2(\boldsymbol{x}-\boldsymbol{z}_A)^a(\boldsymbol{x}-\boldsymbol{z}_A)^b}{|\boldsymbol{x}-\boldsymbol{z}_A|^4} - \frac{\delta^{ab}}{|\boldsymbol{x}-\boldsymbol{z}_A|^2}$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial z_A^a \partial z_B^b} K(\boldsymbol{x};\boldsymbol{z}_A,\boldsymbol{z}_B) \quad = \quad & \frac{1}{S^2}\left[\frac{(\boldsymbol{x}-\boldsymbol{z}_A)^a}{|\boldsymbol{x}-\boldsymbol{z}_A|} - \frac{(\boldsymbol{z}_A-\boldsymbol{z}_B)^a}{|\boldsymbol{z}_A-\boldsymbol{z}_B|}\right]\left[\frac{(\boldsymbol{x}-\boldsymbol{z}_B)^b}{|\boldsymbol{x}-\boldsymbol{z}_B|} + \frac{(\boldsymbol{z}_A-\boldsymbol{z}_B)^b}{|\boldsymbol{z}_A-\boldsymbol{z}_B|}\right] \\
& - \frac{1}{S}\left[\frac{(\boldsymbol{z}_A-\boldsymbol{z}_B)^a(\boldsymbol{z}_A-\boldsymbol{z}_B)^b}{|\boldsymbol{z}_A-\boldsymbol{z}_B|^3} - \frac{\delta^{ab}}{|\boldsymbol{z}_A-\boldsymbol{z}_B|}\right],
\end{aligned}$$

and these results are to be inserted within Eq. (4.2.19).

Our final result is

$$\begin{aligned}
\chi_{\mathscr{N}}^{ab} \quad = \quad & \frac{1}{4}\sum_A \frac{G^2 m_A^2}{|\boldsymbol{x}-\boldsymbol{z}_A|^2}\bigl(n_A^a n_A^b - \delta^{ab}\bigr) \\
& + \sum_A \sum_{B\neq A} \frac{G^2 m_A m_B}{S^2}\bigl(n_A^a - n_{AB}^a\bigr)\bigl(n_B^b + n_{AB}^b\bigr) \\
& - \sum_A \sum_{B\neq A} \frac{G^2 m_A m_B}{S|\boldsymbol{z}_A-\boldsymbol{z}_B|}\bigl(n_{AB}^a n_{AB}^b - \delta^{ab}\bigr) + O(c^{-1}),
\end{aligned} \tag{4.2.23}$$

where we have introduced the unit vectors

$$\boldsymbol{n}_A := \frac{\boldsymbol{x}-\boldsymbol{z}_A}{|\boldsymbol{x}-\boldsymbol{z}_A|}, \qquad \boldsymbol{n}_B := \frac{\boldsymbol{x}-\boldsymbol{z}_B}{|\boldsymbol{x}-\boldsymbol{z}_B|}, \qquad \boldsymbol{n}_{AB} := \frac{\boldsymbol{z}_A-\boldsymbol{z}_B}{|\boldsymbol{z}_A-\boldsymbol{z}_B|}, \tag{4.2.24}$$

and where $S(\boldsymbol{x};\boldsymbol{z}_A,\boldsymbol{z}_B)$ is defined by Eq. (4.2.22).

It is a straightforward exercise to verify that the trace of Eq. (4.2.23) is

$$\begin{aligned}
\chi_{\mathscr{N}} \quad = \quad & -\frac{1}{2}\sum_A \frac{G^2 m_A^2}{|\boldsymbol{x}-\boldsymbol{z}_A|^2} + \frac{1}{2}\sum_A\sum_{B\neq A} G^2 m_A m_B \\
& \times\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{z}_A||\boldsymbol{z}_A-\boldsymbol{z}_B|} + \frac{1}{|\boldsymbol{x}-\boldsymbol{z}_B||\boldsymbol{z}_A-\boldsymbol{z}_B|} - \frac{1}{|\boldsymbol{x}-\boldsymbol{z}_A||\boldsymbol{x}-\boldsymbol{z}_B|}\right) \\
& + O(c^{-1}),
\end{aligned} \tag{4.2.25}$$

and that this is the same statement as in Eq. (4.2.14). This calculation is aided by the identities

$$
\begin{aligned}
\boldsymbol{n}_A \cdot \boldsymbol{n}_B &= \frac{r_A^2 + r_B^2 - z_{AB}^2}{2 r_A r_B}, \\
\boldsymbol{n}_A \cdot \boldsymbol{n}_{AB} &= \frac{r_B^2 - r_A^2 - z_{AB}^2}{2 r_A z_{AB}}, \\
\boldsymbol{n}_B \cdot \boldsymbol{n}_{AB} &= \frac{r_B^2 - r_A^2 + z_{AB}^2}{2 r_B z_{AB}},
\end{aligned}
$$

where $r_A := |\boldsymbol{x} - \boldsymbol{z}_A|$, $r_B := |\boldsymbol{x} - \boldsymbol{z}_B|$, and $z_{AB} := |\boldsymbol{z}_A - \boldsymbol{z}_B|$.

4.2.7 Computation of W^{ab}: $K(\boldsymbol{x}; \boldsymbol{z}_A, \boldsymbol{z}_B)$

To calculate the generating function we note first that Eq. (4.2.20) is a solution to

$$
\nabla^2 K(\boldsymbol{x}; \boldsymbol{z}_A, \boldsymbol{z}_B) = -\frac{1}{|\boldsymbol{x} - \boldsymbol{z}_A||\boldsymbol{x} - \boldsymbol{z}_B|}. \tag{4.2.26}
$$

Strictly speaking, the source term should be multiplied by $\Theta(\mathcal{R} - r)$ to truncate it at the boundary of the near zone, because the domain of integration in Eq. (4.2.20) does not extend beyond this boundary. The step function is not necessary, however, because we require the solution to Eq. (4.2.26) only within the near zone; how the solution extends beyond $r = \mathcal{R}$ is of no concern here.

We will show below that

$$
K_{\mathrm{p}} = -\ln S,
$$

where $S(\boldsymbol{x}; \boldsymbol{z}_A, \boldsymbol{z}_B)$ is defined by Eq. (4.2.22), is a particular solution to Eq. (4.2.26). To this we must add a suitable solution K_{h} to Laplace's equation to obtain the desired solution $K = K_{\mathrm{p}} + K_{\mathrm{h}}$. The solution to the homogeneous equation must be regular in all three variables $\boldsymbol{x}$, $\boldsymbol{z}_A$, and $\boldsymbol{z}_B$, because the singularity structure required by Eq. (4.2.26) is already contained in K_{p}. Furthermore, K_{h} must be dimensionless, and the only possibility is to make it equal to a constant. We are therefore looking for a solution of the form

$$
K = K_0 - \ln\bigl(|\boldsymbol{x} - \boldsymbol{z}_A| + |\boldsymbol{x} - \boldsymbol{z}_B| + |\boldsymbol{z}_A - \boldsymbol{z}_B|\bigr),
$$

where K_0 is a dimensionless constant. To determine this we shall carry out an independent computation of the special value $K(\boldsymbol{x}; \boldsymbol{0}, \boldsymbol{0})$, and compare our result to $K_0 - \ln(2r)$, which follows from the general expression. From Eq. (4.2.20) we have

$$
K(\boldsymbol{x}; \boldsymbol{0}, \boldsymbol{0}) = \frac{1}{4\pi} \int_{\mathscr{M}} \frac{d^3x'}{|\boldsymbol{x} - \boldsymbol{x}'||\boldsymbol{x}'|^2} = \frac{1}{4\pi} \int_0^{\mathcal{R}} \frac{dr'\, d\Omega'}{|\boldsymbol{x} - \boldsymbol{x}'|}.
$$

Invoking the addition theorem for spherical harmonics, this is simply

$$
\int_0^{\mathcal{R}} \frac{dr'}{r_>},
$$

where $r_>$ is the greater of r and r'. The integral evaluates to

$$
K(\boldsymbol{x}; \boldsymbol{0}, \boldsymbol{0}) = 1 + \ln \frac{\mathcal{R}}{r}, \tag{4.2.27}
$$

and we conclude that $K_0 = 1 + \ln(2\mathcal{R})$. All together, this gives us the result displayed in Eq. (4.2.21).

We now wish to verify that $K_\mathrm{p} = -\ln S$ is a solution to Eq. (4.2.26), which we write in the form
$$\nabla^2 K = -\frac{1}{r_A r_B},$$
where r_A and r_B were introduced previously. In this notation, $S = r_A + r_B + z_{AB}$. We first check that
$$\nabla^2 K_\mathrm{p} = -\frac{1}{S^2}\bigl(S\nabla^2 S - \partial_c S \partial^c S\bigr),$$
and we compute the various derivatives of S. We have, for example, $\partial^a S = n_A^a + n_B^a$, from which it follows that
$$\partial^{ab} S = -\frac{n_A^a n_A^b - \delta^{ab}}{r_A} - \frac{n_B^a n_B^b - \delta^{ab}}{r_B}.$$
From this, and the helpful identities that were listed at the end of Sec. 4.2.6, we obtain
$$\nabla^2 S = 2\frac{r_A + r_B}{r_A r_B}, \qquad \partial_c S \partial^c S = \frac{(r_A + r_B - z_{AB})S}{r_A r_B}.$$
Collecting results, we confirm that K_p is indeed a solution to $\nabla^2 K = -1/(r_A r_B)$.

4.2.8 Computation of W^{ab}: $\chi^{ab}_{\mathscr{W}}$

In this subsection we estimate $\chi^{ab}_{\mathscr{W}}$, the wave-zone contribution to the retarded integral associated with Eq. (4.2.10), assuming that the field-point $x = (ct, \boldsymbol{x})$ is within the near zone. The techniques to carry out such a computation were described in Sec. 2.5.3, and crude estimates were obtained in Sec. 2.5.4. These estimates ignore numerical factors and terms that depend explicitly on $\mathcal{R}$, but they are sufficient to allow us to conclude that
$$\chi^{ab}_{\mathscr{W}} = O(c^{-4}). \tag{4.2.28}$$
The wave-zone contribution is therefore much smaller than $\chi^{ab}_{\mathscr{N}}$, which is of order c^0. Recalling that χ^{ab} enters h^{ab} (via $W^{ab}[\mathrm{F}]$) with an additional factor of c^{-4}, we observe that the wave-zone contribution to h^{ab} is of order c^{-8}. Relative to $h^{00} = O(c^{-2})$, this is a correction of order c^{-6}, and we conclude that $\chi^{ab}_{\mathscr{W}}$ contributes a 3PN correction to the near-zone potentials. This post-Newtonian order is far beyond the 1PN accuracy of our calculations in this section, and we shall therefore ignore the wave-zone contribution to χ^{ab}.

The wave-zone integral is
$$\chi^{ab}_{\mathscr{W}} = \frac{1}{4\pi}\int_{\mathscr{W}} G(x,x')\partial^{a'}\Phi\partial^{b'}\Phi\, d^4x',$$
where $\Phi(x')$ is the solution to Eq. (4.1.25) evaluated in the wave zone. A relevant expression was obtained in Sec. 3.3.2, and Eq. (3.3.16) gives
$$\Phi = G\biggl[\frac{I}{r} + \frac{1}{2}\partial_{ab}\biggl(\frac{I^{ab}}{r}\biggr) + \cdots\biggr],$$
where $I = \sum_A m_A$ is the total mass and $I^{ab} = \sum_A m_A z_A^a z_A^b$ is the quadrupole moment, a function of retarded-time $\tau = t - r/c$. (We drop the primes to simplify the notation, and we have set the dipole moment $I^a = \sum_A m_A z_A^a$ to zero by placing the origin of the coordinate system at the system's barycentre.) We recall from Sec. 3.3.3 that the monopole term is formally of 0PN order, that the quadrupole term is formally of 1PN order, and that the discarded terms are of higher post-Newtonian order.

Ignoring all numerical and angle-dependent factors, the source term has a structure given schematically by

$$\partial^a\Phi\partial^b\Phi = G^2\biggl[\frac{I^2}{r^4} + \frac{II^{ab}}{r^6} + \frac{I\dot{I}^{ab}}{cr^5} + \frac{I\ddot{I}^{ab}}{c^2r^4} + \frac{II^{ab(3)}}{c^3r^3} + \cdots\biggr].$$

Each term is of the form $f(\tau)/r^n$ required for the integration techniques of Sec. 2.5.3. For example, for $n = 3$ we have $f = G^2 II^{ab(3)}/c^3$, for $n = 4$ we have $f = G^2(I^2 + I\ddot{I}^{ab}/c^2)$, and so on. According to Eq. (2.5.19), an estimate of $\chi^{ab}_{\mathscr{W}}$ for each contributing n is $c^{-(n-2)}f^{(n-2)} + rc^{-(n-1)}f^{(n-1)}$. (The factors of c appear when the u-derivatives of Sec. 2.5.4 are converted into τ-derivatives; recall that $u = c\tau$.) The dominant term in a post-Newtonian expansion is $c^{-(n-2)}f^{(n-2)}$, and we find that for each n, $\chi^{ab}_{\mathscr{W}}$ is estimated as

$$\frac{G^2}{c^4} I \frac{d^4 I^{ab}}{d\tau^4}.$$

This, as was claimed in Eq. (4.2.28), is of order c^{-4}. This result implies that $\chi^{ab}_{\mathscr{W}}$ is too small to contribute to our 1PN potentials, and for this reason we do not need to calculate it in detail.

4.2.9 Computation of W^{ab}: Final answer

Collecting the results of Sec. 4.2.3–4.2.8, we find that W^{ab} is finally given by

$$W^{ab} = P^{ab} + O(c^{-1}), \tag{4.2.29}$$

where $P^{ab} = W^{ab}[\mathrm{M}] + \chi^{ab} - \frac{1}{2}\delta^{ab}\chi$. Here $W^{ab}[\mathrm{M}]$ is given by Eq. (4.2.12), and $\chi^{ab} = \chi^{ab}_{\mathscr{N}} + O(c^{-4})$, with $\chi^{ab}_{\mathscr{N}}$ and its trace displayed in Eqs. (4.2.23) and (4.2.25), respectively. Explicitly,

$$\begin{aligned} P^{ab}(t,\boldsymbol{x}) \quad = \quad & \sum_A \frac{Gm_Av^a_Av^b_A}{|\boldsymbol{x}-\boldsymbol{z}_A|} + \frac{1}{4}\sum_A \frac{G^2m_A^2}{|\boldsymbol{x}-\boldsymbol{z}_A|^2} n^a_A n^b_A \\ & - 2\sum_A\sum_{B>A} \frac{G^2m_Am_B}{S|\boldsymbol{z}_A-\boldsymbol{z}_B|} n^a_{AB} n^b_{AB} \\ & + 2\sum_A\sum_{B>A}\frac{G^2m_Am_B}{S^2}\biggl[\bigl(n^{(a}_A - n^{(a}_{AB}\bigr)\bigl(n^{b)}_B + n^{b)}_{AB}\bigr) \\ & - \frac{1}{2}\delta^{ab}(\boldsymbol{n}_A - \boldsymbol{n}_{AB})\cdot(\boldsymbol{n}_B + \boldsymbol{n}_{AB})\biggr]. \end{aligned} \tag{4.2.30}$$

The unit vectors $\boldsymbol{n}_A$, $\boldsymbol{n}_B$, and $\boldsymbol{n}_{AB}$ were introduced in Eq. (4.2.24). The trace of P^{ab} is given by $P = W[\mathrm{M}] - \frac{1}{2}\chi$; with Eq. (4.2.25), this is

$$\begin{aligned} P(t,\boldsymbol{x}) \quad = \quad & \sum_A \frac{Gm_Av_A^2}{|\boldsymbol{x}-\boldsymbol{z}_A|} + \frac{1}{4}\sum_A\frac{G^2m_A^2}{|\boldsymbol{x}-\boldsymbol{z}_A|^2} - \frac{1}{2}\sum_A\sum_{B>A} G^2m_Am_B \\ & \times\biggl(\frac{1}{|\boldsymbol{x}-\boldsymbol{z}_A|}\frac{1}{|\boldsymbol{z}_A-\boldsymbol{z}_B|} + \frac{1}{|\boldsymbol{x}-\boldsymbol{z}_B|}\frac{1}{|\boldsymbol{z}_A-\boldsymbol{z}_B|} - \frac{1}{|\boldsymbol{x}-\boldsymbol{z}_A||\boldsymbol{x}-\boldsymbol{z}_B|}\biggr). \end{aligned} \tag{4.2.31}$$

For our final expressions we have chosen to symmetrize the double sums $\sum_A\sum_{A\neq B}$, and to rewrite these as sums over pairs $\sum_A\sum_{B>A}$. Consider, for example, the term

$$\sum_A\sum_{B\neq A}\frac{G^2m_Am_B}{S^2}\bigl(n^a_A - n^a_{AB}\bigr)\bigl(n^b_B + n^b_{AB}\bigr)$$

in $\chi^{ab}_{\mathscr{N}}$. By interchanging the identities of A and B we may write this as

$$\sum_A \sum_{B \neq A} \frac{G^2 m_A m_B}{S^2} \bigl(n^a_B - n^a_{BA}\bigr)\bigl(n^b_A + n^b_{BA}\bigr),$$

where we note that S stays unchanged during this operation. Adding these results together and dividing by 2 gives

$$\sum_A \sum_{B \neq A} \frac{G^2 m_A m_B}{S^2} \bigl(n^{(a}_A - n^{(a}_{AB}\bigr)\bigl(n^{b)}_B + n^{b)}_{AB}\bigr),$$

where we used the fact that $\boldsymbol{n}_{BA} = -\boldsymbol{n}_{AB}$. Each term in the sum is now symmetric under the exchange $A \leftrightarrow B$. Each pair of bodies is counted twice in the sum, and to eliminate this redundancy we write it in its final form as

$$2 \sum_A \sum_{B > A} \frac{G^2 m_A m_B}{S^2} \bigl(n^{(a}_A - n^{(a}_{AB}\bigr)\bigl(n^{b)}_B + n^{b)}_{AB}\bigr).$$

This is a sum over pairs, and since each pair is counted only once, there is a factor of 2 to compensate.

4.2.10 Summary: Near-zone potentials

We may finally collect the results obtained in this section and construct the near-zone expressions for the gravitational potentials. Combining Eqs. (4.1.29)–(4.1.31), (4.2.1), (4.2.5), and (4.2.29), we have

$$h^{00} = \frac{4}{c^2} U + \frac{4}{c^4}\biggl(\psi + \frac{1}{2}\frac{\partial^2 X}{\partial t^2} - P + 2U^2\biggr) + O(c^{-5}), \qquad (4.2.32)$$

$$h^{0a} = \frac{4}{c^3} U^a + O(c^{-5}), \qquad (4.2.33)$$

$$h^{ab} = \frac{4}{c^4} P^{ab} + O(c^{-5}), \qquad (4.2.34)$$

where $P = \delta_{ab} P^{ab}$.

The Newtonian potential is given by Eq. (4.2.2),

$$U(t, \boldsymbol{x}) = \sum_A \frac{G m_A}{|\boldsymbol{x} - \boldsymbol{z}_A|}, \qquad (4.2.35)$$

and the 1PN terms in h^{00} were displayed in Eqs. (4.2.3) and (4.2.4):

$$\psi(t, \boldsymbol{x}) = \sum_A \frac{G m_A \bigl(\frac{3}{2} v_A^2 - \lfloor U \rfloor_A\bigr)}{|\boldsymbol{x} - \boldsymbol{z}_A|} \qquad (4.2.36)$$

and

$$X(t, \boldsymbol{x}) = \sum_A G m_A |\boldsymbol{x} - \boldsymbol{z}_A|. \qquad (4.2.37)$$

We recall from Eq. (4.1.13) that

$$\lfloor U \rfloor_A = \sum_{B \neq A} \frac{G m_B}{|\boldsymbol{z}_A - \boldsymbol{z}_B|} \qquad (4.2.38)$$

is the Newtonian potential evaluated at $\boldsymbol{x} = \boldsymbol{z}_A$, excluding the infinite contribution coming from body A. An alternative expression for ψ is

$$\psi(t, \boldsymbol{x}) = \frac{3}{2} \sum_A \frac{G m_A v_A^2}{|\boldsymbol{x} - \boldsymbol{z}_A|} - \sum_A \sum_{B > A} \frac{G^2 m_A m_B}{|\boldsymbol{z}_A - \boldsymbol{z}_B|} \biggl(\frac{1}{|\boldsymbol{x} - \boldsymbol{z}_A|} + \frac{1}{|\boldsymbol{x} - \boldsymbol{z}_B|}\biggr). \qquad (4.2.39)$$

The vector potential is given by Eq. (4.2.6),

$$U^a(t,\boldsymbol{x}) = \sum_A \frac{Gm_A v_A^a}{|\boldsymbol{x}-\boldsymbol{z}_A|}, \tag{4.2.40}$$

and the tensor potential is displayed in Eq. (4.2.30):

$$\begin{aligned} P^{ab}(t,\boldsymbol{x}) \;=\; & \sum_A \frac{Gm_A v_A^a v_A^b}{|\boldsymbol{x}-\boldsymbol{z}_A|} + \frac{1}{4}\sum_A \frac{G^2 m_A^2}{|\boldsymbol{x}-\boldsymbol{z}_A|^2} n_A^a n_A^b \\ & - 2\sum_A \sum_{B>A} \frac{G^2 m_A m_B}{S|\boldsymbol{z}_A-\boldsymbol{z}_B|} n_{AB}^a n_{AB}^b \\ & + 2\sum_A \sum_{B>A} \frac{G^2 m_A m_B}{S^2}\biggl[\bigl(n_A^{(a} - n_{AB}^{(a}\bigr)\bigl(n_B^{b)} + n_{AB}^{b)}\bigr) \\ & - \frac{1}{2}\delta^{ab}(\boldsymbol{n}_A - \boldsymbol{n}_{AB})\cdot(\boldsymbol{n}_B + \boldsymbol{n}_{AB})\biggr], \end{aligned} \tag{4.2.41}$$

with the unit vectors of Eq. (4.2.24),

$$\boldsymbol{n}_A := \frac{\boldsymbol{x}-\boldsymbol{z}_A}{|\boldsymbol{x}-\boldsymbol{z}_A|}, \qquad \boldsymbol{n}_B := \frac{\boldsymbol{x}-\boldsymbol{z}_B}{|\boldsymbol{x}-\boldsymbol{z}_B|}, \qquad \boldsymbol{n}_{AB} := \frac{\boldsymbol{z}_A-\boldsymbol{z}_B}{|\boldsymbol{z}_A-\boldsymbol{z}_B|}, \tag{4.2.42}$$

and the distance function

$$S = |\boldsymbol{x}-\boldsymbol{z}_A| + |\boldsymbol{x}-\boldsymbol{z}_B| + |\boldsymbol{z}_A-\boldsymbol{z}_B| \tag{4.2.43}$$

defined by Eq. (4.2.22).

Finally, the trace of the tensor potential is given by Eq. (4.2.31),

$$\begin{aligned} P(t,\boldsymbol{x}) \;=\; & \sum_A \frac{Gm_A v_A^2}{|\boldsymbol{x}-\boldsymbol{z}_A|} + \frac{1}{4}\sum_A \frac{G^2 m_A^2}{|\boldsymbol{x}-\boldsymbol{z}_A|^2} \\ & - \frac{1}{2}\sum_A \sum_{B>A} \frac{G^2 m_A m_B}{|\boldsymbol{z}_A-\boldsymbol{z}_B|}\biggl(\frac{1}{|\boldsymbol{x}-\boldsymbol{z}_A|} + \frac{1}{|\boldsymbol{x}-\boldsymbol{z}_B|}\biggr) \\ & + \frac{1}{2}\sum_A \sum_{B>A} \frac{G^2 m_A m_B}{|\boldsymbol{x}-\boldsymbol{z}_A||\boldsymbol{x}-\boldsymbol{z}_B|}. \end{aligned} \tag{4.2.44}$$

It may be noticed that ψ and P have a number of terms in common, and that it is the combination $\psi - P$ that enters h^{00}.

4.3 Conservation identities and equations of motion

It was pointed out in Sec. 1.3, and also in Sec. 1.6, that the gravitational potentials $h^{\alpha\beta}$ will satisfy the harmonic gauge conditions (and will therefore satisfy the full set of Einstein field equations) whenever the effective energy-momentum tensor $\tau^{\alpha\beta}$ satisfies the conservation identities $\partial_\beta \tau^{\alpha\beta} = 0$, or more explicitly,

$$\partial_0 \tau^{00} + \partial_a \tau^{0a} = 0, \qquad \partial_0 \tau^{0a} + \partial_b \tau^{ab} = 0. \tag{4.3.1}$$

Because $\tau^{\alpha\beta}$ depends on both matter and field variables, the conservation identities give rise to equations of motion for the matter. In our specific context in which the matter distribution is a system of N point masses, they give rise to equations of

motion for each particle. We shall show in this section that with the effective energy-momentum pseudo tensor of Sec. 4.1.5, we can derive the *Newtonian equations of motion*

$$\boldsymbol{a}_A = -\sum_{B\neq A} \frac{Gm_B(\boldsymbol{z}_A - \boldsymbol{z}_B)}{|\boldsymbol{z}_A - \boldsymbol{z}_B|^3} + O(c^{-2}), \tag{4.3.2}$$

where $\boldsymbol{a}_A := d\boldsymbol{v}_A/dt = d^2\boldsymbol{z}_A/dt^2$ is the acceleration vector of body A. In Chapter 5 we will use the results obtained in this chapter to obtain 1PN corrections to these equations.

We begin with the second of Eqs. (4.3.1), which we write in the form

$$c^{-1}\partial_t \tau^{0a} + \partial_b \tau^{ab} = 0, \tag{4.3.3}$$

and we recall from Sec. 4.1.5 that

$$c^{-1}\tau^{0a} = \sum_A m_A v_A^a \delta\big(\boldsymbol{x} - \boldsymbol{z}_A\big) + O(c^{-2}) \tag{4.3.4}$$

and

$$\tau^{ab} = \sum_A m_A v_A^a v_A^b \delta\big(\boldsymbol{x} - \boldsymbol{z}_A\big) + \frac{1}{4\pi G}\biggl(\partial^a\Phi\partial^b\Phi - \frac{1}{2}\delta^{ab}\partial_c\Phi\partial^c\Phi\biggr) + O(c^{-2}). \tag{4.3.5}$$

Making the substitutions produces

$$\begin{aligned} 0 \quad = \quad & \sum_A m_A a_A^a \delta\big(\boldsymbol{x} - \boldsymbol{z}_A\big) + \sum_A m_A v_A^a \big(\partial_t + v_A^b\partial_b\big)\delta\big(\boldsymbol{x} - \boldsymbol{z}_A\big) + \frac{1}{4\pi G}\partial^a\Phi\nabla^2\Phi \\ & + O(c^{-2}), \end{aligned}$$

in which we may replace Φ by the Newtonian potential U. The second sum vanishes by virtue of the distributional identity $(\partial_t + v_A^b\partial_b)\delta(\boldsymbol{x} - \boldsymbol{z}_A) = 0$, and we obtain

$$0 = \sum_A m_A \Bigl(a_A^a - \lfloor\partial^a U\rfloor_A\Bigr)\delta\big(\boldsymbol{x} - \boldsymbol{z}_A\big) + O(c^{-2})$$

after involving Poisson's equation $\nabla^2 U = -4\pi G\sum_A m_A\delta(\boldsymbol{x} - \boldsymbol{z}_A)$. Here $\lfloor\partial^a U\rfloor_A$ is formally the derivative of the Newtonian potential evaluated at $\boldsymbol{x} = \boldsymbol{z}_A$. We have

$$\partial^a U = -\sum_B \frac{Gm_B\big(\boldsymbol{x} - \boldsymbol{z}_B\big)^a}{|\boldsymbol{x} - \boldsymbol{z}_B|^3},$$

and this evidently diverges at $\boldsymbol{x} = \boldsymbol{z}_A$. As a plausible extension of our prescription of Eq. (4.1.12), we regularize $\partial^a U$ simply by excluding body A from the sum over B. This yields Eq. (4.3.2), the Newtonian equations of motion for a system of N bodies subjected to their mutual gravitational attractions.

The method of derivation that leads to Eq. (4.3.2) is not entirely satisfactory, because it requires an additional regularization rule beyond the one already introduced in Eq. (4.1.12). We shall do better in Chapter 5, and derive the 1PN equations of motion without invoking additional (and ad-hoc) regularization prescriptions. The considerations of this section are still useful, however, because they reveal in a direct manner the connection between the conservation identities and the concrete form of the equations of motion. They also close a loophole left open in Sec. 4.2.2, in which the Newtonian equations of motion were assumed to hold.

One might ask whether it may not be possible to establish Eq. (4.3.2) more cleanly by dealing instead with the gauge conditions $\partial_0 h^{0a} + \partial_b h^{ab} = 0$. The answer is in the affirmative, but this approach would require lengthy computations, and we choose not to pursue this here.

It may be verified that the first of Eqs. (4.3.1) eventually produces the statement

$$0 = \boldsymbol{v}_A \cdot \biggl[\boldsymbol{a}_A + \sum_{B \neq A} \frac{G m_B (\boldsymbol{z}_A - \boldsymbol{z}_B)}{|\boldsymbol{z}_A - \boldsymbol{z}_B|^3} \biggr] + O(c^{-2}),$$

which is already implied by Eq. (4.3.2). This is evidently a statement of the work-energy theorem for our system of N bodies.

4.4 Wave-zone expressions

4.4.1 Computation of V

The wave equation for V was written down in Eq. (4.1.33), and we copy it as

$$\Box V = -4\pi G \sum_A m_A^* \delta(\boldsymbol{x} - \boldsymbol{z}_A) + O(c^{-4}), \tag{4.4.1}$$

in terms of the "augmented mass parameters"

$$m_A^* = m_A \biggl(1 + \frac{3}{2} \frac{v_A^2}{c^2} - \frac{\lfloor U \rfloor_A}{c^2} \biggr), \tag{4.4.2}$$

which are in fact functions of time. Recall that $\lfloor U \rfloor_A$ is the (regularized) Newtonian potential evaluated at $\boldsymbol{x} = \boldsymbol{z}_A$, as given by Eq. (4.1.13). Notice that the source term is confined to the near zone, so that $V_{\mathscr{W}} = 0$ and $V = V_{\mathscr{N}}$.

Methods to integrate Eq. (4.4.1) in the wave zone were described in Sec. 2.4.1, and according to Eq. (2.4.3) we have

$$V = G \biggl[\frac{I_*}{r} - \partial_a \biggl(\frac{I_*^a}{r} \biggr) + \frac{1}{2} \partial_{ab} \biggl(\frac{I_*^{ab}}{r} \biggr) - \frac{1}{6} \partial_{abc} \biggl(\frac{I_*^{abc}}{r} \biggr) + \cdots \biggr], \tag{4.4.3}$$

where the "augmented multipole moments"

$$I_* = \sum_A m_A^*, \tag{4.4.4}$$

$$I_*^a = \sum_A m_A^* z_A^a, \tag{4.4.5}$$

$$I_*^{ab} = \sum_A m_A^* z_A^a z_A^b, \tag{4.4.6}$$

$$I_*^{abc} = \sum_A m_A^* z_A^a z_A^b z_A^c \tag{4.4.7}$$

are functions of retarded time $\tau := t - r/c$. Recalling the discussion of Sec. 3.3.3, we observe that the monopole term involving I_* gives a contribution at 0PN order to V, but that since I_* includes a 1PN correction to the "bare mass parameters" m_A, this term contributes also at 1PN order. Similarly, the quadrupole term involving I_*^{ab} gives a contribution at 1PN order together with a correction at 2PN order. And the octupole term involving I_*^{abc} gives a contribution at $\frac{3}{2}$PN order together with a correction at $\frac{5}{2}$PN order. For an expression accurate to $\frac{3}{2}$PN order, we can ignore the $O(c^{-2})$ corrections within I_*^{ab} and I_*^{abc}.

The dipole term involving I_*^a requires a separate discussion. This, potentially, would contribute a leading term at $\frac{1}{2}$PN order and a correction at $\frac{3}{2}$PN order. We should expect, however, that conservation identities will eliminate the leading-order contribution. As we shall see presently, this expectation is indeed correct.

Let us examine the multipole moments more closely. We first define

$$M := \sum_A \biggl(1 + \frac{1}{2}\frac{v_A^2}{c^2} - \frac{1}{2}\frac{\lfloor U \rfloor_A}{c^2}\biggr) + O(c^{-4}) \tag{4.4.8}$$

as the *total post-Newtonian gravitational mass* of the N-body system. It is easy to verify that M is conserved by virtue of the Newtonian equations of motion, Eq. (4.3.2). And indeed, Mc^2 is easily recognized as the total energy of the system, including rest-mass energy, kinetic energy, and gravitational potential energy. Next we define

$$\boldsymbol{Z} := \frac{1}{M}\sum_A m_A \boldsymbol{z}_A \biggl(1 + \frac{1}{2}\frac{v_A^2}{c^2} - \frac{1}{2}\frac{\lfloor U \rfloor_A}{c^2}\biggr) + O(c^{-4}) \tag{4.4.9}$$

as the *position vector of the post-Newtonian barycentre* (also known as centre of mass). We shall verify in Chapter 5 that $\boldsymbol{Z}$ is conserved by virtue of the 1PN equations of motion for the N-body system. Placing the origin of the coordinate system at the barycentre, we can set $\boldsymbol{Z} = \boldsymbol{0}$. Next we reintroduce

$$I^{ab}(\tau) := \sum_A m_A z_A^a z_A^b + O(c^{-2}) \tag{4.4.10}$$

and

$$I^{abc}(\tau) := \sum_A m_A z_A^a z_A^b z_A^c + O(c^{-2}) \tag{4.4.11}$$

as the Newtonian quadrupole and multipole moments, respectively. And finally, we reintroduce the Newtonian angular-momentum tensor

$$J^{ab} := \sum_A m_A \bigl(v_A^a z_A^b - z_A^a v_A^b\bigr) + O(c^{-2}) \tag{4.4.12}$$

and its first moment

$$J^{abc}(\tau) := \sum_A m_A \bigl(v_A^a z_A^b - z_A^a v_A^b\bigr) z_A^c + O(c^{-2}); \tag{4.4.13}$$

these were first encountered in Sec. 3.3.1. As was indicated in the definitions, M, $\boldsymbol{Z}$, and J^{ab} are conserved quantities, while I^{ab}, I^{abc}, and J^{abc} are functions of retarded time.

The augmented moments I_* and I_*^a can be expressed in terms of these fundamental quantities. To begin, we find from Eqs. (4.4.2)–(4.4.4) and (4.4.8) that

$$I_* = M + \frac{1}{c^2}\sum_A m_A \biggl(v_A^2 - \frac{1}{2}\lfloor U \rfloor_A\biggr) + O(c^{-4}).$$

The second term, however, can easily be related to the second time derivative of the Newtonian quadrupole moment. From Eq. (4.4.10) we have

$$\ddot{I}^{ab} = \sum_A m_A \bigl(2 v_A^a v_A^b + z_A^a a_A^b + a_A^a z_A^b\bigr) + O(c^{-2}),$$

which leads to

$$\ddot{I}^{cc} = 2\sum_A m_A \bigl(v_A^2 + \boldsymbol{a}_A \cdot \boldsymbol{z}_A\bigr) + O(c^{-2})$$

for its trace (summation over c is implied). Involving now the Newtonian equations of motion, Eq. (4.3.2), we find that

$$\sum_A m_A \boldsymbol{a}_A \cdot \boldsymbol{z}_A = -\sum_A \sum_{B \neq A} \frac{G m_A m_B}{|\boldsymbol{z}_A - \boldsymbol{z}_B|^3}(\boldsymbol{z}_A - \boldsymbol{z}_B)\cdot \boldsymbol{z}_A + O(c^{-2}).$$

After symmetrization of the double sum (see Sec. 4.2.9), this becomes

$$\sum_A m_A \boldsymbol{a}_A \cdot \boldsymbol{z}_A = -\frac{1}{2}\sum_A \sum_{B\neq A} \frac{Gm_A m_B}{|\boldsymbol{z}_A - \boldsymbol{z}_B|} + O(c^{-2}) = -\frac{1}{2}\sum_A m_A \lfloor U \rfloor_A + O(c^{-2}).$$

Collecting results, we have obtained

$$I_* = M + \frac{1}{2c^2}\ddot{I}^{cc} + O(c^{-2}), \tag{4.4.14}$$

in which $I^{cc} := \delta_{bc} I^{bc}$ is the trace of the Newtonian quadrupole moment. Following similar steps we also find that

$$I_*^a = MZ^a + \frac{1}{6c^2}\bigl(\ddot{I}^{acc} + 4\dot{J}^{cac}\bigr) + O(c^{-4}), \tag{4.4.15}$$

in which $I^{acc} := \delta_{bc} I^{abc}$ and $J^{cac} := \delta_{bc} J^{bac}$.

Making these substitutions into our previous expression for V, we finally obtain

$$\begin{aligned} V \;\; = \;\; G\Biggl[& \underbrace{\frac{M}{r}}_{\text{0PN+1PN}} + \underbrace{\frac{1}{2c^2}\frac{\ddot{I}^{cc}}{r}}_{\text{1PN}} + \underbrace{\frac{1}{2}\partial_{ab}\biggl(\frac{I^{ab}}{r}\biggr)}_{\text{1PN}} - \underbrace{\frac{1}{6}\partial_{abc}\biggl(\frac{I^{abc}}{r}\biggr)}_{\frac{3}{2}\text{PN}} \\ & - \underbrace{\frac{1}{6c^2}\partial_a\biggl(\frac{\ddot{I}^{acc} + 4\dot{J}^{cac}}{r}\biggr)}_{\frac{3}{2}\text{PN}} - \underbrace{\partial_a\biggl(\frac{MZ^a}{r}\biggr)}_{=0} + \cdots \Biggr], \end{aligned} \tag{4.4.16}$$

in which we indicate the post-Newtonian order of each term, and the fact that $\boldsymbol{Z}$ can be set equal to zero by placing the origin of the coordinate system at the system's barycentre.

4.4.2 Computation of V^a

The wave equation for V^a was written down in Eq. (4.1.34),

$$\Box V^a = -4\pi G \sum_A m_A v_A^a \delta\bigl(\boldsymbol{x} - \boldsymbol{z}_A\bigr) + O(c^{-2}), \tag{4.4.17}$$

and we notice that this is the same as Eq. (3.1.11) if we replace A^a by V^a and set $j^a = \sum_A m_A v_A^a \delta(\boldsymbol{x} - \boldsymbol{z}_A)$. The wave-zone solution to Eq. (3.1.11) was displayed in Eq. (3.3.17), and expressed in terms of the same multipole moments that were introduced in Eqs. (4.4.10)–(4.4.13). We copy this expression here,

$$V^a = G\biggl[\underbrace{\frac{1}{2}J^{ab}\frac{\Omega_b}{r^2}}_{\text{1PN}} - \underbrace{\frac{1}{2}\partial_b\biggl(\frac{\dot{I}^{ab}}{r}\biggr)}_{\text{1PN}} + \cdots\biggr], \tag{4.4.18}$$

and indicate the post-Newtonian order of each term. (Recall that the relation between h^{0a} and V^a involves a factor of c^{-3}, while the relation between h^{00} and V involves a factor of c^{-2}.)

Once more a conservation identity was invoked to eliminate a potential contribution at $\frac{1}{2}$PN order. Indeed, the leading term in the multipole expansion for V^a should have been

$$\frac{G}{r}\int j^a \, d^3x' = \frac{G}{r}\sum_A m_A v_A^a = \frac{G}{r}P^a,$$

where $\boldsymbol{P}$ is the total Newtonian momentum of the system. Having placed the origin of the coordinate system at the (post-Newtonian) barycentre, we have set $\boldsymbol{P} = \boldsymbol{0} + O(c^{-2})$, and the leading term vanishes. The correction of order c^{-2} contributes a term at $\frac{3}{2}$PN order, and all such terms have been discarded in Eq. (4.4.18).

4.4.3 Computation of W^{ab}: Organization

The wave equation for W^{ab} was written down in Eq. (4.1.35), and we copy it as

$$\Box W^{ab} = -4\pi G \tau^{ab}, \tag{4.4.19}$$

where the effective stress tensor

$$\tau^{ab} = \sum_A m_A v_A^a v_A^b \delta(\boldsymbol{x} - \boldsymbol{z}_A) + \frac{1}{4\pi G}\biggl(\partial^a\Phi\partial^b\Phi - \frac{1}{2}\delta^{ab}\partial_c\Phi\partial^c\Phi\biggr) + O(c^{-2}) \tag{4.4.20}$$

contains both a matter and a field contribution. Recall that Φ is the retarded solution to Eq. (4.1.25). In the near zone,

$$\Phi = U + O(c^{-2}) \qquad \text{(near zone)}, \tag{4.4.21}$$

where U is the Newtonian potential given by Eq. (4.1.10). In the wave zone Φ can be expressed as the multipole expansion of Eq. (3.3.16); for our purposes here, it is sufficient to take

$$\Phi = \frac{GM}{r} + \cdots \qquad \text{(wave zone)}, \tag{4.4.22}$$

in which M differs from the actual monopole moment $I = \sum_A m_A$ by post-Newtonian corrections that are discarded.

The source term of Eq. (4.4.20) is distributed over all space, and as a consequence, the retarded integral for W^{ab} contains both a near-zone and a wave-zone contribution:

$$W^{ab} = W^{ab}_{\mathscr{N}} + W^{ab}_{\mathscr{W}}. \tag{4.4.23}$$

In the following two subsections we will endeavour to compute $W^{ab}_{\mathscr{N}}$ and $W^{ab}_{\mathscr{W}}$ so as to finally obtain a complete expression for W^{ab}.

4.4.4 Computation of W^{ab}: Near-zone integral

Methods to integrate Eq. (4.4.19) in the wave zone were described in Secs. 2.4.1 and 2.5.2, and according to Eq. (2.4.3) the near-zone contribution to the retarded integral is

$$W^{ab}_{\mathscr{N}} = G\biggl[\frac{1}{r}\int_{\mathscr{M}} \tau^{ab}\, d^3x' - \partial_c\biggl(\frac{1}{r}\int_M \tau^{ab}x'^c\, d^3x'\biggr) + \cdots\biggr], \tag{4.4.24}$$

in which τ^{ab} is expressed as a function of retarded time $u := ct - r$ and spatial coordinates $\boldsymbol{x'}$. The domain of integration $\mathscr{M}$ is a surface of constant time bounded externally by the sphere $r' := |\boldsymbol{x'}| = \mathcal{R}$.

Evaluation of the integrals is simplified by involving the conservation identities of Eqs. (1.4.3) and (1.4.4),

$$\tau^{ab} = \frac{1}{2}\partial_{00}\bigl(\tau^{00}x^a x^b\bigr) + \frac{1}{2}\partial_c\bigl(\tau^{ac}x^b + \tau^{bc}x^a - \partial_d\tau^{cd}x^a x^b\bigr) \tag{4.4.25}$$

and

$$\begin{aligned} \tau^{ab}x^c &= \frac{1}{2}\partial_0\bigl(\tau^{0a}x^b x^c + \tau^{0b}x^a x^c - \tau^{0c}x^a x^b\bigr) \\ &\quad + \frac{1}{2}\partial_d\bigl(\tau^{ad}x^b x^c + \tau^{bd}x^a x^c - \tau^{cd}x^a x^b\bigr). \end{aligned} \tag{4.4.26}$$

Here,

$$c^{-2}\tau^{00} = \sum_A m_A \delta(\boldsymbol{x} - \boldsymbol{z}_A) + O(c^{-2}) \tag{4.4.27}$$

and

$$c^{-1}\tau^{0a} = \sum_A m_A v_A^a \delta(\boldsymbol{x} - \boldsymbol{z}_A) + O(c^{-2}) \tag{4.4.28}$$

are the remaining components of the effective energy-momentum pseudotensor, written to a degree of accuracy that will be sufficient for our purposes. Integrating Eqs. (4.4.25) and (4.4.26) over $\mathscr{M}$ and inserting the definitions of Eqs. (4.4.10)–(4.4.13) produces

$$\int_{\mathscr{M}} \tau^{ab}\, d^3x' = \frac{1}{2}\ddot{I}^{ab} + \frac{1}{2}\oint_{\partial\mathscr{M}} \bigl(\tau^{ac}x'^b + \tau^{bc}x'^a - \partial_{d'}\tau^{cd}x'^a x'^b\bigr)\, dS'_c + O(c^{-2}) \tag{4.4.29}$$

and

$$\begin{aligned}\int_{\mathscr{M}} \tau^{ab}x'^c\, d^3x' \quad &= \quad \frac{1}{6}\dddot{I}^{abc} + \frac{1}{3}\bigl(\dot{J}^{acb} + \dot{J}^{bca}\bigr) \\ &\quad + \frac{1}{2}\oint_{\partial\mathscr{M}} \bigl(\tau^{ad}x'^b x'^c + \tau^{bd}x'^a x'^c - \tau^{cd}x'^a x'^b\bigr)\, dS'_d \\ &\quad + O(c^{-2}).\end{aligned} \tag{4.4.30}$$

Here, the multipole moments are expressed in terms of $\tau := t - r/c$ and an overdot indicates differentiation with respect to τ; the surface integrals are over the sphere $r' = \mathcal{R}$, and the surface element is $dS'_a = \mathcal{R}^2\Omega'_a d\Omega'$, in which $\Omega'^a := x'^a/r'$ and $d\Omega' = \sin\theta'\, d\theta' d\phi'$ is an element of solid angle.

The surface integrals in Eqs. (4.4.29) and (4.4.30) are evaluated outside the matter distribution, at the boundary $\partial\mathscr{M}$ between the near and wave zones. They involve only the field contribution to τ^{ab}, and we may use Eq. (4.4.22) to calculate

$$\tau^{ab} = \frac{GM^2}{4\pi r'^4}\left(\Omega'^a\Omega'^b - \frac{1}{2}\delta^{ab}\right) + \cdots. \tag{4.4.31}$$

This expression is valid everywhere in the wave zone, but in order to evaluate the surface integrals we must set $r' = \mathcal{R}$. It is then easy to see that an integral such as $\oint \tau^{ac}x'^b\, dS'_c$ is proportional to $\mathcal{R}^{-1}$, and that an integral such as $\oint \tau^{ad}x'^b x'^c\, dS'_d$, which might have given a result independent of $\mathcal{R}$, actually vanishes because it involves an odd number of angular vectors $\boldsymbol{\Omega}$ (see Sec. 1.8.4). Because we are free to ignore all $\mathcal{R}$-dependent terms in the near-zone and wave-zone contributions to W^{ab} (such terms will cancel out — see the end of Sec. 2.3), we are therefore free to ignore the surface integrals in Eqs. (4.4.29) and (4.4.30).

Inserting Eqs. (4.4.29) and (4.4.30) into Eq. (4.4.24) produces

$$W^{ab}_{\mathscr{N}} = G\Biggl[\underbrace{\frac{1}{2}\frac{\ddot{I}^{ab}}{r}}_{1\mathrm{PN}} - \underbrace{\frac{1}{6}\partial_c\left(\frac{\dddot{I}^{abc} + 2\ddot{J}^{acb} + 2\ddot{J}^{bca}}{r}\right)}_{\frac{3}{2}\mathrm{PN}} + \cdots\Biggr], \tag{4.4.32}$$

in which we indicate the post-Newtonian order of each term. (Recall that the relation between h^{ab} and W^{ab} involves a factor of c^{-4}, while the relation between h^{00} and V involves a factor of c^{-2}.) Notice that Eq. (4.4.32) appears to be identical to Eq. (3.3.18) for the tensor potential B^{ab}, which corresponds to W^{ab} in the first post-Minkowskian approximation to the gravitational potentials. The similarity is deceptive, however, because our $W^{ab}_{\mathscr{N}}$ is truly a second post-Minkowskian approximation; the apparently absent second factor of G appears when the multipole moments are differentiated with respect to τ and Eq. (4.3.2) is inserted within terms involving the Newtonian accelerations $\boldsymbol{a}_A$.

4.4.5 Computation of W^{ab}: Wave-zone integral

Methods to calculate the wave-zone contribution $W^{ab}_{\mathscr{W}}$ were described in Sec. 2.5.2. These methods work for source terms of the form displayed in Eq. (2.5.2), and our first task is to decompose the effective stress tensor of Eq. (4.4.31) in terms of STF angular tensors (see Sec. 1.8.1). We therefore involve the identity $\Omega^a\Omega^b = \Omega^{\langle ab\rangle} + \frac{1}{3}\delta^{ab}$ and rewrite Eq. (4.4.31) as

$$G\tau^{ab} = \frac{G^2M^2}{4\pi r'^4}\biggl(\Omega'^{\langle ab\rangle} - \frac{1}{6}\delta^{ab}\biggr) + \cdots. \tag{4.4.33}$$

This is of the form of Eq. (2.5.2), with $G\tau^{ab}$ playing the role of the source function μ, and we identify $f_{\ell=2} = G^2M^2$ and $f_{\ell=0} = -\frac{1}{6}G^2M^2\delta^{ab}$. In each case we have that f is a constant, and for both contributions we have $n = 4$.

The contribution to $W^{ab}_{\mathscr{W}}$ from each value of ℓ is given by Eq. (2.5.16), which we copy as

$$W^{ab}_{\mathscr{W}} = \frac{\Omega^{\langle L\rangle}}{r}\biggl\{\int_0^{\mathcal{R}} ds f(u-2s)A(s,r) + \int_{\mathcal{R}}^{\infty} ds f(u-2s)B(s,r)\biggr\},$$

where $A(s,r) = \int_{\mathcal{R}}^{r+s} P_\ell(\xi)p^{-(n-1)}\,dp$, $B(s,r) = \int_s^{r+s} P_\ell(\xi)p^{-(n-1)}\,dp$, and $\xi = (r+2s)/r - 2s(r+s)/(rp)$. Because f is a constant it can be taken outside of each integral, and the remaining computations are simple. For $\ell = 2$ we find

$$W^{ab}_{\mathscr{W}} = \biggl(\frac{G^2M^2}{4r^2} - \frac{G^2M^2\mathcal{R}}{5r^3}\biggr)\Omega^{\langle ab\rangle} \qquad (\ell = 2),$$

and for $\ell = 0$ we find

$$W^{ab}_{\mathscr{W}} = \biggl(\frac{G^2M^2}{12r^2} - \frac{G^2M^2}{6\mathcal{R}r}\biggr)\delta^{ab} \qquad (\ell = 0).$$

Adding the results, we arrive at

$$W^{ab}_{\mathscr{W}} = \frac{G^2M^2}{4r^2}\biggl(\Omega^{\langle ab\rangle} + \frac{1}{3}\delta^{ab}\biggr) + \cdots$$

after discarding (as we are free to do) all terms involving $\mathcal{R}$.

We express our final answer as

$$W^{ab}_{\mathscr{W}} = \frac{G^2M^2}{4r^2}\Omega^a\Omega^b + \cdots. \tag{4.4.34}$$

The post-Newtonian order of this contribution to W^{ab} is $\frac{3}{2}$PN. To see this, we multiply $W^{ab}_{\mathscr{W}}$ by c^{-4} to form h^{ab}, and we divide by $h^{00} \sim GM/(c^2r)$ to obtain a quantity of the form $GM/(c^2r)$. We next notice that the Newtonian acceleration GM/r_c^2 is of order r_c/t_c^2, which makes GM of order r_c^3/t_c^2. Setting $r \sim \lambda_c \sim ct_c$, we finally get $h^{ab}/h^{00} \sim r_c^3/(c^3t_c^3) \sim (v_c/c)^3$, and we conclude that Eq. (4.4.34) is indeed a contribution of $\frac{3}{2}$PN order.

4.4.6 Computation of W^{ab}: Final answer

Adding Eq. (4.4.34) to Eq. (4.4.32) yields

$$W^{ab} = G\biggl[\underbrace{\frac{1}{2}\frac{\ddot{I}^{ab}}{r}}_{\text{1PN}} - \underbrace{\frac{1}{6}\partial_c\biggl(\frac{\ddot{I}^{abc} + 2\dot{J}^{acb} + 2\dot{J}^{bca}}{r}\biggr)}_{\frac{3}{2}\text{PN}} + \underbrace{\frac{GM^2}{4r^2}\Omega^a\Omega^b}_{\frac{3}{2}\text{PN}} + \cdots\biggr], \tag{4.4.35}$$

our final expression for the tensor potential. Its trace is given by

$$W = G\biggl[\frac{1}{2}\frac{\ddot{I}^{cc}}{r} - \frac{1}{6}\partial_a\biggl(\frac{\ddot{I}^{acc} + 4\dot{J}^{cac}}{r}\biggr) + \frac{GM^2}{4r^2} + \cdots\biggr]. \tag{4.4.36}$$

4.4.7 Summary: Wave-zone potentials

We may finally collect the results obtained in this section and construct the wave-zone expressions for the gravitational potentials. Combining Eqs. (4.1.29)–(4.1.31), (4.4.16), (4.4.18), (4.4.22), (4.4.35), and (4.4.36), we have

$$\begin{aligned} h^{00} &= \frac{4G}{c^2}\biggl[\underbrace{\frac{M}{r}}_{\text{0PN+1PN}} + \underbrace{\frac{1}{2}\partial_{ab}\biggl(\frac{I^{ab}}{r}\biggr)}_{\text{1PN}} - \underbrace{\frac{1}{6}\partial_{abc}\biggl(\frac{I^{abc}}{r}\biggr)}_{\frac{3}{2}\text{PN}} + \underbrace{\frac{7}{4}\frac{GM^2}{c^2r^2}}_{\frac{3}{2}\text{PN}} \\ &\quad \underbrace{- \partial_a\biggl(\frac{MZ^a}{r}\biggr)}_{=0} + \cdots \biggr], \end{aligned} \tag{4.4.37}$$

$$h^{0a} = \frac{4G}{c^3}\biggl[\underbrace{\frac{1}{2}J^{ab}\frac{\Omega_b}{r^2}}_{\text{1PN}} - \underbrace{\frac{1}{2}\partial_b\biggl(\frac{\dot{I}^{ab}}{r}\biggr)}_{\text{1PN}} + \cdots \biggr], \tag{4.4.38}$$

$$h^{ab} = \frac{4G}{c^4}\biggl[\underbrace{\frac{1}{2}\frac{\ddot{I}^{ab}}{r}}_{\text{1PN}} - \underbrace{\frac{1}{6}\partial_c\biggl(\frac{\ddot{I}^{abc} + 2\dot{J}^{acb} + 2\dot{J}^{bca}}{r}\biggr)}_{\frac{3}{2}\text{PN}} + \underbrace{\frac{GM^2}{4r^2}\Omega^a\Omega^b}_{\frac{3}{2}\text{PN}} + \cdots \biggr]. \tag{4.4.39}$$

The potentials are expressed in terms of $\Omega^a = x^a/r$, and in terms of multipole moments that depend on retarded time $\tau = t - r/c$; overdots indicate differentiation with respect to τ. It is instructive to compare these expressions with Eqs.(1.5.18) and (1.5.19), which give the gravitational potentials for a *static and spherically-symmetric* mass distribution; notice the agreement between all terms that involve the total mass M.

The multipole moments were defined by Eqs. (4.4.8)–(4.4.13): We have the total gravitational mass

$$M = \sum_A \biggl(1 + \frac{1}{2}\frac{v_A^2}{c^2} - \frac{1}{2}\frac{\lfloor U \rfloor_A}{c^2} \biggr) + O(c^{-4}), \tag{4.4.40}$$

the barycentre's position vector

$$\boldsymbol{Z} = \frac{1}{M}\sum_A m_A \boldsymbol{z}_A \biggl(1 + \frac{1}{2}\frac{v_A^2}{c^2} - \frac{1}{2}\frac{\lfloor U \rfloor_A}{c^2} \biggr) + O(c^{-4}), \tag{4.4.41}$$

the mass quadrupole moment

$$I^{ab}(\tau) = \sum_A m_A z_A^a z_A^b + O(c^{-2}), \tag{4.4.42}$$

the mass octupole moment

$$I^{abc}(\tau) = \sum_A m_A z_A^a z_A^b z_A^c + O(c^{-2}), \tag{4.4.43}$$

the angular-momentum tensor

$$J^{ab} = \sum_A m_A \bigl(v_A^a z_A^b - z_A^a v_A^b \bigr) + O(c^{-2}), \tag{4.4.44}$$

and the current moment

$$J^{abc}(\tau) = \sum_A m_A \bigl(v_A^a z_A^b - z_A^a v_A^b \bigr) z_A^c + O(c^{-2}). \tag{4.4.45}$$

We recall that M, $\boldsymbol{Z}$, and J^{ab} are conserved quantities, and that $\lfloor U \rfloor_A$ is defined by Eq. (4.1.13).

The multipole moments must be differentiated a number of times with respect to τ when they are substituted into the gravitational potentials. These operations produce terms involving the acceleration vectors $\boldsymbol{a}_A = d\boldsymbol{v}_A/dt = d^2\boldsymbol{z}_A/dt^2$. These can be expressed in terms of the position vectors via Eq. (4.3.2),

$$\boldsymbol{a}_A = -\sum_{B \neq A} \frac{Gm_B(\boldsymbol{z}_A - \boldsymbol{z}_B)}{|\boldsymbol{z}_A - \boldsymbol{z}_B|^3} + O(c^{-2}), \tag{4.4.46}$$

the Newtonian expression for the acceleration of each body.

Chapter 5
Equations of motion

In this chapter we import the near-zone potentials calculated in Chapter 4 and derive the equations of motion for a system of N bodies moving under their mutual gravitational attraction. We follow the general method devised by Itoh, Futamase, and Asada (2002), which is based on conservation identities formulated on a spherical boundary surrounding each body. Our final expression for the acceleration vector includes the leading-order, Newtonian term, as well as the first post-Newtonian correction, which is smaller by a numerical factor of order $(v/c)^2$. The general framework is described in Sec. 5.1, and its implementation is carried out in Secs. 5.2–5.4; the final answer for the acceleration vector of each body is displayed in Eq. (5.4.17). In Sec. 5.5 we specialize our results to a two-body system, and formulate the equations of motion in terms of barycentric and relative-position variables.

5.1 Conservation identities and laws of motion

Let V_A be a spherical, three-dimensional ball centered on body A, bounded by a two-sphere S_A described by

$$S_A : \qquad |\boldsymbol{x} - \boldsymbol{z}_A| =: s_A = \text{constant}. \tag{5.1.1}$$

The two-sphere moves rigidly with body A, with a velocity $\boldsymbol{v}_A = d\boldsymbol{z}_A/dt$, and our central goal in this chapter is to find equations of motion for $\boldsymbol{z}_A(t)$, the position vector of body A.

Adopting the general strategy developed in Sec. 1.2, we define a momentum four-vector P^α_A associated with body A by

$$P^\alpha_A := \frac{1}{c}\int_{V_A} (-g)\big(T^{0\alpha} + t^{0\alpha}_{\rm LL}\big)\, d^3x, \tag{5.1.2}$$

where $T^{\alpha\beta}$ is the material energy-momentum tensor and $t^{\alpha\beta}_{\rm LL}$ the Landau-Lifshitz pseudotensor. Following the developments leading to Eq. (1.2.2), we use the Einstein field equations in their Landau-Lifshitz form of Eq. (1.1.4) to express this as

$$P^\alpha_A = \frac{c^3}{16\pi G}\oint_{S_A} \partial_\mu H^{\alpha\mu 0c}\, dS_c, \tag{5.1.3}$$

where

$$H^{\alpha\mu\beta\nu} = \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu}, \tag{5.1.4}$$

and where dS_c is an outward-directed surface element on S_A. We recall that the gravitational potentials are related to the "gothic inverse metric" by Eq. (1.3.2), $\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$, where $\eta^{\alpha\beta}$ is the Minkowski metric.

In a time interval dt the volume $V_A(t)$ moves to a new volume $V_A(t+dt)$. In the course of this motion, an element of area $d\boldsymbol{S}$ on S_A sweeps out a volume $dV = (\boldsymbol{v}_A dt)\cdot d\boldsymbol{S}$, because S_A moves with a velocity $\boldsymbol{v}_A$. It follows that in the course of this motion, any quantity defined by

$$F(t) := \int_{V_A} f(t,\boldsymbol{x})\, d^3x \tag{5.1.5}$$

will change according to

$$\frac{dF}{dt} = \int_{V_A} \frac{\partial f}{\partial t}\, d^3x + \oint_{S_A} f\boldsymbol{v}_A\cdot d\boldsymbol{S}; \tag{5.1.6}$$

the first term on the right-hand side accounts for the changes intrinsic to the function $f(t,\boldsymbol{x})$, while the second term accounts for the change in the domain of integration.

We apply Eq. (5.1.6) to the momentum four-vector of Eq. (5.1.2). Recalling that $x^0 = ct$, we have

$$\frac{dP^\alpha_A}{dx^0} = \frac{1}{c}\int_{V_A} \partial_0\Big[(-g)\big(T^{0\alpha} + t^{0\alpha}_{\rm LL}\big)\Big]\, d^3x + \frac{1}{c^2}\oint_{S_A} (-g)\big(T^{0\alpha} + t^{0\alpha}_{\rm LL}\big)v^c_A dS_c,$$

and invoking the conservation statement of Eq. (1.1.7),

$$\partial_\beta\Big[(-g)\big(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}\big)\Big] = 0,$$

we replace the volume integral by another boundary integral. Because $T^{\alpha\beta} = 0$ on S_A, we arrive at

$$\frac{dP^\alpha_A}{dt} = -\oint_{S_A} (-g)\left(t^{\alpha c}_{\rm LL} - t^{0\alpha}_{\rm LL}\frac{v^c_A}{c}\right) dS_c, \tag{5.1.7}$$

a generalization of Eq. (1.2.3) for moving boundaries. Notice that $\boldsymbol{v}_A$ is constant over S_A, and that it can be taken outside the integral.

We next define a dipole-moment vector $\boldsymbol{D}_A$ associated with body A by

$$D^a_A := \frac{1}{c^2}\int_{V_A} (-g)\big(T^{00} + t^{00}_{\rm LL}\big)\big(x^a - z^a_A\big)\, d^3x. \tag{5.1.8}$$

Using the field equations, as well as the symmetry properties of $H^{\alpha\mu\beta\nu}$, we write

$$(-g)\big(T^{00} + t^{00}_{\rm LL}\big) = \frac{c^4}{16\pi G}\partial_{\mu\nu}H^{0\mu0\nu} = \frac{c^4}{16\pi G}\partial_{cd}H^{0c0d}$$

and we integrate by parts. This leads to the alternative expression

$$D^a_A = \frac{c^2}{16\pi G}\oint_{S_A} \Big[\big(\partial_d H^{0c0d}\big)\big(x^a - z^a_A\big) - H^{0a0c}\Big]\, dS_c \tag{5.1.9}$$

for the dipole moment, which involves a surface integral instead of a volume integral.

Returning to Eq. (5.1.8), we differentiate D^a_A with respect to x^0 and involve Eq. (5.1.6). We have

$$\begin{aligned}
\frac{dD^a_A}{dx^0} \quad = \quad & \frac{1}{c^2}\int_{V_A} \partial_0\Big[(-g)\big(T^{00} + t^{00}_{\rm LL}\big)\Big]\big(x^a - z^a_A\big)\, d^3x \\
& - \frac{v^a_A}{c^3}\int_{V_A} (-g)\big(T^{00} + t^{00}_{\rm LL}\big)\, d^3x \\
& + \frac{1}{c^3}\oint_{S_A} (-g)\big(T^{00} + t^{00}_{\rm LL}\big)\big(x^a - z^a_A\big)v^c_A\, dS_c,
\end{aligned}$$

and we convert the first volume integral into a surface integral by invoking the conservation statement of Eq. (1.1.7) and integrating by parts. This produces

$$\begin{aligned}\frac{dD_A^a}{dx^0} &= -\frac{1}{c^2}\oint_{S_A}(-g)\bigl(T^{0c}+t_{\mathrm{LL}}^{0c}\bigr)\bigl(x^a-z_A^a\bigr)\,dS_c+\frac{1}{c^2}\int_{V_A}(-g)\bigl(T^{0a}+t_{\mathrm{LL}}^{0a}\bigr)\,d^3x\\ &\quad -\frac{v_A^a}{c^3}\int_{V_A}(-g)\bigl(T^{00}+t_{\mathrm{LL}}^{00}\bigr)\,d^3x\\ &\quad +\frac{1}{c^3}\oint_{S_A}(-g)\bigl(T^{00}+t_{\mathrm{LL}}^{00}\bigr)\bigl(x^a-z_A^a\bigr)v_A^c\,dS_c.\end{aligned}$$

In the first volume integral we recognize cP_A^a, the spatial component of Eq. (5.1.2), and in the second volume integral we recognize P_A^0, its time component.

At this stage we may set $T^{\alpha\beta}=0$ within all boundary integrals, and we have obtained the identity

$$P_A^a = M_A v_A^a + Q_A^a + \dot{D}_A^a, \tag{5.1.10}$$

in which $M_A := c^{-1}P_A^0$ is a mass parameter associated with body A, and

$$Q_A^a := \frac{1}{c}\oint_{S_A}(-g)\biggl(t_{\mathrm{LL}}^{0b}-t_{\mathrm{LL}}^{00}\frac{v_A^b}{c}\biggr)\bigl(x^a-z_A^a\bigr)\,dS_b. \tag{5.1.11}$$

The overdot in Eq. (5.1.10) indicates differentiation with respect to t (instead of $x^0=ct$).

Equation (5.1.10) is a formal identity that relates a momentum-like quantity $\boldsymbol{P}_A$ to a product of a mass-like quantity M_A with a velocity $\boldsymbol{v}_A$, the time derivative of a dipole-like quantity $\boldsymbol{D}_A$, and an additional vector $\boldsymbol{Q}_A$ that possesses no useful interpretation; each one of these quantities is a function of time only. There is nothing physically meaningful about this identity, but it nevertheless plays a useful mathematical role. We differentiate it with respect to t, and directly obtain a *law of motion* for each body A:

$$M_A\boldsymbol{a}_A = \dot{\boldsymbol{P}}_A - \dot{M}_A\boldsymbol{v}_A - \dot{\boldsymbol{Q}}_A - \ddot{\boldsymbol{D}}_A. \tag{5.1.12}$$

Here, M_A are the mass parameters defined by Eq. (5.1.3),

$$M_A := \frac{c^2}{16\pi G}\oint_{S_A}\partial_c H^{0c0b}\,dS_b, \tag{5.1.13}$$

and these change with time according to Eq. (5.1.7),

$$\dot{M}_A = -\frac{1}{c}\oint_{S_A}(-g)\biggl(t_{\mathrm{LL}}^{0b}-t_{\mathrm{LL}}^{00}\frac{v_A^b}{c}\biggr)\,dS_b, \tag{5.1.14}$$

while

$$\dot{P}_A^a = -\oint_{S_A}(-g)\biggl(t_{\mathrm{LL}}^{ab}-t_{\mathrm{LL}}^{0a}\frac{v_A^b}{c}\biggr)\,dS_b \tag{5.1.15}$$

is the rate of change of the spatial momentum vector. Finally, $\dot{Q}_A^a$ and $\ddot{D}_A^a$ are obtained by differentiating Eqs. (5.1.11) and (5.1.8) with respect to time, respectively.

In the following sections we will endeavour to turn Eq. (5.1.12) into something more explicit. At the end of this calculation, the formal laws of motion will have become concrete *equations of motion* for our system of N bodies; these are listed in Sec. 5.4.4, below.

5.2 Internal and external potentials

To proceed with our calculations we focus on a specific body, the one labeled by $A = 1$, and to simplify the notation we let $m := m_1$, $\boldsymbol{z} := \boldsymbol{z}_1$, $\boldsymbol{v} := \boldsymbol{v}_1$, and so on. In addition, we introduce the vector $\boldsymbol{s} := \boldsymbol{x} - \boldsymbol{z}$, and decompose it as $\boldsymbol{s} = s\boldsymbol{n}$, in terms of its length $s := |\boldsymbol{s}|$ and the unit vector $\boldsymbol{n} := \boldsymbol{s}/s$. As was indicated in Eq. (5.1.1), the two-sphere S that surrounds our body is described by the equation $s = \text{constant}$. The surface element on S is $dS_a = s^2 n_a\, d\Omega$, in which $d\Omega$ is the usual element of solid angle.

We list some useful identities involving s, $\boldsymbol{n}$, and their derivatives:

$$\dot{s} = -\boldsymbol{n}\cdot\boldsymbol{v}, \tag{5.2.1}$$

$$\partial_a s = n_a, \tag{5.2.2}$$

$$\dot{n}_a = -\frac{1}{s}\big(\delta_{ab} - n_a n_b\big)v^b, \tag{5.2.3}$$

$$\partial_a n_b = \frac{1}{s}\big(\delta_{ab} - n_a n_b\big), \tag{5.2.4}$$

$$\ddot{s} = -\boldsymbol{n}\cdot\boldsymbol{a} + \frac{1}{s}\big[v^2 - (\boldsymbol{n}\cdot\boldsymbol{v})^2\big], \tag{5.2.5}$$

$$\partial_a \ddot{s} = -\frac{1}{s}\big(\delta_{ab} - n_a n_b\big)a^b - \frac{1}{s^2}\big(n_a\delta_{bc} + 2\delta_{ab}n_c - 3n_a n_b n_c\big)v^b v^c. \tag{5.2.6}$$

Here, $\boldsymbol{a} := \boldsymbol{a}_1$ is the body's acceleration vector, and $v^2 := \boldsymbol{v}_1 \cdot \boldsymbol{v}_1$.

The near-zone gravitational potentials were calculated in Sec. 4.2. According to Eqs. (4.2.32)–(4.2.34), they are given by

$$h^{00} = \frac{4}{c^2}U + \frac{4}{c^4}\left(\psi + \frac{1}{2}\frac{\partial^2 X}{\partial t^2} - P + 2U^2\right) + O(c^{-5}), \tag{5.2.7}$$

$$h^{0a} = \frac{4}{c^3}U^a + O(c^{-5}), \tag{5.2.8}$$

$$h^{ab} = \frac{4}{c^4}P^{ab} + O(c^{-5}), \tag{5.2.9}$$

where $P = \delta_{ab}P^{ab}$. The potentials U, ψ, X, U^a, and P^{ab} are displayed in Eqs. (4.2.35)–(4.2.44). For our purposes it is useful to decompose them into "internal potentials" that diverge at $\boldsymbol{x} = \boldsymbol{z}$, and "external potentials" that are smooth at $\boldsymbol{x} = \boldsymbol{z}$. We write

$$U = \frac{Gm}{s} + U_{\text{ext}}, \tag{5.2.10}$$

$$\psi = \frac{Gm\mu}{s} + \psi_{\text{ext}}, \tag{5.2.11}$$

$$X = Gms + X_{\text{ext}}, \tag{5.2.12}$$

$$P = \frac{1}{4}U^2 + \tilde{P}, \tag{5.2.13}$$

$$\tilde{P} = \frac{Gm\nu}{s} + \tilde{P}_{\text{ext}}, \tag{5.2.14}$$

$$U^a = \frac{Gmv^a}{s} + U^a_{\text{ext}}, \tag{5.2.15}$$

where we have introduced

$$\mu := \frac{3}{2}v^2 - \sum_{A\neq 1}\frac{Gm_A}{|\boldsymbol{z} - \boldsymbol{z}_A|} \tag{5.2.16}$$

and

$$\nu := v^2 - \frac{1}{2}\sum_{A\neq 1}\frac{Gm_A}{|\boldsymbol{z}-\boldsymbol{z}_A|}. \tag{5.2.17}$$

The tensor potential P^{ab} also can be decomposed into internal and external parts, but this is not required in the following computations. In Eq. (5.2.13) we have indicated that since P approaches $G^2m^2/(4s^2)$ as $s \to 0$, it can neatly be expressed as $\frac{1}{4}U^2$ plus a less-singular quantity $\tilde{P}$, whose behaviour near $s = 0$ is given by Eq. (5.2.14). We also remark that while X does not diverge as $s \to 0$, it is its second time derivative that appears within h^{00}, and Eq. (5.2.5) reveals this is indeed singular at $s = 0$.

The external potentials are given by

$$U_{\rm ext} = \sum_{A\neq 1}\frac{Gm_A}{|\boldsymbol{x}-\boldsymbol{z}_A|}, \tag{5.2.18}$$

$$\begin{aligned}\psi_{\rm ext} &= \frac{3}{2}\sum_{A\neq 1}\frac{Gm_A v_A^2}{|\boldsymbol{x}-\boldsymbol{z}_A|} - \sum_{A\neq 1}\frac{G^2 m m_A}{|\boldsymbol{z}-\boldsymbol{z}_A||\boldsymbol{x}-\boldsymbol{z}_A|} \\ &\quad - \sum_{A\neq 1}\sum_{B>A}\frac{G^2 m_A m_B}{|\boldsymbol{z}_A-\boldsymbol{z}_B|}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{z}_A|}+\frac{1}{|\boldsymbol{x}-\boldsymbol{z}_B|}\right),\end{aligned} \tag{5.2.19}$$

$$X_{\rm ext} = \sum_{A\neq 1} Gm_A|\boldsymbol{x}-\boldsymbol{z}_A|, \tag{5.2.20}$$

$$\begin{aligned}\tilde{P}_{\rm ext} &= \sum_{A\neq 1}\frac{Gm_A v_A^2}{|\boldsymbol{x}-\boldsymbol{z}_A|} \\ &\quad - \frac{1}{2}\sum_{A\neq 1}\sum_{B>A}\frac{G^2 m_A m_B}{|\boldsymbol{z}_A-\boldsymbol{z}_B|}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{z}_A|}+\frac{1}{|\boldsymbol{x}-\boldsymbol{z}_B|}\right),\end{aligned} \tag{5.2.21}$$

$$U^a_{\rm ext} = \sum_{A\neq 1}\frac{Gm_A v_A^a}{|\boldsymbol{x}-\boldsymbol{z}_A|}. \tag{5.2.22}$$

It is interesting to observe that by virtue of the nonlinearity of the Einstein field equations, the "external part" of ψ still carries a dependence on m and $\boldsymbol{z}$.

We shall be interested in the behaviour of the external potentials in the immediate vicinity of our reference body, near $s = 0$. Because the external potentials are all differentiable at $s = 0$, this behaviour is best expressed as a Taylor expansion. We shall write, for example,

$$U_{\rm ext}(\boldsymbol{x}) = U_{\rm ext}(\boldsymbol{z}) + s\partial_a U_{\rm ext}(\boldsymbol{z})n^a + \frac{1}{2}s^2\partial_{ab}U_{\rm ext}(\boldsymbol{z})n^a n^b + O(s^3). \tag{5.2.23}$$

The relevant derivatives of the external potentials will be evaluated at a later stage.

5.3 Computation of M, $\dot{M}$, $\dot{P}^a$, Q^a, and D^a

5.3.1 Computation of M

We begin with the evaluation of $M := M_1$, the mass parameter associated with our reference body, as defined by Eq. (5.1.13). After substituting Eqs. (5.2.7)–(5.2.9) into Eq. (5.1.4), we find that

$$\partial_b H^{0b0a} = -\frac{4}{c^2}\partial^a U - \frac{4}{c^4}\left(\partial^a\psi + \frac{1}{2}\partial^a\ddot{X} - \partial^a P + 4U\partial^a U + \dot{U}^a\right) + O(c^{-4}).$$

Substitution into Eq. (5.1.13) gives

$$\begin{aligned} M &= -\frac{1}{G}\big\langle\!\big\langle s^2 n_a \partial^a U \big\rangle\!\big\rangle - \frac{1}{Gc^2}\big\langle\!\big\langle s^2 n_a \big(\partial^a\psi + \tfrac{1}{2}\partial^a \ddot{X} - \partial^a \tilde{P} + \tfrac{7}{2}U\partial^a U + \dot{U}^a\big)\big\rangle\!\big\rangle \\ &\quad + O(c^{-4}), \end{aligned} \tag{5.3.1}$$

where we return to the notation of Sec. 1.8.4 and let $\langle\!\langle \cdots \rangle\!\rangle := (4\pi)^{-1}\int(\cdots)\,d\Omega$ denote an average over a two-sphere $s =$ constant. To get to Eq. (5.3.1) we wrote $dS_a = s^2 n_a\, d\Omega$ and involved Eq. (5.2.13) to eliminate P in favour of $\tilde{P}$.

To proceed we express each potential in terms of its "internal" and "external" parts, as in Eqs. (5.2.10)–(5.2.15), we use Eqs. (5.2.1)–(5.2.6) to compute the derivatives of the internal contributions, we Taylor-expand the external contributions in powers of s, and finally, we involve Eqs. (1.8.18)–(1.8.21) to carry out the angular averages. For example, from $U = Gm/s + U_{\rm ext}$ we find that $\partial^a U = -Gmn^a/s^2 + \partial^a U_{\rm ext}$, so that $s^2 n_a \partial^a U = -Gm + O(s^2)$, which leads to $\langle\!\langle s^2 n_a \partial^a U\rangle\!\rangle = -Gm + O(s^2)$. In this fashion we obtain

$$\begin{aligned} \big\langle\!\big\langle s^2 n_a \partial^a U \big\rangle\!\big\rangle &= -Gm + O(s^2), \\ \big\langle\!\big\langle s^2 n_a \partial^a \psi \big\rangle\!\big\rangle &= -Gm\mu + O(s^2), \\ \big\langle\!\big\langle s^2 n_a \partial^a \ddot{X} \big\rangle\!\big\rangle &= -\frac{2}{3}Gmv^2 + O(s), \\ \big\langle\!\big\langle s^2 n_a \partial^a \tilde{P} \big\rangle\!\big\rangle &= -Gm\nu + O(s^2), \\ \big\langle\!\big\langle s^2 n_a \dot{U}^a \big\rangle\!\big\rangle &= \frac{1}{3}Gmv^2 + O(s), \\ \big\langle\!\big\langle s^2 n_a U \partial^a U \big\rangle\!\big\rangle &= -\frac{G^2m^2}{s} - GmU_{\rm ext} + O(s). \end{aligned}$$

Notice that the last equation includes a leading term that scales as s^{-1}, and an s-independent term proportional to

$$U_{\rm ext}(\boldsymbol{x} = \boldsymbol{z}) = \sum_{A\neq 1} \frac{Gm_A}{|\boldsymbol{z} - \boldsymbol{z}_A|}. \tag{5.3.2}$$

Recalling the notation employed in Chapter 4, this is $\lfloor U \rfloor_1$, the Newtonian potential evaluated at the position of the reference body, excluding the infinite contribution coming from this very body. Notice that in the context of this chapter, $\lfloor U \rfloor_1$ occurs naturally, and not as a result of an ad-hoc regularization prescription.

Inserting these results into Eq. (5.3.1), we arrive at

$$M = m\bigg\{1 + \frac{1}{c^2}\bigg[\frac{1}{2}v^2 + 3U_{\rm ext} + \frac{7}{2}\frac{Gm}{s} + O(s)\bigg] + O(c^{-4})\bigg\}. \tag{5.3.3}$$

This is the mass parameter of the reference body.

5.3.2 Computation of $\dot{M}$

To compute $\dot{M} := \dot{M}_1$ we refer back to Eq. (5.1.14), which expresses it in terms of the Landau-Lifshitz pseudotensor integrated over the two-sphere S. The relevant components of the pseudotensor were calculated in Sec. 4.1.3, and the results are displayed in Eqs. (4.1.17) and (4.1.18). After substitution of Eqs. (5.2.7)–(5.2.9), we find that

$$\begin{aligned} (-g)\bigg(t^{0a}_{\rm LL} - t^{00}_{\rm LL}\frac{v^a}{c}\bigg) &= \frac{1}{4\pi Gc}\bigg[3\dot{U}\partial^a U + 4\big(\partial^a U^b - \partial^b U^a\big)\partial_b U + \frac{7}{2}v^a \partial_c U \partial^c U\bigg] \\ &\quad + O(c^{-3}). \end{aligned}$$

Substitution into Eq. (5.1.14) gives

$$\dot{M} = -\frac{1}{Gc^2}\langle\!\langle s^2 n_a \big[3\dot{U}\partial^a U + 4(\partial^a U^b - \partial^b U^a)\partial_b U + \tfrac{7}{2} v^a \partial_c U \partial^c U\big]\rangle\!\rangle + O(c^{-4}). \quad (5.3.4)$$

This expression can be evaluated with the same techniques that were employed in the preceding subsection. We obtain, for example,

$$\begin{aligned}
\langle\!\langle s^2 n_a \dot{U}\partial^a U\rangle\!\rangle &= \frac{1}{3} G m v^a \partial_a U_{\text{ext}} - G m \dot{U}_{\text{ext}} + O(s), \\
\langle\!\langle s^2 n_a (\partial^a U^b - \partial^b U^a)\partial_b U\rangle\!\rangle &= -\frac{2}{3} G m v^a \partial_a U_{\text{ext}} + O(s), \\
\langle\!\langle s^2 n_a v^a \partial_c U \partial^c U\rangle\!\rangle &= -\frac{2}{3} G m v^a \partial_a U_{\text{ext}} + O(s),
\end{aligned}$$

in which the external potential U_{ext} is evaluated at $\boldsymbol{x} = \boldsymbol{z}$ after differentiation. Notice that after integration, there are no surviving terms of order s^{-2}, in spite of the fact that something like $s^2 n_a \dot{U}\partial^a U$ contains a contribution that scales as s^{-2}; such a terms disappears because it involves an odd number of unit vectors $\boldsymbol{n}$, which leads to a vanishing integral.

Inserting these results into Eq. (5.3.4), we arrive at

$$\dot{M} = \frac{m}{c^2}\Big[4 v^a \partial_a U_{\text{ext}} + 3\dot{U}_{\text{ext}} + O(s)\Big] + O(c^{-4}), \quad (5.3.5)$$

where, as we indicated before, the derivatives of the external Newtonian potential U_{ext} are to be evaluated at $\boldsymbol{x} = \boldsymbol{z}$, the position of the reference body.

It is an instructive exercise to differentiate Eq. (5.3.3) with respect to t and to verify that the result is compatible with Eq. (5.3.5). This calculation requires an expression for $\boldsymbol{a}$, the acceleration vector of our reference body. Because this is the very quantity that we are in the process of calculating, this exercise must, as a matter of principle, be postponed until the information becomes available. Regardless, anticipating that the leading-order term in the acceleration vector will be the Newtonian acceleration of Eq. (4.3.2), one can easily show that our expressions for M and $\dot{M}$ are indeed compatible.

5.3.3 Computation of $\dot{P}^a$

The computation of $\dot{P}^a := \dot{P}^a_1$ begins with Eq. (5.1.15), which once more involves a surface integration of the Landau-Lifshitz pseudotensor. The relevant components are displayed in Eqs. (4.1.18) and (4.1.20), and after substitution of Eqs. (5.2.7)–(5.2.9), we find the lengthy expression

$$\begin{aligned}
(-g)\left(t^{ab}_{\text{LL}} - t^{0a}_{\text{LL}}\frac{v^b}{c}\right) &= \frac{1}{16\pi G}\Big\{4\partial^a U \partial^b U - 2\delta^{ab}\partial_c U \partial^c U\Big\} \\
&\quad + \frac{1}{16\pi G c^2}\Big\{8\partial^{(a} U \partial^{b)}\psi + 4\partial^{(a} U \partial^{b)}\ddot{X} + 32\partial^{(a} U \dot{U}^{b)} \\
&\quad - 16\big(\partial^a U_c - \partial_c U^a\big)\big(\partial^b U^c - \partial^c U^b\big) \\
&\quad - \delta^{ab}\Big[6\dot{U}^2 + 4\partial_c U \partial^c \psi + 2\partial_c U \partial^c \ddot{X} + 16\partial_c U \dot{U}^c - 8\partial_c U_d(\partial^c U^d - \partial^d U^c)\Big] \\
&\quad - v^b\Big[12\dot{U}\partial^a U + 16(\partial^a U^c - \partial^c U^a)\partial_c U\Big]\Big\} + O(c^{-4}).
\end{aligned}$$

Substitution into Eq. (5.1.15) gives

$$\dot{P}^a = -\frac{1}{4G}\langle\!\langle s^2\big\{4 n_b \partial^a U \partial^b U - 2 n^a \partial_c U \partial^c U\big\}\rangle\!\rangle$$

$$
\begin{aligned}
&- \frac{1}{4Gc^2}\Big\langle\!\!\Big\langle s^2 \Big\{ 8n_b\partial^{(a}U\partial^{b)}\psi - 4n^a\partial_c U\partial^c\psi + 4n_b\partial^{(a}U\partial^{b)}\ddot{X} - 2n^a\partial_c U\partial^c\ddot{X} \\
&+ 32n_b\partial^{(a}U\dot{U}^{b)} - 16n^a\partial_c U\dot{U}^c - 16n_b\big(\partial^a U_c - \partial_c U^a\big)\big(\partial^b U^c - \partial^c U^b\big) \\
&+ 8n^a\partial_c U_d(\partial^c U^d - \partial^d U^c) - 6n^a\dot{U}^2 \\
&- 12n_b v^b\dot{U}\partial^a U - 16n_b v^b(\partial^a U^c - \partial^c U^a)\partial_c U \Big\} \Big\rangle\!\!\Big\rangle + O(c^{-4}). \qquad (5.3.6)
\end{aligned}
$$

The relevant angular averages are

$$
\begin{aligned}
\big\langle\!\big\langle s^2 n_b\partial^a U\partial^b U\big\rangle\!\big\rangle &= -\frac{4}{3}Gm\partial^a U_{\text{ext}} + O(s), \\
\big\langle\!\big\langle s^2 n^a\partial_c U\partial^c U\big\rangle\!\big\rangle &= -\frac{2}{3}Gm\partial^a U_{\text{ext}} + O(s), \\
\big\langle\!\big\langle s^2 n_b\partial^a U\partial^b \psi\big\rangle\!\big\rangle &= -Gm\mu\partial^a U_{\text{ext}} - \frac{1}{3}Gm\partial^a\psi_{\text{ext}} + O(s), \\
\big\langle\!\big\langle s^2 n_b\partial^a \psi\partial^b U\big\rangle\!\big\rangle &= -\frac{1}{3}Gm\mu\partial^a U_{\text{ext}} - Gm\partial^a\psi_{\text{ext}} + O(s), \\
\big\langle\!\big\langle s^2 n^a\partial_c U\partial^c \psi\big\rangle\!\big\rangle &= -\frac{1}{3}Gm\mu\partial^a U_{\text{ext}} - \frac{1}{3}Gm\partial^a\psi_{\text{ext}} + O(s), \\
\big\langle\!\big\langle s^2 n_b\partial^a U\partial^b \ddot{X}\big\rangle\!\big\rangle &= -\frac{2}{3}Gmv^2\partial^a U_{\text{ext}} - \frac{1}{3}Gm\partial^a\ddot{X}_{\text{ext}} + O(s), \\
\big\langle\!\big\langle s^2 n_b\partial^a U\partial^b \ddot{X}\big\rangle\!\big\rangle &= -\frac{2}{3}Gmv^2\partial^a U_{\text{ext}} - \frac{1}{3}Gm\partial^a\ddot{X}_{\text{ext}} + O(s), \\
\big\langle\!\big\langle s^2 n_b\partial^a \ddot{X}\partial^b U\big\rangle\!\big\rangle &= \frac{2}{3}\frac{G^2m^2}{s}a^a - \frac{2}{15}Gm\big(v^2\partial^a U_{\text{ext}} + 2v^a v^b\partial_b U_{\text{ext}}\big) \\
&\quad - Gm\partial^a\ddot{X}_{\text{ext}} + O(s), \\
\big\langle\!\big\langle s^2 n^a\partial_c U\partial^c \ddot{X}\big\rangle\!\big\rangle &= -\frac{2}{15}Gm\big(v^2\partial^a U_{\text{ext}} + 2v^a v^b\partial_b U_{\text{ext}}\big) - \frac{1}{3}Gm\partial^a\ddot{X}_{\text{ext}} + O(s), \\
\big\langle\!\big\langle s^2 n_b\partial^a U\dot{U}^b\big\rangle\!\big\rangle &= -\frac{1}{3}\frac{G^2m^2}{s}a^a + \frac{1}{3}Gmv^2\partial^a U_{\text{ext}} - \frac{1}{3}Gm\dot{U}^a_{\text{ext}} + O(s), \\
\big\langle\!\big\langle s^2 n_b\dot{U}^a\partial^b U\big\rangle\!\big\rangle &= -\frac{G^2m^2}{s}a^a + \frac{1}{3}Gmv^a v^b\partial_b U_{\text{ext}} - Gm\dot{U}^a_{\text{ext}} + O(s), \\
\big\langle\!\big\langle s^2 n^a\partial_c U\dot{U}^c\big\rangle\!\big\rangle &= -\frac{1}{3}\frac{G^2m^2}{s}a^a + \frac{1}{3}Gmv^a v^b\partial_b U_{\text{ext}} - \frac{1}{3}Gm\dot{U}^a_{\text{ext}} + O(s), \\
\big\langle\!\big\langle s^2 n_b\big(\partial^a U_c - \partial_c U^a\big)\big(\partial^b U^c - \partial^c U^b\big)\big\rangle\!\big\rangle &= -Gmv_b\big(\partial^a U^b_{\text{ext}} - \partial^b U^a_{\text{ext}}\big) + O(s), \\
\big\langle\!\big\langle s^2 n^a\partial_c U_d\big(\partial^c U^d - \partial^d U^c\big)\big\rangle\!\big\rangle &= -\frac{2}{3}Gmv_b\big(\partial^a U^b_{\text{ext}} - \partial^b U^a_{\text{ext}}\big) + O(s), \\
\big\langle\!\big\langle s^2 n^a\dot{U}^2\big\rangle\!\big\rangle &= \frac{2}{3}Gmv^a\dot{U}_{\text{ext}} + O(s), \\
\big\langle\!\big\langle s^2 n_b v^b\dot{U}\partial^a U\big\rangle\!\big\rangle &= \frac{1}{3}Gmv^2\partial^a U_{\text{ext}} - \frac{1}{3}Gmv^a\dot{U}_{\text{ext}} + O(s), \\
\big\langle\!\big\langle s^2 n_b v^b\big(\partial^a U^c - \partial^c U^a\big)\partial_c U\big\rangle\!\big\rangle &= -\frac{1}{3}Gmv_b\big(\partial^a U^b_{\text{ext}} - \partial^b U^a_{\text{ext}}\big) + O(s).
\end{aligned}
$$

Inserting these results within Eq. (5.3.6), we arrive at

$$
\begin{aligned}
\dot{P}^a &= m\Big[\partial^a U_{\text{ext}} + O(s)\Big] + \frac{m}{c^2}\Big[\Big(\frac{3}{2}v^2 - U_{\text{ext}}\Big)\partial^a U_{\text{ext}} + \partial^a\psi_{\text{ext}} + \frac{1}{2}\partial^a\ddot{X}_{\text{ext}} \\
&\quad + 4\dot{U}^a_{\text{ext}} - 4\big(\partial^a U^b_{\text{ext}} - \partial^b U^a_{\text{ext}}\big)v_b + \frac{11}{3}\frac{Gm}{s}a^a + O(s)\Big] + O(c^{-4}), \qquad (5.3.7)
\end{aligned}
$$

in which the external potentials and their derivatives are evaluated at $\boldsymbol{x} = \boldsymbol{z}$, the position of the reference body.

5.3.4 Computation of Q^a

The computation of $Q^a := Q^a_1$ begins with Eq. (5.1.11), in which we substitute $x^a - z^a = sn^a$, $dS_b = s^2 n_b\, d\Omega$, as well as the components of the Landau-Lifshitz pseudotensor that were obtained at the beginning of Sec. 5.3.2. This gives

$$Q^a = \frac{1}{Gc^2}\big\langle\!\big\langle s^3 n^a n_b \big[3\dot{U}\partial^b U + 4(\partial^b U^c - \partial^c U^b)\partial_c U + \tfrac{7}{2} v^b \partial_c U \partial^c U\big]\big\rangle\!\big\rangle + O(c^{-4}). \quad (5.3.8)$$

We have

$$\begin{aligned}
\big\langle\!\big\langle s^3 n^a n_b \dot{U}\partial^b U \big\rangle\!\big\rangle &= -\frac{1}{3}\frac{G^2m^2}{s} v^a + O(s), \\
\big\langle\!\big\langle s^3 n^a n_b (\partial^b U^c - \partial^c U^b)\partial_c U \big\rangle\!\big\rangle &= O(s), \\
\big\langle\!\big\langle s^3 n^a n_b v^b \partial_c U \partial^c U \big\rangle\!\big\rangle &= \frac{1}{3}\frac{G^2m^2}{s} v^a + O(s),
\end{aligned}$$

and this gives

$$Q^a = \frac{m}{c^2}\frac{Gm}{6s} v^a + O(s) + O(c^{-4}). \quad (5.3.9)$$

From this we immediately obtain

$$\dot{Q}^a = \frac{m}{c^2}\frac{Gm}{6s} a^a + O(s) + O(c^{-4}), \quad (5.3.10)$$

because s is set equal to a constant during the integration over S.

5.3.5 Computation of D^a

We return to Eq. (5.1.9), which we write in the form

$$D^a = \frac{c^2}{4G}\big\langle\!\big\langle s^2 n_b \big(s n^a \partial_c H^{0b0c} - H^{0a0b}\big)\big\rangle\!\big\rangle.$$

This becomes

$$\begin{aligned}
D^a \;=\;& \frac{1}{G}\big\langle\!\big\langle s^2 n^a \big(U - s n_b \partial^b U\big)\big\rangle\!\big\rangle \\
&+ \frac{1}{Gc^2}\Big\langle\!\!\Big\langle s^2 n^a \Big[\big(\psi - s n_b \partial^b \psi\big) + \tfrac{1}{2}\big(\ddot{X} - s n_b \partial^b \ddot{X}\big) - \big(\tilde{P} - s n_b \partial^b \tilde{P}\big) \\
&+ \tfrac{7}{4}\big(U^2 - s n_b \partial^b U^2\big)\Big]\Big\rangle\!\!\Big\rangle - \frac{1}{Gc^2}\big\langle\!\big\langle s^2 n_b \big(P^{ab} - s n^a \partial_c P^{bc}\big)\big\rangle\!\big\rangle \\
&+ O(c^{-4})
\end{aligned} \quad (5.3.11)$$

after inserting Eqs. (5.2.7)–(5.2.9) into Eq. (5.1.4). Evaluation of the angular integrals reveals that there are no surviving terms of order s^{-1} or s^0, and we conclude that

$$D^a = O(s) + O(c^{-4}). \quad (5.3.12)$$

This conclusion is a consequence of the fact that the internal contributions to the potentials U, ψ, $\ddot{X}$, and $\tilde{P}$ are spherically symmetric. Consider, for example, the Newtonian potential $U = Gm/s + U_{\rm ext}$. The combination of terms that appears within the angular integral is $U - s n_b \partial^b U = 2Gm/s + U_{\rm ext} - \frac{1}{2} s^2 n^b n^c \partial_{bc} U_{\rm ext} + O(s^3)$, in which $U_{\rm ext}$ and its derivatives are evaluated at $s = 0$. After multiplication by $s^2 n^a$ and angular integration, we get a result of order s^5. Examination of the terms in U^2 reveals that these contribute a result of order s^2. The terms in P^{ab} must be examined more carefully. According to Eq. (4.2.41), the most singular term in the tensor potential is equal to $G^2m^2 n^a n^b/(4s^2)$, and this vanishes after multiplication

by $s^2 n_b$ and angular integration; less singular terms would give a $O(s)$ contribution to D^a. Finally, the term involving $\partial_c P^{bc}$ can be written in terms of $\dot{U}^b$ by invoking the harmonic gauge conditions, and this contribution also can be shown to be of order s. All in all, we arrive at Eq. (5.3.12).

It follows immediately from Eq. (5.3.12) that

$$\ddot{D}^a = O(s) + O(c^{-4}). \tag{5.3.13}$$

5.4 First post-Newtonian equations of motion

5.4.1 Acceleration in terms of external potentials

The results obtained in the preceding section, namely Eqs. (5.3.3), (5.3.5), (5.3.7), (5.3.10), and (5.3.13), may now be substituted into Eq. (5.1.12), which we write in the specialized form $M\boldsymbol{a} = \dot{P} - \dot{M}\boldsymbol{v} - \dot{Q} - \ddot{D}$ that applies to the reference body. On each side of the equation we have terms of order s^{-1}, terms independent of s, and terms of higher order in s that have not been calculated. Because the equation is an identity, and because s has been set equal to an arbitrary constant, *the equality must be true order-by-order in s*. It is easy to verify that as expected, all terms of order s^{-1} cancel out of the equation. It is then clear that an expression for the acceleration vector will come from the s-independent terms, and that the $O(s)$ terms would generate redundant information.

In this way, we obtain

$$\begin{aligned} a^a &= \partial^a U_{\rm ext} + \frac{1}{c^2}\biggl[(v^2 - 4U_{\rm ext})\partial^a U_{\rm ext} - 4v^a v^b \partial_b U_{\rm ext} - 3v^a \dot{U}_{\rm ext} + \partial^a \psi_{\rm ext} \\ &\quad + \frac{1}{2}\partial^a \ddot{X}_{\rm ext} + 4\dot{U}^a_{\rm ext} - 4\bigl(\partial^a U^b_{\rm ext} - \partial^b U^a_{\rm ext}\bigr)v_b\biggr] + O(c^{-4}), \end{aligned} \tag{5.4.1}$$

an expression for the acceleration vector of the reference body, in terms of the external potentials of Eqs. (5.2.18)–(5.2.22); it is understood that these are to be evaluated at $\boldsymbol{x} = \boldsymbol{z}$, the position of the reference body. To arrive at Eq. (5.4.1) we have moved the factor $1 + c^{-2}(\frac{1}{2}v^2 + 3U_{\rm ext}) + O(c^{-4})$, which originates from the mass parameter M, from the left-hand side of the equation to its right-hand side. We recognize, in Eq. (5.4.1), the Newtonian acceleration field $\partial^a U_{\rm ext}$; the terms of order c^{-2} are 1PN corrections to the acceleration vector.

5.4.2 Geodesic equation

It is instructive to work out the geodesic equations for a test mass moving in a spacetime whose geometry is determined by our original system of N bodies. The metric of this spacetime is determined by the gravitational potentials of Eqs. (5.2.7)–(5.2.9), and Eq. (1.6.4) allows us to obtain an explicit expression. To the appropriate post-Newtonian order, we have

$$g_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}\left(\psi + \frac{1}{2}\ddot{X} - U^2\right) + O(c^{-5}), \tag{5.4.2}$$

$$g_{0a} = -\frac{4}{c^3}U_a + O(c^{-5}), \tag{5.4.3}$$

$$\begin{aligned} g_{ab} &= \delta_{ab} + \frac{2}{c^2}U\delta_{ab} + \frac{2}{c^4}\left[2P_{ab} + \left(\psi + \frac{1}{2}\ddot{X} - 2P + U^2\right)\delta_{ab}\right] \\ &\quad + O(c^{-5}), \end{aligned} \tag{5.4.4}$$

in which $x^0 = ct$ and an overdot indicates differentiation with respect to t. The potentials U, ψ, X, U^a, and P^{ab} were introduced in Sec. 4.2, and explicit expressions are given in Eqs. (4.2.35)–(4.2.44).

An action functional for geodesic motion is

$$S = -c \int \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}\, d\lambda, \tag{5.4.5}$$

in which the parametric relations $x^\alpha(\lambda)$ give a description of the particle's world line. The action is invariant under reparameterizations, and we choose t as the parameter λ. This gives us the Lagrangian

$$L = -c\sqrt{-g_{\alpha\beta} v^\alpha v^\beta}, \tag{5.4.6}$$

in which $v^\alpha = dx^\alpha/dt = (c, \boldsymbol{v})$ is the velocity four-vector of the test mass. More explicitly, using the metric of Eqs. (5.4.2)–(5.4.4), we have

$$\begin{aligned} L &= -c^2 + \frac{1}{2}v^2 + U + \frac{1}{c^2}\left(\frac{1}{8}v^4 + \frac{3}{2}Uv^2 - 4U_a v^a + \psi + \frac{1}{2}\ddot{X} - \frac{1}{2}U^2\right) \\ &\quad + O(c^{-4}), \end{aligned} \tag{5.4.7}$$

after truncation at the appropriate post-Newtonian order.

Substitution of L into the Euler-Lagrange equations produces

$$\begin{aligned} \left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + 3U\right) + O(c^{-4})\right] a^a &= \partial^a U + \frac{1}{c^2}\biggl[\left(\frac{3}{2}v^2 - U\right)\partial^a U - v^a v_b a^b \\ - 3v^a v^b \partial_b U - 3v^a \dot{U} + \partial^a \psi + \frac{1}{2}\partial^a \ddot{X} &+ 4\dot{U}^a - 4\bigl(\partial^a U^b - \partial^b U^a\bigr) v_b\biggr] + O(c^{-4}). \end{aligned}$$

After moving the factor multiplying a^a from the left-hand side to the right-hand side of the equation (as we did in the preceding subsection), and after substituting $a^a = \partial^a U + O(c^{-2})$ within the 1PN term on the right-hand side, we finally arrive at

$$\begin{aligned} a^a &= \partial^a U + \frac{1}{c^2}\biggl[(v^2 - 4U)\partial^a U - 4v^a v^b \partial_b U - 3v^a \dot{U} + \partial^a \psi \\ &\quad + \frac{1}{2}\partial^a \ddot{X} + 4\dot{U}^a - 4\bigl(\partial^a U^b - \partial^b U^a\bigr) v_b\biggr] + O(c^{-4}). \end{aligned} \tag{5.4.8}$$

This is the acceleration vector of a test mass moving on a geodesic in the post-Newtonian spacetime.

Equation (5.4.8) looks virtually identical to Eq. (5.4.1), and for this reason it might be said that the reference body moves on a geodesic in a spacetime whose geometry is determined by all remaining (external) bodies; the metric of this spacetime would be given by Eqs. (5.4.2)–(5.4.4), with all potentials replaced by the *external potentials* of Eqs. (5.2.18)–(5.2.22). This interpretation of Eq. (5.4.1) is attractive and perhaps technically correct, but it is also misleading. It gives the incorrect impression that the acceleration of the reference body should be determined by the external masses only, and should not contain a dependence on m itself. This is indeed incorrect: As Eq. (5.2.19) reveals, the "external potential" $\psi_{\rm ext}$ does, in fact, depend explicitly on m and $\boldsymbol{z}$. The nonlinear nature of Einstein's theory implies that the reference body exerts a force on itself, and this effect appears at the first post-Newtonian order; one must therefore be careful with the geodesic-equation interpretation of Eq. (5.4.1).

5.4.3 Derivatives of the external potentials

The external potentials $U_{\rm ext}$, $\psi_{\rm ext}$, $X_{\rm ext}$, and $U^a_{\rm ext}$ were listed in Sec. 5.2, Eqs. (5.2.18)–(5.2.22), and their derivatives may be computed by involving Eqs. (5.2.1)–(5.2.6),

in which we change the identity of s from its old value $|\boldsymbol{x}-\boldsymbol{z}|$ to a new value $s := |\boldsymbol{x}-\boldsymbol{z}_A|$. After evaluating the results at $\boldsymbol{x}=\boldsymbol{z} := \boldsymbol{z}_1$, we obtain

$$
\begin{aligned}
U_{\rm ext} &= \sum_{A\neq 1} \frac{Gm_A}{z_{1A}}, \\
\partial^a U_{\rm ext} &= -\sum_{A\neq 1} \frac{Gm_A}{z_{1A}^2} n_{1A}^a, \\
\dot{U}_{\rm ext} &= \sum_{A\neq 1} \frac{Gm_A}{z_{1A}^2} (\boldsymbol{n}_{1A}\cdot\boldsymbol{v}_A), \\
\partial^a \psi_{\rm ext} &= -\frac{3}{2}\sum_{A\neq 1} \frac{Gm_A v_A^2}{z_{1A}^2} n_{1A}^a + \sum_{A\neq 1} \frac{G^2 m m_A}{z_{1A}^3} n_{1A}^a \\
&\quad + \sum_{A\neq 1}\sum_{B>A} \frac{G^2 m_A m_B}{z_{AB}} \biggl(\frac{n_{1A}^a}{z_{1A}^2} + \frac{n_{1B}^a}{z_{1B}^2} \biggr), \\
\partial^a \ddot{X}_{\rm ext} &= -\sum_{A\neq 1} \frac{Gm_A}{z_{1A}} \Bigl[a_A^a - (\boldsymbol{n}_{1A}\cdot\boldsymbol{a}_A) n_{1A}^a \Bigr] \\
&\quad - \sum_{A\neq 1} \frac{Gm_A}{z_{1A}^2} \Bigl[v_A^2 - 3(\boldsymbol{n}_{1A}\cdot\boldsymbol{v}_A)^2 \Bigr] n_{1A}^a \\
&\quad - 2\sum_{A\neq 1} \frac{Gm_A}{z_{1A}^2} (\boldsymbol{n}_{1A}\cdot\boldsymbol{v}_A) v_A^a, \\
\dot{U}^a_{\rm ext} &= \sum_{A\neq 1} \frac{Gm_A}{z_{1A}} a_A^a + \sum_{A\neq 1} \frac{Gm_A}{z_{1A}^2} (\boldsymbol{n}_{1A}\cdot\boldsymbol{v}_A) v_A^a, \\
\partial^a U^b_{\rm ext} &= -\sum_{A\neq 1} \frac{Gm_A}{z_{1A}^2} n_{1A}^a v_A^b,
\end{aligned}
$$

where $z_{1A} := |\boldsymbol{z}-\boldsymbol{z}_A|$ and $n_{1A}^a := (\boldsymbol{z}-\boldsymbol{z}_A)^a/|\boldsymbol{z}-\boldsymbol{z}_A|$.

Substitution of these results into Eq. (5.4.1) leads to an explicit expression for $\boldsymbol{a}$, the acceleration vector of our reference body. At the leading order we get

$$
a^a = \partial^a U_{\rm ext} + O(c^{-2}) = -\sum_{A\neq 1} \frac{Gm_A}{z_{1A}^2} n_{1A}^a + O(c^{-2}),
$$

the expected expression for the Newtonian acceleration. At the next order we obtain the 1PN corrections to the acceleration vector. At this stage it is useful to take our attention away from the reference body $A=1$, and to start writing down general expressions that are valid for each mass within the N-body system. We shall therefore rewrite our previous result as

$$
\boldsymbol{a}_A = -\sum_{B\neq A} \frac{Gm_B}{z_{AB}^2} \boldsymbol{n}_{AB} + O(c^{-2}). \tag{5.4.9}
$$

This is the acceleration vector of body A, accurate to 0PN order, and expressed in terms of the interbody distance $z_{AB} := |\boldsymbol{z}_A - \boldsymbol{z}_B|$ and the unit vector $\boldsymbol{n}_{AB} := (\boldsymbol{z}_A-\boldsymbol{z}_B)/|\boldsymbol{z}_A-\boldsymbol{z}_B|$.

To obtain the post-Newtonian corrections to this result we must perform a similar change of notation in the derivatives of the external potentials (which are now external to body A). We will also insert Eq. (5.4.9), written as

$$
\boldsymbol{a}_B = \frac{Gm_A}{z_{AB}^2}\boldsymbol{n}_{AB} - \sum_{C\neq A,B} \frac{Gm_C}{z_{BC}^2} \boldsymbol{n}_{BC} + O(c^{-2})
$$

to isolate the term $C = A$ in the original sum over $C \neq B$ (notice that $\boldsymbol{n}_{AB} = -\boldsymbol{n}_{BA}$), within our previous expressions for $\partial^a \ddot{X}$ and $\dot{U}^a$. After some simple algebra, we obtain

$$\begin{aligned}
U_{\rm ext} &= \sum_{C\neq A} \frac{Gm_C}{z_{AC}}, && (5.4.10)\\
\partial^a U_{\rm ext} &= -\sum_{B\neq A} \frac{Gm_B}{z_{AB}^2} n^a_{AB}, && (5.4.11)\\
\dot{U}_{\rm ext} &= \sum_{B\neq A} \frac{Gm_B}{z_{AB}^2} \bigl(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_B\bigr), && (5.4.12)\\
\partial^a \psi_{\rm ext} &= -\frac{3}{2}\sum_{B\neq A} \frac{Gm_B v_B^2}{z_{AB}^2} n^a_{AB} + \sum_{B\neq A} \frac{G^2 m_A m_B}{z_{AB}^3} n^a_{AB} \\
&\quad + \sum_{B\neq A}\sum_{C\neq A,B} \frac{G^2 m_B m_C}{z_{AB}^2 z_{BC}} n^a_{AB}, && (5.4.13)\\
\partial^a \ddot{X}_{\rm ext} &= -\sum_{B\neq A}\sum_{C\neq A,B} \frac{G^2 m_B m_C}{z_{AB} z_{BC}^2} \bigl(\boldsymbol{n}_{AB}\cdot\boldsymbol{n}_{BC}\bigr) n^a_{AB} \\
&\quad + \sum_{B\neq A}\sum_{C\neq A,B} \frac{G^2 m_B m_C}{z_{AB} z_{BC}^2} n^a_{BC} \\
&\quad - \sum_{B\neq A} \frac{Gm_B}{z_{AB}^2} \Bigl[v_B^2 - 3\bigl(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_B\bigr)^2\Bigr] n^a_{AB} \\
&\quad - 2\sum_{B\neq A} \frac{Gm_B}{z_{AB}^2} \bigl(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_B\bigr) v^a_B, && (5.4.14)\\
\dot{U}^a_{\rm ext} &= \sum_{B\neq A} \frac{G^2 m_A m_B}{z_{AB}^3} n^a_{AB} - \sum_{B\neq A}\sum_{C\neq A,B} \frac{G^2 m_B m_C}{z_{AB} z_{BC}^2} n^a_{BC} \\
&\quad + \sum_{B\neq A} \frac{Gm_B}{z_{AB}^2} \bigl(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_B\bigr) v^a_B, && (5.4.15)\\
\partial^a U^b_{\rm ext} &= -\sum_{B\neq A} \frac{Gm_B}{z_{AB}^2} n^a_{AB} v^b_B. && (5.4.16)
\end{aligned}$$

The derivatives of the external potentials are now written explicitly in terms of the masses m_A, the position vectors $\boldsymbol{z}_A$, and the velocity vectors $\boldsymbol{v}_A$. We recall that $z_{AB} := |\boldsymbol{z}_A - \boldsymbol{z}_B|$ is the distance between bodies A and B, and that $\boldsymbol{n}_{AB} := (\boldsymbol{z}_A - \boldsymbol{z}_B)/|\boldsymbol{z}_A - \boldsymbol{z}_B|$ is a unit vector that points from body B to body A.

5.4.4 Equations of motion: Final answer

It is now straightforward to substitute Eqs. (5.4.10)–(5.4.16) into Eq. (5.4.1) for $\boldsymbol{a}_A$. Noting that we must also let $\boldsymbol{v}$ become $\boldsymbol{v}_A$ in this equation, we obtain our final expression

$$\begin{aligned}
\boldsymbol{a}_A &= -\sum_{B\neq A} \frac{Gm_B}{z_{AB}^2} \boldsymbol{n}_{AB} \\
&\quad + \frac{1}{c^2}\Biggl\{ -\sum_{B\neq A} \frac{Gm_B}{z_{AB}^2} \biggl[v_A^2 + 2v_B^2 - 4\bigl(\boldsymbol{v}_A\cdot\boldsymbol{v}_B\bigr) - \frac{3}{2}\bigl(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_B\bigr)^2 \\
&\qquad - \frac{5Gm_A}{z_{AB}} - \frac{4Gm_B}{z_{AB}} - 4\sum_{C\neq A,B} \frac{Gm_C}{z_{AC}} - \sum_{C\neq A,B} \frac{Gm_C}{z_{BC}}
\end{aligned}$$

$$+ \frac{1}{2} \sum_{C \neq A,B} \frac{G m_C z_{AB}}{z_{BC}^2} (\boldsymbol{n}_{AB} \cdot \boldsymbol{n}_{BC}) \Big] \boldsymbol{n}_{AB}$$

$$+ \sum_{B \neq A} \frac{G m_B}{z_{AB}^2} \boldsymbol{n}_{AB} \cdot (4\boldsymbol{v}_A - 3\boldsymbol{v}_B)(\boldsymbol{v}_A - \boldsymbol{v}_B)$$

$$- \frac{7}{2} \sum_{B \neq A} \sum_{C \neq A,B} \frac{G^2 m_B m_C}{z_{AB} z_{BC}^2} \boldsymbol{n}_{BC} \Big\} + O(c^{-4}), \tag{5.4.17}$$

where

$$z_{AB} := |\boldsymbol{z}_A - \boldsymbol{z}_B| \tag{5.4.18}$$

is the distance between bodies A and B, and

$$\boldsymbol{n}_{AB} := \frac{\boldsymbol{z}_A - \boldsymbol{z}_B}{|\boldsymbol{z}_A - \boldsymbol{z}_B|} \tag{5.4.19}$$

is a unit vector that points from body B to body A.

Equations (5.4.17) are the standard post-Newtonian equations of motion for a system of N point masses. These equations were first obtained in 1917 by Lorentz and Droste, and they were made famous by Einstein, Infeld, and Hoffmann, who rederived them in 1938. Our expression in Eq. (5.4.17) differs only superficially from the equation displayed in Exercise 39.15 of Misner, Thorne, and Wheeler; it is easy to show that these are equivalent by rearranging some of the double sums.

5.4.5 Post-Newtonian barycentre

In Sec. 4.4.1 we introduced

$$M := \sum_A m_A \biggl(1 + \frac{1}{2} \frac{v_A^2}{c^2} - \frac{1}{2} \frac{\lfloor U \rfloor_A}{c^2} \biggr) + O(c^{-4}) \tag{5.4.20}$$

as the total post-Newtonian gravitational mass of the N-body system, and

$$\boldsymbol{Z} := \frac{1}{M} \sum_A m_A \biggl(1 + \frac{1}{2} \frac{v_A^2}{c^2} - \frac{1}{2} \frac{\lfloor U \rfloor_A}{c^2} \biggr) \boldsymbol{z}_A + O(c^{-4}) \tag{5.4.21}$$

as the position vector of the post-Newtonian barycentre. Here,

$$\lfloor U \rfloor_A := \sum_{B \neq A} \frac{G m_B}{z_{AB}} \tag{5.4.22}$$

is the Newtonian potential external to body A.

It is easy to show that M is conserved by virtue of the *Newtonian* equations of motion,

$$\dot{M} = O(c^{-4}). \tag{5.4.23}$$

It is also possible to show that the barycentre's acceleration vanishes by virtue of the *post-Newtonian* equations of motion,

$$\ddot{\boldsymbol{Z}} = O(c^{-4}). \tag{5.4.24}$$

The barycentre therefore moves freely, and $\boldsymbol{Z}$ can be set equal to zero by placing the origin of the coordinate system at the post-Newtonian barycentre.

To see how Eq. (5.4.24) comes about, we differentiate Eq. (5.4.21) with respect to time and replace, within the terms of order c^{-2}, all occurrences of the acceleration

vector $\boldsymbol{a}_A$ by its Newtonian expression. This produces the equation

$$\begin{aligned} M\dot{\boldsymbol{Z}} &= \sum_A m_a \boldsymbol{v}_A + \frac{1}{c^2}\biggl\{\frac{1}{2}\sum_A m_A v_A^2 \boldsymbol{v}_A \\ &\quad - \frac{1}{2}\sum_A\sum_{B\neq A}\frac{Gm_Am_B}{z_{AB}}\Bigl[\boldsymbol{v}_A + \bigl(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_A\bigr)\boldsymbol{n}_{AB}\Bigr]\biggr\} \\ &\quad + O(c^{-4}) \end{aligned} \tag{5.4.25}$$

for the barycentre's velocity vector. An additional differentiation gives $M\ddot{\boldsymbol{Z}}$, and substitution of Eq. (5.4.17) reveals (after a long computation) that indeed, $\ddot{\boldsymbol{Z}}$ vanishes at 1PN order.

5.5 Two-body dynamics

5.5.1 Two-body equations

In the special case in which the system contains only two bodies, Eq. (5.4.17) reduces to

$$\begin{aligned} \boldsymbol{a}_1 &= -\frac{Gm_2}{z^2}\boldsymbol{n} \\ &\quad + \frac{1}{c^2}\biggl\{-\frac{Gm_2}{z^2}\biggl[v_1^2 + 2v_2^2 - 4\bigl(\boldsymbol{v}_1\cdot\boldsymbol{v}_2\bigr) - \frac{3}{2}\bigl(\boldsymbol{n}\cdot\boldsymbol{v}_2\bigr)^2 - \frac{5Gm_1}{z} - \frac{4Gm_2}{z}\biggr]\boldsymbol{n} \\ &\qquad + \frac{Gm_2}{z^2}\boldsymbol{n}\cdot\bigl(4\boldsymbol{v}_1 - 3\boldsymbol{v}_2\bigr)\bigl(\boldsymbol{v}_1 - \boldsymbol{v}_2\bigr)\biggr\} + O(c^{-4}), \end{aligned} \tag{5.5.1}$$

and

$$\begin{aligned} \boldsymbol{a}_2 &= \frac{Gm_1}{z^2}\boldsymbol{n} \\ &\quad + \frac{1}{c^2}\biggl\{\frac{Gm_1}{z^2}\biggl[v_2^2 + 2v_1^2 - 4\bigl(\boldsymbol{v}_1\cdot\boldsymbol{v}_2\bigr) - \frac{3}{2}\bigl(\boldsymbol{n}\cdot\boldsymbol{v}_1\bigr)^2 - \frac{4Gm_1}{z} - \frac{5Gm_2}{z}\biggr]\boldsymbol{n} \\ &\qquad + \frac{Gm_1}{z^2}\boldsymbol{n}\cdot\bigl(4\boldsymbol{v}_2 - 3\boldsymbol{v}_1\bigr)\bigl(\boldsymbol{v}_1 - \boldsymbol{v}_2\bigr)\biggr\} + O(c^{-4}), \end{aligned} \tag{5.5.2}$$

where

$$\boldsymbol{z} := \boldsymbol{z}_1 - \boldsymbol{z}_2 \tag{5.5.3}$$

is the position of body 1 relative to body 2,

$$z := |\boldsymbol{z}| = |\boldsymbol{z}_1 - \boldsymbol{z}_2| \tag{5.5.4}$$

is the distance between the two bodies, and

$$\boldsymbol{n} := \frac{\boldsymbol{z}}{z} = \frac{\boldsymbol{z}_1 - \boldsymbol{z}_2}{|\boldsymbol{z}_1 - \boldsymbol{z}_2|} \tag{5.5.5}$$

is a unit vector that points from body 2 to body 1.

In this two-body context, Eq. (5.4.20) becomes

$$M = m_1 + m_2 + \frac{1}{c^2}\biggl\{\frac{1}{2}\bigl(m_1v_1^2 + m_2v_2^2\bigr) - \frac{Gm_1m_2}{z}\biggr\} + O(c^{-4}), \tag{5.5.6}$$

and Eq. (5.4.21) reduces to

$$\begin{aligned} M\boldsymbol{Z} &= m_1\boldsymbol{z}_1 + m_2\boldsymbol{z}_2 + \frac{1}{c^2}\biggl\{\frac{1}{2}\bigl(m_1v_1^2\boldsymbol{z}_1 + m_2v_2^2\boldsymbol{z}_2\bigr) - \frac{Gm_1m_2}{2z}\bigl(\boldsymbol{z}_1 + \boldsymbol{z}_2\bigr)\biggr\} \\ &\quad + O(c^{-4}). \end{aligned} \tag{5.5.7}$$

5.5.2 Dynamics of the relative system

It is useful at this stage to impose the barycentre condition

$$\boldsymbol{Z} = \boldsymbol{0}, \tag{5.5.8}$$

and to express $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ in terms of $\boldsymbol{z} := \boldsymbol{z}_1 - \boldsymbol{z}_2$, the relative separation vector. At Newtonian order we get the usual relations $\boldsymbol{z}_1 = (m_2/m)\boldsymbol{z} + O(c^{-2})$ and $\boldsymbol{z}_2 = -(m_1/m)\boldsymbol{z} + O(c^{-2})$, where $m := m_1 + m_2$. These imply $\boldsymbol{v}_1 = (m_2/m)\boldsymbol{v} + O(c^{-2})$ and $\boldsymbol{v}_2 = -(m_1/m)\boldsymbol{v} + O(c^{-2})$, where $\boldsymbol{v} := d\boldsymbol{z}/dt = \boldsymbol{v}_1 - \boldsymbol{v}_2$ is the relative velocity vector. These Newtonian relations can be inserted within the post-Newtonian terms in Eqs. (5.5.6) and (5.5.7), and this produces

$$M = m\biggl\{1 + \frac{1}{c^2}\biggl[\frac{1}{2}\eta v^2 - \frac{G\eta m}{z}\biggr] + O(c^{-4})\biggr\} \tag{5.5.9}$$

for the total gravitational mass, and

$$\boldsymbol{0} = m_1\boldsymbol{z}_1 + m_2\boldsymbol{z}_2 - \frac{\eta\Delta m}{2c^2}\biggl(v^2 - \frac{Gm}{z}\biggr)\boldsymbol{z} + O(c^{-4}) \tag{5.5.10}$$

for the barycentre condition.

We have introduced the mass parameters

$$m \;:=\; m_1 + m_2 \quad : \quad \text{total mass,} \tag{5.5.11}$$

$$\eta \;:=\; \frac{m_1 m_2}{(m_1+m_2)^2} \quad : \quad \text{dimensionless reduced mass,} \tag{5.5.12}$$

$$\Delta \;:=\; \frac{m_1 - m_2}{m_1 + m_2} \quad : \quad \text{dimensionless mass difference.} \tag{5.5.13}$$

The solutions to Eq. (5.5.10) and $\boldsymbol{z} = \boldsymbol{z}_1 - \boldsymbol{z}_2$ are

$$\boldsymbol{z}_1 = \frac{m_2}{m}\boldsymbol{z} + \frac{\eta\Delta}{2c^2}\biggl(v^2 - \frac{Gm}{z}\biggr)\boldsymbol{z} + O(c^{-4}) \tag{5.5.14}$$

and

$$\boldsymbol{z}_2 = -\frac{m_1}{m}\boldsymbol{z} + \frac{\eta\Delta}{2c^2}\biggl(v^2 - \frac{Gm}{z}\biggr)\boldsymbol{z} + O(c^{-4}). \tag{5.5.15}$$

We recall the definitions

$$\boldsymbol{z} := \boldsymbol{z}_1 - \boldsymbol{z}_2, \qquad \boldsymbol{v} := \boldsymbol{v}_1 - \boldsymbol{v}_2 \tag{5.5.16}$$

for the relative position and velocity vectors, respectively.

Subtracting Eq. (5.5.2) from Eq. (5.5.1) produces an expression for

$$\boldsymbol{a} := \boldsymbol{a}_1 - \boldsymbol{a}_2, \tag{5.5.17}$$

the relative acceleration vector. After some simple manipulations, we obtain

$$\begin{aligned}\boldsymbol{a} \;=\; & -\frac{Gm}{z^2}\boldsymbol{n} \\ & + \frac{1}{c^2}\biggl\{-\frac{Gm}{z^2}\biggl[(1+3\eta)v^2 - \frac{3}{2}\eta(\boldsymbol{n}\cdot\boldsymbol{v})^2 - 2(2+\eta)\frac{Gm}{z}\biggr]\boldsymbol{n} \\ & \qquad + 2(2-\eta)\frac{Gm}{z^2}(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{v}\biggr\} + O(c^{-4}).\end{aligned} \tag{5.5.18}$$

This, together with the definition $\boldsymbol{a} = d^2\boldsymbol{z}/dt^2$, gives us an evolution equation for $\boldsymbol{z}$, the relative position vector. Once $\boldsymbol{z}(t)$ is known, the position of each body is determined by Eqs. (5.5.14) and (5.5.15).

5.5.3 Lagrangian and conserved quantities

The post-Newtonian equations of motion for $\boldsymbol{z}$, given by Eq. (5.5.18), can be reproduced on the basis of the Lagrangian $L = \eta m \tilde{L}$, where

$$\begin{aligned}\tilde{L} &= \frac{1}{2}v^2 + \frac{Gm}{z} \\ &\quad + \frac{1}{c^2}\left\{\frac{1}{8}(1-3\eta)v^4 + \frac{1}{2}(3+\eta)\frac{Gm}{z}v^2 + \frac{G\eta m}{2z}(\boldsymbol{n}\cdot\boldsymbol{v})^2 - \frac{G^2m^2}{2z^2}\right\} \\ &\quad + O(c^{-4}) \end{aligned} \tag{5.5.19}$$

is the Lagrangian per unit of reduced mass ηm.

The generalized momentum $\boldsymbol{p} := \partial\tilde{L}/\partial\boldsymbol{v}$ associated with this Lagrangian is

$$\boldsymbol{p} = \boldsymbol{v} + \frac{1}{c^2}\left[\frac{1}{2}(1-3\eta)v^2 + (3+\eta)\frac{Gm}{z}\right]\boldsymbol{v} + \frac{G\eta m}{c^2 z}(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{n} + O(c^{-4}), \tag{5.5.20}$$

and it follows that

$$\begin{aligned}\frac{d\boldsymbol{p}}{dt} &= \boldsymbol{a} - \frac{Gm}{c^2z^2}\left[\frac{1}{2}(1-5\eta)v^2 + 3\eta\dot{z}^2 + (3+2\eta)\frac{Gm}{z}\right]\boldsymbol{n} - \frac{Gm}{c^2z^2}(4-3\eta)\dot{z}\boldsymbol{v} \\ &\quad + O(c^{-4}), \end{aligned} \tag{5.5.21}$$

where $\dot{z} := dz/dt = \boldsymbol{n}\cdot\boldsymbol{v}$. To arrive at Eq. (5.5.21) we involved the identity $\dot{n}^a = (v^a - \dot{z}n^a)/z$ and inserted the Newtonian expression for the acceleration, $\boldsymbol{a} = -Gm\boldsymbol{n}/z^2 + O(c^{-2})$, within the post-Newtonian terms; as a consequence of this equation, we have that $\ddot{z} = (v^2 - \dot{z}^2 - Gm/z)/z + O(c^{-2})$, and this also was required to obtain Eq. (5.5.21).

On the other hand,

$$\frac{\partial\tilde{L}}{\partial\boldsymbol{z}} = -\frac{Gm}{z^2}\boldsymbol{n} - \frac{Gm}{c^2z^2}\left[\frac{1}{2}(3+\eta)v^2 + \frac{3}{2}\eta\dot{z}^2 - \frac{Gm}{z}\right]\boldsymbol{n} + \frac{G\eta m}{c^2z^2}\dot{r}\boldsymbol{v} + O(c^{-4}), \tag{5.5.22}$$

and Eq. (5.5.18) follows directly from the Euler-Lagrange equations, $d\boldsymbol{p}/dt = \partial\tilde{L}/\partial\boldsymbol{z}$.

The conserved energy associated with $\tilde{L}$ is $\tilde{E} = \boldsymbol{p}\cdot\boldsymbol{v} - \tilde{L}$, and according to Eqs. (5.5.19) and (5.5.20), this is

$$\begin{aligned}\tilde{E} &= \frac{1}{2}v^2 - \frac{Gm}{z} \\ &\quad + \frac{1}{c^2}\left\{\frac{3}{8}(1-3\eta)v^4 + \frac{Gm}{2z}\Big[(3+\eta)v^2 + \eta(\boldsymbol{n}\cdot\boldsymbol{v})^2\Big] + \frac{G^2m^2}{2z^2}\right\} \\ &\quad + O(c^{-4}). \end{aligned} \tag{5.5.23}$$

The system's actual conserved energy is $E = \eta m\tilde{E}$, and this *excludes* the rest-mass energy of each body. The angular momentum associated with $\tilde{L}$ is $\boldsymbol{J} = \boldsymbol{z}\times\boldsymbol{p}$, and this is given by

$$\boldsymbol{J} = \left\{1 + \frac{1}{c^2}\left[\frac{1}{2}(1-3\eta)v^2 + (3+\eta)\frac{Gm}{z}\right] + O(c^{-4})\right\}\boldsymbol{z}\times\boldsymbol{v}; \tag{5.5.24}$$

this must be multiplied by ηm to obtain the actual angular-momentum vector of the two-body system. It is a straightforward exercise to show that $d\boldsymbol{J}/dt = \boldsymbol{0}$, so that the angular-momentum vector is a constant of the post-Newtonian motion.

5.5.4 *Orbital equations*

The facts that $\boldsymbol{J}$, as defined by Eq. (5.5.24), is a constant vector, and that it is at all times orthogonal to both $\boldsymbol{z}$ and $\boldsymbol{v}$, imply that the motion of each body must take place within a fixed orbital plane. We take this plane to be the x-y plane, and we use polar coordinates z and ψ to describe the orbital motion (z should not be confused with the third coordinate of the Cartesian system; it continues to stand for the distance between the two bodies). We write $\boldsymbol{z} = [z\cos\psi, z\sin\psi, 0]$, and we resolve all vectors in the basis $\boldsymbol{n} = [\cos\psi, \sin\psi, 0]$ and $\boldsymbol{\psi} = [-\sin\psi, \cos\psi, 0]$ associated with the polar coordinates. We have

$$\boldsymbol{z} = z\boldsymbol{n}, \tag{5.5.25}$$

$$\boldsymbol{v} = \dot{z}\boldsymbol{n} + (z\dot{\psi})\boldsymbol{\psi}, \tag{5.5.26}$$

$$\boldsymbol{a} = \left(\ddot{z} - z\dot{\psi}^2\right)\boldsymbol{n} + \frac{1}{z}\frac{d}{dt}(z^2\dot{\psi})\boldsymbol{\psi}, \tag{5.5.27}$$

and Eq. (5.5.18) gives rise to the set of equations

$$\ddot{z} - z\dot{\psi}^2 + \frac{Gm}{z^2} = \frac{Gm}{c^2 z^2}\left[\left(3 - \frac{7\eta}{2}\right)\dot{z}^2 - (1+3\eta)\left(z\dot{\psi}\right)^2 + 2(2+\eta)\frac{Gm}{z}\right] + O(c^{-4}) \tag{5.5.28}$$

and

$$\frac{d}{dt}(z^2\dot{\psi}) = 2(2-\eta)\frac{Gm}{c^2}\dot{z}\dot{\psi} + O(c^{-4}). \tag{5.5.29}$$

In Newtonian theory, the right-hand sides of Eqs. (5.5.28) and (5.5.29) are zero, and the orbital equations are those of *Kepler's problem.* In post-Newtonian theory, the orbital equations contain small corrections of fractional order $(v/c)^2 \sim Gm/(c^2 z)$, and the motion is no longer Keplerian. Because the corrections are small, the equations can be handled by any suitable perturbation technique of celestial mechanics. This leads, for example, to the well-known prediction that the angular position of the post-Newtonian periastron advances by an amount equal to $6\pi Gm/(c^2 p)$ per orbit, where p is the semilatus rectum of the Keplerian orbit.

Chapter 6
Gravitational waves

In this chapter we formulate a post-Newtonian theory of gravitational waves. We follow closely the general methods devised by Will and Wiseman (1996), and rely on many results obtained in preceding chapters. We begin in Sec. 6.1 with an examination of the gravitational potentials in the far-away wave zone. We show that it is always possible to refine the harmonic gauge into a transverse-tracefree (TT) gauge that is ideally suited to the description of gravitational radiation, and we introduce efficient techniques to carry out the TT projection. In Sec. 6.2 we launch a calculation of the gravitational-wave field to $\frac{3}{2}$PN order beyond the leading, Newtonian expression. This calculation is extremely lengthy, and it occupies the bulk of the chapter, from Sec. 6.3 to Sec. 6.10. The results obtained in these sections apply to a general N-body system, but they are stated in somewhat abstract terms. In Sec. 6.11 we convert them into more concrete expressions by specializing to a two-body system, and in Sec. 6.12 we specialize them even further to the case of circular orbits. At this stage our results are fully explicit, and expressions for h_+ and $h_\times$, the two gravitational-wave polarizations, are given in Eqs. (6.12.17) and (6.12.18), respectively.

6.1 Far-away wave zone and TT gauge

6.1.1 Gravitational potentials in the far-away wave zone

The notion of a wave zone was introduced back in Sec. 2.2; this is the region of three-dimensional space in which $r := |\boldsymbol{x}|$ is much larger than λ_c, the characteristic wavelength of the gravitational-wave field. The notion of a far-away wave zone was introduced back in Sec. 2.4.1; this is a neighbourhood of future null infinity in which the r^{-1} part of the gravitational potentials $h^{\alpha\beta}$ dominates over the parts

that fall off as r^{-2} and faster. As we shall see below, the gravitational-wave field is the transverse-tracefree (TT) piece of these r^{-1} potentials; this is what we aim to calculate in this chapter.

We constructed wave-zone potentials back in Sec. 4.4 (in the second post-Minkowskian approximation), and from the summary provided in Sec. 4.4.7 we gather that their behaviour in the far-away wave zone is given by

$$h^{00} = \frac{4GM}{c^2 r} + \frac{G}{c^4 r} C(\tau, \boldsymbol{\Omega}) + O(r^{-2}), \tag{6.1.1}$$

$$h^{0a} = \frac{G}{c^4 r} D^a(\tau, \boldsymbol{\Omega}) + O(r^{-2}), \tag{6.1.2}$$

$$h^{ab} = \frac{G}{c^4 r} A^{ab}(\tau, \boldsymbol{\Omega}) + O(r^{-2}). \tag{6.1.3}$$

Here, M is the total gravitational mass of Eq. (4.4.40), while C, D^a, and A^{ab} are functions of retarded-time

$$\tau := t - r/c \tag{6.1.4}$$

and of the unit vector $\boldsymbol{\Omega} := \boldsymbol{x}/r$. The functions C, D^a, and A^{ab} were calculated in the second post-Minkowskian approximation in Sec. 4.4, but we shall not need their precise form here. In fact, the validity of Eqs. (6.1.1)–(6.1.3) extends beyond the post-Minkowskian domain of Sec. 4.4. Indeed, it is easy to show that these equations provide solutions to the wave equations $\Box h^{\alpha\beta} = 16\pi G \tau^{\alpha\beta}/c^4$ provided only that $\tau^{\alpha\beta}$, the effective energy-momentum pseudotensor, falls off at least as fast as r^{-2}. The impact of the harmonic gauge conditions $\partial_\beta h^{\alpha\beta} = 0$ on these solutions will be examined below.

6.1.2 Decomposition into irreducible components

To proceed it is useful to decompose the vector D^a and the tensor A^{ab} into their irreducible components. We write

$$D^a = D\Omega^a + D^a_{\mathrm{T}}, \tag{6.1.5}$$

with $D\Omega^a$ representing the longitudinal part of D^a, and D^a_{T} its transverse part; this is required to satisfy

$$\Omega_a D^a_{\mathrm{T}} = 0. \tag{6.1.6}$$

The three components of D^a are therefore partitioned into one longitudinal component D, and two transverse components contained in D^a_{T}; these are functions of τ and $\boldsymbol{\Omega}$. Similarly, we write

$$A^{ab} = \frac{1}{3}\delta^{ab} A + \Bigl(\Omega^a\Omega^b - \frac{1}{3}\delta^{ab}\Bigr) B + \Omega^a A^b_{\mathrm{T}} + \Omega^b A^a_{\mathrm{T}} + A^{ab}_{\mathrm{TT}}, \tag{6.1.7}$$

which is a breakdown of A^{ab} into a trace part $\frac{1}{3}\delta^{ab}A$, a longitudinal-tracefree part $(\Omega^a\Omega^b - \frac{1}{3}\delta^{ab})B$, a longitudinal-transverse part $\Omega^a A^b_{\mathrm{T}} + \Omega^b A^a_{\mathrm{T}}$, and a transverse-tracefree part A^{ab}_{TT}; these also are functions of τ and $\boldsymbol{\Omega}$. We impose the constraints

$$\Omega_a A^a_{\mathrm{T}} = 0 \tag{6.1.8}$$

and

$$\Omega_a A^{ab}_{\mathrm{TT}} = 0 = \delta_{ab} A^{ab}_{\mathrm{TT}}, \tag{6.1.9}$$

so that the six independent components of A^{ab} are contained in two scalars A and B, two components of a transverse vector A^a_{T}, and two components a transverse-tracefree tensor A^{ab}_{TT}. The last term in Eq. (6.1.7) is called the transverse-tracefree part, or TT part, of A^{ab}. As we shall see, the radiative part of the gravitational potentials are contained entirely within A^{ab}_{TT}.

6.1.3 Harmonic gauge conditions

The harmonic gauge conditions are

$$c^{-1}\dot{h}^{00} + \partial_b h^{0b} = 0, \qquad c^{-1}\dot{h}^{0a} + \partial_b h^{ab} = 0,$$

in which an overdot indicates differentiation with respect to τ. Spatial derivatives have a simple structure in the far-away wave zone. Going back to Eq. (1.8.5), we see that $\partial_a r^{-1} = O(r^{-2})$ and $\partial_a \Omega_b = O(r^{-1})$, but that $\partial_a r = \Omega_a$. It follows that when ∂_a is acting on the gravitational potentials of Eq. (6.1.1)–(6.1.3), the only term that survives comes from

$$\partial_a \tau = -c^{-1}\Omega_a. \tag{6.1.10}$$

We therefore have $\partial_a h^{\alpha\beta} = -c^{-1}\dot{h}^{\alpha\beta}\Omega_a + O(r^{-2})$, and the gauge conditions reduce to

$$\dot{h}^{00} - \Omega_b \dot{h}^{0b} = 0, \qquad \dot{h}^{0a} - \Omega_b \dot{h}^{ab} = 0 \tag{6.1.11}$$

in the far-away wave zone.

After substituting Eqs. (6.1.5) and (6.1.7) into Eqs. (6.1.1)–(6.1.3), and these into Eqs. (6.1.11), we find that the harmonic gauge conditions imply

$$C = D, \tag{6.1.12}$$
$$D = \frac{1}{3}A + \frac{2}{3}B, \tag{6.1.13}$$
$$D^a_{\mathrm{T}} = A^a_{\mathrm{T}}. \tag{6.1.14}$$

We have set the constants of integration to zero, because an eventual τ-independent term in C would correspond to an unphysical shift in the gravitational mass M, while a τ-independent term in D^a would be incompatible with Eq. (4.4.38) — the time-independent part of h^{0a} is associated with the spacetime's total angular momentum, and it must fall off as r^{-2}.

Incorporating these constraints, the gravitational potentials become

$$h^{00} = \frac{4GM}{c^2 r} + \frac{G}{c^4 r}\frac{1}{3}\big(A + 2B\big) + O(r^{-2}), \tag{6.1.15}$$
$$h^{0a} = \frac{G}{c^4 r}\left[\frac{1}{3}(A+2B)\Omega^a + A^a_{\mathrm{T}}\right] + O(r^{-2}), \tag{6.1.16}$$
$$h^{ab} = \frac{G}{c^4 r}\left[\frac{1}{3}\delta^{ab}A + \Big(\Omega^a\Omega^b - \frac{1}{3}\delta^{ab}\Big)B + \Omega^a A^b_{\mathrm{T}} + \Omega^b A^a_{\mathrm{T}} + A^{ab}_{\mathrm{TT}}\right] + O(r^{-2}), \tag{6.1.17}$$

in which A, B, A^a_{T}, and A^{ab}_{TT} are functions of τ and $\boldsymbol{\Omega}$. We have now a total of six independent quantities: one in A, another in B, two in A^a_{T}, and two more in A^{ab}_{TT}. The harmonic gauge conditions have eliminated four redundant quantities.

6.1.4 Transformation to the TT gauge

It is possible, in the far-away wave zone, to specialize the harmonic gauge even further, and to eliminate four additional redundant quantities. We wish to implement a gauge transformation that is generated by a four-vector field $\xi^\alpha(t, \boldsymbol{x})$. It is well-known that if the spacetime metric is expressed as $g_{\alpha\beta} = \eta_{\alpha\beta} + \delta g_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is the Minkowski metric and $\delta g_{\alpha\beta}$ is a perturbation, then the gauge transformation produces the change

$$\delta g_{\alpha\beta} \to \delta g_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha,$$

where $\xi_\alpha := \eta_{\alpha\beta}\xi^\beta$. In the far-away wave zone we can neglect terms quadratic in $h^{\alpha\beta}$ (because they fall off as r^{-2}), and Eq. (1.6.4) reduces to

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta} + O(r^{-2}),$$

where $h_{\alpha\beta} = \eta_{\alpha\mu}\eta_{\beta\nu}h^{\mu\nu}$ and $h = \eta_{\mu\nu}h^{\mu\nu}$. It follows that $h_{\alpha\beta} = \delta g_{\alpha\beta} - \frac{1}{2}\delta g\eta_{\alpha\beta}$, where $\delta g = \eta^{\mu\nu}\delta g_{\mu\nu}$, and that the gauge transformation produces the change

$$h^{\alpha\beta} \to h^{\alpha\beta} - \partial^\alpha\xi^\beta - \partial^\beta\xi^\alpha + \bigl(\partial_\mu\xi^\mu\bigr)\eta_{\alpha\beta} \tag{6.1.18}$$

in the gravitational potentials. It follows from Eq. (6.1.18) that the harmonic gauge conditions will be preserved whenever the vector field satisfies the wave equation $\Box\xi^\alpha = 0$, because $\partial_\beta h^{\alpha\beta} \to \partial_\beta h^{\alpha\beta} - \Box\xi^\alpha$.

We wish to preserve the harmonic gauge, and we construct a solution to the wave equation by writing

$$\xi^0 = \frac{G}{c^3 r}\alpha(\tau, \mathbf{\Omega}) + O(r^{-2}), \tag{6.1.19}$$

$$\xi^a = \frac{G}{c^3 r}\beta^a(\tau, \mathbf{\Omega}) + O(r^{-2}), \tag{6.1.20}$$

where α and β^a are arbitrary functions of their arguments. As before we decompose the vector in terms of its irreducible components,

$$\beta^a = \beta\Omega^a + \beta^a_{\mathrm{T}}, \qquad \Omega_a\beta^a_{\mathrm{T}} = 0. \tag{6.1.21}$$

We differentiate ξ^0 and ξ^a using the rules spelled out in Sec. 6.1.3, and we insert the results within Eq. (6.1.18). After also involving Eqs. (6.1.15)–(6.1.17), we eventually deduce that the gauge transformation produces the changes

$$A \to A + 3\dot\alpha - \dot\beta, \tag{6.1.22}$$

$$B \to B + 2\dot\beta, \tag{6.1.23}$$

$$A^a_{\mathrm{T}} \to A^a_{\mathrm{T}} + \dot\beta^a_{\mathrm{T}}, \tag{6.1.24}$$

$$A^{ab}_{\mathrm{TT}} \to A^{ab}_{\mathrm{TT}} \tag{6.1.25}$$

in the irreducible pieces of the gravitational potentials.

We see that the transverse-tracefree part of A^{ab} is invariant under the gauge transformation. We see also that α, β, and β^a_{T} can be chosen so as to set A, B, and A^a_{T} equal to zero. Implementing this gauge transformation, we arrive at the simplest expressions for the gravitational potentials in the far-away wave zone:

$$h^{00} = \frac{4GM}{c^2 r} + O(r^{-2}), \tag{6.1.26}$$

$$h^{0a} = O(r^{-2}), \tag{6.1.27}$$

$$h^{ab} = \frac{G}{c^4 r}A^{ab}_{\mathrm{TT}}(\tau, \mathbf{\Omega}) + O(r^{-2}). \tag{6.1.28}$$

By virtue of the conditions imposed in Eq. (6.1.9),

$$\Omega_a A^{ab}_{\mathrm{TT}} = 0 = \delta_{ab}A^{ab}_{\mathrm{TT}},$$

the number of time-dependent quantities has been reduced to two. The gravitational potentials of Eqs. (6.1.26)–(6.1.28) are said to be in the *transverse-tracefree gauge*, or *TT gauge*, a specialization of the harmonic gauge that can be achieved in the far-away wave zone. It is clear that the radiative degrees of freedom of the gravitational field must be contained in the two independent components of A^{ab}_{TT}.

6.1.5 Geodesic deviation

This conclusion, that $A^{ab}_{\rm TT}$ contains the radiative degrees of freedom, is reinforced by the following argument. Suppose that a gravitational-wave detector consists of two test masses that are moving freely in the far-away wave zone. The masses are separated by a spacetime vector η^α, and they move with a four-velocity u^α. Assuming that the distance between the masses is small compared with the radiation's characteristic wavelength (this defines a short gravitational-wave detector such as the LIGO instrument), the behaviour of the separation vector is governed by the equation of geodesic deviation

$$\frac{D^2\eta^\alpha}{ds^2} = -R^\alpha_{\ \beta\gamma\delta} u^\beta \eta^\gamma u^\delta,$$

in which D/ds indicates covariant differentiation in the direction of u^α, and where $R^\alpha_{\ \beta\gamma\delta}$ is the Riemann tensor. Assuming in addition that the test masses are moving slowly, this equation reduces to

$$\frac{d^2\eta^a}{dt^2} = -c^2 R^a_{\ 0b0}\eta^b,$$

which involves ordinary differentiations with respect to $t = x^0/c$, as well as the spatial components of the separation vector.

It is a straightforward exercise to compute the Riemann tensor associated with the metric $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta}$, even when the gravitational potentials are expressed in their general form of Eqs. (6.1.15)–(6.1.17). Alternatively, one can proceed instead from Eqs. (6.1.26)–(6.1.28) and appeal to the fact that the Riemann tensor is invariant under gauge transformations. In any event, the computation reveals that

$$c^2 R^a_{\ 0b0} = -\frac{G}{2c^4 r}\ddot{A}^{ab}_{\rm TT} + O(r^{-2}),$$

and the equation of geodesic deviation becomes

$$\frac{d^2\eta^a}{dt^2} = \frac{G}{2c^4 r}\ddot{A}^{ab}_{\rm TT}\eta_b + O(r^{-2}) = \frac{1}{2}\ddot{h}^{ab}_{\rm TT} + O(c^{-2}).$$

We conclude that our gravitational-wave detector is driven by the TT part of the gravitational potentials, and that the other pieces of the potentials contain no radiative information.

6.1.6 Extraction of the TT part

Given gravitational potentials presented in the general form of Eq. (6.1.3),

$$h^{ab} = \frac{G}{c^4 r}A^{ab}(\tau, \mathbf{\Omega}) + O(r^{-2}), \tag{6.1.29}$$

the radiative pieces can be extracted by isolating the transverse-tracefree part of A^{ab}. This can be done efficiently by introducing the TT projector $(\textsc{tt})^{ab}_{\ \ cd}$, and by writing

$$A^{ab}_{\rm TT} = (\textsc{tt})^{ab}_{\ \ cd}A^{cd}. \tag{6.1.30}$$

The TT projector is constructed as follows. We first introduce the transverse projector

$$P^a_{\ b} := \delta^a_{\ b} - \Omega^a\Omega_b, \tag{6.1.31}$$

which removes the longitudinal components of vectors and tensors. For example, for a vector $A^a = A\Omega^a + A^a_{\rm T}$ with $\Omega_a A^a_{\rm T} = 0$, we have that $P^a_{\ b}A^b = A^a_{\rm T}$. The transverse projector satisfies

$$P^a_{\ b}\Omega^b = 0, \qquad P^a_{\ a} = 2, \qquad P^a_{\ c}P^c_{\ b} = P^a_{\ b}. \tag{6.1.32}$$

The TT projector is obtained by acting with the transverse projector twice and removing the trace:

$$(\mathrm{TT})^{ab}{}_{cd} := P^a{}_c P^b{}_d - \frac{1}{2} P^{ab} P_{cd}. \tag{6.1.33}$$

It is easy to see that this possesses the required properties. First, $(\mathrm{TT})^{ab}{}_{cd}\Omega^d = 0$; second, $(\mathrm{TT})^{ab}{}_{cd}\delta^{cd} = 0$; and third, $(\mathrm{TT})^{ab}{}_{cd}A^{cd}_{\mathrm{TT}} = A^{ab}_{\mathrm{TT}}$ if the tensor A^{ab}_{TT} is already transverse and tracefree. For a general symmetric tensor A^{ab} decomposed as in Eq. (6.1.7), it is easy to verify that

$$(\mathrm{TT})^{ab}{}_{cd} A^{cd} = A^{ab}_{\mathrm{TT}}. \tag{6.1.34}$$

This equation informs us that the TT part of any symmetric tensor A^{ab} can be extracted by acting with the TT projector defined by Eq. (6.1.33).

It is useful to introduce a vectorial basis in the transverse subspace. Having previously introduced

$$\boldsymbol{\Omega} := [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta] \tag{6.1.35}$$

as the unit vector that points in the longitudinal direction, we introduce now the unit vectors

$$\boldsymbol{\theta} := [\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta] \tag{6.1.36}$$

and

$$\boldsymbol{\phi} := [-\sin\phi, \cos\phi, 0], \tag{6.1.37}$$

which are both orthogonal to $\boldsymbol{\Omega}$, and also orthogonal to each other. The vector $\boldsymbol{\theta}$ points in the direction of increasing θ on the two-sphere, while $\boldsymbol{\phi}$ points in the direction of increasing ϕ; they span the transverse subspace orthogonal to $\boldsymbol{\Omega}$, which points in the direction of increasing r. The basis gives us the completeness relations

$$\delta^{ab} = \Omega^a\Omega^b + \theta^a\theta^b + \phi^a\phi^b, \tag{6.1.38}$$

and it follows from Eq. (6.1.31) that the transverse projector is given by

$$P^{ab} = \theta^a\theta^b + \phi^a\phi^b. \tag{6.1.39}$$

This can be inserted within Eq. (6.1.33) to form the TT projector.

Any symmetric, transverse, and tracefree tensor A^{ab}_{TT} can be decomposed in a tensorial basis that is built entirely from the vectors $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$. Such a tensor has two independent components, which we denote A_+ and $A_\times$. We write

$$A^{ab}_{\mathrm{TT}} = A_+\big(\theta^a\theta^b - \phi^a\phi^b\big) + A_\times\big(\theta^a\phi^b + \phi^a\theta^b\big), \tag{6.1.40}$$

so that A_+ represents the θ-θ component of the tensor (and also minus the ϕ-ϕ component, in order to satisfy the tracefree condition), while $A_\times$ represents its θ-ϕ component. It is easy to check that Eq. (6.1.40) implies

$$\begin{aligned} A_+ &= \frac{1}{2}\big(\theta_a\theta_b - \phi_a\phi_b\big)A^{ab}_{\mathrm{TT}}, \\ A_\times &= \frac{1}{2}\big(\theta_a\phi_b + \phi_a\theta_b\big)A^{ab}_{\mathrm{TT}}. \end{aligned}$$

Because the tensorial operators acting on A^{ab}_{TT} are already transverse and tracefree, this can also be written as

$$A_+ = \frac{1}{2}\big(\theta_a\theta_b - \phi_a\phi_b\big)A^{ab}, \tag{6.1.41}$$

$$A_\times = \frac{1}{2}\big(\theta_a\phi_b + \phi_a\theta_b\big)A^{ab}, \tag{6.1.42}$$

in which the projection operators are acting on the original tensor A^{ab} instead of its TT part $A^{ab}_{\rm TT}$.

Equations (6.1.41) and (6.1.42), together with the definitions of Eqs. (6.1.36) and (6.1.37), provide an efficient way of extracting the transverse-tracefree components of a general tensor A^{ab}. With A_+ and $A_\times$ known, the TT part of the original tensor can be constructed with the help of Eq. (6.1.40).

6.2 Computation of h^{ab}: Strategy and requirements

We wish to integrate the wave equation

$$\Box h^{ab} = -\frac{16\pi G}{c^4}\tau^{ab} \tag{6.2.1}$$

for the spatial components of the gravitational potentials, and evaluate the solution in the far-away wave zone. Here,

$$\tau^{ab} = (-g)\bigl(T^{ab} + t^{ab}_{\rm LL} + t^{ab}_{\rm H}\bigr) \tag{6.2.2}$$

are the spatial components of the effective energy-momentum pseudotensor first introduced in Sec. 1.3, decomposed into a material contribution T^{ab}, the Landau-Lifshitz pseudotensor $t^{\alpha\beta}_{\rm LL}$, and the harmonic-gauge contribution $t^{ab}_{\rm H}$. We wish to integrate the wave equation to a degree of accuracy that surpasses what was achieved in Chapter 4, and we wish to extract from h^{ab} the transverse-tracefree pieces that truly represent the gravitational-wave field.

Techniques to integrate Eq. (6.2.1) were developed in Chapter 2. In Sec. 2.3 we learned to express the solution as

$$h^{ab} = h^{ab}_{\mathscr{N}} + h^{ab}_{\mathscr{W}}, \tag{6.2.3}$$

in terms of a near-zone retarded integral $h^{ab}_{\mathscr{N}}$ and a wave-zone integral $h^{ab}_{\mathscr{W}}$. In Sec. 2.4.1 we derived an expression for $h^{ab}_{\mathscr{N}}$ that is valid in the far-away wave zone; this is given by Eq. (2.4.5), which we copy as

$$h^{ab}_{\mathscr{N}} = \frac{4G}{c^4 r}\sum_{q=0}^{\infty}\frac{1}{q!}\Omega_Q\left(\frac{\partial}{\partial u}\right)^q\int_{\mathscr{M}}\tau^{ab}(u,\boldsymbol{x}')x'^Q\,d^3x' + O(r^{-2}), \tag{6.2.4}$$

where $u := ct - r = c(t - r/c) =: c\tau$ is a retarded-time variable, $\Omega_Q x'^Q := \Omega_{a_1}\Omega_{a_2}\cdots\Omega_{a_q}x'^{a_1}x'^{a_2}\cdots x'^{a_q} = (\boldsymbol{\Omega}\cdot\boldsymbol{x}')^q$, and where the domain of integration $\mathscr{M}$ is defined by $r' := |\boldsymbol{x}'| < \mathcal{R}$, with $r' = \mathcal{R}$ representing the boundary $\partial\mathscr{M}$ of the near and wave zones. And in Sec. 2.5.2 we devised a method to calculate $h^{ab}_{\mathscr{W}}$ when τ^{ab} can be expressed as a sum of terms of the form

$$\tau^{ab}[\ell,n] = \frac{1}{4\pi}\frac{f(u)}{r^n}\Omega^{\langle L\rangle}, \tag{6.2.5}$$

in which f is an arbitrary function of u, n is an arbitrary integer, and $\Omega^{\langle L\rangle}$ is an angular STF tensor of degree ℓ, of the sort introduced in Sec. 1.8.1. According to Eq. (2.5.16), $h^{ab}_{\mathscr{W}}$ is a sum of terms of the form

$$h^{ab}_{\mathscr{W}}[\ell,n] = \frac{4G}{c^4 r}\Omega^{\langle L\rangle}\left\{\int_0^{\mathcal{R}} ds f(u-2s)A(s,r) + \int_{\mathcal{R}}^{\infty} ds f(u-2s)B(s,r)\right\}, \tag{6.2.6}$$

where

$$A(s,r) = \int_{\mathcal{R}}^{r+s}\frac{P_\ell(\xi)}{p^{n-1}}\,dp, \qquad B(s,r) = \int_s^{r+s}\frac{P_\ell(\xi)}{p^{n-1}}\,dp, \tag{6.2.7}$$

in which P_ℓ is a Legendre polynomial of argument $\xi = (r+2s)/r - 2s(r+s)/(rp)$.

We shall return to $h^{ab}_{\mathscr{W}}$ at a later stage. For the time being we focus our attention on the near-zone contribution $h^{ab}_{\mathscr{N}}$, and we write Eq. (6.2.4) in a form that reveals the early terms of the sum:

$$\begin{aligned} h^{ab}_{\mathscr{N}} &= \frac{4G}{c^4 r}\biggl\{ \int_{\mathscr{M}} \tau^{ab}\, d^3x' + \Omega_c \frac{\partial}{\partial u} \int_{\mathscr{M}} \tau^{ab} x'^c\, d^3x' + \frac{1}{2}\Omega_c\Omega_d \frac{\partial^2}{\partial u^2} \int_{\mathscr{M}} \tau^{ab} x'^c x'^d\, d^3x' \\ &\quad + \frac{1}{6}\Omega_c\Omega_d\Omega_e \frac{\partial^3}{\partial u^3} \int_{\mathscr{M}} \tau^{ab} x'^c x'^d x'^d\, d^3x' + [q \geq 4] \biggr\} + O(r^{-2}), \end{aligned}$$

in which $[q \geq 4]$ stands for the remaining terms in the sum over q. To evaluate the first two integrals we invoke the conservation identities of Sec. 1.4. According to Eq. (1.4.3), for example,

$$\tau^{ab} = \frac{1}{2}\frac{\partial^2}{\partial u^2}\bigl(\tau^{00} x^a x^b\bigr) + \frac{1}{2}\partial_c\bigl(\tau^{ac}x^b + \tau^{bc}x^a - \partial_d \tau^{cd} x^a x^b\bigr)$$

and τ^{ab} can be replaced by $\frac{1}{2}\tau^{00}x^a x^b$ inside the volume integral, at the price of adding an integral over $\partial\mathscr{M}$ to account for the total divergence. We involve Eq. (1.4.4),

$$\tau^{ab}x^c = \frac{1}{2}\frac{\partial}{\partial u}\bigl(\tau^{0a}x^b x^c + \tau^{0b}x^a x^c - \tau^{0c}x^a x^b\bigr) + \frac{1}{2}\partial_d\bigl(\tau^{ad}x^b x^c + \tau^{bd}x^a x^c - \tau^{cd}x^a x^b\bigr),$$

in a similar way.

Introducing some notation to simplify the writing, we have

$$\begin{aligned} h^{ab}_{\mathscr{N}} &= \frac{2G}{c^4 r}\frac{\partial^2}{\partial \tau^2}\biggl\{ Q^{ab} + Q^{abc}\Omega_c + Q^{abcd}\Omega_c\Omega_d + \frac{1}{3}Q^{abcde}\Omega_c\Omega_d\Omega_e + [q \geq 4] \biggr\} \\ &\quad + \frac{2G}{c^4 r}\biggl\{ P^{ab} + P^{abc}\Omega_c \biggr\} + O(r^{-2}), \end{aligned} \tag{6.2.8}$$

in which the radiative multipole moments are defined by

$$Q^{ab} := \frac{1}{c^2}\int_{\mathscr{M}} \tau^{00} x'^a x'^b\, d^3x', \tag{6.2.9}$$

$$Q^{abc} := \frac{1}{c^2}\int_{\mathscr{M}} \bigl(\tau^{0a}x'^b x'^c + \tau^{0b}x'^a x'^c - \tau^{0c}x'^a x'^b\bigr)\, d^3x', \tag{6.2.10}$$

$$Q^{abcd} := \frac{1}{c^2}\int_{\mathscr{M}} \tau^{ab} x'^c x'^d\, d^3x', \tag{6.2.11}$$

$$Q^{abcde} := \frac{1}{c^3}\frac{\partial}{\partial \tau}\int_{\mathscr{M}} \tau^{ab} x'^c x'^d x'^e\, d^3x', \tag{6.2.12}$$

and where

$$P^{ab} := \oint_{\partial\mathscr{M}} \bigl(\tau^{ac}x'^b + \tau^{bc}x'^a - \partial'_d \tau^{cd} x'^a x'^b\bigr)\, dS_c, \tag{6.2.13}$$

$$P^{abc} := \frac{1}{c}\frac{\partial}{\partial \tau}\oint_{\partial\mathscr{M}} \bigl(\tau^{ad}x'^b x'^c + \tau^{bd}x'^a x'^c - \tau^{cd}x'^a x'^b\bigr)\, dS_d. \tag{6.2.14}$$

In the volume integrals, the components of the effective energy-momentum pseudotensor are expressed as functions of $\tau := t - r/c$ and $\boldsymbol{x'}$. The same is true within surface integrals, except for the fact that x'^a is now set equal to $\mathcal{R}\Omega'^a$; the surface element on $\partial\mathscr{M}$ is $dS_a := \mathcal{R}^2\Omega'_a d\Omega'$. The multipole moments Q^{ab}, Q^{abc}, Q^{abcd}, and Q^{abcde}, as well as the surface integrals P^{ab} and P^{abc}, are functions of τ only.

In the following sections we will endeavour to calculate the quantities that appear within Eq. (6.2.8), and we will extract the transverse-tracefree part of h^{ab}. In

Chapter 4 (see Sec. 4.4.7 for a summary) we saw that at leading order in a post-Newtonian expansion,

$$h^{ab} = \frac{2G}{c^4 r}\ddot{I}^{ab},$$

where I^{ab} is the Newtonian quadrupole moment of the mass distribution, and an overdot indicates differentiation with respect to τ. This leading term was labeled as a 1PN term, and it could be calculated on the basis of the leading-order (Newtonian) contribution to the effective mass density τ^{00}. For our purposes in this chapter, it is useful to reset the post-Newtonian counter, and to call the leading term in h^{ab} the *Newtonian contribution* to the gravitational potentials; additional terms will be labeled $\frac{1}{2}$PN, 1PN, $\frac{3}{2}$PN, and so on. This new convention will have the virtue of keeping the post-Newtonian orders of the solution in step with those of the source, and those of the multipole moments.

In this new post-Newtonian counting, we wish to compute h^{ab} accurately to $\frac{3}{2}$PN order. Schematically, we want

$$h^{ab} = \frac{G}{c^4 r}\Bigl(c^0 + c^{-1} + c^{-2} + c^{-3} + \cdots\Bigr),$$

in which the leading term is called a 0PN contribution, the correction of order c^{-1} a $\frac{1}{2}$PN term, and so on. To achieve this we need to calculate:

$$\begin{aligned}\tau^{00} &= c^2 + c^0 + \cdots \text{ to obtain } Q^{ab} = c^0 + c^{-2} + \cdots,\\ \tau^{0a} &= c + c^{-1} + \cdots \text{ to obtain } Q^{abc} = c^{-1} + c^{-3} + \cdots,\end{aligned}$$

and

$$\tau^{ab} = c^0 + \cdots \text{ to obtain } Q^{abcd} = c^{-2} + \cdots \text{ and } Q^{abcde} = c^{-3} + \cdots.$$

And on $\partial\mathscr{M}$ we need to calculate

$$\tau^{ab} = c^0 + c^{-2} + \cdots \text{ to obtain } P^{ab} = c^0 + c^{-2} + \cdots \text{ and } P^{abc} = c^{-1} + c^{-3} + \cdots.$$

All in all, this will give us the $\frac{3}{2}$PN accuracy that we demand for h^{ab}.

Our considerations have so far excluded $h^{ab}_{\mathscr{W}}$. We postpone a detailed discussion until Sec. 6.10, in which this contribution to the gravitational potentials is computed. For the time being it will suffice to say that $h^{ab}_{\mathscr{W}}$ contributes to the gravitational potentials at $\frac{3}{2}$PN order. It is therefore needed to achieve the required level of accuracy for h^{ab}.

The calculations that follow are lengthy. They are considerably simplified by the observation that ultimately, we wish to extract the transverse-tracefree part of h^{ab}. It is therefore not necessary to calculate any term that will not survive the TT projection introduced in Sec. 6.1.6. For example, a term in h^{ab} that is proportional to δ^{ab}, or to Ω^a, will not survive the projection, and does not need to be computed (there are many such terms, and ignoring them will be a substantial time saver). As another example, terms in Q^{abcd} that are proportional to δ^{ac}, or δ^{bc}, or δ^{ad}, or δ^{bd} (but not δ^{cd}!), can all be ignored because they will produce contributions to h^{ab} that are proportional to Ω^a or Ω^b, and these will not survive the TT projection. To indicate equality modulo terms that do not survive the transverse-tracefree projection, we introduce the notation $\overset{\text{TT}}{=}$, so that

$$A^{ab} \overset{\text{TT}}{=} B^{ab}$$

whenever

$$(\text{TT})^{ab}{}_{cd}A^{cd} = (\text{TT})^{ab}{}_{cd}B^{cd};$$

in other words, A^{ab} and B^{ab} differ by a tensor C^{ab} that contains no TT part: $(\mathrm{TT})^{ab}{}_{cd}C^{cd} = 0$.

An additional source of simplification — an important one — was exploited previously, back in Sec. 4.4.4, with a justification provided in Sec. 2.3: We are free to ignore all $\mathcal{R}$-dependent terms in $h^{ab}_{\mathscr{N}}$, and all $\mathcal{R}$-dependent terms in $h^{ab}_{\mathscr{W}}$, because any dependence on the arbitrary cutoff parameter $\mathcal{R}$ (the radius of the artificial boundary between the near zone and the wave zone) is guaranteed to cancel out after $h^{ab}_{\mathscr{N}}$ and $h^{ab}_{\mathscr{W}}$ are added together to form the complete potentials h^{ab}. We shall, therefore, feel completely free to drop all $\mathcal{R}$-dependent terms from our expressions, and this is another significant time saver.

6.3 Integration techniques for field integrals

In the course of our calculations we shall encounter a number of field integrals, an example of which is

$$E^{ab} := \frac{1}{4\pi}\int_{\mathscr{M}} U\partial^a U x^b\, d^3x, \tag{6.3.1}$$

where $\mathscr{M}$ is the domain of integration $r := |\boldsymbol{x}| < \mathcal{R}$, and where

$$U := \sum_A \frac{Gm_A}{|\boldsymbol{x} - \boldsymbol{z}_A|} \tag{6.3.2}$$

is the Newtonian potential. In this section we introduce techniques to evaluate such integrals. We will examine the specific case of Eq. (6.3.1), but the techniques are quite general, and they will apply equally well to many similar field integrals.

6.3.1 *Explicit form of E^{ab}; Change of integration variables*

After evaluating $\partial^a U$ we find that the field integral can be expressed in the more explicit form

$$E^{ab} = -\sum_A G^2 m_A^2 E^{ab}_A - \sum_A \sum_{B\neq A} G^2 m_A m_B E^{ab}_{AB}, \tag{6.3.3}$$

where

$$E^{ab}_A := \frac{1}{4\pi}\int_{\mathscr{M}} \frac{(\boldsymbol{x} - \boldsymbol{z}_A)^a x^b}{|\boldsymbol{x} - \boldsymbol{z}_A|^4}\, d^3x \tag{6.3.4}$$

and

$$E^{ab}_{AB} := \frac{1}{4\pi}\int_{\mathscr{M}} \frac{(\boldsymbol{x} - \boldsymbol{z}_B)^a x^b}{|\boldsymbol{x} - \boldsymbol{z}_A||\boldsymbol{x} - \boldsymbol{z}_B|^3}\, d^3x. \tag{6.3.5}$$

To evaluate the first integral we make the substitution

$$\boldsymbol{x} = \boldsymbol{z}_A + \boldsymbol{y}, \tag{6.3.6}$$

and integrate with respect to the new variables $\boldsymbol{y}$. This leads to

$$E^{ab}_A = \frac{1}{4\pi}\int_{\mathscr{M}} \frac{y^a y^b}{y^4}\, d^3y + \frac{z^b_A}{4\pi}\int_{\mathscr{M}} \frac{y^a}{y^4}\, d^3y, \tag{6.3.7}$$

where $y := |\boldsymbol{y}|$. To evaluate the second integral we use instead

$$\boldsymbol{x} = \boldsymbol{z}_B + \boldsymbol{y} \tag{6.3.8}$$

and integrate with respect to $\boldsymbol{y}$. This leads to

$$E^{ab}_{AB} = \frac{1}{4\pi}\int_{\mathscr{M}} \frac{1}{|\boldsymbol{y} - \boldsymbol{z}_{AB}|}\frac{y^a y^b}{y^3}\, d^3y + \frac{z^b_B}{4\pi}\int_{\mathscr{M}} \frac{1}{|\boldsymbol{y} - \boldsymbol{z}_{AB}|}\frac{y^a}{y^3}\, d^3y, \tag{6.3.9}$$

where $\boldsymbol{z}_{AB} := \boldsymbol{z}_A - \boldsymbol{z}_B$.

6.3.2 Translation of the domain of integration

The schematic form of each integral that appears in Eqs. (6.3.7) and (6.3.9) is

$$\int_{\mathscr{M}} f(\boldsymbol{y})\, d^3y,$$

where f is a function of the vector $\boldsymbol{y}$, which is related to the original set of variables $\boldsymbol{x}$ by a relation of the form $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$, with $\boldsymbol{z}$ independent of $\boldsymbol{x}$. The domain of integration $\mathscr{M}$ is defined by $|\boldsymbol{x}| < \mathcal{R}$, or $|\boldsymbol{y} + \boldsymbol{z}| < \mathcal{R}$, and it will be convenient to replace it by the simpler domain $\mathscr{M}_y$ defined by $y := |\boldsymbol{y}| < \mathcal{R}$.

To effect this replacement we note that the cutoff radius $\mathcal{R}$ can be assumed to be large compared with $z := |\boldsymbol{z}|$. (Recall the discussion of Sec. 2.3, in which $\mathcal{R}$ is chosen to be comparable to λ_c, the characteristic wavelength of the gravitational radiation. Recall also the discussion of Sec. 3.3.3, in which λ_c is shown to be large compared with both $|\boldsymbol{z}_A|$ and $|\boldsymbol{z}_{AB}|$, because in a slow-motion situation the matter distribution is always situated deep within the near zone. Conclude from these observations that $z/\mathcal{R} \ll 1$, as claimed.) The condition that defines $\mathscr{M}$ is $y^2 + 2\boldsymbol{z}\cdot\boldsymbol{y} + z^2 < \mathcal{R}^2$, and this can be expressed more simply as

$$y < \mathcal{R} - z\cos\gamma + O(z^2/\mathcal{R})$$

when $z/\mathcal{R} \ll 1$; here γ is the angle between the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$, defined by the statement $zy\cos\gamma := \boldsymbol{z}\cdot\boldsymbol{y}$.

Switching to the spherical polar coordinates (y, θ, ϕ) associated with the vector $\boldsymbol{y}$, the integral is

$$\begin{aligned}
\int_{\mathscr{M}} f(\boldsymbol{y})\, d^3y &= \int d\Omega \int_0^{\mathcal{R} - z\cos\gamma + \cdots} f(y,\theta,\phi)\, y^2 dy \\
&= \int d\Omega \int_0^{\mathcal{R}} f(y,\theta,\phi)\, y^2 dy + \int d\Omega \int_{\mathcal{R}}^{\mathcal{R} - z\cos\gamma + \cdots} f(y,\theta,\phi)\, y^2 dy,
\end{aligned}$$

where $d\Omega = \sin\theta\, d\theta d\phi$ is the element of solid angle. In the second line, the first integral is over the domain $\mathscr{M}_y$, while the second integral is estimated as

$$\int (-z\cos\gamma)\mathcal{R}^2 f(\mathcal{R},\theta,\phi)\, d\Omega = -\oint_{\partial\mathscr{M}_y} f(\boldsymbol{y})\, \boldsymbol{z}\cdot d\boldsymbol{S}$$

to first order in $z/\mathcal{R}$, where $dS^a := \mathcal{R}^2 \Omega^a\, d\Omega$ (with $\boldsymbol{\Omega} := \boldsymbol{y}/y$) is the surface element on $\partial\mathscr{M}_y$, the boundary of $\mathscr{M}_y$ described by the equation $y = \mathcal{R}$.

We have obtained the useful approximation

$$\int_{\mathscr{M}} f(\boldsymbol{y})\, d^3y = \int_{\mathscr{M}_y} f(\boldsymbol{y})\, d^3y - \oint_{\partial\mathscr{M}_y} f(\boldsymbol{y})\, \boldsymbol{z}\cdot d\boldsymbol{S} + \cdots, \tag{6.3.10}$$

in which the domain of integration $\mathscr{M}_y$ is defined by $y := |\boldsymbol{y}| < \mathcal{R}$, and $\partial\mathscr{M}_y$ is its boundary at $y = \mathcal{R}$. In Eq. (6.3.10), the integration variables $\boldsymbol{y}$ are related to the original variables $\boldsymbol{x}$ by the translation $\boldsymbol{x} = \boldsymbol{z} + \boldsymbol{y}$, and $\mathscr{M}$ is the original domain of integration defined by $|\boldsymbol{x}| < \mathcal{R}$. It is clear that the surface integral in Eq. (6.3.10) is smaller than the volume integral by a factor of order $|\boldsymbol{z}|/\mathcal{R} \ll 1$; the neglected terms are smaller still.

6.3.3 Evaluation of E_A^{ab}

We now return to the field integral of Eq. (6.3.7). We begin by working on the first term, which we copy as

$$\frac{1}{4\pi} \int_{\mathscr{M}} \frac{y^a y^b}{y^4}\, d^3y.$$

Inserting this within Eq. (6.3.10), we find that the volume integral is

$$\frac{1}{4\pi}\int_{\mathscr{M}_y}\frac{y^a y^b}{y^4}\,d^3y = \frac{1}{4\pi}\int_{\mathscr{M}_y}\Omega^a\Omega^b\,dyd\Omega = \langle\!\langle\Omega^a\Omega^b\rangle\!\rangle\int_0^{\mathcal{R}} dy = \frac{1}{3}\delta^{ab}\mathcal{R},$$

in which $\langle\!\langle\cdots\rangle\!\rangle := (4\pi)^{-1}\int(\cdots)\,d\Omega$ denotes an angular average; the identity $\langle\!\langle\Omega^a\Omega^b\rangle\!\rangle = \frac{1}{3}\delta^{ab}$ was established back in Sec. 1.8.4, along with other similar results. This contribution to E_A^{ab} can be discarded, because it is proportional to $\mathcal{R}$, and it was agreed near the end of Sec. 6.2 that all $\mathcal{R}$-dependent terms can indeed be ignored. With the understanding that $\boldsymbol{z}$ stands for $\boldsymbol{z}_A$, the surface integral is

$$-\frac{1}{4\pi}\oint_{\partial\mathscr{M}_y}\frac{y^a y^b}{y^4}\,\boldsymbol{z}\cdot\boldsymbol{dS} = -\frac{1}{4\pi}\int\Omega^a\Omega^b z_c\Omega^c\,d\Omega = -z_c\langle\!\langle\Omega^a\Omega^b\Omega^c\rangle\!\rangle = 0.$$

The neglected terms in Eq. (6.3.10) are of order $\mathcal{R}^{-1}$ and smaller, and because they depend on $\mathcal{R}$, they can be freely discarded. We conclude that the first term in Eq. (6.3.7) evaluates to zero.

We next set to work on the second term, which involves the integral

$$\frac{1}{4\pi}\int_{\mathscr{M}}\frac{y^a}{y^4}\,d^3y.$$

Inserting this within Eq. (6.3.10), we find that the volume integral is

$$\frac{1}{4\pi}\int_{\mathscr{M}_y}\frac{y^a}{y^4}\,d^3y = \langle\!\langle\Omega^a\rangle\!\rangle\int_0^{\mathcal{R}}\frac{dy}{y} = 0.$$

The volume integral is zero, and it is a fortunate outcome that the logarithmic divergence at $y=0$ (which occurs because the matter distribution is modeled as a collection of point masses) requires no explicit regularization, because the angular integration vanishes identically. The surface integral is

$$-\frac{1}{4\pi}\oint_{\mathscr{M}_y}\frac{y^a}{y^4}\boldsymbol{z}\cdot\boldsymbol{dS} = -\frac{z_c}{\mathcal{R}}\langle\!\langle\Omega^a\Omega^c\rangle\!\rangle = -\frac{1}{3}\frac{z^a}{\mathcal{R}},$$

in which $\boldsymbol{z}$ stands for $\boldsymbol{z}_A$. The additional terms in Eq. (6.3.10) are smaller by additional powers of $z/\mathcal{R} \ll 1$, and because they all depend on $\mathcal{R}$, they can be freely discarded. We conclude that the second term in Eq. (6.3.7) evaluates to zero.

We have arrived at

$$E_A^{ab} = 0, \tag{6.3.11}$$

modulo $\mathcal{R}$-dependent terms that can be freely discarded.

6.3.4 Evaluation of E_{AB}^{ab}

To evaluate the right-hand side of Eq. (6.3.9) we continue to make use of Eq. (6.3.10) to express an integral over the domain $\mathscr{M}$ in terms of a volume integral over $\mathscr{M}_y$ and a surface integral over $\partial\mathscr{M}_y$. We also make use of the addition theorem for spherical harmonics,

$$\frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\frac{r_<^{\ell}}{r_>^{\ell+1}}\bar{Y}_{\ell m}(\boldsymbol{n}_{AB})Y^{\ell m}(\boldsymbol{\Omega}), \tag{6.3.12}$$

in which $r_< := \min(y, z_{AB})$, $r_> = \max(y, z_{AB})$, $\boldsymbol{\Omega} := \boldsymbol{y}/y$, and $\boldsymbol{n}_{AB} := \boldsymbol{z}_{AB}/z_{AB}$. We recall that $\boldsymbol{z}_{AB} := \boldsymbol{z}_A - \boldsymbol{z}_B$.

We insert Eq. (6.3.12) within the first integral on the right-hand side of Eq. (6.3.9). Recalling Eq. (6.3.10), we approximate this by

$$\begin{aligned}\frac{1}{4\pi}\int_{\mathscr{M}_y}\frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^a y^b}{y^3}\,d^3y &= \frac{1}{4\pi}\int_{\mathscr{M}_y}\frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\Omega^a\Omega^b y\,dyd\Omega\\ &= \sum_\ell \frac{1}{2\ell+1}\int_0^{\mathcal{R}} dy\,y\frac{r_<^\ell}{r_>^{\ell+1}}\sum_m\int \bar{Y}_{\ell m}(\boldsymbol{n}_{AB})Y^{\ell m}(\boldsymbol{\Omega})\Omega^a\Omega^b\,d\Omega.\end{aligned}$$

To evaluate the angular integral we express $\Omega^a\Omega^b$ as

$$\Omega^a\Omega^b = \Omega^{\langle ab\rangle} + \frac{1}{3}\delta^{ab},$$

where $\Omega^{\langle ab\rangle}$ is an STF tensor of the sort introduced in Sec. 1.8.1, and we invoke the identity displayed in Eq. (1.8.16),

$$\sum_{m=-\ell}^{\ell}\bar{Y}_{\ell m}(\boldsymbol{n}_{AB})\int Y_{\ell m}(\boldsymbol{\Omega})\Omega^{\langle L'\rangle}\,d\Omega = \delta_{\ell\ell'}\,n_{AB}^{\langle L\rangle}. \tag{6.3.13}$$

This produces

$$\frac{1}{4\pi}\int_{\mathscr{M}_y}\frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^a y^b}{y^3}\,d^3y = \frac{1}{5}K(2,1)\,n_{AB}^{\langle ab\rangle} + \frac{1}{3}K(0,1)\,\delta^{ab},$$

where the radial integrals

$$K(\ell,n) := \int_0^{\mathcal{R}} y^n\frac{r_<^\ell}{r_>^{\ell+1}}\,dy \tag{6.3.14}$$

will be evaluated in the next subsection. This expression must be corrected by the surface integral of Eq. (6.3.10). We have

$$\frac{1}{4\pi}\oint_{\partial\mathscr{M}_y}\frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^a y^b}{y^3}\,\boldsymbol{z}\cdot\boldsymbol{dS} = \frac{\mathcal{R}z_c}{4\pi}\int\frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\Omega^a\Omega^b\Omega^c\,d\Omega,$$

in which $\boldsymbol{z}$ stands for $\boldsymbol{z}_B$. Because the leading term of $|\boldsymbol{y}-\boldsymbol{z}_{AB}|^{-1}$ in an expansion in powers of $z_{AB}/\mathcal{R} \ll 1$ is equal to $\mathcal{R}^{-1}$, the surface integral potentially gives rise to an $\mathcal{R}$-independent contribution to E_{AB}^{ab}. But this leading term is proportional to $\langle\!\langle \Omega^a\Omega^b\Omega^c\rangle\!\rangle = 0$, and we find that the surface integral does not actually contribute. At this stage we have obtained

$$\frac{1}{4\pi}\int_{\mathscr{M}}\frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^a y^b}{y^3}\,d^3y = \frac{1}{5}K(2,1)\,n_{AB}^{\langle ab\rangle} + \frac{1}{3}K(0,1)\,\delta^{ab}$$

for the first integral on the right-hand side of Eq. (6.3.9).

We next set to work on the second integral, and we begin by evaluating

$$\begin{aligned}\frac{1}{4\pi}\int_{\mathscr{M}_y}\frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^a}{y^3}\,d^3y &= \frac{1}{4\pi}\int_{\mathscr{M}_y}\frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\Omega^a\,dyd\Omega\\ &= \sum_\ell \frac{1}{2\ell+1}\int_0^{\mathcal{R}} dy\frac{r_<^\ell}{r_>^{\ell+1}}\sum_m\int \bar{Y}_{\ell m}(\boldsymbol{n}_{AB})Y^{\ell m}(\boldsymbol{\Omega})\Omega^a\,d\Omega.\end{aligned}$$

Using Eqs. (6.3.13) and (6.3.14), this is

$$\frac{1}{4\pi}\int_{\mathscr{M}_y}\frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^a}{y^3}\,d^3y = \frac{1}{3}K(1,0)\,n_{AB}^a.$$

This must be corrected by the surface integral of Eq. (6.3.10), and it is easy to show that in this case also, the result scales as $\mathcal{R}^{-1}$ and does not contribute. We have therefore obtained

$$\frac{1}{4\pi}\int_{\mathscr{M}}\frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^a}{y^3}\,d^3y=\frac{1}{3}K(1,0)\,n^a_{AB}$$

for the second integral on the right-hand side of Eq. (6.3.9).

Altogether we find that

$$E^{ab}_{AB}=\frac{1}{5}K(2,1)\,n^{\langle ab\rangle}_{AB}+\frac{1}{3}K(0,1)\,\delta^{ab}+\frac{1}{3}K(1,0)\,n^a_{AB}z^b_B, \tag{6.3.15}$$

where

$$\boldsymbol{n}_{AB}=\frac{\boldsymbol{z}_{AB}}{z_{AB}}=\frac{\boldsymbol{z}_A-\boldsymbol{z}_B}{|\boldsymbol{z}_A-\boldsymbol{z}_B|} \tag{6.3.16}$$

is a unit vector that points from body B to body A, and where $n^{\langle ab\rangle}_{AB}:=n^a_{AB}n^b_{AB}-\frac{1}{3}\delta^{ab}$ is an STF tensor.

6.3.5 Radial integrals

To complete the computation we must now evaluate the radial integrals defined by Eq. (6.3.14),

$$K(\ell,n):=\int_0^{\mathcal{R}}y^n\frac{r^\ell_<}{r^{\ell+1}_>}\,dy, \tag{6.3.17}$$

in which $r_<:=\min(y,z)$ and $r_>=\max(y,z)$, with z standing for $z_{AB}:=|\boldsymbol{z}_A-\boldsymbol{z}_B|$.

Excluding the case $n=\ell$, which never occurs in applications, we have

$$\begin{aligned}K(\ell,n)&=\frac{1}{z^{\ell+1}}\int_0^z y^{\ell+n}\,dy+z^\ell\int_z^{\mathcal{R}}y^{n-\ell-1}\,dy\\&=\frac{z^n}{\ell+n+1}-\frac{z^n}{n-\ell}\Big[1-(z/\mathcal{R})^{\ell-n}\Big].\end{aligned}$$

We discard the last term, because it depends on the cutoff radius $\mathcal{R}$, and we conclude that

$$K(\ell,n)=\frac{2\ell+1}{(\ell-n)(\ell+n+1)}|\boldsymbol{z}_{AB}|^n,\qquad(\ell\neq n). \tag{6.3.18}$$

In particular, $K(2,1)=\frac{5}{4}z_{AB}$, $K(0,1)=-\frac{1}{2}z_{AB}$, and $K(1,0)=\frac{3}{2}$.

6.3.6 E^{ab}: Final answer

Substituting Eq. (6.3.18) into Eq. (6.3.15), we find that E^{ab}_{AB} becomes

$$E^{ab}_{AB}=\frac{1}{4}z_{AB}n^{\langle ab\rangle}_{AB}-\frac{1}{6}z_{AB}\delta^{ab}+\frac{1}{2}n^a_{AB}z^b_B.$$

This, together with Eq. (6.3.11) for E^{ab}_A, can now be inserted within Eq. (6.3.3). We arrive at

$$E^{ab}=-\sum_A\sum_{B\neq A}G^2m_Am_B\biggl(\frac{1}{4}z_{AB}n^{\langle ab\rangle}_{AB}-\frac{1}{6}z_{AB}\delta^{ab}+\frac{1}{2}n^a_{AB}z^b_B\biggr),$$

and this can also be expressed as

$$E^{ab}=-\sum_A\sum_{B\neq A}G^2m_Am_B\biggl(\frac{1}{4}z_{AB}n^{\langle ab\rangle}_{AB}-\frac{1}{6}z_{AB}\delta^{ab}-\frac{1}{2}n^a_{AB}z^b_A\biggr)$$

if we interchange the identities of A and B and recall that $\boldsymbol{n}_{BA} = -\boldsymbol{n}_{AB}$. When we add these expressions and divide by two, we obtain the symmetrized form

$$E^{ab} = -\sum_A \sum_{B\neq A} G^2 m_A m_B \left(\frac{1}{4} z_{AB} n_{AB}^{\langle ab\rangle} - \frac{1}{6} z_{AB}\delta^{ab} - \frac{1}{4} z_{AB} n_{AB}^a n_{AB}^b \right).$$

This becomes

$$E^{ab} = \frac{1}{4}\delta^{ab} \sum_A \sum_{B\neq A} G^2 m_A m_B |\boldsymbol{z}_A - \boldsymbol{z}_B| \tag{6.3.19}$$

after simplification, and this is our final answer.

6.3.7 Summary

To sum up, let us retrace the main steps that led us from the definition

$$E^{ab} = \frac{1}{4\pi} \int_{\mathscr{M}} U \partial^a U x^b \, d^3x,$$

to its evaluation

$$E^{ab} = \frac{1}{4}\delta^{ab} \sum_A \sum_{B\neq A} G^2 m_A m_B |\boldsymbol{z}_A - \boldsymbol{z}_B|.$$

These steps will allow us, in the following sections, to evaluate many similar field integrals.

After inserting the Newtonian potential and its derivative within the integral, we change the variables of integration from $\boldsymbol{x}$ to $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{z}$, in which $\boldsymbol{z}$ stands for either $\boldsymbol{z}_A$ or $\boldsymbol{z}_B$, depending on the context. We also translate the domain of integration from $\mathscr{M}$ (defined by $|\boldsymbol{x}| < \mathcal{R}$) to $\mathscr{M}_y$ (defined by $|\boldsymbol{y}| < \mathcal{R}$), and we make use of the identity

$$\int_{\mathscr{M}} f(\boldsymbol{y})\, d^3y = \int_{\mathscr{M}_y} f(\boldsymbol{y})\, d^3y - \oint_{\partial\mathscr{M}_y} f(\boldsymbol{y})\, \boldsymbol{z}\cdot d\boldsymbol{S} + \cdots, \tag{6.3.20}$$

in which the surface integral is smaller than the volume integral by a factor of order $|\boldsymbol{z}|/\mathcal{R} \ll 1$ (and the dotted terms are smaller still).

Next we invoke the addition theorem for spherical harmonics,

$$\frac{1}{|\boldsymbol{y} - \boldsymbol{z}_{AB}|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} \bar{Y}_{\ell m}(\boldsymbol{n}_{AB}) Y^{\ell m}(\boldsymbol{\Omega}), \tag{6.3.21}$$

in which $\boldsymbol{z}_{AB} = \boldsymbol{z}_A - \boldsymbol{z}_B$, $r_< := \min(y, z_{AB})$, $r_> = \max(y, z_{AB})$, $\boldsymbol{\Omega} := \boldsymbol{y}/y$, and $\boldsymbol{n}_{AB} := \boldsymbol{z}_{AB}/z_{AB}$. After expressing all factors such as Ω^L in terms of STF tensors, the angular integrations are carried out with the help of the identity

$$\sum_{m=-\ell}^{\ell} \bar{Y}_{\ell m}(\boldsymbol{n}_{AB}) \int Y_{\ell m}(\boldsymbol{\Omega}) \Omega^{\langle L'\rangle}\, d\Omega = \delta_{\ell\ell'}\, n_{AB}^{\langle L\rangle}. \tag{6.3.22}$$

This leaves us with a number of radial integrations to work out, and these are given by

$$K(\ell, n) := \int_0^{\mathcal{R}} y^n \frac{r_<^\ell}{r_>^{\ell+1}}\, dy = \frac{2\ell+1}{(\ell-n)(\ell+n+1)} |\boldsymbol{z}_{AB}|^n, \tag{6.3.23}$$

provided that $\ell \neq n$.

And at last, after simplification, we obtain our final expression for the field integral. All the while we are justified to throw away any term that contains an explicit dependence on the arbitrary cutoff radius $\mathcal{R}$.

6.4 Computation of Q^{ab}

We launch our calculation of the gravitational-wave field with a computation of Q^{ab}, the radiative quadrupole moment. According to Eq. (6.2.9), this is defined by

$$Q^{ab}(\tau) := \frac{1}{c^2}\int_{\mathscr{M}} \tau^{00}(\tau,\boldsymbol{x})x^a x^b\, d^3x, \tag{6.4.1}$$

in which $\tau := t - r/c$ is retarded time, and where we suppress the primes on the integration variables to simplify the notation. (It should be kept in mind that r is the distance from the origin to the field point, which is very distinct from the source point now identified by the vector $\boldsymbol{x}$.)

According to the discussion of Sec. 6.2, to calculate Q^{ab} to the required degree of accuracy we need an expression for $c^{-2}\tau^{00}$ that includes terms of order c^0 (Newtonian, or 0PN) and terms of order c^{-2} (1PN). Such an expression was obtained in Sec. 4.1, and it can be read off the right-hand side of Eq. (4.1.26). We have

$$c^{-2}\tau^{00} = \sum_A m_A\biggl(1+\frac{v_A^2}{2c^2}+\frac{3\lfloor U\rfloor_A}{c^2}\biggr)\delta\bigl(\boldsymbol{x}-\boldsymbol{z}_A\bigr) - \frac{14}{16\pi G c^2}\partial_c U\partial^c U + O(c^{-4}), \tag{6.4.2}$$

where we used the fact that in the near zone, and at this order of accuracy, the gravitational potential Φ defined by Eq. (4.1.25) can be set equal to the Newtonian potential

$$U = \sum_A \frac{G m_A}{|\boldsymbol{x}-\boldsymbol{z}_A|}. \tag{6.4.3}$$

We recall that

$$\lfloor U\rfloor_A := \sum_{B\neq A}\frac{G m_B}{|\boldsymbol{z}_A-\boldsymbol{z}_B|} \tag{6.4.4}$$

is the potential external to body A, evaluated at $\boldsymbol{x} = \boldsymbol{z}_A$. In Eqs. (6.4.2)–(6.4.4), the position vectors $\boldsymbol{z}_A$ and velocity vectors $\boldsymbol{v}_A$ evaluated at the retarded time τ.

The quadrupole moment contains both a matter contribution that comes from the δ-functions in τ^{00}, and a field contribution that comes from the term involving $\partial_c U\partial^c U$. The matter contribution can be calculated at once:

$$Q^{ab}[\mathrm{M}] = \sum_A m_A\biggl(1+\frac{v_A^2}{2c^2}+\frac{3\lfloor U\rfloor_A}{c^2}\biggr) z_A^a z_A^b. \tag{6.4.5}$$

The field contribution is

$$Q^{ab}[\mathrm{F}] = -\frac{14}{16\pi G c^2}\int_{\mathscr{M}} \partial_c U\partial^c U x^a x^b\, d^3x, \tag{6.4.6}$$

and its computation requires a lot more work. The complete radiative quadrupole moment is

$$Q^{ab} = Q^{ab}[\mathrm{M}] + Q^{ab}[\mathrm{F}] + O(c^{-4}), \tag{6.4.7}$$

and it will be calculated accurately through 1PN order.

To evaluate the field integral of Eq. (6.4.6) we first express the integrand in the equivalent form

$$\begin{aligned}\partial_c U\partial^c U x^a x^b &= \partial_c\bigl(U\partial^c U x^a x^b\bigr) - \frac{1}{2}\partial^a\bigl(U^2 x^b\bigr) - \frac{1}{2}\partial^b\bigl(U^2 x^a\bigr)\\ &\quad + U^2\delta^{ab} - U(\nabla^2 U)x^a x^b,\end{aligned}$$

which allows us to integrate by parts. We may discard the term $U^2\delta^{ab}$ on the grounds that it will not survive the TT projection introduced in Sec. 6.1.6. We may also replace $\nabla^2 U$ by $-4\pi G \sum_A m_A \delta(\boldsymbol{x} - \boldsymbol{z}_A)$, and write

$$\int_{\mathscr{M}} \partial_c U \partial^c U x^a x^b \, d^3x \overset{\text{TT}}{=} \oint_{\partial\mathscr{M}} U \partial^c U x^a x^b \, dS_c - \oint_{\partial\mathscr{M}} U^2 x^{(a} dS^{b)} + 4\pi G \sum_A m_A \lfloor U \rfloor_A z_A^a z_A^b,$$

where the notation $\overset{\text{TT}}{=}$ was introduced near the end of Sec. 6.2, and where $dS^a = \mathcal{R}^2 \Omega^a \, d\Omega$ is the surface element on $\partial\mathscr{M}$. Making this substitution, we obtain

$$Q^{ab}[\mathrm{F}] \overset{\text{TT}}{=} -\frac{7}{2Gc^2}\Big(\mathcal{R}^4 \langle\!\langle U \partial_c U \Omega^a \Omega^b \Omega^c \rangle\!\rangle - \mathcal{R}^3 \langle\!\langle U^2 \Omega^a \Omega^b \rangle\!\rangle\Big) - \frac{7}{2c^2} \sum_A m_A \lfloor U \rfloor_A z_A^a z_A^b,$$

in which the angular brackets denote an average over the unit two-sphere.

We must now think of evaluating the surface integrals, on which $\boldsymbol{x}$ is set equal to $\mathcal{R}\boldsymbol{\Omega}$. Recalling that $\mathcal{R}$ is large compared with $\boldsymbol{z}_A$ (refer back to Sec. 6.3.2), it is appropriate to expand U in powers of r^{-1} before we insert it within the surface integrals. We have

$$U = \frac{Gm}{r} + \frac{1}{2} G I^{ab} \partial_{ab} r^{-1} + O(r^{-3}), \tag{6.4.8}$$

where $m := \sum_A m_A$ is the total mass, and $I^{ab} := \sum_A m_A z_A^a z_A^b$ is the Newtonian quadrupole moment of the mass distribution. It is important to notice that the Newtonian dipole moment, $I^a := \sum_A m_A z_A^a$, has been set equal to zero. This is allowed, because $\boldsymbol{I} = m\boldsymbol{Z} + O(c^{-2})$, where $\boldsymbol{Z}$ is the post-Newtonian barycentre (refer back to Sec. 5.4.5), and we work in a coordinate system such that $\boldsymbol{Z} \equiv \boldsymbol{0}$. From Eq. (6.4.8) we also get

$$\partial_a U = Gm \partial_a r^{-1} + \frac{1}{2} G I^{bc} \partial_{abc} r^{-1} + O(r^{-4}). \tag{6.4.9}$$

These results indicate that on $\partial\mathscr{M}$, the potential and its gradient are given schematically by $U = \mathcal{R}^{-1} + \mathcal{R}^{-3} + \cdots$ and $\partial_a U = \mathcal{R}^{-2} + \mathcal{R}^{-4} + \cdots$. This implies, for example, that $\mathcal{R}^4 U \partial_c U = \mathcal{R} + \mathcal{R}^{-1} + \cdots$ and $\mathcal{R}^3 U^2 = \mathcal{R} + \mathcal{R}^{-1} + \cdots$. This reveals, finally, that the surface integrals produce no $\mathcal{R}$-independent contributions to $Q^{ab}[\mathrm{F}]$.

We have obtained

$$Q^{ab}[\mathrm{F}] \overset{\text{TT}}{=} -\frac{7}{2c^2} \sum_A m_A \lfloor U \rfloor_A z_A^a z_A^b, \tag{6.4.10}$$

and combining this with Eqs. (6.4.5) and (6.4.7), we conclude that the radiative quadrupole moment is given by

$$Q^{ab} \overset{\text{TT}}{=} \sum_A m_A \left(1 + \frac{1}{2}\frac{v_A^2}{c^2} - \frac{1}{2}\frac{\lfloor U \rfloor_A}{c^2}\right) z_A^a z_A^b + O(c^{-4}). \tag{6.4.11}$$

This expression leaves out a term proportional to δ^{ab} that would not survive the action of the transverse-tracefree projector $(\mathrm{TT})^{ab}{}_{cd}$, as well as $\mathcal{R}$-dependent terms that can be freely discarded.

6.5 Computation of Q^{abc}

6.5.1 Definition of auxiliary quantities

We turn next to the computation of Q^{abc}, the radiative octupole moment. According to Eq. (6.2.10), this is defined by

$$Q^{abc} := A^{abc} + A^{bac} - A^{cab}, \tag{6.5.1}$$

where

$$A^{abc}(\tau) := \frac{1}{c^2}\int_{\mathscr{M}} \tau^{0a}(\tau,\boldsymbol{x})x^b x^c\,d^3x, \tag{6.5.2}$$

in which $\tau := t - r/c$ is retarded time, and where we suppress the primes on the integration variables to simplify the notation.

According to the discussion of Sec. 6.2, to calculate Q^{abc} to the required degree of accuracy we need an expression for $c^{-2}\tau^{0a}$ that includes terms of order c^{-1} ($\frac{1}{2}$PN) and terms of order c^{-3} ($\frac{3}{2}$PN). Such an expression was worked out in Sec. 4.1, and it can be obtained by adding the contributions provided by Eqs. (4.1.11), (4.1.18), and (4.1.22). After inserting $h^{00} = 4U/c^2 + O(c^{-4})$ and $h^{0a} = 4U^a/c^3 + O(c^{-5})$ within Eq. (4.1.18), we find that

$$\begin{aligned} c^{-2}\tau^{0a} &= \frac{1}{c}\sum_A m_A v_A^a\biggl(1 + \frac{v_A^2}{2c^2} + \frac{3\lfloor U\rfloor_A}{c^2}\biggr)\delta(\boldsymbol{x}-\boldsymbol{z}_A) \\ &\quad + \frac{1}{16\pi G c^3}\Bigl[12\dot{U}\partial^a U + 16\bigl(\partial^a U^d - \partial^d U^a\bigr)\partial_d U\Bigr] + O(c^{-5}). \end{aligned} \tag{6.5.3}$$

Here, U is the Newtonian potential of Eq. (6.4.3), $\dot{U}$ is its derivative with respect to τ, $\lfloor U\rfloor_A$ is its regularized value at $\boldsymbol{x} = \boldsymbol{z}_A$, and

$$U^a := \sum_A \frac{G m_A v_A^a}{|\boldsymbol{x}-\boldsymbol{z}_A|} \tag{6.5.4}$$

is the gravitational vector potential. We recall that the potentials satisfy the Poisson equations $\nabla^2 U = -4\pi G\sum_A m_A\delta(\boldsymbol{x}-\boldsymbol{z}_A)$ and $\nabla^2 U^a = -4\pi G\sum_A m_A v_A^a\delta(\boldsymbol{x}-\boldsymbol{z}_A)$.

The octupole moment contains a contribution $Q^{abc}[\mathrm{M}]$ that comes directly from the matter distribution, and another contribution $Q^{abc}[\mathrm{F}]$ that comes from the gravitational field. They are obtained from $A^{abc} = A^{abc}[\mathrm{M}] + A^{abc}[\mathrm{F}] + O(c^{-5})$, which is then substituted into Eq. (6.5.1). We have introduced

$$A^{abc}[\mathrm{M}] := \frac{1}{c}\sum_A m_A v_A^a\biggl(1 + \frac{v_A^2}{2c^2} + \frac{3\lfloor U\rfloor_A}{c^2}\biggr)z_A^b z_A^c \tag{6.5.5}$$

and

$$A^{abc}[\mathrm{F}] := \frac{1}{16\pi G c^3}\int_{\mathscr{M}}\Bigl[12\dot{U}\partial^a U + 16\bigl(\partial^a U^d - \partial^d U^a\bigr)\partial_d U\Bigr]x^b x^c\,d^3x, \tag{6.5.6}$$

and the remainder of this section will be mostly devoted to the computation of $A^{abc}[\mathrm{F}]$.

6.5.2 Computation of the field integral: Organization

To simplify our computations, we invoke the harmonic gauge conditions, specifically its near-zone consequence $\dot{U} + \partial_d U^d = 0$, to eliminate $\dot{U}$ from Eq. (6.5.6). It becomes

$$A^{abc}[\mathrm{F}] = -\frac{3}{Gc^3}B_1^{abc} + \frac{4}{Gc^3}B_2^{abc} - \frac{4}{Gc^3}B_3^{abc}, \tag{6.5.7}$$

where

$$B_1^{abc} := \frac{1}{4\pi}\int_{\mathscr{M}} \partial^a U\partial_d U^d x^b x^c\,d^3x, \tag{6.5.8}$$

$$B_2^{abc} := \frac{1}{4\pi}\int_{\mathscr{M}} \partial_d U\partial^a U^d x^b x^c\,d^3x, \tag{6.5.9}$$

$$B_3^{abc} := \frac{1}{4\pi}\int_{\mathscr{M}} \partial_d U\partial^d U^a x^b x^c\,d^3x. \tag{6.5.10}$$

After integration by parts, which is designed to leave one factor of U undifferentiated, we find that each field integral B^{abc} breaks down into a volume integral $B^{abc}[\mathscr{M}]$ and a surface integral $B^{abc}[\partial\mathscr{M}]$. A number of terms are found to be proportional to δ^{ab}, or δ^{ac}, or δ^{bc}. All such terms will not survive the transverse-tracefree projection effected by Eq. (6.1.34), and according to our discussion near the end of Sec. 6.2, they can all be dropped. For example, if B^{abc} contains a term $\delta^{ab}B^c$, then its contribution to Q^{abc} will be of the form $2\delta^{ab}B^c - \delta^{ac}B^b$. The first term is a pure trace, and the second term is longitudinal, because it becomes proportional to Ω^a after Q^{abc} is multiplied by Ω_c; in each case the contribution does not survive the TT projection.

After eliminating all such terms, we find that

$$\begin{aligned}
B_1^{abc} &\overset{\text{TT}}{=} B_1^{abc}[\mathscr{M}] + B_1^{abc}[\partial\mathscr{M}], && (6.5.11)\\
B_1^{abc}[\mathscr{M}] &:= -\frac{1}{4\pi}\int_{\mathscr{M}} U\partial^a{}_d U^d\, x^b x^c\, d^3x, && (6.5.12)\\
B_1^{abc}[\partial\mathscr{M}] &:= \frac{1}{4\pi}\oint_{\partial\mathscr{M}} U\partial_d U^d\, x^b x^c\, dS^a, && (6.5.13)
\end{aligned}$$

that

$$\begin{aligned}
B_2^{abc} &\overset{\text{TT}}{=} B_2^{abc}[\mathscr{M}] + B_2^{abc}[\partial\mathscr{M}], && (6.5.14)\\
B_2^{abc}[\mathscr{M}] &:= -\frac{1}{4\pi}\int_{\mathscr{M}} U\big(\partial^a{}_d U^d\, x^b x^c + \partial^a U^b\, x^c + \partial^a U^c\, x^b\big)\, d^3x, && (6.5.15)\\
B_2^{abc}[\partial\mathscr{M}] &:= \frac{1}{4\pi}\oint_{\partial\mathscr{M}} U\partial^a U^d\, x^b x^c\, dS_d, && (6.5.16)
\end{aligned}$$

and that

$$\begin{aligned}
B_3^{abc} &\overset{\text{TT}}{=} B_3^{abc}[\mathscr{M}] + B_3^{abc}[\partial\mathscr{M}], && (6.5.17)\\
B_3^{abc}[\mathscr{M}] &:= -\frac{1}{4\pi}\int_{\mathscr{M}} U\big(\nabla^2 U^a\, x^b x^c + \partial^b U^a\, x^c + \partial^c U^a\, x^b\big)\, d^3x, && (6.5.18)\\
B_3^{abc}[\partial\mathscr{M}] &:= \frac{1}{4\pi}\oint_{\partial\mathscr{M}} U\partial^d U^a\, x^b x^c\, dS_d. && (6.5.19)
\end{aligned}$$

We recall that $\mathscr{M}$ is the domain $r := |\boldsymbol{x}| \le \mathcal{R}$, with a boundary $\partial\mathscr{M}$ described by $r = \mathcal{R}$, and that $dS^a = \mathcal{R}^2\Omega^a\, d\Omega$ is the surface element on $\partial\mathscr{M}$.

There are many volume integrals to evaluate, but they are all of the form of

$$C^{mnpbc} := -\frac{1}{4\pi}\int_{\mathscr{M}} U\partial^{mn}U^p\, x^b x^c\, d^3x \qquad (6.5.20)$$

and

$$D^{mnp} := -\frac{1}{4\pi}\int_{\mathscr{M}} U\partial^m U^n\, x^p\, d^3x. \qquad (6.5.21)$$

Specifically,

$$\begin{aligned}
B_1^{abc}[\mathscr{M}] &= C^a{}_d{}^{dbc},\\
B_2^{abc}[\mathscr{M}] &= C^a{}_d{}^{dbc} + D^{abc} + D^{acb},\\
B_3^{abc}[\mathscr{M}] &= C^d{}_d{}^{abc} + D^{bac} + D^{cab}.
\end{aligned} \qquad (6.5.22)$$

Similarly, the surface integrals are of the form of

$$E^{mnbcp} := \frac{1}{4\pi}\oint_{\partial\mathscr{M}} U\partial^m U^n\, x^b x^c\, dS^p, \qquad (6.5.23)$$

with

$$B_1^{abc}[\partial\mathscr{M}] = E_d{}^{dbca}, \qquad B_2^{abc}[\partial\mathscr{M}] = E^{adbc}{}_d, \qquad B_3^{abc}[\partial\mathscr{M}] = E^{dabc}{}_d. \quad (6.5.24)$$

The key is therefore the evaluation of the generic volume integrals of Eqs. (6.5.20) and (6.5.21), as well as the evaluation of the surface integral of Eq. (6.5.23). Once these are in hand, the computation of B_1^{abc}, B_2^{abc}, and B_3^{abc} is soon completed, and Eq. (6.5.7) gives us $A^{abc}[\mathrm{F}]$. Adding the $A^{abc}[\mathrm{M}]$ of Eq. (6.5.5) produces A^{abc}, and from Eq. (6.5.1) we get our final answer for Q^{abc}.

6.5.3 Computation of C^{mnpbc}

We follow the general methods described in Sec. 6.3. We begin by differentiating Eq. (6.5.4) twice, which gives

$$\partial^{mn}U^p = -\sum_B Gm_B v_B^p\biggl[-3\frac{(\boldsymbol{x}-\boldsymbol{z}_B)^m(\boldsymbol{x}-\boldsymbol{z}_B)^n}{|\boldsymbol{x}-\boldsymbol{z}_B|^5} + \frac{\delta^{mn}}{|\boldsymbol{x}-\boldsymbol{z}_B|^3} + \frac{4\pi}{3}\delta^{mn}\delta(\boldsymbol{x}-\boldsymbol{z}_A)\biggr]$$

after inserting the distributional term to ensure that U^p satisfies the appropriate Poisson equation. After insertion of Eq. (6.4.3) and some algebra, Eq. (6.5.20) becomes

$$\begin{aligned} C^{mnpbc} &= \sum_A G^2 m_A^2 v_A^p\Bigl(\delta^{mn}F_A^{bc} - 3F_A^{mnbc}\Bigr) \\ &+ \sum_A\sum_{B\neq A} G^2 m_A m_B v_B^p\biggl(\delta^{mn}F_{AB}^{bc} - 3F_{AB}^{mnbc} + \frac{1}{3}\delta^{mn}\frac{z_B^b z_B^c}{z_{AB}}\biggr), \end{aligned} \quad (6.5.25)$$

where

$$F_A^{mnbc} := \frac{1}{4\pi}\int_{\mathscr{M}} \frac{(\boldsymbol{x}-\boldsymbol{z}_A)^m(\boldsymbol{x}-\boldsymbol{z}_A)^n}{|\boldsymbol{x}-\boldsymbol{z}_A|^6}\, x^b x^c\, d^3x \quad (6.5.26)$$

and

$$F_{AB}^{mnbc} := \frac{1}{4\pi}\int_{\mathscr{M}} \frac{1}{|\boldsymbol{x}-\boldsymbol{z}_A|}\frac{(\boldsymbol{x}-\boldsymbol{z}_B)^m(\boldsymbol{x}-\boldsymbol{z}_B)^n}{|\boldsymbol{x}-\boldsymbol{z}_B|^5}\, x^b x^c\, d^3x, \quad (6.5.27)$$

and where

$$F_A^{bc} := \delta_{mn}F_A^{mnbc}, \qquad F_{AB}^{bc} := \delta_{mn}F_{AB}^{mnbc}. \quad (6.5.28)$$

The term involving $z_B^b z_B^c/z_{AB}$ in Eq. (6.5.25), where $z_{AB} := |\boldsymbol{z}_A - \boldsymbol{z}_B|$, originates from the distributional term in $\partial^{mn}U^p$; a similar term that would involve $z_A^b z_A^c/z_{AA}$ has been set equal to zero by invoking the regularization prescription of Eq. (4.1.12), according to which $\delta(\boldsymbol{x}-\boldsymbol{z}_A)/|\boldsymbol{x}-\boldsymbol{z}_A| \equiv 0$.

We first set to work on F_A^{mnbc}. Following the general strategy summarized in Sec. 6.3.7, we substitute $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}_A$ inside the integral, and get

$$\begin{aligned} F_A^{mnbc} &= \frac{1}{4\pi}\int_{\mathscr{M}} \frac{y^m y^n y^b y^c}{y^6}\, d^3y + \frac{z_B^b}{4\pi}\int_{\mathscr{M}} \frac{y^m y^n y^c}{y^6}\, d^3y \\ &\quad + \frac{z_B^c}{4\pi}\int_{\mathscr{M}} \frac{y^m y^n y^b}{y^6}\, d^3y + \frac{z_B^b z_B^c}{4\pi}\int_{\mathscr{M}} \frac{y^m y^n}{y^6}\, d^3y. \end{aligned}$$

According to Eq. (6.3.20), each integral over $\mathscr{M}$ can be expressed as a volume integral over the simpler domain $\mathscr{M}_y$ defined by $y := |\boldsymbol{y}| < \mathcal{R}$, plus a correction of fractional order $|bmz_B|/\mathcal{R}$ given by a surface integral over $\partial\mathscr{M}_y$.

The first integral produces

$$\begin{aligned} \frac{1}{4\pi}\int_{\mathscr{M}_y} \frac{y^m y^n y^b y^c}{y^6}\, d^3y &= \langle\!\langle \Omega^m\Omega^n\Omega^b\Omega^c \rangle\!\rangle \int_0^{\mathcal{R}} dy \\ &= \frac{1}{15}\mathcal{R}\bigl(\delta^{mn}\delta^{bc} + \delta^{mb}\delta^{nc} + \delta^{mc}\delta^{nb}\bigr), \end{aligned}$$

where we involved Eq. (1.8.21). Because it is proportional to $\mathcal{R}$, this contribution to F_A^{mnbc} can be discarded. The surface integral that corrects this will potentially give rise to an $\mathcal{R}$-independent contribution, and it must be evaluated carefully. It turns out, however, that it is proportional to $z_B^d \langle\langle \Omega^m \Omega^n \Omega^b \Omega^c \Omega^d \rangle\rangle$, and it vanishes because the angular average of an odd number of vectors $\boldsymbol{\Omega}$ is necessarily zero. The neglected terms in Eq. (6.3.21) are of order $\mathcal{R}^{-1}$ and higher, and we conclude that the first integral in F_A^{mnbc} makes no contribution to C^{mnpbc}.

The second and third integrals produce terms such as

$$\frac{1}{4\pi}\int_{\mathscr{M}_y} \frac{y^m y^n y^c}{y^6}\, d^3y = \langle\!\langle \Omega^m \Omega^n \Omega^c \rangle\!\rangle \int_0^{\mathcal{R}} \frac{dy}{y},$$

and this vanishes by virtue of Eq. (1.8.20); the logarithmic divergence of the radial integration requires no explicit regularization. The surface integral that corrects this is easily shown to be of order $\mathcal{R}^{-1}$, and we conclude that the second and third integrals do not contribute to C^{mnpbc}.

The fourth integral produces

$$\frac{1}{4\pi}\int_{\mathscr{M}_y} \frac{y^m y^n}{y^6}\, d^3y = \langle\!\langle \Omega^m \Omega^n \rangle\!\rangle \int_0^{\mathcal{R}} \frac{dy}{y^2} = \frac{1}{3}\delta^{mn} \int_0^{\mathcal{R}} \frac{dy}{y^2},$$

and this involves a radial integration that is formally divergent. Once more the surface integral does not contribute, and we have obtained

$$F_A^{mnbc} = \frac{1}{3}\delta^{mn} z_A^b z_A^c \int_0^{\mathcal{R}} \frac{dy}{y^2} \tag{6.5.29}$$

for the field integral of Eq. (6.5.26), modulo $\mathcal{R}$-dependent terms that can be freely discarded. It is disturbing to see that F_A^{mnbc} is proportional to a diverging integral, but it is a fortunate outcome that the combination $\delta^{mn}F_A^{bc} - 3F_A^{mnbc}$ that appears into C^{mnbc} happens to vanish by virtue of the fact that F_A^{mnbc} is also proportional to δ^{mn}. The divergence does not require explicit regularization, and all in all we find that F_A^{mnbc} makes no contribution to C^{mnbc}.

We next set to work on F_{AB}^{mnbc}. Once more we follow the general strategy summarized in Sec. 6.3.7, and we substitute $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}_B$ inside the integral. We get

$$\begin{aligned} F_{AB}^{mnbc} \quad = \quad & \frac{1}{4\pi}\int_{\mathscr{M}} \frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^m y^n y^b y^c}{y^5}\, d^3y + \frac{z_B^b}{4\pi}\int_{\mathscr{M}} \frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^m y^n y^c}{y^5}\, d^3y \\ & + \frac{z_B^c}{4\pi}\int_{\mathscr{M}} \frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^m y^n y^b}{y^5}\, d^3y + \frac{z_B^b z_B^c}{4\pi}\int_{\mathscr{M}} \frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^m y^n}{y^5}\, d^3y. \end{aligned}$$

Relying once more on Eq. (6.3.20), each one of the four integrals over $\mathscr{M}$ is approximated as a volume integral over $\mathscr{M}_y$, and this is evaluated by utilizing Eqs. (6.3.21), (6.3.22), and (6.3.23). This expression is then corrected by evaluating the corresponding surface integral over $\partial\mathscr{M}_y$.

We begin with the first integral, which produces

$$\frac{1}{4\pi}\int_{\mathscr{M}_y} \frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^m y^n y^b y^c}{y^5}\, d^3y.$$

To evaluate this we involve Eq. (6.3.21), and we express $\Omega^m\Omega^n\Omega^b\Omega^c$ as

$$\begin{aligned} \Omega^m\Omega^n\Omega^b\Omega^c \quad = \quad & \Omega^{\langle mnbc\rangle} + \frac{1}{7}\Big(\delta^{mn}\Omega^{\langle bc\rangle} + \delta^{mb}\Omega^{\langle nc\rangle} + \delta^{mc}\Omega^{\langle nb\rangle} + \delta^{nb}\Omega^{\langle mc\rangle} \\ & + \delta^{nc}\Omega^{\langle mb\rangle} + \delta^{bc}\Omega^{\langle mn\rangle}\Big) + \frac{1}{15}\Big(\delta^{mn}\delta^{bc} + \delta^{mb}\delta^{nc} + \delta^{mc}\delta^{nb}\Big), \end{aligned}$$

in terms of the angular STF tensors $\Omega^{\langle abcd\rangle}$ and $\Omega^{\langle ab\rangle}$. We evaluate the angular integrations with the help of Eq. (6.3.22), and the remaining radial integrations are in the form of Eq. (6.3.23). After some algebra, we obtain the expression

$$\frac{1}{9}K(4,1)n_{AB}^{\langle mnbc\rangle} + \frac{1}{35}K(2,1)\Bigl(\delta^{mn}n_{AB}^{\langle bc\rangle} + \text{permutations}\Bigr) + \frac{1}{15}K(0,1)\Bigl(\delta^{mn}\delta^{bc} + \delta^{mb}\delta^{nc} + \delta^{mc}\delta^{nb}\Bigr)$$

for the volume integral. The corresponding surface integral is easily seen to be of order $\mathcal{R}^{-1}$, and we arrive at

$$\begin{aligned}\frac{1}{4\pi}\int_{\mathscr{M}} \frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^m y^n y^b y^c}{y^5}\,d^3y &= \frac{1}{18}z_{AB}n_{AB}^{\langle mnbc\rangle} + \frac{1}{28}z_{AB}\Bigl(\delta^{mn}n_{AB}^{\langle bc\rangle} \\ &\quad + \delta^{mb}n_{AB}^{\langle nc\rangle} + \delta^{mc}n_{AB}^{\langle nb\rangle} + \delta^{nb}n_{AB}^{\langle mc\rangle} + \delta^{nc}n_{AB}^{\langle mb\rangle} + \delta^{bc}n_{AB}^{\langle mn\rangle}\Bigr) \\ &\quad - \frac{1}{30}z_{AB}\Bigl(\delta^{mn}\delta^{bc} + \delta^{mb}\delta^{nc} + \delta^{mc}\delta^{nb}\Bigr)\end{aligned}$$

after using Eq. (6.3.23) to evaluate the radial integrals.

We next turn to the second and third integrals, which are both approximated by

$$\frac{1}{4\pi}\int_{\mathscr{M}_y} \frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^m y^n y^b}{y^5}\,d^3y.$$

To evaluate this we involve Eq. (6.3.21), and we express $\Omega^m\Omega^n\Omega^b$ as

$$\Omega^m\Omega^n\Omega^b = \Omega^{\langle mnb\rangle} + \frac{1}{5}\Bigl(\delta^{mn}\Omega^b + \delta^{mb}\Omega^n + \delta^{nb}\Omega^m\Bigr),$$

in terms of the angular STF tensor $\Omega^{\langle mnb\rangle}$. We carry out the angular integrations with the help of Eq. (6.3.22), and the remaining radial integrations are in the form of Eq. (6.3.23). After some algebra, we obtain the expression

$$\frac{1}{7}K(3,0)n_{AB}^{\langle mnb\rangle} + \frac{1}{15}K(1,0)\Bigl(\delta^{mn}n_{AB}^b + \delta^{mb}n_{AB}^n + \delta^{nb}n_{AB}^m\Bigr)$$

for the volume integral. The corresponding surface integral is once more of order $\mathcal{R}^{-1}$, and we arrive at

$$\frac{1}{4\pi}\int_{\mathscr{M}} \frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^m y^n y^b}{y^5}\,d^3y = \frac{1}{12}n_{AB}^{\langle mnb\rangle} + \frac{1}{10}\Bigl(\delta^{mn}n_{AB}^b + \delta^{mb}n_{AB}^n + \delta^{nb}n_{AB}^m\Bigr)$$

after using Eq. (6.3.23) to evaluate the radial integrals.

The final step in the computation of F_{AB}^{mnbc} is the evaluation of the fourth integral, which is approximated by

$$\frac{1}{4\pi}\int_{\mathscr{M}_y} \frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^m y^n}{y^5}\,d^3y.$$

After following the familiar steps, this becomes

$$\frac{1}{5}K(2,-1)n_{AB}^{\langle mn\rangle} + \frac{1}{3}K(0,-1)\delta^{mn},$$

and the corresponding surface integral is of order $\mathcal{R}^{-2}$. We arrive at

$$\frac{1}{4\pi}\int_{\mathscr{M}} \frac{1}{|\boldsymbol{y}-\boldsymbol{z}_{AB}|}\frac{y^m y^n}{y^5}\,d^3y = \frac{1}{6z_{AB}}n_{AB}^{\langle mn\rangle} + \frac{1}{3}K(0,-1)\delta^{mn},$$

and we note that $K(0,-1)$ is formally a divergent integral of the same sort that was encountered in Eq. (6.5.29). We shall see that this divergence requires no explicit regularization, because (as it happened before) it eventually drops out of the calculation.

Collecting results, we have obtained

$$\begin{aligned} F^{mnbc}_{AB} &= \frac{1}{18} z_{AB} n^{\langle mnbc\rangle}_{AB} + \frac{1}{28} z_{AB} \Big(\delta^{mn} n^{\langle bc\rangle}_{AB} + \delta^{mb} n^{\langle nc\rangle}_{AB} + \delta^{mc} n^{\langle nb\rangle}_{AB} + \delta^{nb} n^{\langle mc\rangle}_{AB} \\ &\quad + \delta^{nc} n^{\langle mb\rangle}_{AB} + \delta^{bc} n^{\langle mn\rangle}_{AB} \Big) - \frac{1}{30} z_{AB} \Big(\delta^{mn}\delta^{bc} + \delta^{mb}\delta^{nc} + \delta^{mc}\delta^{nb} \Big) \\ &\quad + \frac{1}{12} n^{\langle mnb\rangle}_{AB} z^c_B + \frac{1}{10} \Big(\delta^{mn} n^b_{AB} + \delta^{mb} n^n_{AB} + \delta^{nb} n^m_{AB} \Big) z^c_B \\ &\quad + \frac{1}{12} n^{\langle mnc\rangle}_{AB} z^b_B + \frac{1}{10} \Big(\delta^{mn} n^c_{AB} + \delta^{mc} n^n_{AB} + \delta^{nc} n^m_{AB} \Big) z^b_B \\ &\quad + \frac{1}{6 z_{AB}} n^{\langle mn\rangle}_{AB} z^b_B z^c_B + \frac{1}{3} K(0,-1) \delta^{mn} z^b_B z^c_B \end{aligned} \tag{6.5.30}$$

for the field integral of Eq. (6.5.27), modulo $\mathcal{R}$-dependent terms that can be freely discarded. The trace of this is

$$F^{bc}_{AB} = \frac{1}{4} z_{AB} n^{\langle bc\rangle}_{AB} - \frac{1}{6} z_{AB} \delta^{bc} + \frac{1}{2} n^b_{AB} z^c_B + \frac{1}{2} n^c_{AB} z^b_B + K(0,-1) z^b_B z^c_B, \tag{6.5.31}$$

and we see that, as claimed, the terms involving $K(0,-1)$ cancel out in the combination $\delta^{mn} F^{bc}_{AB} - 3F^{mnab}_{AB}$ that appears in Eq. (6.5.25); these terms make no contribution to C^{mnpbc}.

We may now substitute Eqs. (6.5.29)–(6.5.31) into Eq. (6.5.25). After simplification, our final result is

$$\begin{aligned} C^{mnpbc} &= \sum_A \sum_{B\neq A} G^2 m_A m_B v^p_B \biggl[-\frac{1}{6} z_{AB} n^{\langle mnbc\rangle}_{AB} \\ &\quad - \frac{3}{28} z_{AB} \Big(\delta^{mb} n^{\langle nc\rangle}_{AB} + \delta^{mc} n^{\langle nb\rangle}_{AB} + \delta^{nb} n^{\langle mc\rangle}_{AB} + \delta^{nc} n^{\langle mb\rangle}_{AB} + \delta^{bc} n^{\langle mn\rangle}_{AB} \Big) \\ &\quad + z_{AB} \delta^{mn} \Big(\frac{1}{7} n^{\langle bc\rangle}_{AB} - \frac{1}{15} \delta^{bc} \Big) + \frac{1}{10} z_{AB} \Big(\delta^{mb}\delta^{nc} + \delta^{mc}\delta^{nb} \Big) \\ &\quad - \frac{1}{4} n^{\langle mnb\rangle}_{AB} z^c_B - \frac{1}{4} n^{\langle mnc\rangle}_{AB} z^b_B - \frac{3}{10} \Big(\delta^{mb} n^n_{AB} + \delta^{nb} n^m_{AB} \Big) z^c_B \\ &\quad - \frac{3}{10} \Big(\delta^{mc} n^n_{AB} + \delta^{nc} n^m_{AB} \Big) z^b_B + \frac{1}{5} \delta^{mn} \Big(n^b_{AB} z^c_B + n^c_{AB} z^b_B \Big) \\ &\quad + \frac{1}{z_{AB}} \Big(-\frac{1}{2} n^{\langle mn\rangle}_{AB} + \frac{1}{3} \delta^{mn} \Big) z^b_B z^c_B \biggr]. \end{aligned} \tag{6.5.32}$$

We recall that $z_{AB} := |\boldsymbol{z}_A - \boldsymbol{z}_B|$ is the distance between bodies A and B, and that $\boldsymbol{n}_{AB} := (\boldsymbol{z}_A - \boldsymbol{z}_B)/|\boldsymbol{z}_A - \boldsymbol{z}_B|$ is a unit vector that points from body B to body A.

6.5.4 Computation of D^{mnp}

After inserting U from Eq. (6.4.3) and U^p from Eq. (6.5.4) within Eq. (6.5.21), we obtain

$$D^{mnp} = \sum_A G^2 m_A^2 v^n_A E^{mp}_A + \sum_A \sum_{B\neq A} G^2 m_A m_B v^n_B E^{mp}_{AB}, \tag{6.5.33}$$

where

$$E^{mp}_A := \frac{1}{4\pi} \int_{\mathscr{M}} \frac{(\boldsymbol{x} - \boldsymbol{z}_A)^m x^p}{|\boldsymbol{x} - \boldsymbol{z}_A|^4}\, d^3x \tag{6.5.34}$$

and

$$E^{mp}_{AB} := \frac{1}{4\pi} \int_{\mathscr{M}} \frac{(\boldsymbol{x} - \boldsymbol{z}_B)^m x^p}{|\boldsymbol{x} - \boldsymbol{z}_A||\boldsymbol{x} - \boldsymbol{z}_B|^3}\, d^3x \tag{6.5.35}$$

were already introduced in Eqs. (6.3.4) and (6.3.5), respectively. These integrals were evaluated in Secs. 6.3.3 and 6.3.4, respectively, and we obtained

$$E^{mp}_A = 0 \tag{6.5.36}$$

and

$$E^{mp}_{AB} = \frac{1}{4} z_{AB} n^{\langle mp \rangle}_{AB} - \frac{1}{6} z_{AB} \delta^{mp} + \frac{1}{2} n^m_{AB} z^p_B; \tag{6.5.37}$$

these are Eqs. (6.3.11) and (6.3.15), respectively. Making the substitutions, we arrive at

$$D^{mnp} = \sum_A \sum_{B \neq A} G^2 m_A m_B v^n_B \biggl(\frac{1}{4} z_{AB} n^{\langle mp \rangle}_{AB} - \frac{1}{6} z_{AB} \delta^{mp} + \frac{1}{2} n^m_{AB} z^p_B \biggr). \tag{6.5.38}$$

6.5.5 Computation of E^{mnbcp}

The surface integrals

$$E^{mnbcp} = \frac{1}{4\pi} \oint_{\partial \mathscr{M}} U \partial^m U^n\, x^b x^c\, dS^p$$

are evaluated at $r := |\boldsymbol{x}| = \mathcal{R}$, and to do this we may substitute Eq. (6.4.8) for U, which has the schematic form $U = \mathcal{R}^{-1} + \mathcal{R}^{-3} + \cdots$. Similarly, we may expand Eq. (6.5.4) in powers of r^{-1}, and express the result schematically as $U^a = \mathcal{R}^{-2} + \mathcal{R}^{-3} + \cdots$, which implies that $\partial^a U^b = \mathcal{R}^{-3} + \mathcal{R}^{-4} + \cdots$. We recall that U does not include a term in $\mathcal{R}^{-2}$ because the Newtonian dipole moment $I^a := \sum_A m_A z^a_A$ can be set equal to zero, and similarly, U^n does not contain a term in $\mathcal{R}^{-1}$ because $\dot{I}^a = \sum_A m_A v^a_A = 0$. With $x^a = \mathcal{R}\Omega^a$ and $dS^a = \mathcal{R}^2 \Omega^a\, d\Omega$, we find that the leading term in the surface integral is of order $\mathcal{R}^0$, and that it must be evaluated carefully. Further investigation reveals that at this order, $\partial^a U^b$ involves an even number of angular vectors Ω^a, which implies that the surface integral involves an odd number of such vectors. This guarantees that

$$E^{mnbcp} = 0, \tag{6.5.39}$$

modulo $\mathcal{R}$-dependent terms that can be freely discarded.

6.5.6 Computation of $A^{abc}[\mathrm{F}]$

It is now a straightforward task to substitute Eq. (6.5.32) for C^{mnpbc}, Eq. (6.5.38) for D^{mnp}, and Eq. (6.5.39) for E^{mnbcp} into Eqs. (6.5.22) and (6.5.24). These results, in turn, can be inserted within Eq. (6.5.11) for B^{abc}_1, Eq. (6.5.14) for B^{abc}_2, and Eq. (6.5.17) for B^{abc}_3. The final step is to substitute these expressions into the right-hand side of Eq. (6.5.7). The end result, after much simplification, and after discarding terms that will not survive the TT projection, is

$$\begin{aligned} A^{abc}[\mathrm{F}] \overset{\scriptscriptstyle\mathrm{TT}}{=} & \frac{1}{c^3} \sum_A \sum_{B \neq A} G m_A m_B \biggl\{ z_{AB} \biggl[-\frac{1}{6} (\boldsymbol{n}_{AB} \cdot \boldsymbol{v}_B) n^a_{AB} n^b_{AB} n^c_{AB} \\ & + \frac{11}{12} n^a_{AB} \bigl(n^b_{AB} v^c_B + v^b_B n^c_{AB} \bigr) - \frac{11}{6} v^a_B n^b_{AB} n^c_{AB} \biggr] \\ & - \frac{1}{4} (\boldsymbol{n}_{AB} \cdot \boldsymbol{v}_B) n^a_{AB} \bigl(n^b_{AB} z^c_B + z^b_B n^c_{AB} \bigr) + \frac{7}{4} n^a_{AB} \bigl(v^b_B z^c_B + z^b_B v^c_B \bigr) \end{aligned}$$

$$- \frac{7}{4} v_B^a \bigl(n_{AB}^b z_B^c + z_B^b n_{AB}^c \bigr) - \frac{1}{z_{AB}} \biggl[\frac{1}{2} (\boldsymbol{n}_{AB} \cdot \boldsymbol{v}_B) n_{AB}^a z_B^b z_B^c + \frac{7}{2} v_B^a z_B^b z_B^c \biggr] \biggr\}. \qquad (6.5.40)$$

6.5.7 Q^{abc}: Final answer

Equation (6.5.40) for $A^{abc}[\mathrm{F}]$ and Eq. (6.5.5) for $A^{abc}[\mathrm{M}]$ can finally be combined to form A^{abc}, as defined by Eq. (6.5.2). After inserting $\sum_{B \neq A} Gm_B/z_{AB}$ for $\lfloor U \rfloor_A$, and after additional simplification, we obtain

$$\begin{aligned}
A^{abc} \overset{\scriptscriptstyle\mathrm{TT}}{=} \; & \frac{1}{c} \sum_A m_A \biggl(1 + \frac{v_A^2}{2c^2} \biggr) v_A^a z_A^b z_A^c \\
& - \frac{1}{2c^3} \sum_A \sum_{B \neq A} \frac{G m_A m_B}{z_{AB}} \Bigl[(\boldsymbol{n}_{AB} \cdot \boldsymbol{v}_A) n_{AB}^a z_A^b z_A^c + v_A^a z_A^b z_A^c \Bigr] \\
& + \frac{1}{2c^3} \sum_A \sum_{B \neq A} G m_A m_B \Bigl[(\boldsymbol{n}_{AB} \cdot \boldsymbol{v}_A) n_{AB}^a n_{AB}^{(b} z_A^{c)} - 7 n_{AB}^a v_A^{(b} z_A^{c)} \\
& \qquad + 7 v_A^a n_{AB}^{(b} z_A^{c)} \Bigr] \\
& - \frac{1}{6c^3} \sum_A \sum_{B \neq A} G m_A m_B z_{AB} \Bigl[(\boldsymbol{n}_{AB} \cdot \boldsymbol{v}_A) n_{AB}^a n_{AB}^b n_{AB}^c - 11 n_{AB}^a n_{AB}^{(b} v_A^{c)} \\
& \qquad + 11 v_A^a n_{AB}^b n_{AB}^c \Bigr] \\
& + O(c^{-5}).
\end{aligned} \qquad (6.5.41)$$

To arrive at this result we have rearranged the sums in Eq. (6.5.40), and switched the identities of bodies A and B; this permutation affects the signs of some terms, because $\boldsymbol{n}_{BA} = -\boldsymbol{n}_{AB}$.

The final expression for Q^{abc} is obtained by inserting Eq. (6.5.41) within Eq. (6.5.1), which we copy as

$$Q^{abc} = A^{abc} + A^{bac} - A^{cab}. \qquad (6.5.42)$$

We shall not display this result here, as it is more convenient to perform the substitution at a later stage.

In Eq. (6.5.41), all position and velocity vectors are evaluated at the retarded time $\tau := t - r/c$, and A^{abc} is a function of τ only. We recall that $z_{AB} = |\boldsymbol{z}_A - \boldsymbol{z}_B|$ is the distance between bodies A and B, and that $\boldsymbol{n}_{AB} = (\boldsymbol{z}_A - \boldsymbol{z}_B)/z_{AB}$ is a unit vector that points from body B to body A.

6.6 Computation of Q^{abcd}

Our next step is the computation of Q^{abcd}, the radiative 4-pole moment. This is defined by Eq. (6.2.11),

$$Q^{abcd}(\tau) := \frac{1}{c^2} \int_{\mathscr{M}} \tau^{ab}(\tau, \boldsymbol{x}) x^c x^d \, d^3x, \qquad (6.6.1)$$

in which $\tau := t - r/c$ is retarded time, and where we suppress the primes on the integration variables to simplify the notation.

According to the discussion of Sec. 6.2, to calculate Q^{abcd} to the required degree of accuracy we need an expression for $c^{-2}\tau^{ab}$ that includes terms of order c^{-2} (1PN

terms), but we do not need higher-order terms. Such an expression was obtained in Sec. 4.1, and it can be read off the right-hand side of Eq. (4.1.28). We have

$$c^{-2}\tau^{ab} = \frac{1}{c^2}\sum_A m_A v_A^a v_A^b \delta(\boldsymbol{x}-\boldsymbol{z}_A) + \frac{1}{4\pi G c^2}\biggl(\partial^a U \partial^b U - \frac{1}{2}\delta^{ab}\partial_c U \partial^c U\biggr) + O(c^{-4}), \tag{6.6.2}$$

where we used the fact that in the near zone, and at this order of accuracy, the gravitational potential Φ defined by Eq. (4.1.25) can be set equal to the Newtonian potential

$$U = \sum_A \frac{G m_A}{|\boldsymbol{x}-\boldsymbol{z}_A|}. \tag{6.6.3}$$

The multipole moment contains both a matter contribution that comes from the δ-functions in τ^{ab}, and a field contribution that comes from the terms involving the Newtonian potential. The matter contribution can be calculated at once:

$$Q^{abcd}[\mathrm{M}] = \frac{1}{c^2}\sum_A m_A v_A^a v_A^b z_A^c z_A^d. \tag{6.6.4}$$

The field contribution is

$$Q^{abcd}[\mathrm{F}] = \frac{1}{4\pi G c^2}\int_{\mathscr{M}} \partial^a U \partial^b U x^c x^d\, d^3x - \frac{1}{8\pi G c^2}\delta^{ab}\int_{\mathscr{M}} \partial_e U \partial^e U x^c x^d\, d^3x, \tag{6.6.5}$$

and the second integral, because it comes with a factor δ^{ab} in front, will not survive a TT projection; it does not need to be evaluated. The complete 4-pole moment is $Q^{abcd} = Q^{abcd}[\mathrm{M}] + Q^{abcd}[\mathrm{F}] + O(c^{-4})$.

To evaluate the first integral we employ our usual strategy of integrating by parts so as to leave one factor of U undifferentiated. We find that the integral splits into a volume integral over the domain $\mathscr{M}$ and a surface integral over $\partial\mathscr{M}$, and that Eq. (6.6.5) becomes

$$Q^{abcd}[\mathrm{F}] \overset{\mathrm{TT}}{=} Q^{abcd}[\mathrm{F},\mathscr{M}] + Q^{abcd}[\mathrm{F},\partial\mathscr{M}], \tag{6.6.6}$$

where

$$Q^{abcd}[\mathrm{F},\mathscr{M}] = -\frac{1}{4\pi G c^2}\int_{\mathscr{M}} U \partial^{ab} U\, x^c x^d\, d^3x \tag{6.6.7}$$

and

$$Q^{abcd}[\mathrm{F},\partial\mathscr{M}] = \frac{1}{4\pi G c^2}\oint_{\partial\mathscr{M}} U \partial^b U\, x^c x^d\, dS^a. \tag{6.6.8}$$

To arrive at Eq. (6.6.6) we have discarded additional terms that will not survive a TT projection. For example, a contribution to Q^{abcd} of the form $\delta^{ac}A^{bd}$ would become $\Omega^a A^{bd}\Omega_d$ after contraction with $\Omega_c\Omega_d$, and this would make an irrelevant, longitudinal contribution to h^{ab}.

To evaluate the volume integral in Eq. (6.6.7) we substitute Eq. (6.6.3) for U, as well as

$$\partial^{ab}U = -\sum_A G m_A\biggl[-3\frac{(\boldsymbol{x}-\boldsymbol{z}_A)^a(\boldsymbol{x}-\boldsymbol{z}_A)^b}{|\boldsymbol{x}-\boldsymbol{z}_A|^5} + \frac{\delta^{ab}}{|\boldsymbol{x}-\boldsymbol{z}_A|^3} + \frac{4\pi}{3}\delta^{ab}\delta(\boldsymbol{x}-\boldsymbol{z}_A)\biggr].$$

Once more we can ignore the terms in δ^{ab}, and we find that

$$Q^{abcd}[\mathrm{F},\mathscr{M}] \overset{\mathrm{TT}}{=} -\frac{3}{c^2}\sum_A G m_A^2 F_A^{abcd} - \frac{3}{c^2}\sum_A\sum_{B\neq A} G m_A m_B F_{BA}^{abcd},$$

where the field integrals F_A^{abcd} and F_{BA}^{abcd} were already introduced in Sec. 6.5.3, and are defined by Eqs. (6.5.26) and (6.5.27), respectively. From Eq. (6.5.29) we learn

that F_A^{abcd} is proportional to δ^{ab} and therefore will not survive a TT projection, and Eq. (6.5.30) gives us an explicit expression for F_{BA}^{abcd}. After discarding additional terms that will eventually be projected out, and after some further simplification, we find that

$$F_{BA}^{abcd} \overset{\text{TT}}{=} \frac{1}{36} z_{AB} n_{AB}^a n_{AB}^b \bigl(2n_{AB}^c n_{AB}^d + \delta^{cd}\bigr) - \frac{1}{6} n_{AB}^a n_{AB}^b n_{AB}^{(c} z_A^{d)} + \frac{1}{6z_{AB}} n_{AB}^a n_{AB}^b z_A^c z_A^d .$$

Inserting these results within $Q^{abcd}[\mathrm{F}, \mathscr{M}]$, we arrive at

$$\begin{aligned} Q^{abcd}[\mathrm{F}, \mathscr{M}] \overset{\text{TT}}{=} & -\frac{1}{12c^2} \sum_A \sum_{B\neq A} G m_A m_B z_{AB} n_{AB}^a n_{AB}^b \bigl(2n_{AB}^c n_{AB}^d + \delta^{cd}\bigr) \\ & + \frac{1}{2c^2} \sum_A \sum_{B\neq A} G m_A m_B n_{AB}^a n_{AB}^b n_{AB}^{(c} z_A^{d)} \\ & - \frac{1}{2c^2} \sum_A \sum_{B\neq A} \frac{G m_A m_B}{z_{AB}} n_{AB}^a n_{AB}^b z_A^c z_A^d . \end{aligned}$$

This expression can be simplified. Consider the second line, which we write as

$$\frac{1}{4c^2} \sum_A \sum_{B\neq A} G m_A m_B n_{AB}^a n_{AB}^b n_{AB}^c z_A^d + (c \leftrightarrow d).$$

By rearranging the sums, we see that this is also

$$\frac{1}{4c^2} \sum_A \sum_{B>A} G m_A m_B n_{AB}^a n_{AB}^b \bigl(n_{AB}^c z_A^d + n_{BA}^c z_B^d\bigr) + (c \leftrightarrow d),$$

or

$$\frac{1}{4c^2} \sum_A \sum_{B>A} G m_A m_B n_{AB}^a n_{AB}^b n_{AB}^c \bigl(z_A^d - z_B^d\bigr) + (c \leftrightarrow d).$$

The term within brackets is $z_{AB} n_{AB}^d$, and we see that the second line in $Q^{abcd}[\mathrm{F}, \mathscr{M}]$ can be joined with the first line. Our final expression is

$$\begin{aligned} Q^{abcd}[\mathrm{F}, \mathscr{M}] \overset{\text{TT}}{=} & \frac{1}{12c^2} \sum_A \sum_{B\neq A} G m_A m_B z_{AB} n_{AB}^a n_{AB}^b \bigl(n_{AB}^c n_{AB}^d - \delta^{cd}\bigr) \\ & - \frac{1}{2c^2} \sum_A \sum_{B\neq A} \frac{G m_A m_B}{z_{AB}} n_{AB}^a n_{AB}^b z_A^c z_A^d . \end{aligned} \tag{6.6.9}$$

Moving on the surface integral of Eq. (6.6.8), we recall our previous work in Sec. 6.2, in which U was seen to have the schematic structure $U = \mathcal{R}^{-1} + \mathcal{R}^{-3} + \cdots$ when evaluated at $r = \mathcal{R}$, while $\partial^a U$ is given by $\partial^a U = \mathcal{R}^{-2} + \mathcal{R}^{-4} + \cdots$. With $x^a = \mathcal{R}\Omega^a$ and $dS^a = \mathcal{R}^2 \Omega^a \, d\Omega$, these statements imply that $Q^{abcd}[\mathrm{F}, \partial\mathscr{M}]$ contains terms at orders $\mathcal{R}$, $\mathcal{R}^{-1}$, and so on, but that there is no $\mathcal{R}$-independent contribution. For this reason, we may set

$$Q^{abcd}[\mathrm{F}, \partial\mathscr{M}] = 0,$$

modulo $\mathcal{R}$-dependent terms that can be freely discarded.

Collecting results, we find that the radiative 4-pole moment is given by

$$\begin{aligned} Q^{abcd} \overset{\text{TT}}{=} & \frac{1}{c^2} \sum_A m_A v_A^a v_A^b z_A^c z_A^d \\ & - \frac{1}{2c^2} \sum_A \sum_{B\neq A} \frac{G m_A m_B}{z_{AB}} n_{AB}^a n_{AB}^b z_A^c z_A^d \\ & + \frac{1}{12c^2} \sum_A \sum_{B\neq A} G m_A m_B z_{AB} n_{AB}^a n_{AB}^b \bigl(n_{AB}^c n_{AB}^d - \delta^{cd}\bigr) \\ & + O(c^{-4}). \end{aligned} \tag{6.6.10}$$

In this equation, all position and velocity vectors are evaluated at the retarded time $\tau := t - r/c$, and Q^{abcd} is a function of τ only. We recall once more that $z_{AB} = |\boldsymbol{z}_A - \boldsymbol{z}_B|$ is the distance between bodies A and B, and that $\boldsymbol{n}_{AB} = (\boldsymbol{z}_A - \boldsymbol{z}_B)/z_{AB}$ is a unit vector that points from body B to body A.

6.7 Computation of Q^{abcde}

Next on our list of radiative multipole moments is Q^{abcde}, the 5-pole moment defined by Eq. (6.2.12),

$$Q^{abcde}(\tau) := \frac{1}{c^3}\frac{\partial}{\partial\tau}\int_{\mathscr{M}} \tau^{ab}(\tau,\boldsymbol{x})x^c x^d x^e\, d^3x. \tag{6.7.1}$$

To compute this, with the effective stress tensor displayed in Eq. (6.6.2), requires the same familiar steps that were followed in the preceding sections. We shall not labour through the details here, and simply present the final answer:

$$\begin{aligned} Q^{abcde} &\overset{\text{TT}}{=} \frac{1}{c^3}\frac{\partial}{\partial\tau}\biggl[\sum_A m_A v_A^a v_A^b z_A^c z_A^d z_A^e - \frac{1}{2}\sum_A\sum_{B\neq A}\frac{Gm_Am_B}{z_{AB}} n_{AB}^a n_{AB}^b z_A^c z_A^d z_A^e \\ &\quad + \frac{1}{4}\sum_A\sum_{B\neq A} Gm_Am_B z_{AB} n_{AB}^a n_{AB}^b z_A^{(c}\Bigl(n_{AB}^d n_{AB}^{e)} - \delta^{de)}\Bigr)\biggr] \\ &\quad + O(c^{-5}). \end{aligned} \tag{6.7.2}$$

In the last sum, the index symmetrization is over the trio of indices cde. We shall leave the differentiation with respect to τ unevaluated for the time being; it is advantageous to take care of this at a later stage.

6.8 Computation of P^{ab} and P^{abc}

The multipole expansion of Eq. (6.2.8) involves also a pair of surface integrals, P^{ab} and P^{abc}, which are defined by Eqs. (6.2.13) and (6.2.14), respectively. Our task in this section is to evaluate them.

We begin with

$$P^{ab} := \oint_{\partial\mathscr{M}} \bigl(\tau^{ac}x^b + \tau^{bc}x^a - \partial_d\tau^{cd}x^a x^b\bigr)\, dS_c, \tag{6.8.1}$$

in which τ^{ab} is expressed as a function of τ and $\boldsymbol{x}$, and where we suppress the primes on the integration variables to simplify the notation. The effective stress tensor τ^{ab} is given to leading order — order c^0 — by Eq. (6.6.2), but to achieve the required degree of accuracy (as specified in Sec. 6.2), we must also incorporate terms of order c^{-2}. A sufficiently accurate expression can be obtained from Eqs. (4.1.20) and (4.1.23); there is no need to include also the material contribution of Eq. (4.1.11), because τ^{ab} is evaluated on $\partial\mathscr{M}$, away from the matter distribution. In this we must substitute the near-zone gravitational potentials of Eqs. (5.2.7)–(5.2.9), and such a computation was already carried out at the beginning of Sec. 5.3.3. We obtain, finally,

$$\begin{aligned} \tau^{ab} &\overset{\text{TT}}{=} \frac{1}{4\pi G}\partial^a U\partial^b U + \frac{1}{4\pi Gc^2}\Bigl[2\partial^{(a}U\partial^{b)}\psi + \partial^{(a}U\partial^{b)}\ddot{X} + 8\partial^{(a}U\dot{U}^{b)} \\ &\quad - 4\bigl(\partial^a U_c - \partial_c U^a\bigr)\bigl(\partial^b U^c - \partial^c U^b\bigr)\Bigr] + O(c^{-4}), \end{aligned} \tag{6.8.2}$$

after discarding all terms proportional to δ^{ab}, for the usual reason that they will not survive a TT projection.

To calculate P^{ab} we also need $\partial_d \tau^{cd}$, which we express as $-c^{-1}\dot{\tau}^{0c}$ by involving the conservation identities $\partial_\beta \tau^{\alpha\beta} = 0$. With Eq. (6.5.3), this is

$$\partial_d \tau^{cd} = \frac{1}{4\pi G c^2} \frac{\partial}{\partial \tau} \Big[3\partial_d U^d \partial^c U - 4\big(\partial^c U^d - \partial^d U^c\big)\partial_d U \Big], \tag{6.8.3}$$

where we have also used the gauge condition $\dot{U} = -\partial_d U^d$. The derivative operator can be taken outside of the surface integral.

From the explicit expressions obtained in Sec. 5.2 for U, U^a, ψ, and $\ddot{X}$, we observe that each one of these quantities has an expansion of the schematic form $r^{-1} + r^{-2} + \cdots$. It follows that when $\partial^a U$, $\partial^a \psi$, $\partial^a \ddot{X}$, $\partial^b U^a$, and $\dot{U}^a$ are evaluated on $\partial\mathscr{M}$, they each have the schematic form $\mathcal{R}^{-2} + \mathcal{R}^{-3} + \cdots$. This means that $\tau^{ab} = O(\mathcal{R}^{-4})$, and it follows that a quantity such as $\tau^{ac} x^b dS_c$ must scale as $\mathcal{R}^{-1}$; this will give no $\mathcal{R}$-independent contribution to the surface integral. A similar argument reveals that $\partial_d \tau^{cd} = O(\mathcal{R}^{-5})$, so that $\partial_d \tau^{cd} x^a x^b dS_c$ scales as $\mathcal{R}^{-1}$; this also makes no contribution. We conclude that

$$P^{ab} = 0, \tag{6.8.4}$$

modulo $\mathcal{R}$-dependent terms that can be freely discarded.

We next evaluate

$$P^{abc} := \frac{1}{c}\frac{\partial}{\partial \tau} \oint_{\partial\mathscr{M}} \big(\tau^{ad} x^b x^c + \tau^{bd} x^a x^c - \tau^{cd} x^a x^b\big)\, dS_d, \tag{6.8.5}$$

using the effective stress tensor displayed in Eq. (6.8.2). Relative to P^{ab}, this surface integral involves an additional power of $\boldsymbol{x}$, and therefore an additional power of $\mathcal{R}$; because P^{ab} was seen to be of order $\mathcal{R}^{-1}$, there is a chance that the surface integral might contain an $\mathcal{R}$-independent contribution. As we shall see presently, however, this does not happen, and as a matter of fact,

$$P^{abc} = 0, \tag{6.8.6}$$

modulo $\mathcal{R}$-dependent terms that can be freely discarded. This conclusion emerges as a result of a closer examination of the terms that make up τ^{ab}. It was stated previously that at leading order, $\partial^a U$, $\partial^a \psi$, $\partial^a \ddot{X}$, $\partial^b U^a$, and $\dot{U}^a$ all scale as $\mathcal{R}^{-2}$ when they are evaluated on $\partial\mathscr{M}$, so that $\tau^{ab} = O(\mathcal{R}^{-4})$. With the four powers of $\mathcal{R}$ that are contained in the position vectors and the surface element, we find that the integral does indeed scale as $\mathcal{R}^0$. It can be verified, however, that $\partial^a U$, $\partial^a \psi$, $\partial^a \ddot{X}$, $\partial^b U^a$, and $\dot{U}^a$ are all proportional to an *odd number* of angular vectors $\boldsymbol{\Omega} := \boldsymbol{x}/\mathcal{R}$. This implies that τ^{ab} is proportional to an *even number* of such vectors, and this, in turn, implies that the integrand in Eq. (6.8.5) contains an *odd number* of angular vectors. Integration gives zero, and we have established the statement of Eq. (6.8.6).

6.9 Summary: $h^{ab}_{\mathscr{N}}$

Our computation of the near-zone contribution to h^{ab} is essentially complete, and for easy reference we copy our main results in this section. The gravitational potentials are expressed as a multipole expansion in Eq. (6.2.8),

$$h^{ab}_{\mathscr{N}} = \frac{2G}{c^4 r}\frac{\partial^2}{\partial \tau^2}\bigg\{ Q^{ab} + Q^{abc}\Omega_c + Q^{abcd}\Omega_c\Omega_d + \frac{1}{3}Q^{abcde}\Omega_c\Omega_d\Omega_e + O(c^{-4}) \bigg\}, \tag{6.9.1}$$

in which the surface integrals P^{ab} and P^{abc} were set equal to zero by virtue of Eqs. (6.8.4) and (6.8.6).

According to Eqs. (6.4.11), (6.5.41), (6.5.42), (6.6.10), and (6.7.2), respectively, the radiative multipole moments are given by

$$
\begin{aligned}
Q^{ab} &\overset{\text{TT}}{=} \sum_A m_A\biggl(1+\frac{v_A^2}{2c^2}\biggr)z_A^a z_A^b - \frac{1}{2c^2}\sum_A\sum_{B\neq A}\frac{Gm_Am_B}{z_{AB}}z_A^a z_A^b + O(c^{-4}), &(6.9.2)\\
Q^{abc} &= A^{abc}+A^{bac}-A^{cab}, &(6.9.3)\\
A^{abc} &\overset{\text{TT}}{=} \frac{1}{c}\sum_A m_A\biggl(1+\frac{v_A^2}{2c^2}\biggr)v_A^a z_A^b z_A^c\\
&\quad - \frac{1}{2c^3}\sum_A\sum_{B\neq A}\frac{Gm_Am_B}{z_{AB}}\Bigl[(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_A)n_{AB}^a z_A^b z_A^c + v_A^a z_A^b z_A^c\Bigr]\\
&\quad + \frac{1}{2c^3}\sum_A\sum_{B\neq A}Gm_Am_B\Bigl[(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_A)n_{AB}^a n_{AB}^{(b}z_A^{c)} - 7n_{AB}^a v_A^{(b}z_A^{c)}\\
&\qquad\qquad + 7v_A^a n_{AB}^{(b}z_A^{c)}\Bigr]\\
&\quad - \frac{1}{6c^3}\sum_A\sum_{B\neq A}Gm_Am_Bz_{AB}\Bigl[(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_A)n_{AB}^a n_{AB}^b n_{AB}^c - 11n_{AB}^a n_{AB}^{(b}v_A^{c)}\\
&\qquad\qquad + 11v_A^a n_{AB}^b n_{AB}^c\Bigr]\\
&\quad + O(c^{-5}), &(6.9.4)\\
Q^{abcd} &\overset{\text{TT}}{=} \frac{1}{c^2}\sum_A m_A v_A^a v_A^b z_A^c z_A^d\\
&\quad - \frac{1}{2c^2}\sum_A\sum_{B\neq A}\frac{Gm_Am_B}{z_{AB}}n_{AB}^a n_{AB}^b z_A^c z_A^d\\
&\quad + \frac{1}{12c^2}\sum_A\sum_{B\neq A}Gm_Am_Bz_{AB}n_{AB}^a n_{AB}^b\bigl(n_{AB}^c n_{AB}^d - \delta^{cd}\bigr)\\
&\quad + O(c^{-4}), &(6.9.5)\\
Q^{abcde} &\overset{\text{TT}}{=} \frac{1}{c^3}\frac{\partial}{\partial\tau}\biggl[\sum_A m_A v_A^a v_A^b z_A^c z_A^d z_A^e - \frac{1}{2}\sum_A\sum_{B\neq A}\frac{Gm_Am_B}{z_{AB}}n_{AB}^a n_{AB}^b z_A^c z_A^d z_A^e\\
&\quad + \frac{1}{4}\sum_A\sum_{B\neq A}Gm_Am_Bz_{AB}n_{AB}^a n_{AB}^b z_A^{(c}\Bigl(n_{AB}^d n_{AB}^{e)} - \delta^{de)}\Bigr)\biggr]\\
&\quad + O(c^{-5}). &(6.9.6)
\end{aligned}
$$

In these equations, all position and velocity vectors are evaluated at the retarded time $\tau := t - r/c$, and the multipole moments are functions of τ only. We recall that

$$z_{AB} = |\boldsymbol{z}_A - \boldsymbol{z}_B| \tag{6.9.7}$$

is the distance between bodies A and B, and that

$$\boldsymbol{n}_{AB} = \frac{\boldsymbol{z}_A - \boldsymbol{z}_B}{|\boldsymbol{z}_A - \boldsymbol{z}_B|} \tag{6.9.8}$$

is a unit vector that points from body B to body A.

6.10 Computation of $h^{ab}_{\mathscr{W}}$

6.10.1 Construction of the source term

The wave-zone contribution to h^{ab} is obtained by evaluating the integrals displayed in Eq. (6.2.6), and this relies on a decomposition of τ^{ab} into irreducible pieces of the form of Eq. (6.2.5). Our first order of business, therefore, is to obtain an expression for the effective stress tensor; this expression must be valid everywhere in the wave zone.

The source term is constructed from the gravitational potentials, and wave-zone expressions for these were obtained in Sec. 4.4. According to Eqs. (4.4.37)–(4.4.39), we have

$$h^{00} = \frac{4G}{c^2}\biggl[\frac{M}{r} + \frac{1}{2}\partial_{ab}\biggl(\frac{I^{ab}}{r}\biggr) + O\biggl(\frac{M}{r}\frac{v^3}{c^3}\biggr)\biggr], \tag{6.10.1}$$

$$h^{0a} = \frac{4G}{c^2}\biggl[\frac{1}{2c}J^{ab}\frac{\Omega_b}{r^2} - \frac{1}{2c}\partial_b\biggl(\frac{\dot{I}^{ab}}{r}\biggr) + O\biggl(\frac{M}{r}\frac{v^3}{c^3}\biggr)\biggr], \tag{6.10.2}$$

$$h^{ab} = \frac{4G}{c^2}\biggl[\frac{1}{2c^2}\frac{\ddot{I}^{ab}}{r} + O\biggl(\frac{M}{r}\frac{v^3}{c^3}\biggr)\biggr]. \tag{6.10.3}$$

The potentials are expressed in terms of $\Omega^a := x^a/r$, and in terms of multipole moments that were introduced in Sec. 4.4. We have the total gravitational mass

$$M = \sum_A \biggl(1 + \frac{1}{2}\frac{v_A^2}{c^2} - \frac{1}{2}\frac{\lfloor U \rfloor_A}{c^2}\biggr) + O(c^{-4}), \tag{6.10.4}$$

the angular-momentum tensor

$$J^{ab} = \sum_A m_A\bigl(v_A^a z_A^b - z_A^a v_A^b\bigr) + O(c^{-2}), \tag{6.10.5}$$

and the Newtonian quadrupole moment

$$I^{ab}(\tau) = \sum_A m_A z_A^a z_A^b + O(c^{-2}). \tag{6.10.6}$$

We have indicated that the mass and angular momentum are conserved quantities, while I^{ab} depends on retarded-time $\tau := t - r/c$.

The post-Newtonian order of each term in Eqs. (6.10.1)–(6.10.3) was clearly indicated in Sec. 4.4.7: Relative to $GM/(c^2 r)$, each term involving I^{ab} is of 1PN order, and the term involving the angular-momentum tensor also is of 1PN order; the expressions are therefore truncated at 1PN order, and the neglected terms are of $\frac{3}{2}$PN order. The rules to count the post-Newtonian order of wave-zone potentials were derived back in Sec. 3.3.3. It is useful to recall that in the wave zone, r is larger than $\lambda_c = ct_c$, the characteristic wavelength of the gravitational radiation (which is defined in terms of t_c, the characteristic time scale of the source); it follows that if r_c is a characteristic length scale of the source, then $r_c/r \sim (r_c/t_c)/c = v_c/c$, where v_c is the source's characteristic velocity. It is also useful to recall that for gravitationally bound sources, $GM/r_c \sim v_c^2$.

In the wave zone, away from the matter distribution, the effective stress tensor τ^{ab} is made up of the Landau-Lifshitz pseudotensor $(-g)t^{ab}_{\rm LL}$ and the harmonic-gauge contribution $(-g)t^{ab}_{\rm H}$. Sufficiently accurate expressions for these quantities were obtained in Secs. 4.1.3 and 4.1.4, respectively. The leading term comes from the Landau-Lifshitz pseudotensor of Eq. (4.1.19), which is

$$\frac{c^4}{64\pi G}\partial^a h^{00}\partial^b h^{00};$$

here we ignore the term proportional to δ^{ab} because, as we observed many times before, it will not survive a TT projection. Using Eq. (6.10.1), we find that this is equal to

$$\frac{G}{4\pi}\biggl[\frac{M^2}{r^4}\Omega^a\Omega^a - \frac{M}{r^2}\Omega^{(a}\partial^{b)}{}_{cd}\biggl(\frac{I^{cd}}{r}\biggr) + \cdots\biggr].$$

It is easy to show that relative to GM^2/r^4, the second term is of order $(v_c/c)^2$, and the neglected terms are smaller by an additional power of v_c/c.

We wish our expression for τ^{ab} to be as accurate as what was displayed previously. In particular, we want to be sure that our expression contains all occurrences of terms that involve a product of M with derivatives of I^{ab}; all such terms will contribute at order $(v_c/c)^2$ relative to GM^2/r^4, and they must all be included. A careful examination of Eq. (4.1.20) reveals that the relevant terms are contained in

$$\begin{aligned}(-g)t^{ab}_{\text{LL}} &= \frac{c^4}{16\pi G}\biggl[\frac{1}{4}\partial^a h^{00}\partial^b h^{00} + \partial^a h^{00}\partial_0 h^{0b} + \partial^b h^{00}\partial_0 h^{0a} \\ &\quad + \frac{1}{4}\partial^a h^{00}\partial^b h^c{}_c + \frac{1}{4}\partial^b h^{00}\partial^a h^c{}_c + \cdots\biggr],\end{aligned}$$

and that the additional terms are smaller by additional powers of v_c/c.

A careful examination of Eq. (4.1.24) reveals that

$$(-g)t^{ab}_{\text{H}} = \frac{c^4}{16\pi G}\Bigl[-h^{00}\partial_{00}h^{ab} + \cdots\Bigr]$$

also is a relevant term. It is easy to see why: After writing $\partial_{00} = c^{-2}\partial_{\tau\tau}$, we find that this contribution to τ^{ab} is schematically

$$\frac{c^2}{G}h^{00}\ddot{h}^{ab} \sim \frac{G}{c^4}\frac{MI^{(4)}}{r^2},$$

in which $I^{(4)}$ stands for four derivatives of the quadrupole moment tensor. We have that $I^{ab} \sim Mr_c^2$, so that $I^{(4)} \sim Mr_c^2/t_c^4$, and in the wave zone $r > \lambda_c = ct_c$. All together, these scalings imply that this term is of order $(v_c/c)^2$ relative to GM^2/r^4, and that it contributes at the required post-Newtonian order.

This is the first time that $(-g)t^{\alpha\beta}_{\text{H}}$ explicitly enters a computation. As we saw in Sec. 1.3, this contribution to $\tau^{\alpha\beta}$ comes from the difference between $\partial_{\mu\nu}H^{\alpha\mu\beta\nu}$ and $-\Box h^{\alpha\beta}$ on the left-hand side of the Einstein field equations. It is this term that informs us that the gravitational waves are propagating not in flat spacetime, but in a curved spacetime whose metric $g_{\alpha\beta}$ must be obtained self-consistently from the gravitational potentials. It is this contribution to $\tau^{\alpha\beta}$, therefore, that will reveal the differences between the light cones of the mathematical flat spacetime, and those of the physical curved spacetime. And as we shall see, this term will generate interesting physical consequences.

Collecting results, we find that the appropriate starting expression for the source term is

$$\begin{aligned}\tau^{ab} &= \frac{c^4}{16\pi G}\biggl[\frac{1}{4}\partial^a h^{00}\partial^b h^{00} + \frac{1}{c}\partial^a h^{00}\dot{h}^{0b} + \frac{1}{c}\partial^b h^{00}\dot{h}^{0a} \\ &\quad + \frac{1}{4}\partial^a h^{00}\partial^b h^c{}_c + \frac{1}{4}\partial^b h^{00}\partial^a h^c{}_c - \frac{1}{c^2}h^{00}\ddot{h}^{ab} + \cdots\biggr]. \qquad (6.10.7)\end{aligned}$$

In the next subsection we will turn this into something more concrete, a set of expressions that will be ready for substitution within Eq. (6.2.6).

6.10.2 Evaluation of the source term

The first step is to insert Eqs. (6.10.1)–(6.10.3) within Eq. (6.10.7). We need

$$\begin{aligned}
\partial^a h^{00} &= \frac{4G}{c^2}\biggl[-\frac{M}{r^2}\Omega^a + \frac{1}{2}\partial^a{}_{cd}\biggl(\frac{I^{cd}}{r}\biggr) + \cdots\biggr], \\
\dot{h}^{0a} &= \frac{4G}{c^2}\biggl[-\frac{1}{2c}\partial_c\biggl(\frac{\dddot{I}^{ac}}{r}\biggr) + \cdots\biggr], \\
\partial^a h^c{}_c &= \frac{4G}{c^2}\biggl[-\frac{1}{2c^2}\frac{\ddot{I}}{r^2}\Omega^a + \cdots\biggr], \\
\ddot{h}^{ab} &= \frac{4G}{c^2}\biggl[\frac{1}{2c^2}\frac{I^{ab(4)}}{r} + \cdots\biggr],
\end{aligned}$$

in which $\ddot{I} := \ddot{I}^c{}_c$ and $I^{ab(4)}$ stands for the fourth derivative of I^{ab} with respect to τ. After some algebra, we obtain

$$\begin{aligned}
\tau^{ab} &= \frac{GM}{4\pi r^2}\biggl[\frac{M}{r^2}\Omega^a\Omega^a - \Omega^{(a}\partial^{b)}{}_{cd}\biggl(\frac{I^{cd}}{r}\biggr) + \frac{4}{c^2}\Omega^{(a}\partial_c\biggl(\frac{\ddot{I}^{b)c}}{r}\biggr) \\
&\quad + \frac{1}{c^2}\biggl(\frac{\ddot{I}}{r^2} + \frac{1}{c}\frac{I^{(3)}}{r}\biggr)\Omega^a\Omega^b - \frac{2}{c^4}I^{ab(4)} + \cdots\biggr].
\end{aligned} \tag{6.10.8}$$

The next step is to evaluate the derivatives. From Sec. 1.8.1 we recall that $\partial_a r = \Omega_a$ and $\partial_a\Omega_b = r^{-1}(\delta_{ab} - \Omega_a\Omega_b)$. We recall also that I^{ab} depends on the spatial coordinates through $\tau = t - r/c$, so that $\partial_c I^{ab} = -c^{-1}\dot{I}^{ab}\Omega_c$. Using these rules, we calculate that

$$\partial_c\biggl(\frac{\ddot{I}^{ac}}{r}\biggr) = -\biggl(\frac{\ddot{I}^{ac}}{r^2} + \frac{1}{c}\frac{I^{ac(3)}}{r}\biggr)\Omega_c$$

and

$$\begin{aligned}
\partial^a{}_{cd}\biggl(\frac{I^{cd}}{r}\biggr) &= -\biggl(15\frac{I^{cd}}{r^4} + \frac{15}{c}\frac{\dot{I}^{cd}}{r^3} + \frac{6}{c^2}\frac{\ddot{I}^{cd}}{r^2} + \frac{1}{c^3}\frac{I^{cd(3)}}{r}\biggr)\Omega^a\Omega_c\Omega_d \\
&\quad + \biggl(3\frac{I^{cd}}{r^4} + \frac{3}{c}\frac{\dot{I}^{cd}}{r^3} + \frac{1}{c^2}\frac{\ddot{I}^{cd}}{r^2}\biggr)\bigl(\Omega^a\delta_{cd} + \delta^a{}_c\Omega_d + \delta^a{}_d\Omega_c\bigr).
\end{aligned}$$

With these results, Eq. (6.10.8) becomes

$$\begin{aligned}
\tau^{ab} &= \frac{GM^2}{4\pi r^4}\Omega^a\Omega^b + \frac{GM}{4\pi r^2}\biggl[\biggl(15\frac{I^{cd}}{r^4} + \frac{15}{c}\frac{\dot{I}^{cd}}{r^3} + \frac{6}{c^2}\frac{\ddot{I}^{cd}}{r^2} + \frac{1}{c^3}\frac{I^{cd(3)}}{r}\biggr)\Omega^a\Omega^b\Omega_c\Omega_d \\
&\quad - \biggl(3\frac{I}{r^4} + \frac{3}{c}\frac{\dot{I}}{r^3} - \frac{1}{c^3}\frac{\ddot{I}^{(3)}}{r}\biggr)\Omega^a\Omega^b \\
&\quad - \biggl(3\frac{I^{ac}}{r^4} + \frac{3}{c}\frac{\dot{I}^{ac}}{r^3} + \frac{3}{c^2}\frac{\ddot{I}^{ac}}{r^2} + \frac{2}{c^3}\frac{I^{ac(3)}}{r}\biggr)\Omega^b\Omega_c \\
&\quad - \biggl(3\frac{I^{bc}}{r^4} + \frac{3}{c}\frac{\dot{I}^{bc}}{r^3} + \frac{3}{c^2}\frac{\ddot{I}^{bc}}{r^2} + \frac{2}{c^3}\frac{I^{bc(3)}}{r}\biggr)\Omega^a\Omega_c - \frac{2}{c^4}I^{ab(4)}\biggr].
\end{aligned} \tag{6.10.9}$$

The final step is to express the angular dependence of τ^{ab} in terms of STF tensors $\Omega^{\langle L\rangle}$. We involve the definitions of Eqs. (1.8.2)–(1.8.4), and write $\Omega^a\Omega^b\Omega^c\Omega^d$ in terms of $\Omega^{\langle abcd\rangle}$, $\Omega^a\Omega^b\Omega^c$ in terms of $\Omega^{\langle abc\rangle}$, and $\Omega^a\Omega^b$ in terms of $\Omega^{\langle ab\rangle}$. After discarding all terms proportional to δ^{ab}, our final expression for the effective stress

tensor is

$$\begin{aligned}\tau^{ab} &= \frac{GM^2}{4\pi r^4}\Omega^{\langle ab\rangle} + \frac{GM}{4\pi r^2}\biggl[\biggl(15\frac{I_{cd}}{r^4} + \frac{15}{c}\frac{\dot{I}_{cd}}{r^3} + \frac{6}{c^2}\frac{\ddot{I}_{cd}}{r^2} + \frac{1}{c^3}\frac{I^{(3)}_{cd}}{r}\biggr)\Omega^{\langle abcd\rangle} \\ &\quad + \biggl(-\frac{6}{7}\frac{I}{r^4} - \frac{6}{7c}\frac{\dot{I}}{r^3} + \frac{6}{7c^2}\frac{\ddot{I}}{r^2} + \frac{8}{7c^3}\frac{I^{(3)}}{r}\biggr)\Omega^{\langle ab\rangle} \\ &\quad + 2\biggl(\frac{9}{7}\frac{I^{(a}{}_c}{r^4} + \frac{9}{7c}\frac{\dot{I}^{(a}{}_c}{r^3} - \frac{9}{7c^2}\frac{\ddot{I}^{(a}{}_c}{r^2} - \frac{12}{7c^3}\frac{I^{(a(3)}{}_c}{r}\biggr)\Omega^{\langle b)c\rangle} \\ &\quad - \frac{6}{5c^2}\frac{\ddot{I}^{\langle ab\rangle}}{r^2} - \frac{6}{5c^3}\frac{I^{\langle ab\rangle(3)}}{r} - \frac{2}{c^4}I^{\langle ab\rangle(4)}\biggr].\end{aligned} \tag{6.10.10}$$

This expression is a sum of terms that have the structure of Eq. (6.2.5),

$$\tau^{ab}[\ell, n] = \frac{1}{4\pi}\frac{f(u)}{r^n}\Omega^{\langle L\rangle}. \tag{6.10.11}$$

For example, the first group of terms inside the square brackets has $\ell = 4$, and it consists of four terms with $n = -6$, $n = -5$, $n = -4$, and $n = -3$; for each of these contributions we can easily read off the appropriate function f (which is currently expressed in terms of $\tau = t - r/c$ instead of $u = ct - r$).

We shall keep in mind that it is the last term of Eq. (6.10.10), the one involving four derivatives of $I^{\langle ab\rangle}(\tau)$, that originated from $(-g)t^{ab}_{\mathrm{H}}$. It is this term that will reveal the differences between the light cones of the mathematical flat spacetime and those of the physical curved spacetime.

6.10.3 Evaluation of the wave-zone integrals

Each term $\tau^{ab}[\ell, n]$ in Eq. (6.10.10) makes a contribution to the gravitational-wave field h^{ab}, and according to Eq. (6.2.6), this is given by

$$h^{ab}_{\mathscr{W}}[\ell, n] = \frac{4G}{c^4 r}\Omega^{\langle L\rangle}\biggl\{\int_0^{\mathcal{R}} ds f(u-2s)A(s,r) + \int_{\mathcal{R}}^{\infty} ds f(u-2s)B(s,r)\biggr\}, \tag{6.10.12}$$

where

$$A(s,r) = \int_{\mathcal{R}}^{r+s}\frac{P_\ell(\xi)}{p^{n-1}}\,dp, \qquad B(s,r) = \int_s^{r+s}\frac{P_\ell(\xi)}{p^{n-1}}\,dp. \tag{6.10.13}$$

Here, P_ℓ is a Legendre polynomial of argument $\xi = (r+2s)/r - 2s(r+s)/(rp)$.

To begin, we shall work through the specific, but representative, case of $\ell = 0$ and $n = 3$. Extracting this piece of τ^{ab} from Eq. (6.10.10) and comparing with Eq. (6.10.11), we find that in this case the function f is given by

$$f(u) = -\frac{6}{5}GMI^{\langle ab\rangle\prime\prime\prime},$$

in which a prime indicates differentiation with respect to $u = c\tau$.

We must first evaluate the functions A and B. With $\ell = 0$ and $n = 3$, the computations are elementary, and the results are

$$A(s,r) = \frac{1}{\mathcal{R}} - \frac{1}{r+s}$$

and

$$B(s,r) = \frac{1}{s} - \frac{1}{r+s}.$$

We next set to work on the integrals that appear in Eq. (6.10.12). The first is

$$F_A := \int_0^{\mathcal{R}} ds f(u-2s) A(s,r) = \int_0^{\mathcal{R}} ds f(u-2s)\biggl(\frac{1}{\mathcal{R}} - \frac{1}{r+s}\biggr),$$

and we rewrite it as

$$F_A = \frac{1}{\mathcal{R}} \int_0^{\mathcal{R}} f(u-2s)\, ds - \int_0^{\mathcal{R}} f(u-2s)\, d\ln(r+s).$$

After integrating the second term by parts, our final expression is

$$\begin{aligned} F_A &= -f(u-2\mathcal{R})\ln(r+\mathcal{R}) + f(u)\ln r + \frac{1}{\mathcal{R}}\int_0^{\mathcal{R}} f(u-2s)\, ds \\ &\quad - 2\int_0^{\mathcal{R}} f'(u-2s)\ln\frac{r+s}{s}\, ds + 2\int_0^{\mathcal{R}} f'(u-2s)\ln s\, ds. \end{aligned}$$

The second integral is

$$F_B := \int_{\mathcal{R}}^{\infty} ds f(u-2s) B(s,r) = \int_{\mathcal{R}}^{\infty} ds f(u-2s)\biggl(\frac{1}{s} - \frac{1}{r+s}\biggr),$$

and we rewrite it as

$$F_B = -\int_{\mathcal{R}}^{\infty} f(u-2s)\, d\ln\frac{r+s}{s}.$$

Integration by parts yields

$$F_B = f(u-2\mathcal{R})\ln\frac{r+\mathcal{R}}{\mathcal{R}} - 2\int_{\mathcal{R}}^{\infty} f'(u-2s)\ln\frac{r+s}{s}\, ds,$$

assuming that $f(u-2s)$ goes to zero sufficiently rapidly as $s \to \infty$ to ensure that there is no boundary term at $s=\infty$. (Physically, this condition implies that the system is only weakly dynamical in the infinite past.)

The sum of F_A and F_B is

$$\begin{aligned} F &= -f(u-2\mathcal{R})\ln\mathcal{R} + f(u)\ln r + \frac{1}{\mathcal{R}}\int_0^{\mathcal{R}} f(u-2s)\, ds \\ &\quad + 2\int_0^{\mathcal{R}} f'(u-2s)\ln s\, ds - 2\int_0^{\infty} f'(u-2s)\ln\frac{r+s}{s}\, ds. \end{aligned}$$

This result is exact, but to simplify it we use the fact that we may remove from this all $\mathcal{R}$-dependent pieces. As a formal tool to achieve this, we express $f(u-2s)$ and its derivative as infinite Taylor series in powers of s, and we evaluate the two integrals from $s=0$ to $s=\mathcal{R}$. We find that they combine to give $f(u)$, plus terms that can be discarded because they come with explicit factors of $\mathcal{R}$. After also expanding $f(u-2\mathcal{R})$ in powers of $\mathcal{R}$, we find that

$$F = f(u)\Bigl[1 + \ln(r/\mathcal{R})\Bigr] - 2\int_0^{\infty} f'(u-2s)\ln\frac{r+s}{s}\, ds,$$

modulo $\mathcal{R}$-dependent terms that can be freely discarded. This still contains a logarithmic dependence on $\mathcal{R}$, but it could be removed by writing $\ln(r/\mathcal{R}) = \ln(r/r_0) + \ln(r_0/\mathcal{R})$ and discarding the second term. This alternate expression would then contain a dependence on an arbitrary constant r_0, and it is perhaps preferable to stick with the original form, in spite of the residual $\mathcal{R}$-dependence.

Our final answer is obtained by inserting our expressions for $f(u)$ and F within Eq. (6.10.12). After also changing the primes into overdots, we get

$$h^{ab}_{\mathscr{W}}[0,3] = \frac{4GM}{c^4 r}\biggl\{ -\frac{6G}{5c^3}\Bigl[1+\ln(r/\mathcal{R})\Bigr] I^{\langle ab\rangle(3)} + \frac{12}{5}K^{ab} \biggr\}, \tag{6.10.14}$$

in which the *tail integral*

$$K^{ab}(\tau, r) := \frac{G}{c^4}\int_0^\infty I^{\langle ab\rangle(4)}(\tau - 2s/c)\ln\frac{r+s}{s}\, ds \tag{6.10.15}$$

must be left unevaluated. Notice that the tail integral involves the entire past history of the system, from the infinite past (at $s=\infty$) to the current retarded time (at $s=0$). We recall the notation

$$I^{\langle ab\rangle(q)} = \frac{d^q}{d\tau^q} I^{\langle ab\rangle},$$

and we shall see what fate awaits the logarithmic term $\ln(r/\mathcal{R})$ in Eq. (6.10.14), when this contribution to $h^{ab}_{\mathscr{W}}$ is combined with others.

The same techniques are employed to calculate all other contributions to $h^{ab}_{\mathscr{W}}$. We shall not labour through the details here, but simply list the final results for $h^{ab}_{\mathscr{W}}[\ell, n]$:

$$h^{ab}_{\mathscr{W}}[0,2] = \frac{4GM}{c^4 r}\biggl\{ -2K^{ab} \biggr\}, \tag{6.10.16}$$

$$h^{ab}_{\mathscr{W}}[0,3] = \frac{4GM}{c^4 r}\biggl\{ -\frac{6G}{5c^3}\Bigl[1+\ln(r/\mathcal{R})\Bigr] I^{\langle ab\rangle(3)} + \frac{12}{5}K^{ab} \biggr\}, \tag{6.10.17}$$

$$h^{ab}_{\mathscr{W}}[0,4] = \frac{4GM}{c^4 r}\biggl\{ \frac{6G}{5c^3}\Bigl[\frac{3}{2}+\ln(r/\mathcal{R})\Bigr] I^{\langle ab\rangle(3)} - \frac{12}{5}K^{ab} \biggr\}, \tag{6.10.18}$$

$$h^{ab}_{\mathscr{W}}[2,3] = \frac{4GM}{c^4 r}\biggl\{ -\frac{2G}{7c^3} I^{a(3)}_{\ c} \biggr\}\Omega^{\langle cb\rangle} + (a \leftrightarrow b), \tag{6.10.19}$$

$$h^{ab}_{\mathscr{W}}[2,4] = \frac{4GM}{c^4 r}\biggl\{ -\frac{3G}{28c^3} I^{a(3)}_{\ c} \biggr\}\Omega^{\langle cb\rangle} + (a \leftrightarrow b), \tag{6.10.20}$$

$$h^{ab}_{\mathscr{W}}[2,5] = \frac{4GM}{c^4 r}\biggl\{ \frac{G}{c^3}\Bigl[\frac{47}{700}+\frac{3}{35}\ln(r/\mathcal{R})\Bigr] I^{a(3)}_{\ c} - \frac{6}{35}K^{a}_{\ c} \biggr\}\Omega^{\langle cb\rangle} + (a \leftrightarrow b), \tag{6.10.21}$$

$$h^{ab}_{\mathscr{W}}[2,6] = \frac{4GM}{c^4 r}\biggl\{ \frac{G}{c^3}\Bigl[-\frac{97}{700}-\frac{3}{35}\ln(r/\mathcal{R})\Bigr] I^{a(3)}_{\ c} + \frac{6}{35}K^{a}_{\ c} \biggr\}\Omega^{\langle cb\rangle} + (a \leftrightarrow b), \tag{6.10.22}$$

$$h^{ab}_{\mathscr{W}}[4,3] = \frac{4GM}{c^4 r}\biggl\{ \frac{G}{20c^3} I^{(3)}_{cd} \biggr\}\Omega^{\langle abcd\rangle}, \tag{6.10.23}$$

$$h^{ab}_{\mathscr{W}}[4,4] = \frac{4GM}{c^4 r}\biggl\{ \frac{G}{30c^3} I^{(3)}_{cd} \biggr\}\Omega^{\langle abcd\rangle}, \tag{6.10.24}$$

$$h^{ab}_{\mathscr{W}}[4,5] = \frac{4GM}{c^4 r}\biggl\{ \frac{G}{42c^3} I^{(3)}_{cd} \biggr\}\Omega^{\langle abcd\rangle}, \tag{6.10.25}$$

$$h^{ab}_{\mathscr{W}}[4,6] = \frac{4GM}{c^4 r}\biggl\{ \frac{G}{56c^3} I^{(3)}_{cd} \biggr\}\Omega^{\langle abcd\rangle}. \tag{6.10.26}$$

To arrive at these results we have freely discarded all $\mathcal{R}$-dependent terms, except when the dependence is logarithmic. In some cases we have also removed terms that fall off as r^{-2}, r^{-3}, or faster, because these are negligible in the far-away wave zone.

From the preceding listing of results we find that the sum of contributions for $\ell = 0$ is

$$h^{ab}_{\mathscr{W}}[\ell=0] = \frac{4GM}{c^4 r}\left\{\frac{3G}{5c^3} I^{\langle ab\rangle(3)} - 2K^{ab}\right\}.$$

Similarly,

$$h^{ab}_{\mathscr{W}}[\ell=2] = \frac{4GM}{c^4 r}\left\{-\frac{13G}{28c^3} I^{a(3)}_{\ c}\Omega^{\langle cb\rangle} + (a \leftrightarrow b)\right\}$$

and

$$h^{ab}_{\mathscr{W}}[\ell=4] = \frac{4GM}{c^4 r}\left\{\frac{G}{8c^3} I^{(3)}_{cd}\Omega^{\langle abcd\rangle}\right\}.$$

Notice that the logarithmic terms have all canceled out, and that the tail integral appears only within the contribution from $\ell = 0$. Tracing the origin of the tail integral, we see that it comes from $\tau^{ab}[0,2]$, the term in τ^{ab} that involves four derivatives of the Newtonian quadrupole moment with respect to τ. This term, the last one in Eq. (6.10.10), originates from $(-g)t^{ab}_{\mathrm{H}}$, and it reveals the differences between the light cones of the mathematical flat spacetime and those of the physical curved spacetime. The tail integral, therefore, informs us that the gravitational waves propagate in a curved spacetime instead of the fictitious flat spacetime of the post-Minkowski expansion.

6.10.4 Final answer

Adding the contributions from $\ell = 0$, $\ell = 2$, and $\ell = 3$, we find that the wave-zone part of the gravitational-wave field is given by

$$\begin{aligned} h^{ab}_{\mathscr{W}} &= \frac{4GM}{c^4 r}\left\{\frac{3G}{5c^3} I^{\langle ab\rangle(3)} - 2K^{ab} - \frac{13G}{28c^3}\left(I^{a(3)}_{\ c}\Omega^{\langle cb\rangle} + I^{b(3)}_{\ c}\Omega^{\langle ca\rangle}\right)\right. \\ &\quad \left. + \frac{G}{8c^3} I^{(3)}_{cd}\Omega^{\langle abcd\rangle}\right\}. \end{aligned}$$

From this we may remove any term that will not survive a TT projection. In particular,

$$I^{a(3)}_{\ c}\Omega^{\langle cb\rangle} = I^{a(3)}_{\ c}\left(\Omega^c\Omega^b - \frac{1}{3}\delta^{cb}\right) \overset{\mathrm{TT}}{=} -\frac{1}{3} I^{\langle ab\rangle(3)}$$

and similarly,

$$I^{(3)}_{cd}\Omega^{\langle abcd\rangle} \overset{\mathrm{TT}}{=} \frac{2}{35} I^{\langle ab\rangle(3)}.$$

Collecting these results, we find that $h^{ab}_{\mathscr{W}}$ reduces to

$$h^{ab}_{\mathscr{W}} \overset{\mathrm{TT}}{=} \frac{4GM}{c^4 r}\left\{\frac{11G}{12c^3} I^{\langle ab\rangle(3)} - 2K^{ab}\right\}.$$

To arrive at our final expression we substitute Eq. (6.10.15) for the tail integral, and we clean it up by setting $s = \frac{1}{2}c\,\zeta$ and adopting ζ as an integration variable.

We obtain, finally,

$$h^{ab}_{\mathscr{W}} \overset{\mathrm{TT}}{=} \frac{4G^2 M}{c^7 r}\left\{\frac{11}{12} I^{\langle ab\rangle(3)}(\tau) + \int_0^\infty I^{\langle ab\rangle(4)}(\tau - \zeta)\ln\frac{\zeta}{\zeta + 2r/c}\,d\zeta\right\}. \tag{6.10.27}$$

To get a useful alternative expression, we differentiate the first term and insert it within the integral. This produces

$$h^{ab}_{\mathscr{W}} \overset{\mathrm{TT}}{=} \frac{4G^2 M}{c^7 r}\int_0^\infty I^{\langle ab\rangle(4)}(\tau - \zeta)\left(\ln\frac{\zeta}{\zeta + 2r/c} + \frac{11}{12}\right) d\zeta. \tag{6.10.28}$$

The wave-zone contribution to h^{ab} depends on r and $\tau = t - r/c$, and we notice that it is isotropic — the angular dependence has been eliminated by the TT projection. It is expressed in terms of the total gravitational mass of Eq. (6.10.4),

$$M = \sum_A m_A + O(c^{-2}), \tag{6.10.29}$$

as well as the Newtonian quadrupole moment of Eq. (6.10.6),

$$I^{ab} = \sum_A m_A z_A^a z_A^b + O(c^{-2}). \tag{6.10.30}$$

Recalling our discussion near the end of Sec. 6.2, we see from Eq. (6.10.28) that $h^{ab}_{\mathscr{W}}$ is a correction of order c^{-3} relative to the leading term in h^{ab}, which is of order c^{-4}. The wave-zone contribution to the gravitational-wave field is therefore a term of $\frac{3}{2}$PN order. And we recall from the end of Sec. 6.10.3 that the tail integral originates from $(-g)t^{ab}_{\mathrm{H}}$, the harmonic-gauge contribution to the effective stress tensor; it reveals the differences between the light cones of the mathematical flat spacetime and those of the physical curved spacetime.

6.11 Specialization to a two-body system

The gravitational-wave field is given by the sum of $h^{ab}_{\mathscr{N}}$, given by Eq. (6.9.1) and the following equations, and $h^{ab}_{\mathscr{W}}$, given by Eq. (6.10.28). These expressions are still fairly implicit, and to make the results more concrete we specialize them to a two-body system.

6.11.1 Motion in the barycentric frame

We shall work in the post-Newtonian barycentric frame ($\boldsymbol{Z} = \boldsymbol{0}$), and according to Eqs. (5.5.14) and (5.5.15), the position vectors of the two bodies are given by

$$\boldsymbol{z}_1 = \frac{m_2}{m}\boldsymbol{z} + \frac{\eta\Delta}{2c^2}\left(v^2 - \frac{Gm}{z}\right)\boldsymbol{z} + O(c^{-4}) \tag{6.11.1}$$

and

$$\boldsymbol{z}_2 = -\frac{m_1}{m}\boldsymbol{z} + \frac{\eta\Delta}{2c^2}\left(v^2 - \frac{Gm}{z}\right)\boldsymbol{z} + O(c^{-4}). \tag{6.11.2}$$

They are expressed in terms of the relative position

$$\boldsymbol{z} := \boldsymbol{z}_1 - \boldsymbol{z}_2 \tag{6.11.3}$$

and the relative velocity

$$\boldsymbol{v} := \boldsymbol{v}_1 - \boldsymbol{v}_2; \tag{6.11.4}$$

these vectors have a length $z = |\boldsymbol{z}|$ and $v = |\boldsymbol{v}|$, respectively. We also have re-introduced the mass parameters

$$m := m_1 + m_2 \tag{6.11.5}$$

$$\eta := \frac{m_1 m_2}{(m_1 + m_2)^2} \tag{6.11.6}$$

$$\Delta := \frac{m_1 - m_2}{m_1 + m_2}. \tag{6.11.7}$$

Differentiation of Eqs. (6.11.1) and (6.11.2) returns the velocity vectors of each body:

$$\boldsymbol{v}_1 = \frac{m_2}{m}\boldsymbol{v} + \frac{\eta\Delta}{2c^2}\left[\left(v^2 - \frac{Gm}{z}\right)\boldsymbol{v} - \frac{Gm}{z}\dot{z}\boldsymbol{n}\right] + O(c^{-4}) \tag{6.11.8}$$

and

$$\boldsymbol{v}_2 = -\frac{m_1}{m}\boldsymbol{v} + \frac{\eta\Delta}{2c^2}\biggl[\Bigl(v^2 - \frac{Gm}{z}\Bigr)\boldsymbol{v} - \frac{Gm}{z}\dot{z}\boldsymbol{n}\biggr] + O(c^{-4}), \tag{6.11.9}$$

where

$$\dot{z} = \boldsymbol{n}\cdot\boldsymbol{v} \tag{6.11.10}$$

is the radial component of the velocity vector, and

$$\boldsymbol{n} := \frac{\boldsymbol{z}}{z} = \frac{\boldsymbol{z}_1 - \boldsymbol{z}_2}{|\boldsymbol{z}_1 - \boldsymbol{z}_2|} \tag{6.11.11}$$

is a unit vector that points from body 2 to body 1. To arrive at these expressions we had to involve the relative acceleration of the two bodies, which according to Eq. (5.5.18) is given by

$$\begin{aligned}\boldsymbol{a} &= -\frac{Gm}{z^2}\boldsymbol{n} \\ &\quad + \frac{1}{c^2}\biggl\{-\frac{Gm}{z^2}\biggl[(1+3\eta)v^2 - \frac{3}{2}\eta(\boldsymbol{n}\cdot\boldsymbol{v})^2 - 2(2+\eta)\frac{Gm}{z}\biggr]\boldsymbol{n} \\ &\qquad + 2(2-\eta)\frac{Gm}{z^2}(\boldsymbol{n}\cdot\boldsymbol{v})\boldsymbol{v}\biggr\} + O(c^{-4}).\end{aligned} \tag{6.11.12}$$

6.11.2 Radiative multipole moments

We make these substitutions into Eqs. (6.9.2)–(6.9.6) and simplify the resulting expressions. The sums that appear in these equations must be specialized to two bodies, and in these we set $z_{12} = z_{21} = z$ and $\boldsymbol{n}_{12} = -\boldsymbol{n}_{21} = \boldsymbol{n}$. In the course of these (lengthy, but straightforward) computations we encounter various functions of m_1 and m_2 that can be re-written in terms of the mass parameters of Eqs. (6.11.5)–(6.11.7). For example, it is easy to show that

$$\frac{m_1^2 + m_2^2}{(m_1+m_2)^2} = 1 - 2\eta, \qquad \frac{m_1^3 + m_2^3}{(m_1+m_2)^3} = 1 - 3\eta, \qquad \frac{m_1^4 - m_2^4}{(m_1+m_2)^4} = \Delta(1-2\eta),$$

and we make many such substitutions while simplifying our expressions.

We obtain

$$Q^{ab} = m\eta\biggl[1 + \frac{1}{2}(1-3\eta)\frac{v^2}{c^2} - \frac{1}{2}(1-2\eta)\frac{Gm}{c^2 z} + O(c^{-4})\biggr]z^a z^b, \tag{6.11.13}$$

$$\begin{aligned}A^{abc} &= \frac{m\eta\Delta}{c}\biggl\{-\biggl[1 + \frac{1}{2}(1-5\eta)\frac{v^2}{c^2} + \Bigl(\frac{7}{6} + 2\eta\Bigr)\frac{Gm}{c^2 z}\biggr]v^a z^b z^c \\ &\quad + \Bigl(\frac{1}{6} - \eta\Bigr)\frac{Gm}{c^2 z}\dot{z}n^a z^b z^c + \frac{5}{3}\frac{Gm}{c^2 z}z^a v^{(b}z^{c)} + O(c^{-4})\biggr\},\end{aligned} \tag{6.11.14}$$

$$\begin{aligned}Q^{abc} &= \frac{m\eta\Delta}{c}\biggl\{z^a z^b v^c - \bigl(v^a z^b + z^a v^b\bigr)z^c \\ &\quad - \biggl[\frac{1}{2}(1-5\eta)\frac{v^2}{c^2} + \frac{1}{6}(7+12\eta)\frac{Gm}{c^2 z}\biggr]\Bigl[\bigl(v^a z^b + z^a v^b\bigr)z^c - z^a z^b v^c\Bigr] \\ &\quad + \frac{1}{6}(1-6\eta)\frac{Gm}{c^2 z}\dot{z}n^a z^b z^c + O(c^{-4})\biggr\},\end{aligned} \tag{6.11.15}$$

$$\begin{aligned}Q^{abcd} &= \frac{m\eta}{c^2}\biggl\{(1-3\eta)v^a v^b z^c z^d - \frac{1}{3}(1-3\eta)\frac{Gm}{z}n^a n^b z^c z^d \\ &\quad - \frac{1}{6}\frac{Gm}{z}z^a z^b \delta^{cd} + O(c^{-2})\biggr\},\end{aligned} \tag{6.11.16}$$

$$Q^{abcde} = \frac{m\eta\Delta}{c^3}\frac{\partial}{\partial\tau}\biggl\{-(1-2\eta)v^a v^b z^c z^d z^e + \frac{1}{4}(1-2\eta)\frac{Gm}{z}n^a n^b z^c z^d z^e + \frac{1}{4}\frac{Gm}{z}z^a z^b z^{(c}\delta^{de)} + O(c^{-2})\biggr\}. \qquad (6.11.17)$$

Equation (6.11.15) is obtained from Eq. (6.11.14) by involving Eq. (6.9.3), $Q^{abc} = A^{abc} + A^{bac} - A^{cab}$. We also observe that to simplify the writing, we have replaced the qualified equality sign $\stackrel{\text{TT}}{=}$ ("equal after a TT projection") by the usual equality sign.

6.11.3 Computation of retarded-time derivatives

The near-zone contribution to h^{ab} is given by Eq. (6.9.1), and in this we must substitute the radiative multipole moments displayed in the preceding subsection; the computation involves taking two retarded-time derivatives of these moments. Similarly, the wave-zone contribution to h^{ab} is given by Eq. (6.10.28), and this involves four retarded-time derivatives of $I^{ab} = m\eta z^a z^b + O(c^{-2})$, which is equal to the Newtonian piece of Q^{ab}. Our task in this subsection is to compute these derivatives.

The general strategy is clear. The radiative multipole moments of Eqs. (6.11.13)–(6.11.17) are expressed explicitly in terms of the position and velocity vectors, and these are functions of the retarded time τ. Differentiating one of these moments with respect to τ therefore involves taking derivatives of the position and velocity vectors. Differentiating $\boldsymbol{z}$ gives $\boldsymbol{v}$, and differentiating $\boldsymbol{v}$ gives $\boldsymbol{a}$, the post-Newtonian acceleration vector of Eq. (6.11.12). After making this substitution, the result is once more expressed in terms of $\boldsymbol{z}$ and $\boldsymbol{v}$, and it is ready for further differentiation.

More concretely, consider the task of computing $\ddot{Q}^{ab}$. The quadrupole moment is a function of $\boldsymbol{z}$ at order c^0, and a function of $\boldsymbol{z}$ and $\boldsymbol{v}$ at order c^{-2}. Taking a first derivative with respect to τ produces terms in $\boldsymbol{z}$ and $\boldsymbol{v}$ at order c^0, and terms in $\boldsymbol{z}$, $\boldsymbol{v}$, and $\boldsymbol{a}$ at order c^{-2}. In the post-Newtonian term we may substitute the *Newtonian expression* for the acceleration vector, $\boldsymbol{a} = -Gm\boldsymbol{z}/z^3 + O(c^{-2})$, because the error incurred occurs at order c^{-4} in $\dot{Q}^{ab}$. The end result is a function of $\boldsymbol{z}$ and $\boldsymbol{v}$ at order c^0, another function of $\boldsymbol{z}$ and $\boldsymbol{v}$ at order c^{-2}, and neglected terms at order c^{-4}. Taking a second derivative introduces the acceleration vector at orders c^0 and c^{-2}. In the Newtonian term we must now substitute the *post-Newtonian expression* for the acceleration vector, because its PN term will influence the c^{-2} piece of $\ddot{Q}^{ab}$; we are still, however, allowed to insert the Newtonian acceleration within the c^{-2} piece of the second derivative. The end result for $\ddot{Q}^{ab}$ is a function of $\boldsymbol{z}$ and $\boldsymbol{v}$ at order c^0, and another function of $\boldsymbol{z}$ and $\boldsymbol{v}$ at order c^{-2}.

Derivatives of higher multipole moments are computed in a similar way. These computations are tedious and lengthy, but they are completely straightforward. They are aided by the identities

$$v\dot{v} = -\frac{Gm}{z^2}\dot{z} + O(c^{-2}) \qquad (6.11.18)$$

and

$$z\ddot{z} = v^2 - \dot{z}^2 - \frac{Gm}{z} + O(c^{-2}), \qquad (6.11.19)$$

which are consequences of the Newtonian expression for the acceleration vector.

We display the final results:

$$\ddot{Q}^{ab} = m\eta\biggl\{2\biggl(v^a v^b - \frac{Gm}{z}n^a n^b\biggr)\biggr\} + \frac{m\eta}{c^2}\biggl\{\biggl[-\frac{1}{2}(7+2\eta)v^2 + \frac{3}{2}(1-2\eta)\dot{z}^2 + \frac{19}{2}\frac{Gm}{z}\biggr]\frac{Gm}{z}n^a n^b$$

$$
\begin{aligned}
& + \biggl[(1-3\eta)v^2 - (1-2\eta)\frac{Gm}{z} \biggr] v^a v^b + (3+2\eta)\frac{Gm}{z}\dot{z}\bigl(v^a n^b + n^a v^b\bigr) \biggr\} \\
& + O(c^{-4}), \qquad (6.11.20)
\end{aligned}
$$

$$
\begin{aligned}
\ddot{Q}^{abc} &= \frac{m\eta\Delta}{c}\biggl\{ -3\frac{Gm}{z}\dot{z} n^a n^b n^c + 3\frac{Gm}{z}\bigl(v^a n^b + n^a v^b\bigr) n^c \\
&\quad + \frac{Gm}{z} n^a n^b v^c - 2 v^a v^b v^c \biggr\} + \frac{m\eta\Delta}{c^3}\biggl\{ \biggl[\frac{3}{2}(2-\eta)v^2 + \frac{9}{2}(1+\eta)\dot{z}^2 \\
&\quad - \frac{1}{3}(31-9\eta)\frac{Gm}{z} \biggr] \frac{Gm}{z}\bigl(v^a n^b + n^a v^b\bigr) n^c - (15+2\eta)\frac{Gm}{z}\dot{z}\, v^a v^b n^c \\
&\quad + \biggl[-\frac{3}{2}(4-3\eta)v^2 + \frac{5}{2}(1-3\eta)\dot{z}^2 + \frac{2}{3}(29-3\eta)\frac{Gm}{z} \biggr] \frac{Gm}{z}\dot{z}\, n^a n^b n^c \\
&\quad + \biggl[\frac{1}{2}(4-\eta)v^2 - \frac{3}{2}(1-\eta)\dot{z}^2 - \frac{1}{3}(25-3\eta)\frac{Gm}{z} \biggr] \frac{Gm}{z} n^a n^b v^c \\
&\quad - (3+2\eta)\frac{Gm}{z}\dot{z}\bigl(v^a n^b + n^a v^b\bigr) v^c \\
&\quad + \biggl[-(1-5\eta)v^2 + (1-4\eta)\frac{Gm}{z} \biggr] v^a v^b v^c \biggr\} \\
&\quad + O(c^{-5}), \qquad (6.11.21)
\end{aligned}
$$

$$
\begin{aligned}
\ddot{Q}^{abcd} &= \frac{m\eta}{c^2}\biggl\{ 5(1-3\eta)\frac{Gm}{z}\dot{z}\bigl(v^a n^b + n^a v^b\bigr) n^c n^d \\
&\quad + (1-3\eta)\biggl(v^2 - 5\dot{z}^2 + \frac{7}{3}\frac{Gm}{z} \biggr) \frac{Gm}{z} n^a n^b n^c n^d \\
&\quad - \frac{14}{3}(1-3\eta)\frac{Gm}{z} v^a v^b n^c n^d \\
&\quad - \frac{8}{3}(1-3\eta)\frac{Gm}{z}\bigl(v^a n^b + n^a v^b\bigr)\bigl(v^c n^d + n^c v^d\bigr) \\
&\quad + 2(1-3\eta) v^a v^b v^c v^d + 2(1-3\eta)\frac{Gm}{z}\dot{z}\, n^a n^b \bigl(v^c n^d + n^c v^d\bigr) \\
&\quad - \frac{2}{3}(1-3\eta)\frac{Gm}{z} n^a n^b v^c v^d + \frac{1}{6}\frac{Gm}{z}\biggl(v^2 - 3\dot{z}^2 + \frac{Gm}{z} \biggr) n^a n^b \delta^{cd} \\
&\quad + \frac{1}{3}\frac{Gm}{z}\dot{z}\bigl(v^a n^b + n^a v^b\bigr)\delta^{cd} - \frac{1}{3}\frac{Gm}{z} v^a v^b \delta^{cd} \biggr\} \\
&\quad + O(c^{-4}), \qquad (6.11.22)
\end{aligned}
$$

$$
\begin{aligned}
\ddot{Q}^{abcde} &= \frac{m\eta\Delta}{c^3}\biggl\{ -\frac{1}{4}(1-2\eta)\biggl(21v^2 - 105\dot{z}^2 + 44\frac{Gm}{z} \biggr) \frac{Gm}{z}\bigl(v^a n^b + n^a v^b\bigr) n^c n^d n^e \\
&\quad + \frac{1}{4}(1-2\eta)\biggl(45v^2 - 105\dot{z}^2 + 90\frac{Gm}{z} \biggr) \frac{Gm}{z}\dot{z}\, n^a n^b n^c n^d n^e \\
&\quad - \frac{51}{2}(1-2\eta)\frac{Gm}{z}\dot{z}\, v^a v^b n^c n^d n^e \\
&\quad - \frac{27}{2}(1-2\eta)\frac{Gm}{z}\dot{z}\bigl(v^a n^b + n^a v^b\bigr)\bigl(v^c n^d n^e + n^c v^d n^e + n^c n^d v^e\bigr) \\
&\quad - \frac{1}{4}(1-2\eta)\biggl(9v^2 - 45\dot{z}^2 + 28\frac{Gm}{z} \biggr) \frac{Gm}{z} n^a n^b \bigl(v^c n^d n^e + n^c v^d n^e + n^c n^d v^e\bigr) \\
&\quad + \frac{29}{2}(1-2\eta)\frac{Gm}{z} v^a v^b \bigl(v^c n^d n^e + n^c v^d n^e + n^c n^d v^e\bigr) \\
&\quad + \frac{15}{2}(1-2\eta)\frac{Gm}{z}\bigl(v^a n^b + n^a v^b\bigr)\bigl(v^c n^d n^e + n^c v^d n^e + n^c n^d v^e\bigr) \\
&\quad - 6(1-2\eta) v^a v^b v^c v^d v^e - \frac{9}{2}(1-2\eta)\frac{Gm}{z}\dot{z}\, n^a n^b \bigl(v^c v^d n^e + v^c n^d v^e + n^c v^d v^e\bigr)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3}{2}(1-2\eta)\frac{Gm}{z}n^a n^b v^c v^d v^e + \frac{1}{4}\biggl(9v^2 - 15\dot{z}^2 + 10\frac{Gm}{z}\biggr)\frac{Gm}{z}\dot{z}\,n^a n^b n^{(c}\delta^{de)} \\
&-\frac{1}{4}\biggl(3v^2 - 9\dot{z}^2 + 4\frac{Gm}{z}\biggr)\frac{Gm}{z}\bigl(v^a n^b + n^a v^b\bigr)n^{(c}\delta^{de)} \\
&-\frac{1}{4}\biggl(3v^2 - 9\dot{z}^2 + 4\frac{Gm}{z}\biggr)\frac{Gm}{z}n^a n^b v^{(c}\delta^{de)} \\
&-\frac{3}{2}\frac{Gm}{z}\dot{z}\,v^a v^b n^{(c}\delta^{de)} - \frac{3}{2}\frac{Gm}{z}\dot{z}\bigl(v^a n^b + n^a v^b\bigr)v^{(c}\delta^{de)} + \frac{3}{2}\frac{Gm}{z}\dot{z}\,v^a v^b v^{(c}\delta^{de)}\biggr\} \\
&+ O(c^{-5}). \qquad (6.11.23)
\end{aligned}
$$

In addition, we have that

$$
\begin{aligned}
I^{ab(4)} &= 2m\eta\frac{Gm}{z^3}\biggl[\biggl(3v^2 - 15\dot{z}^2 + \frac{Gm}{z}\biggr)n^a n^b + 9\dot{z}\bigl(v^a n^b + n^a v^b\bigr) - 4v^a v^b\biggr] \\
&\quad + O(c^{-2}). \qquad (6.11.24)
\end{aligned}
$$

6.11.4 Gravitational-wave field

We may now substitute Eqs. (6.11.20)–(6.11.24) into Eqs. (6.9.1) and (6.10.28) and obtain the gravitational-wave field. These computations are straightforward, and we express the result as

$$
h^{ab} = \frac{2Gm\eta}{c^4 r}\biggl(H^{ab}[\mathrm{N}] + A^{ab}[\tfrac{1}{2}\mathrm{PN}] + A^{ab}[1\mathrm{PN}] + A^{ab}[\tfrac{3}{2}\mathrm{PN}] + A^{ab}[\mathrm{tail}] + O(c^{-4})\biggr), \qquad (6.11.25)
$$

in which we group terms according to their post-Newtonian order (the last term, with the label "tail," is also of $\frac{3}{2}$PN order). We have

$$
A^{ab}[\mathrm{N}] = 2\biggl(v^a v^b - \frac{Gm}{z}n^a n^b\biggr), \qquad (6.11.26)
$$

$$
\begin{aligned}
A^{ab}[\tfrac{1}{2}\mathrm{PN}] &= \frac{\Delta}{c}\biggl(3\frac{Gm}{z}(\boldsymbol{n}\cdot\boldsymbol{\Omega})\bigl(v^a n^b + n^a v^b - \dot{z}\,n^a n^b\bigr) \\
&\quad + (\boldsymbol{v}\cdot\boldsymbol{\Omega})\biggl(-2v^a v^b + \frac{Gm}{z}n^a n^b\biggr)\biggr), \qquad (6.11.27)
\end{aligned}
$$

$$
\begin{aligned}
A^{ab}[1\mathrm{PN}] &= \frac{1}{c^2}\biggl(\frac{1}{3}\biggl[3(1-3\eta)v^2 - 2(2-3\eta)\frac{Gm}{z}\biggr]v^a v^b \\
&\quad + \frac{2}{5}(5+3\eta)\frac{Gm}{z}\dot{z}\bigl(v^a n^b + n^a v^b\bigr) \\
&\quad + \frac{1}{3}\frac{Gm}{z}\biggl[-(10+3\eta)v^2 + 3(1-3\eta)\dot{z}^2 + 29\frac{Gm}{z}\biggr]n^a n^b \\
&\quad + \frac{2}{3}(1-3\eta)(\boldsymbol{v}\cdot\boldsymbol{\Omega})^2\biggl(3v^a v^b - \frac{Gm}{z}n^a n^b\biggr) \\
&\quad + \frac{4}{3}(1-3\eta)(\boldsymbol{v}\cdot\boldsymbol{\Omega})(\boldsymbol{z}\cdot\boldsymbol{\Omega})\frac{Gm}{z}\biggl[-4\bigl(v^a n^b + n^a v^b\bigr) + 3\dot{z}n^a n^b\biggr] \\
&\quad + \frac{1}{3}(1-3\eta)(\boldsymbol{z}\cdot\boldsymbol{\Omega})^2\frac{Gm}{z}\biggl[-14v^a v^b + 15\dot{z}\bigl(v^a n^b + n^a v^b\bigr) \\
&\quad + \biggl(3v^2 - 15\dot{z}^2 + 7\frac{Gm}{z}\biggr)n^a n^b\biggr]\biggr), \qquad (6.11.28)
\end{aligned}
$$

$$
A^{ab}[\tfrac{3}{2}\mathrm{PN}] = \frac{\Delta}{c^3}\biggl(\frac{1}{12}(\boldsymbol{v}\cdot\boldsymbol{\Omega})\biggl\{-6\biggl[2(1-5\eta)v^2 - (3-8\eta)\frac{Gm}{z}\biggr]v^a v^b
$$

$$
\begin{aligned}
&- 6(7+4\eta)\frac{Gm}{z}\dot{z}\big(v^a n^b + n^a v^b\big) \\
&+ \frac{Gm}{z}\bigg[3(7-2\eta)v^2 - 9(1-2\eta)\dot{z}^2 - 4(26-3\eta)\frac{Gm}{z}\bigg]n^a n^b\bigg\} \\
&+ \frac{1}{12}(\boldsymbol{z}\cdot\boldsymbol{\Omega})\frac{Gm}{z}\bigg\{-6(31+4\eta)\dot{z}\,v^a v^b \\
&+ \bigg[3(11-6\eta)v^2 + 9(7+6\eta)\dot{z}^2 - 4(32-9\eta)\frac{Gm}{z}\bigg]\big(v^a n^b + n^a v^b\big) \\
&- \dot{z}\bigg[9(7-6\eta)v^2 - 15(1-6\eta)\dot{z}^2 - 2(121-12\eta)\frac{Gm}{z}\bigg]n^a n^b\bigg\} \\
&+ \frac{1}{2}(1-2\eta)(\boldsymbol{v}\cdot\boldsymbol{\Omega})^3\bigg\{-4v^a v^b + \frac{Gm}{z}n^a n^b\bigg\} \\
&+ \frac{3}{2}(1-2\eta)(\boldsymbol{v}\cdot\boldsymbol{\Omega})^2(\boldsymbol{n}\cdot\boldsymbol{\Omega})\frac{Gm}{z}\Big\{5\big(v^a n^b + n^a v^b\big) - 3\dot{z}n^a n^b\Big\} \\
&+ \frac{1}{4}(1-2\eta)(\boldsymbol{v}\cdot\boldsymbol{\Omega})(\boldsymbol{n}\cdot\boldsymbol{\Omega})^2\frac{Gm}{z}\bigg\{58v^a v^b - 54\dot{z}\big(v^a n^b + n^a v^b\big) \\
&- \bigg[9v^2 - 45\dot{z}^2 + 28\frac{Gm}{z}\bigg]n^a n^b\bigg\} \\
&+ \frac{1}{12}(1-2\eta)(\boldsymbol{n}\cdot\boldsymbol{\Omega})^3\frac{Gm}{z}\bigg\{-102\dot{z}\,v^a v^b \\
&- \bigg[21v^2 - 105\dot{z}^2 + 44\frac{Gm}{z}\bigg]\big(v^a n^b + n^a v^b\big) \\
&+ 15\dot{z}\bigg[3v^2 - 7\dot{z}^2 + 6\frac{Gm}{z}\bigg]n^a n^b\bigg\}\Bigg), \qquad (6.11.29)
\end{aligned}
$$

$$
\begin{aligned}
A^{ab}[\text{tail}] &= \frac{4Gm}{c^3}\int_0^\infty \Bigg(\frac{Gm}{z^3}\bigg[\bigg(3v^2 - 15\dot{z}^2 + \frac{Gm}{z}\bigg)n^a n^b + 9\dot{z}\big(v^a n^b + n^a v^b\big) \\
&\quad - 4v^a v^b\bigg]\Bigg)_{\tau-\zeta}\bigg[\ln\bigg(\frac{\zeta}{\zeta + 2r/c}\bigg) + \frac{11}{12}\bigg]\,d\zeta. \qquad (6.11.30)
\end{aligned}
$$

The gravitational-wave field is expressed in terms of the relative position vector $\boldsymbol{z} = \boldsymbol{z}_1 - \boldsymbol{z}_2$, the relative velocity vector $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$, the radial velocity $\dot{z} = \boldsymbol{z}\cdot\boldsymbol{v}$, and the mass parameters $m = m_1 + m_2$, $\eta = m_1 m_2/m^2$, and $\Delta = (m_1 - m_2)/m$. In addition, h^{ab} depends on retarded time $\tau = t - r/c$ as well as the angular vector $\boldsymbol{\Omega} := \boldsymbol{x}/r$, which specifies the direction from the barycentre to the field point $\boldsymbol{x}$. In the tail integral of Eq. (6.11.30), the terms within the large round brackets are evaluated at $\tau - \zeta$ instead of τ, and the integration from $\zeta = 0$ to $\zeta = -\infty$ involves the entire past history of the two-body system.

The expressions listed here are not fully optimal, because it is still necessary to extract the transverse-tracefree part of h^{ab}. While the TT projection has been invoked repeatedly in the preceding sections to discard irrelevant terms and simplify expressions, Eqs. (6.11.26)–(6.11.30) still contain unwanted traces and longitudinal pieces. As was discussed in Sec. 6.1.6, these are removed by subjecting h^{ab} to the TT projection operator of Eq. (6.1.33): We must write

$$
h^{ab}_{\mathrm{TT}} = (\mathrm{TT})^{ab}{}_{cd}h^{cd}, \qquad (6.11.31)
$$

and subject each post-Newtonian contribution to the TT projection. Because the final expressions are rather large, we shall not display them here.

6.12 Specialization to circular orbits

6.12.1 Circular motion

In this section we make a further specialization to circular orbital motion. This is defined by the condition

$$\dot{z} = 0, \tag{6.12.1}$$

so that the two bodies move while maintaining a constant relative separation. This is undoubtedly a restriction on all possible motions, but more than that, Eq. (6.12.1) is also an approximation, because as the system loses energy to gravitational radiation (an effect that will be examined in Chapter 7), the orbital separation slowly decreases, and even for circular orbits, $\dot{z}$ should actually be negative. But because this radiation-reaction effect appears at $\frac{2}{5}$PN order in the equations of motion, we are justified to neglect it here.

We refer back to the orbital equations of Sec. 5.5.4, and describe the motion in terms of the polar coordinates z and ψ, where z is the (now constant) distance between the two bodies, and ψ is an angular coordinate in the fixed orbital plane (chosen here to be the x-y plane). The position and velocity vectors are given by Eqs. (5.5.25) and (5.5.26), respectively:

$$\boldsymbol{z} = z\,\boldsymbol{n}, \qquad \boldsymbol{v} = \omega z\,\boldsymbol{\psi}, \tag{6.12.2}$$

where $\omega := \dot{\psi}$ is the angular velocity, and

$$\boldsymbol{n} = [\cos\psi, \sin\psi, 0], \qquad \boldsymbol{\psi} = [-\sin\psi, \cos\psi, 0] \tag{6.12.3}$$

are the basis vectors. Equation (5.5.29) informs us that ω is constant when the motion is circular, and Eq. (5.5.28) gives rise to a relation between ω and z. After setting $\dot{z} = \ddot{z} = 0$ and solving for ω^2, we obtain

$$\omega^2 = \frac{Gm}{z^3}\left[1 - (3-\eta)\frac{Gm}{c^2 z} + O(c^{-4})\right], \tag{6.12.4}$$

the post-Newtonian generalization of the usual Keplerian relation $\omega^2 = Gm/z^3$. (When radiation-reaction effects are included, z decreases as time increases, and this causes ω to increase.)

The orbital velocity is $v = \omega z$, and according to Eq. (6.12.4) we have

$$v^2 = \frac{Gm}{z}\left[1 - (3-\eta)\frac{Gm}{c^2 z} + O(c^{-4})\right]. \tag{6.12.5}$$

Making the substitution into Eq. (5.5.23), we find that the orbital energy per unit mass is

$$\tilde{E} = -\frac{Gm}{2z}\left[1 - \frac{1}{4}(7-\eta)\frac{Gm}{c^2 z} + O(c^{-4})\right]. \tag{6.12.6}$$

The system's actual energy is $E = \eta m\tilde{E}$; this includes kinetic energy and gravitational potential energy, but excludes the rest-mass energy of each body.

6.12.2 Post-Newtonian expansion parameter

The post-Newtonian expansion is formally an expansion in powers of c^{-2}, but physically it is an expansion in powers of a dimensionless quantity such as v^2/c^2. There are many such quantities that could be adopted as an expansion parameter. Equation (6.12.5) suggests, for example, that $Gm/(c^2 z)$ could be selected, and this would indeed be a valid substitute to v^2/c^2. Another choice is

$$x := \left(\frac{Gm\omega}{c^3}\right)^{2/3}, \tag{6.12.7}$$

and this has the important advantage of directly involving ω, a quantity that is directly measurable in the gravitational-wave signal. As we shall see in Sec. 6.12.4, the orbital frequency ω is directly related to the frequency of the gravitational waves, and it can therefore be measured directly. This is unlike v or z, which are coordinate-dependent and cannot be measured directly. It is easy to show, using Eqs. (6.12.4) and (6.12.5), that

$$(v/c)^2 = x\biggl[1 - \frac{2}{3}(3-\eta)x + O(x^2)\biggr] \tag{6.12.8}$$

and

$$\frac{Gm}{c^2 z} = x\biggl[1 + \frac{1}{3}(3-\eta)x + O(x^2)\biggr]. \tag{6.12.9}$$

We shall henceforth adopt x as a meaningful post-Newtonian parameter, and reexpress Eq. (6.11.25) as an expansion in powers of x.

6.12.3 TT projection

The transverse-tracefree projection of h^{ab} is accomplished with the techniques developed in Sec. 6.1.6. We re-introduce the vectorial basis $(\boldsymbol{\Omega}, \boldsymbol{\theta}, \boldsymbol{\phi})$, with

$$\begin{aligned} \boldsymbol{\Omega} &= [S\cos\phi, S\sin\phi, C], && (6.12.10)\\ \boldsymbol{\theta} &= [C\cos\phi, C\sin\phi, -S], && (6.12.11)\\ \boldsymbol{\phi} &= [-\sin\phi, \cos\phi, 0], && (6.12.12) \end{aligned}$$

where

$$C := \cos\theta, \qquad S := \sin\theta. \tag{6.12.13}$$

Here, the angles (θ, ϕ) determine the direction of the field point $\boldsymbol{x}$ at which the gravitational wave is measured. The polar angle θ refers to the z direction, which is normal to the orbital plane. The azimuthal angle ϕ, and also the angular position ψ of the relative orbit, refer to the x direction, which is arbitrary within the orbital plane; as we shall see, the gravitational-wave polarizations depend on the combination $\Psi := \psi - \phi$, and this is invariant under rotations within the plane. The unit vector $\boldsymbol{\Omega} = \boldsymbol{x}/r$ points in the longitudinal direction, and the transverse space is spanned by $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$. In terms of these, the two independent components of the transverse-tracefree piece of h^{ab} are given by Eqs. (6.1.41) and (6.1.42),

$$h_+ = \frac{1}{2}\bigl(\theta_a\theta_b - \phi_a\phi_b\bigr)h^{ab} \tag{6.12.14}$$

and

$$h_\times = \frac{1}{2}\bigl(\theta_a\phi_b + \phi_a\theta_b\bigr)h^{ab}. \tag{6.12.15}$$

The tensorial field is then constructed as in Eq. (6.1.40),

$$h^{ab}_{\mathrm{TT}} = h_+\bigl(\theta^a\theta^b - \phi^a\phi^b\bigr) + h_\times\bigl(\theta^a\phi^b + \phi^a\theta^b\bigr), \tag{6.12.16}$$

so that h_+ represents the θ-θ component of the tensor (and also minus the ϕ-ϕ component, in order to satisfy the tracefree condition), while $h_\times$ represents the θ-ϕ component.

6.12.4 Gravitational-wave polarizations

We take the gravitational-wave field of Eqs. (6.11.25)–(6.11.30) and specialize it to circular orbits by substituting Eqs. (6.12.1)–(6.12.3). Next we expand it in powers

of x by involving Eqs. (6.12.8) and (6.12.9). And finally, we extract its TT part by making use of Eqs. (6.12.10)–(6.12.15). After simplification, and after evaluation of the tail integrals (as described in the next subsection), we arrive at

$$\begin{aligned} h_+ &= \frac{2Gm\eta}{c^2 r}\left(\frac{Gm\omega}{c^3}\right)^{2/3}\Big\{ H_+^{[0]} + \Delta x^{1/2} H_+^{[1/2]} + x H_+^{[1]} + \Delta x^{3/2} H_+^{[3/2]} \\ &\quad + x^{3/2} H_+^{\rm tail} + O(x^2)\Big\} \end{aligned} \tag{6.12.17}$$

and

$$\begin{aligned} h_\times &= \frac{2Gm\eta}{c^2 r}\left(\frac{Gm\omega}{c^3}\right)^{2/3}\Big\{ H_\times^{[0]} + \Delta x^{1/2} H_\times^{[1/2]} + x H_\times^{[1]} + \Delta x^{3/2} H_\times^{[3/2]} \\ &\quad + x^{3/2} H_\times^{\rm tail} + O(x^2)\Big\}, \end{aligned} \tag{6.12.18}$$

where

$$H_+^{[0]} = -(1+C^2)\cos 2\Psi, \tag{6.12.19}$$

$$H_+^{[1/2]} = -\frac{1}{8}S(5+C^2)\sin\Psi - \frac{9}{8}S(1+C^2)\sin 3\Psi, \tag{6.12.20}$$

$$\begin{aligned} H_+^{[1]} &= \frac{1}{6}\Big[(19+9C^2-2C^4) - (19-11C^2-6C^4)\eta\Big]\cos 2\Psi \\ &\quad + \frac{4}{3}(1-3\eta)S^2(1+C^2)\cos 4\Psi, \end{aligned} \tag{6.12.21}$$

$$\begin{aligned} H_+^{[3/2]} &= \frac{1}{192}S\Big[(57+60C^2-C^4) - 2(49-12C^2-C^4)\eta\Big]\sin\Psi \\ &\quad + \frac{9}{128}S\Big[(73+40C^2-9C^4) - 2(25-8C^2-9C^4)\eta\Big]\sin 3\Psi \\ &\quad + \frac{625}{384}(1-2\eta)S^3(1+C^2)\sin 5\Psi, \end{aligned} \tag{6.12.22}$$

$$H_+^{\rm tail} = -4(1+C^2)\left[\frac{\pi}{2}\cos 2\Psi + (\gamma + \ln 4\omega r/c)\sin 2\Psi\right] \tag{6.12.23}$$

and

$$H_\times^{[0]} = -2C\sin 2\Psi, \tag{6.12.24}$$

$$H_\times^{[1/2]} = \frac{3}{4}CS\cos\Psi + \frac{9}{4}CS\cos 3\Psi, \tag{6.12.25}$$

$$\begin{aligned} H_\times^{[1]} &= \frac{1}{3}C\Big[(17-4C^2) - (13-12C^2)\eta\Big]\sin 2\Psi \\ &\quad + \frac{8}{3}(1-3\eta)CS^2\sin 4\Psi, \end{aligned} \tag{6.12.26}$$

$$\begin{aligned} H_\times^{[3/2]} &= -\frac{1}{96}CS\Big[(63-5C^2) - 2(23-5C^2)\eta\Big]\cos\Psi \\ &\quad - \frac{9}{64}CS\Big[(67-15C^2) - 2(19-15C^2)\eta\Big]\cos 3\Psi \\ &\quad - \frac{625}{192}(1-2\eta)CS^3\cos 5\Psi, \end{aligned} \tag{6.12.27}$$

$$H_\times^{\rm tail} = -8C\left[\frac{\pi}{2}\sin 2\Psi - (\gamma + \ln 4\omega r/c)\cos 2\Psi\right]. \tag{6.12.28}$$

We recall that $m = m_1 + m_2$ is the total mass of the two-body system, $\eta = m_1 m_2/m^2$ is the dimensionless reduced mass, $\Delta = (m_1 - m_2)/m$ is the dimensionless mass

difference, r is the distance from the barycentre to the detector, θ and ϕ give the angular position of the detector relative to the orbital plane, $C = \cos\theta$, $S = \sin\theta$, and $x = (Gm\omega/c^3)^{2/3}$ is the post-Newtonian expansion parameter, expressed in terms of ω, the orbital angular velocity. The phase of the wave is determined by

$$\Psi := \psi - \phi = \omega(t - r/c) - \phi, \tag{6.12.29}$$

where $\psi = \omega\tau$ is the (retarded) angular position of the relative orbit. Equations (6.12.20) and (6.12.24) imply that at leading order, the gravitational-wave signal oscillates at twice the orbital frequency; the post-Newtonian corrections contribute additional frequencies and the signal is modulated. Our results for the gravitational-wave polarizations agree with the expressions listed in Blanchet, Iyer, Will, and Wiseman (1996), except for a different convention regarding angles and phases.

The tails terms of Eqs. (6.12.23) and (6.12.28) are interesting. They involve the mathematical constants π and $\gamma \simeq 0.5772$ (Euler's constant), and they also involve a logarithmic term that depends on $\omega r/c$. The tail terms are best interpreted as giving rise to a correction to Ψ, the quantity that determines the phase of the gravitational wave. Indeed, it is a simple matter to show that the Newtonian and tail contributions to h_+ and $h_\times$ can be combined and expressed as

$$H_+^{[0]} + x^{3/2} H_+^{\rm tail} = -(1 + C^2)\bigl(1 + 2\pi x^{3/2}\bigr)\cos 2\Psi^*, \tag{6.12.30}$$

$$H_\times^{[0]} + x^{3/2} H_\times^{\rm tail} = -2C\bigl(1 + 2\pi x^{3/2}\bigr)\sin 2\Psi^*. \tag{6.12.31}$$

These expressions involve an amplitude correction equal to $2\pi x^{3/2}$, and a new phase function given by

$$\Psi^* = \Psi - 2x^{3/2}(\gamma + \ln 4\omega r/c) = \omega\biggl(t - r/c - \frac{2Gm}{c^3}\ln\frac{4\omega r}{c} + \text{constant}\biggr). \tag{6.12.32}$$

It is this shifted phase function that informs us, at long last, that the radiation propagates not along the mathematical light cones of Minkowski spacetime, but along the true, physical light cones of a curved spacetime. Indeed, the logarithmic term in Eq. (6.12.32) represents the well known *Shapiro time delay*, the extra time required by a light wave, or a gravitational wave, to climb up a gravitational potential well created by a distribution of matter with total mass m.

6.12.5 Evaluation of the tail integrals

We must still evaluate the tail integrals, and show that they lead to Eqs. (6.12.23) and (6.12.28). We start with Eq. (6.11.30), which we specialize to circular orbits by involving Eqs. (6.12.1)–(6.12.3), and we extract its TT part by making use of Eqs. (6.12.10)–(6.12.15). After converting Eq. (6.11.25) to the notation of Eqs. (6.12.17) and (6.12.18), we find that

$$H_+^{\rm tail} = 8(1 + C^2)\omega \int_0^\infty \cos(2\Psi - 2\omega\zeta)\biggl[\ln\frac{\zeta}{\zeta + 2r/c} + \frac{11}{12}\biggr]\,d\zeta$$

and

$$H_\times^{\rm tail} = 16C\omega \int_0^\infty \sin(2\Psi - 2\omega\zeta)\biggl[\ln\frac{\zeta}{\zeta + 2r/c} + \frac{11}{12}\biggr]\,d\zeta.$$

We next change the variable of integration to $y := 2\omega\zeta$ and we introduce $k := 4\omega r/c$. The tail integrals become

$$H_+^{\rm tail} = 4(1 + C^2) \int_0^\infty \cos(2\Psi - y)\biggl[\ln\frac{y}{y + k} + \frac{11}{12}\biggr]\,dy$$

and

$$H_\times^{\rm tail} = 8C \int_0^\infty \sin(2\Psi - y) \biggl[\ln \frac{y}{y+k} + \frac{11}{12} \biggr]\, dy.$$

Expanding the trigonometric functions, this is

$$H_+^{\rm tail} = 4(1+C^2)\bigl(J_{\rm c} \cos 2\Psi + J_{\rm s} \sin 2\Psi \bigr)$$

and

$$H_\times^{\rm tail} = 8C\bigl(J_{\rm c} \sin 2\Psi - J_{\rm s} \cos 2\Psi \bigr),$$

where

$$J_{\rm c} := \int_0^\infty \cos(y) \biggl[\ln \frac{y}{y+k} + \frac{11}{12} \biggr]\, dy$$

and

$$J_{\rm s} := \int_0^\infty \sin(y) \biggl[\ln \frac{y}{y+k} + \frac{11}{12} \biggr]\, dy.$$

These integrals are ill-defined, because the function within the square brackets behaves as $\frac{11}{12} - k/y$ for large y, and the constant term prevents each integral from converging. This, however, is an artificial problem that comes as a consequence of our (unphysical) approximation $\omega = \text{constant}$. In reality, the two-body system undergoes radiation reaction, and ω slowly decreases as ζ increases toward ∞. (Recall that z decreases as time increases, which causes ω to increase as time increases; but recall also that the tail term integrates towards the past, so that ω decreases as ζ increases.) This effect does not alter substantially the logarithmic portion of the integral, but it is sufficient to ensure the convergence of the constant term.

The integrals can be defined properly by inserting a convergence factor within the integrand. Alternatively, and this practice is consistent with what was done back in Sec. 6.10.3, we can integrate by parts and simply discard an ambiguous (and unphysical) boundary term at $y = \infty$. Proceeding along those lines, we find that our integrals are equivalent to

$$J_{\rm c} = -\int_0^\infty \frac{k \sin y}{y(y+k)}\, dy$$

and

$$J_{\rm s} = \int_0^\infty \frac{k(\cos y - 1)}{y(y+k)}\, dy,$$

and we observe that these integrals are indeed well defined. They can be evaluated in closed form. We have

$$\begin{aligned} J_{\rm c} &= -\frac{\pi}{2} + \frac{\pi}{2} \cos k + \mathrm{Ci}(k) \sin k - \mathrm{Si}(k) \cos k \\ &= -\frac{\pi}{2} + O(k^{-1}) \end{aligned}$$

and

$$\begin{aligned} J_{\rm s} &= -\gamma - \ln k - \frac{\pi}{2} \sin k + \mathrm{Ci}(k) \cos k + \mathrm{Si}(k) \sin k \\ &= -\gamma - \ln k + O(k^{-2}), \end{aligned}$$

where γ is Euler's constant, $\mathrm{Ci}(k)$ is the cosine integral, and $\mathrm{Si}(k)$ is the sine integral (these are defined, for example, in Sec. 5.2 of Abramowitz and Stegun's *Handbook of mathematical functions*). The approximate forms neglect terms of order $k^{-1} = (4\omega r/c)^{-1} \sim (\lambda_c/r)$ and smaller, and these are small by virtue of the fact that the gravitational-wave field is evaluated in the far-away wave zone, where $r \gg \lambda_c$.

Collecting results, we find that

$$H_+^{\rm tail} = -4(1+C^2)\biggl[\frac{\pi}{2}\cos 2\Psi + (\gamma + \ln 4\omega r/c)\sin 2\Psi\biggr]$$

and

$$H_\times^{\rm tail} = -8C\biggl[\frac{\pi}{2}\sin 2\Psi - (\gamma + \ln 4\omega r/c)\cos 2\Psi\biggr],$$

and these expressions were already presented in Eqs. (6.12.23) and (6.12.28).

Chapter 7
Energy radiated and radiation reaction

In this final chapter we calculate the energy carried off by gravitational waves, and we construct a radiation-reaction force that acts on each body within an N-body system. We begin by constructing an expression for the rate at which energy is dissipated by gravitational waves. We provide two very distinct derivations. In Sec. 7.1 we use the Landau-Lifshitz pseudotensor as a basis for the calculation, and in Sec. 7.2 we recreate the Bondi-Sachs argument, which is based on a careful integration of the Einstein field equations in the far-away wave zone. Both approaches lead to the same result, expressed by Eq. (7.1.4) or Eq. (7.1.5). In Sec. 7.3 we give two applications of this result. First, we derive Eq. (7.3.6) or Eq. (7.3.8), the celebrated quadrupole formula of gravitational-wave physics. Second, we calculate the energy radiated by a two-body system in circular, post-Newtonian motion; this is expressed by Eq. (7.3.18). In Sec. 7.4 we calculate the gravitational potentials that are required in the computation of the radiation-reaction force, which is carried out in Sec. 7.5. The final result for the radiation-reaction force is given by Eq. (7.5.32) for the general N-body system, and by Eq. (7.5.49) for a two-body system. This final section also provides a discussion of energy balance; we show that the radiation-reaction force does work on the N bodies, and we verify that in a coarse-grained sense, the work done is equal to the energy radiated.

7.1 Energy radiated: Landau and Lifshitz

The most direct way of calculating the rate at which energy is radiated by a source of gravitational waves is based on the conservation identities of Sec. 1.2. These, we recall, are a direct consequence of the Landau-Lifshitz formulation of the Einstein field equations, which was reviewed in Sec. 1.1.

Recall from Eq. (1.2.2) that

$$P^0[V] = \frac{c^3}{16\pi G} \oint_S \partial_\mu H^{0\mu 0c}\, dS_c$$

represents the zeroth component (the energy divided by c) of the total momentum four-vector associated to a three-dimensional volume V bounded by a two-dimensional surface S; the integrand is related to the gravitational potentials $h^{\alpha\beta}$ via the relations $H^{\alpha\mu\beta\nu} = \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu}$ and $\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$. The rate at which this quantity changes with time is given by Eq. (1.2.3), which we write as

$$\frac{dE[V]}{dt} = -c\oint_S (-g)t^{0c}_{\rm LL}\, dS_c,$$

having set $E[V] = P^0[V]c$ and $x^0 = ct$. In the limit in which V becomes infinitely large, this must become equal to (minus) the rate at which the gravitational waves carry energy away, and we have

$$\frac{dE_{\rm gw}}{dt} = c\oint_\infty (-g)t^{0c}_{\rm LL}\, dS_c, \tag{7.1.1}$$

an equation that relates $\dot{E}_{\rm gw}$ to the surface integral of the normal component of $(-g)t^{0a}_{\rm LL}$, the Landau-Lifshitz flux vector.

We evaluate Eq. (7.1.1) in the far-away wave zone, and we take the limit $r \to \infty$ at the end of the calculation. We work in the TT gauge of Sec. 6.1.4, and we use the gravitational potentials of Eqs. (6.1.26)–(6.1.28),

$$h^{00} = \frac{4GM}{c^2 r}, \qquad h^{0a} = 0, \qquad h^{ab} = h^{ab}_{\rm TT}. \tag{7.1.2}$$

Here, M is the total gravitational mass of the spacetime, and $h^{ab}_{\rm TT}$ depends on the retarded time $\tau := t - r/c$, the angular vector $\boldsymbol{\Omega} := \boldsymbol{x}/r$, and falls off as r^{-1}; it also satisfies the transverse-tracefree conditions

$$\Omega_b h^{ab}_{\rm TT} = 0 = \delta_{ab} h^{ab}_{\rm TT}.$$

Equation (7.1.2) is valid to leading order in r^{-1}, and the neglected terms are of order r^{-2}.

We need an expression for $(-g)t^{0c}_{\rm LL}$ that is sufficiently accurate in the far-away wave zone. Because the Landau-Lifshitz pseudotensor is dominantly quadratic in the gravitational potentials, the leading-order terms fall off as r^{-2}, and we may neglect cubic and higher-order terms that will not survive the limit $r \to \infty$. Going back to the original definition of Eq. (1.1.5), we substitute $\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$, $g^{\alpha\beta} = \eta^{\alpha\beta} + O(h)$, and we obtain

$$\begin{aligned} (-g)t^{0c}_{\rm LL} &= \frac{c^4}{16\pi G}\biggl\{ \frac{1}{2}\partial^0 h_{\mu\nu}\partial^c h^{\mu\nu} - \frac{1}{4}\partial^0 h \partial^c h - \partial^0 h_{\mu\nu}\partial^\nu h^{\mu c} \\ &\qquad - \partial^c h_{\mu\nu}\partial^\nu h^{0\mu} + \partial_\mu h^0{}_\nu \partial^\mu h^{\nu c} \biggr\}, \end{aligned}$$

in which indices are lowered with $\eta_{\mu\nu}$, and $h = \eta_{\mu\nu}h^{\mu\nu}$. When we substitute Eqs. (7.1.2) into this expression, we notice first that h^{00} does not participate, because its time derivative is zero, and because its spatial derivatives fall off as r^{-2}. We notice also that $h = 0$ in the TT gauge, and we recall that in the far-away wave zone, spatial derivatives can be expressed in terms of retarded-time derivatives according to $\partial_c = -c^{-1}\Omega_c \partial_\tau$, a statement that follows from Eq. (6.1.10).

With these simplifications, we find that the Landau-Lifshitz flux vector reduces to

$$(-g)t^{0c}_{\rm LL} = \frac{c^2}{32\pi G}\dot{h}^{\rm TT}_{ab}\dot{h}^{ab}_{\rm TT}\Omega^c, \tag{7.1.3}$$

in which an overdot indicates differentiation with respect to retarded-time τ. If we take S to be a surface of constant r, then $dS_c = r^2\Omega_c\,d\Omega$, where $d\Omega = \sin\theta\,d\theta d\phi$ is an element of solid angle, and Eq. (7.1.1) becomes

$$\dot{E}_{\rm gw} = \frac{c^3}{32\pi G}\lim_{r\to\infty}\oint r^2 \dot{h}^{\rm TT}_{ab}\dot{h}^{ab}_{\rm TT}\,d\Omega. \tag{7.1.4}$$

The energy flux can also be expressed in terms of the gravitational-wave polarizations h_+ and $h_\times$, by involving Eq. (6.1.40) and the orthonormality of the basis vectors $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$. We obtain

$$\dot{E}_{\rm gw} = \frac{c^3}{16\pi G}\lim_{r\to\infty}\oint r^2\Big(\dot{h}_+^2 + \dot{h}_\times^2\Big)\,d\Omega. \tag{7.1.5}$$

It is understood that $h^{ab}_{\rm TT}$, h_+, or $h_\times$ are expressed as functions of τ and $\boldsymbol{\Omega}$, and that they fall off as r^{-1}; as a consequence, the factors of r disappear from both Eqs. (7.1.4) and (7.1.5).

This derivation of $\dot{E}_{\rm gw}$ leaves much room for criticism. To begin, the calculation is based on the (fairly arbitrary) definitions for momentum and momentum flux introduced by Landau and Lifshitz. While the conservation identities that follow from these definitions are perfectly rigourous, the interpretation of $cP^0[V]$ as a physical energy is not, and it becomes meaningful only when the spacetime is static, and when V is infinitely large. There is no guarantee that this quantity should provide a sound description of total gravitational energy in dynamical situations, and the current foundation of Eqs. (7.1.4) and (7.1.5) is not as solid as one might wish. In addition, the calculation of $\dot{E}_{\rm gw}$ was carried out in the TT gauge, and there is no guarantee that the result should be gauge invariant. In view of this criticism, we provide in the next section an alternative, more rigourous derivation of Eqs. (7.1.4) and (7.1.5).

7.2 Energy radiated: Bondi and Sachs

The derivation of Eqs. (7.1.4) and (7.1.5) presented in this section is based on a careful integration of the Einstein field equations in a neighbourhood of $r=\infty$, in the far-away wave zone. The method goes back to the celebrated work of Bondi, van der Burg, and Metzner (1962), and of Sachs (1962). The presentation here follows closely the paper by Brown, Lau, and York (1997). We shall establish Eq. (7.1.5) directly, and this is sufficient, because Eq. (7.1.4) can be recovered from it by involving the equations that precede Eqs. (6.1.41) and (6.1.42).

7.2.1 Bondi-Sachs metric

We work in a system of coordinates (u, r, θ, ϕ), with the usual relation to a Cartesian system (t, x, y, z) given by $u = ct - r$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$. We denote the angular coordinates collectively by $\theta^A = (\theta,\phi)$, with an index A that runs from 2 to 3. The metric is put in the form

$$ds^2 = -UV\,du^2 - 2U\,dudr + r^2\Omega_{AB}\big(d\theta^A + W^A\,du\big)\big(d\theta^B + W^B\,du\big), \tag{7.2.1}$$

in which U, V, W^A, and Ω_{AB} are functions of u, r, and θ^A. To reduce the number of independent components from seven to six, we impose the condition

$$\Omega := \det[\Omega_{AB}] = \sin^2\theta. \tag{7.2.2}$$

This ensures that a two-surface of constant u and r has a proper area equal to $4\pi r^2$; Eq. (7.2.2) is therefore a normalization of the radial coordinate r, which is interpreted as an areal radius.

The geometrical meaning of the coordinates is revealed by an examination of the inverse metric, which is given by $g^{uu} = 0 = g^{uA}$, $g^{ur} = -1/U$, $g^{rr} = V/U$, $g^{rA} = W^A/U$, and $g^{AB} = \Omega^{AB}/r^2$, in which Ω^{AB} is the inverse to Ω_{AB}. The vector

$$k_\alpha := -\partial_\alpha u$$

is normal to hypersurfaces of constant u, and from the inverse metric we find that $g^{\alpha\beta}k_\alpha k_\beta = g^{uu} = 0$; the vector is null, and each surface $u =$ constant is therefore a null hypersurface. These surfaces are generated by null geodesics to which the vector k^α is tangent. We can show that the angular coordinates θ^A are constant along the null generators: $k^\beta\partial_\beta\theta^A = g^{\alpha\beta}k_\alpha\partial_\beta\theta^A = g^{uA} = 0$. And we can show that the change in r along each generator is determined by the metric function U: $k^\beta\partial_\beta r = g^{\alpha\beta}k_\alpha\partial_\beta r = g^{ur} = -1/U$. The meaning of the coordinates is therefore clear: The retarded-time coordinate u labels a family of null hypersurfaces, the angular coordinates θ^A label the null geodesics that generate the hypersurfaces, and the areal radius r runs along each generator. Because U is positive, r decreases toward the future (because k^α is future directed), and the generators converge toward $r = 0$; this implies that the null hypersurfaces are converging light cones.

The metric of Eq. (7.2.1) is required to be asymptotically flat, and this implies that the metric functions must satisfy the conditions

$$U \to 1, \qquad V \to 1, \qquad W^A \to 0, \qquad \Omega_{AB} \to \mathrm{diag}[1, \sin^2\theta]$$

when $r \to \infty$.

To integrate the Einstein field equations in a neighbourhood of $r = \infty$ we introduce the asymptotic expansions

$$U = 1 + \frac{A(u,\theta^A)}{r} + \frac{B(u,\theta^A)}{r^2} + O(r^{-3}), \tag{7.2.3}$$

$$V = 1 - \frac{2Gm(u,\theta^A)}{c^2 r} + O(r^{-2}), \tag{7.2.4}$$

$$W^A = \frac{C^A(u,\theta^A)}{r} + \frac{D^A(u,\theta^A)}{r^2} + O(r^{-3}), \tag{7.2.5}$$

$$\Omega_{\theta\theta} = 1 + \frac{X(u,\theta^A)}{r} + \frac{X^2 + Y^2 + P(u,\theta^A)}{2r^2} + O(r^{-3}), \tag{7.2.6}$$

$$\Omega_{\theta\phi} = \sin\theta\biggl[\frac{Y(u,\theta^A)}{r} + \frac{Q(u,\theta^A)}{r^2} + O(r^{-3})\biggr], \tag{7.2.7}$$

$$\Omega_{\phi\phi} = \sin^2\theta\biggl[1 - \frac{X(u,\theta^A)}{r} + \frac{X^2 + Y^2 - P(u,\theta^A)}{2r^2} + O(r^{-3})\biggr]. \tag{7.2.8}$$

We have introduced a number of functions of u and θ^A (such as A, B, m, C^A, D^A, P, and Q) that will be determined by the field equations. The functions X and Y will remain free, however, and will be seen to represent the gravitational-wave degrees of freedom of the solution. The function $m(u,\theta^A)$, called the *mass aspect* of the spacetime, will play an important role below. And finally, we remark that the specific forms introduced in Eqs. (7.2.6)–(7.2.8) for Ω_{AB} are designed to enforce the normalization condition of Eq. (7.2.2).

7.2.2 Integration of the field equations

The strategy is to substitute the expansions of Eqs. (7.2.3)–(7.2.8) into the metric of Eq. (7.2.1), and then to use this metric to calculate the Ricci tensor. Because we wish to construct a solution to the vacuum field equations, we set $R_{\alpha\beta} = 0$ and examine the consequences. (These computations are best carried out with a tensor manipulation package such as GRTensorII running under Maple.)

The computation returns the Ricci tensor expressed as an expansion in powers of r^{-1}, and we must set the coefficient of each term equal to zero. The leading terms are $R_{uu} = O(r^{-1})$, $R_{ur} = O(r^{-2})$, $R_{uA} = O(1)$, $R_{rr} = O(r^{-3})$, $R_{rA} = O(r^{-1})$, and $R_{AB} = O(1)$.

We begin by enforcing $R_{rr} = 0$ at leading order, and this immediately implies that $A = 0$. Next we set $R_{rA} = 0$, and this produces $C^A = 0$. With these assignments, we find that all the leading terms in $R_{\alpha\beta}$ vanish.

Moving on, we now enforce $R_{rA} = 0$ at the next order, and we deduce that

$$D^\theta = \frac{1}{2}\biggl(\frac{\partial X}{\partial\theta} + 2\frac{\cos\theta}{\sin\theta}X + \frac{1}{\sin\theta}\frac{\partial Y}{\partial\phi}\biggr)$$

and

$$D^\phi = \frac{1}{2\sin\theta}\biggl(\frac{\partial Y}{\partial\theta} + 2\frac{\cos\theta}{\sin\theta}Y - \frac{1}{\sin\theta}\frac{\partial X}{\partial\phi}\biggr).$$

Continuing like this, we also produce the relations $B = -\frac{1}{8}(X^2+Y^2)$, $P = 0$, and $Q = 0$.

The final piece of information comes from setting the $O(r^{-2})$ term in R_{uu} to zero. This reveals that

$$\frac{\partial m}{\partial u} = -\frac{c^2}{4G}\biggl[\biggl(\frac{\partial X}{\partial u}\biggr)^2 + \biggl(\frac{\partial Y}{\partial u}\biggr)^2\biggr] + \frac{c^2}{4G}\frac{\partial F}{\partial u}, \tag{7.2.9}$$

where

$$F := \frac{\partial^2 X}{\partial\theta^2} + 3\frac{\cos\theta}{\sin\theta}\frac{\partial X}{\partial\theta} - 2X - \frac{1}{\sin^2\theta}\frac{\partial^2 X}{\partial\phi^2} + \frac{2}{\sin\theta}\frac{\partial^2 Y}{\partial\theta\partial\phi} + 2\frac{\cos\theta}{\sin^2\theta}\frac{\partial Y}{\partial\phi}. \tag{7.2.10}$$

Equation (7.2.9) determines how the the mass aspect changes with time, assuming that the functions $X(u,\theta^A)$ and $Y(u,\theta^A)$ are known. Notice that these two functions are not determined by the field equations; they represent unconstrained degrees of freedom, and in Sach's treatment, they are combined into a single complex quantity known as the *news function*. It is an important fact then when there is no news, that is, when $X = Y = 0$, the mass aspect becomes independent of the retarded-time u. And what's more, it can be also shown (by involving additional pieces of the field equations) that when $X = Y = 0$, the mass aspect must be independent of the angles. Under these conditions, m is a constant, the asymptotic spacetime is spherically symmetric, and m remains as the sole characterization of the spacetime.

7.2.3 Mass-loss formula

Our results in the preceding subsection imply that the time-time component of the metric tensor is given by

$$-g_{uu} = UV = 1 - \frac{2Gm(u,\theta^A)}{c^2 r} + O(r^{-2}), \tag{7.2.11}$$

and the origin of the name "mass aspect" for $m(u,\theta^A)$ becomes clear. As we have just seen, the interpretation of m as a mass parameter is firm when the functions $X(u,\theta^A)$ and $Y(u,\theta^A)$ vanish; in these circumstances m is independent of both u and θ^A, the asymptotic spacetime is spherically symmetric, and Eq. (7.2.11) informs us that m is the total gravitational mass of the spacetime.

The angular average of the mass aspect is what is known as the *Bondi-Sachs mass*,

$$M_{\rm BS}(u) = \frac{1}{4\pi}\int m(u,\theta,\phi)\,d\Omega. \tag{7.2.12}$$

This depends on u only, and its rate of change is obtained by integrating Eq. (7.2.9) over a unit two-sphere. We shall prove below (in Sec. 7.2.5) that $\int F\, d\Omega = 0$, and we find that

$$\frac{M_{\rm BS}}{du} = -\frac{c^2}{16\pi G} \int \biggl[\Bigl(\frac{\partial X}{\partial u} \Bigr)^2 + \Bigl(\frac{\partial Y}{\partial u} \Bigr)^2 \biggr]\, d\Omega. \tag{7.2.13}$$

This is the celebrated *Bondi-Sachs mass-loss formula*. We shall attempt to give an interpretation to this formula in terms of a flux of gravitational-wave energy (represented by the right-hand side) producing a decrease in the energy function of the source (represented by the Bondi-Sachs mass).

The interpretation is especially clear when the spacetime proceeds from an initial stationary state, becomes dynamical for a while, and settles down to a final stationary state. In the initial stationary state there is no news ($X = Y = 0$), the mass aspect m is a constant, and according to Eq. (7.2.12), it is equal to the initial value $M_1 := M_{\rm BS}(u_1)$ of the Bondi-Sachs mass (here u_1 denotes the initial retarded time); the initial mass content of the spacetime is therefore measured by M_1. Much of the same is true for the final stationary state: The news has turned off, the mass aspect m is once again constant, and it is equal to the final value $M_2 := M_{\rm BS}(u_2)$ of the Bondi-Sachs mass (u_2 is the final retarded time); the final mass content of the spacetime is therefore measured by M_2.

Between $u = u_1$ and $u = u_2$ the spacetime is dynamical, and the functions $X(u, \theta^A)$, $Y(u, \theta^A)$ are nonzero. According to Eq. (7.2.13), *the Bondi-Sachs mass must decrease while there is news*, and we find that M_2 is necessarily smaller than M_1. The spacetime has lost some of its mass, and it must be the gravitational waves (represented by the news) that have transported this energy away from the source. The rate at which the waves carry energy must therefore be given by the right-hand side of Eq. (7.2.13).

It is important to notice that we are introducing here a notion of *coarse-grained rate*: What we can say with full certainty is that in the time interval $\Delta u = u_2 - u_1$, the spacetime has lost an amount of mass given by $\Delta M = M_2 - M_1$, and that the *averaged rate* at which the waves carry energy must be given by $\Delta M/\Delta u$. This coarse-grained rate can be calculated by integrating the right-hand side of Eq. (7.2.13) between $u = u_1$ and $u = u_2$, and dividing the result by Δu.

The scenario elaborated here is based on the idea that the spacetime is dynamical for a period of time Δu, and the mass-loss formula allows us to calculate the accumulated change in mass over that period. The scenario does not allow us to take the limit $\Delta u \to 0$ and to conclude that in this limit, $\Delta M/\Delta u$ becomes $dM_{\rm BS}/du$ as given by Eq. (7.2.13). The reason is that while the limit is mathematically well defined, the *physical interpretation* of the result, in terms of an operationally well-defined mass function, does not survive the limiting procedure. We must therefore learn to live with a coarse-grained notion of gravitational-wave energy flux, and abandon the idea that there might exist a precise, physically meaningful, notion of fine-grained energy flux in general relativity. This said, we shall nevertheless allow ourselves to view Eq. (7.2.13) as a plausible expression for the energy flux, keeping in mind that the interpretation is valid only after coarse graining.

7.2.4 Gravitational-wave flux

The interpretation of Eq. (7.2.13) as an energy-flux formula relies on an identification of X and Y with the spacetime's gravitational-wave degrees of freedom. There are many ways of establishing this connection, but to avoid the many dangers associated with different coordinate systems and gauge conditions, we rely on the discussion of Sec. 6.1.5, in which $h^{ab}_{\rm TT}$ is related to certain components of the spacetime's asymptotic Riemann tensor (a gauge-invariant quantity in the far-away

wave zone). We recall the relation

$$\ddot{h}^{\rm TT}_{ab} = -2R_{tatb} + O(r^{-2}),$$

in which an overdot indicates differentiation with respect to $\tau := u/c$. This equation is expressed in harmonic coordinates, but it can easily be written in covariant form if instead of dealing with the tensor $h^{\rm TT}_{ab}$, we work with the polarizations h_+ and $h_\times$, which are scalar quantities. Recalling Eqs. (6.1.41) and (6.1.42), we have that

$$\ddot{h}_+ = -\bigl(\theta^a\theta^b - \phi^a\phi^b\bigr)R_{tatb}$$

and

$$\ddot{h}_\times = -\bigl(\theta^a\phi^b + \phi^a\theta^b\bigr)R_{tatb},$$

and these equations can be written in fully covariant form as

$$\ddot{h}_+ = -R_{\mu\alpha\nu\beta}t^\mu\bigl(\theta^\alpha\theta^\beta - \phi^\alpha\phi^\beta\bigr)t^\nu + O(r^{-2}) \tag{7.2.14}$$

and

$$\ddot{h}_\times = -R_{\mu\alpha\nu\beta}t^\mu\bigl(\theta^\alpha\phi^\beta + \phi^\alpha\theta^\beta\bigr)t^\nu + O(r^{-2}). \tag{7.2.15}$$

Here, t^α is a timelike vector that asymptotically coincides with the timelike Killing vector of Minkowski spacetime at $r = \infty$, and θ^α and ϕ^α are vectors that asymptotically coincide with unit vectors pointing in the θ and ϕ directions, respectively.

We are interested in the quantities

$$A_+ := \lim_{r\to\infty} rh_+, \qquad A_\times := \lim_{r\to\infty} rh_\times,$$

which can be evaluated with the help of Eqs. (7.2.14) and (7.2.15). Performing the calculation with the Bondi-Sachs metric of Eqs. (7.2.1)–(7.2.8), we arrive at $\ddot{A}_+ = \ddot{X}$ and $\ddot{A}_\times = \ddot{Y}$, and we conclude that

$$A_+ = X, \qquad A_\times = Y.$$

The components of the complex news function do indeed represent the spacetime's gravitational-wave degrees of freedom.

We insert these results within Eq. (7.2.13), which we write in terms of the Bondi-Sachs energy $E_{\rm BS} := M_{\rm BS}c^2$. The result is

$$\frac{dE_{\rm BS}}{du} = -\frac{c^4}{16\pi G}\int r^2\biggl[\Bigl(\frac{\partial h_+}{\partial u}\Bigr)^2 + \Bigl(\frac{\partial h_\times}{\partial u}\Bigr)^2\biggr]\,d\Omega.$$

This is the rate at which the spacetime is losing energy, and this must be equal to (minus) the rate at which the gravitational waves carry energy away from the source. Writing $u = c\tau$, we have arrived at

$$\dot{E}_{\rm gw} = \frac{c^3}{16\pi G}\int r^2\biggl[\Bigl(\frac{\partial h_+}{\partial \tau}\Bigr)^2 + \Bigl(\frac{\partial h_\times}{\partial \tau}\Bigr)^2\biggr]\,d\Omega, \tag{7.2.16}$$

the same statement as Eq. (7.1.5), which was obtained on the basis of the Landau-Lifshitz pseudotensor. It is comforting that we get the same expression from two radically different approaches. We shall keep in mind, however, the lesson that was learned in the preceding subsection, that Eq. (7.2.16) is meant to be involved in a coarse-graining procedure whereby it is averaged over an interval of time $\delta\tau$ during which the spacetime is strongly dynamical.

7.2.5 Integration of F

We still have to show that $\int F\, d\Omega = 0$, where the quantity F is given by Eq. (7.2.10). Here X and Y are to be viewed as arbitrary functions of θ and ϕ, and their dependence on u is irrelevant.

We begin with a formal proof of the statement. We first place a metric Ω_{AB} on a topological two-sphere, with components

$$\begin{aligned}
\Omega_{\theta\theta} &= 1 + \epsilon X + O(\epsilon^2), \\
\Omega_{\theta\phi} &= \sin\theta\big[\epsilon Y + O(\epsilon^2)\big], \\
\Omega_{\phi\phi} &= \sin^2\theta\big[1 - \epsilon X + O(\epsilon^2)\big],
\end{aligned}$$

where X and Y are arbitrary functions of θ and ϕ, and where $\epsilon \ll 1$ is a parameter that measures the deformation of the two-sphere relative to a perfectly round shape. [These equations are the same as Eqs. (7.2.6)–(7.2.8), with the dependence on u removed and with r^{-1} replaced by ϵ.] We next calculate the Ricci scalar associated with this metric,

$$\mathsf{R} = 2 + \epsilon F + O(\epsilon^2),$$

where F is defined by Eq. (7.2.10), and we integrate this over the manifold:

$$\int \mathsf{R}\,\sqrt{\Omega}\, d\theta d\phi = 8\pi + \epsilon \int F\, d\Omega + O(\epsilon^2).$$

The *Gauss-Bonnet theorem* states that the integral of R over the two-dimensional manifold is a topological invariant; its value depends on the topology of the manifold (through its Euler characteristic, which depends on the genus of the surface), but it must independent of the metric. In our case the integral of R must be equal to 8π, the value of the topological invariant that is appropriate for a two-sphere, and it must be independent of ϵ. We conclude that

$$\int F\, d\Omega = 0, \tag{7.2.17}$$

irrespective of the form of the functions $X(\theta,\phi)$ and $Y(\theta,\phi)$.

To see how this "miracle" happens, we examine the specific form of the function F. Equation (7.2.10) can be written as

$$F = \biggl(\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\biggr)X + 2\biggl(\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} - 1\biggr)X - \frac{1}{\sin^2\theta}\frac{\partial^2 X}{\partial\phi^2} + \frac{2}{\sin\theta}\frac{\partial}{\partial\phi}\biggl(\frac{\partial}{\partial\theta} + \frac{\cos\theta}{\sin\theta}\biggr)Y,$$

which is the same as

$$F = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\biggl(\sin\theta\frac{\partial X}{\partial\theta}\biggr) + \frac{2}{\sin\theta}\frac{\partial}{\partial\theta}\Bigl(\cos\theta X\Bigr) - \frac{1}{\sin^2\theta}\frac{\partial^2 X}{\partial\phi^2} + \frac{2}{\sin\theta}\frac{\partial}{\partial\phi}\biggl(\frac{\partial}{\partial\theta} + \frac{\cos\theta}{\sin\theta}\biggr)Y.$$

Integration of all derivatives with respect to ϕ gives zero, and integration of the first terms yields

$$\int_0^\pi \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\biggl(\sin\theta\frac{\partial X}{\partial\theta}\biggr)\sin\theta\, d\theta = \sin\theta\frac{\partial X}{\partial\theta}\bigg|_0^\pi = 0.$$

The only remaining term is

$$2\int_0^\pi \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Bigl(\cos\theta X\Bigr)\sin\theta\, d\theta = -2\big[X(\pi,\phi) + X(0,\phi)\big].$$

This is actually independent of ϕ, because a regular function $X(\theta,\phi)$ cannot depend on ϕ when it is evaluated at the poles ($\theta = 0$ or $\theta = \pi$). Furthermore, elementary flatness at the poles requires that $X(\theta = 0) = X(\theta = \pi) = 0$, and we find that the second integral must vanish also. The statement of Eq. (7.2.17) is therefore verified.

7.3 Energy radiated: Quadrupole formula and circular orbits

7.3.1 *Quadrupole formula*

In this first subsection we calculate $\dot{E}_{\rm gw}$ to leading order in a post-Newtonian expansion, for an arbitrary source of gravitational waves. Our end result will be the celebrated *quadrupole formula* for the energy radiated by gravitational waves.

We recall from Sec. 6.9 that to leading order in a post-Newtonian expansion, the gravitational potentials are given by

$$h^{ab} = \frac{2G}{c^4 r}\ddot{I}^{ab}, \tag{7.3.1}$$

where

$$I^{ab} = \int \rho x^a x^b \, d^3x \tag{7.3.2}$$

is the Newtonian quadrupole moment of a mass distribution with density $\rho := T^{00}/c^2$. For a system of N bodies with masses m_A and positions $\boldsymbol{z}_A(t)$, this is $I^{ab} = \sum_A m_A z_A^a z_A^b$. These are Eqs. (6.9.1) and (6.9.2), respectively, with all expressions truncated at Newtonian order. The transverse-tracefree part of this is

$$h^{ab}_{\rm TT} = ({\rm TT})^{ab}{}_{cd} h^{cd}, \tag{7.3.3}$$

in which the TT projector

$$({\rm TT})^{ab}{}_{cd} = P^a{}_c P^b{}_d - \frac{1}{2}P^{ab}P_{cd}, \tag{7.3.4}$$

with

$$P^a{}_b = \delta^a{}_b - \Omega^a\Omega_b, \tag{7.3.5}$$

was first introduced in Sec. 6.1.6. To calculate the energy flux, at this order of accuracy, we must substitute Eq. (7.3.1) into Eq. (7.3.3), and that into Eq. (7.1.4).

Using the properties $P^a{}_c P^c{}_b = P^a{}_b$ and $P^a{}_a = 2$ of the transverse projector, it is easy to show that

$$\dot{h}^{\rm TT}_{ab}\dot{h}^{ab}_{\rm TT} = \left(P_{ac}P_{bd} - \frac{1}{2}P_{ab}P_{cd}\right)\dot{h}^{ab}\dot{h}^{cd}.$$

This becomes

$$\begin{aligned}\dot{h}^{\rm TT}_{ab}\dot{h}^{ab}_{\rm TT} \;=\; & \frac{4G^2}{c^8 r^2}\biggl(\delta_{ac}\delta_{bd} - \frac{1}{2}\delta_{ab}\delta_{cd} - \delta_{ac}\Omega_b\Omega_d - \delta_{bd}\Omega_a\Omega_c \\ & + \frac{1}{2}\delta_{ab}\Omega_c\Omega_d + \frac{1}{2}\delta_{cd}\Omega_a\Omega_b + \frac{1}{2}\Omega_a\Omega_b\Omega_c\Omega_d\biggr) I^{ab(3)} I^{cd(3)}\end{aligned}$$

after substitution of Eqs. (7.3.1) and (7.3.5).

Putting this into Eq. (7.1.4),

$$\dot{E}_{\rm gw} = \frac{c^3}{32\pi G}\int r^2 \dot{h}^{\rm TT}_{ab}\dot{h}^{ab}_{\rm TT}\, d\Omega,$$

we find that the gravitational-wave luminosity is given by

$$\begin{aligned}\dot{E}_{\rm gw} \;=\; & \frac{G}{2c^5}\biggl(\delta_{ac}\delta_{bd} - \frac{1}{2}\delta_{ab}\delta_{cd} - \delta_{ac}\langle\!\langle \Omega_b\Omega_d\rangle\!\rangle - \delta_{bd}\langle\!\langle \Omega_a\Omega_c\rangle\!\rangle \\ & + \frac{1}{2}\delta_{ab}\langle\!\langle \Omega_c\Omega_d\rangle\!\rangle + \frac{1}{2}\delta_{cd}\langle\!\langle \Omega_a\Omega_b\rangle\!\rangle + \frac{1}{2}\langle\!\langle \Omega_a\Omega_b\Omega_c\Omega_d\rangle\!\rangle\biggr) I^{ab(3)} I^{cd(3)},\end{aligned}$$

in which $\langle\!\langle \cdots \rangle\!\rangle := (4\pi)^{-1}\int(\cdots)\, d\Omega$ indicates an angular average. Importing the relevant results from Sec. 1.8.4, we eventually arrive at

$$\dot{E}_{\rm gw} = \frac{1}{5}\frac{G}{c^5}\Bigl(I^{ab(3)}I^{(3)}_{ab} - \frac{1}{3}I^{(3)\,2}\Bigr), \tag{7.3.6}$$

which expresses $\dot{E}_{\rm gw}$ in terms of the third derivative of the quadrupole moment with respect to retarded-time $\tau := t - r/c$.

An alternative expression involves

$$I^{\langle ab\rangle} := I^{ab} - \frac{1}{3}\delta^{ab}I, \qquad I := \delta_{ab}I^{ab} \tag{7.3.7}$$

the tracefree version of the Newtonian quadrupole moment. It is easy to show that $I^{\langle ab\rangle}I_{\langle ab\rangle} = I^{ab}I_{ab} - \frac{1}{3}I^2$, so that Eq. (7.3.6) can also be written as

$$\dot{E}_{\rm gw} = \frac{1}{5}\frac{G}{c^5}I^{\langle ab\rangle(3)}I^{(3)}_{\langle ab\rangle}. \tag{7.3.8}$$

Equation (7.3.6), or its alternate form of Eq. (7.3.8), is the well-known quadrupole formula of gravitational-wave physics.

To illustrate the content of the quadrupole formula, we apply it to a Newtonian two-body system. Working in the reference frame of the barycentre, we have from Eq. (6.11.13) that

$$I^{ab} = m\eta z^a z^b, \tag{7.3.9}$$

where $m := m_1 + m_2$ is the total mass, $\eta := m_1 m_2/m^2$ is the dimensionless reduced mass, and $\boldsymbol{z} := \boldsymbol{z}_1 - \boldsymbol{z}_2$ is the relative position vector. The Newtonian relative acceleration is

$$\boldsymbol{a} = -\frac{Gm}{z^2}\boldsymbol{n}, \tag{7.3.10}$$

in which $z := |\boldsymbol{z}|$ and $\boldsymbol{n} := \boldsymbol{z}/z$. After differentiating three times with the help of Eq. (7.3.10), Eq. (7.3.9) gives

$$I^{ab(3)} = \frac{2Gm^2\eta}{z^2}\Bigl[-2\bigl(v^a n^b + n^a v^b\bigr) + 3\dot{z}n^a n^b\Bigr],$$

where $\boldsymbol{v} := \boldsymbol{v}_1 - \boldsymbol{v}_2$ is the relative velocity vector, and $\dot{z} = \boldsymbol{n}\cdot\boldsymbol{v}$ is the radial velocity. Substitution into Eq. (7.3.6) gives

$$\dot{E}_{\rm gw} = \frac{8}{15}\frac{G}{c^5}\frac{(m\eta)^2(Gm)^2}{z^4}\bigl(12v^2 - 11\dot{z}^2\bigr). \tag{7.3.11}$$

This is the quadrupole formula applied to any Newtonian two-body system.

For circular orbits we have $\dot{z} = 0$ and $v^2 = Gm/z$, so that Eq. (7.3.11) becomes

$$\dot{E}_{\rm gw} = \frac{32\eta^2}{5}\frac{c^5}{G}(v/c)^{10}. \tag{7.3.12}$$

The Newtonian orbital energy is $E = -Gm^2\eta/(2z)$, and the relation $\dot{E} = -\dot{E}_{\rm gw}$ implies that the orbital radius must decrease according to

$$\dot{z} = -\frac{64\eta}{5}\frac{G^3m^3}{c^5z^3}. \tag{7.3.13}$$

This equation can easily be integrated for $z(t)$. It implies that the orbital velocity v, and the angular velocity $\omega = \sqrt{Gm/z^3}$, increase with time. This reaction of the orbital motion to the emission of gravitational waves will be examined more closely in Secs. 7.4 and 7.5.

7.3.2 Post-Newtonian circular orbits

In this subsection we use Eqs. (7.1.5) or (7.2.16) to calculate the energy radiated by a two-body system in a post-Newtonian circular orbit. The gravitational-wave polarizations were obtained in Sec. 6.12.4, and according to Eqs. (6.12.17) and (6.12.18), we have that

$$h_+ = \frac{2Gm\eta}{c^2 r} x H_+, \qquad h_\times = \frac{2Gm\eta}{c^2 r} x H_\times, \tag{7.3.14}$$

where

$$x := \left(\frac{Gm\omega}{c^3}\right)^{2/3} \tag{7.3.15}$$

is the post-Newtonian expansion parameter, written in terms of the total mass $m := m_1 + m_2$ and the orbital angular velocity ω. The functions H_+ and $H_\times$ admit post-Newtonian expansions of the form

$$H = H^{[0]} + \Delta x^{1/2} H^{[1/2]} + x H_+^{[1]} + \Delta x^{3/2} H^{[3/2]} + x^{3/2} H^{\rm tail} + O(x^2), \tag{7.3.16}$$

where $\Delta := (m_1 - m_2)/m^2$, $\eta := m_1 m_2/m^2$, and the various terms are listed in Eqs. (6.12.19)–(6.12.28). These depend on θ via $C := \cos\theta$ and $S := \sin\theta$, and they depend on τ and ϕ through the phase variable $\Psi := \omega\tau - \phi$.

Differentiation of h_+ and $h_\times$ with respect to τ involves differentiating H_+ and $H_\times$ with respect to Ψ, and we indicate this with a prime. After squaring, we get something of the form

$$\dot{h}^2 = \frac{4(Gm\eta)^2}{c^4 r^2} \omega^2 x^2 H'^2.$$

Using Eq. (7.3.15) we express $\omega^2 x^2$ as $c^6 x^5/(Gm)^2$ and rewrite the previous expression as

$$\dot{h}^2 = \frac{4c^2\eta^2}{r^2} x^5 H'^2.$$

Substitution into Eq. (7.2.16) gives

$$\dot{E}_{\rm gw} = \frac{c^5}{4\pi G}\eta^2 x^5 \int \Big[\big(H'_+\big)^2 + \big(H'_\times\big)^2 \Big] \sin\theta d\theta\, d\Psi, \tag{7.3.17}$$

where we have replaced an integration with respect to ϕ by an integration with respect to the phase variable Ψ.

The computation of H'_+ and $H'_\times$ and the evaluation of the integral is tedious but straightforward. After expanding the result in powers of x and eliminating Δ^2 in favour of $1 - 4\eta$, we obtain

$$\dot{E}_{\rm gw} = \frac{32}{5}\frac{c^5}{G}\eta^2 x^5 \left[1 - \left(\frac{1247}{336} + \frac{35}{12}\eta \right) x + 4\pi x^{3/2} + O(x^2) \right]. \tag{7.3.18}$$

We observe that $\dot{E}_{\rm gw}$ contains no correction term at order $x^{1/2}$, in spite of the fact that the gravitational-wave polarizations do possess such terms; a $\frac{1}{2}$PN correction to the energy flux would have to come from an interaction between the $H^{[0]}$ and $H^{[1/2]}$ terms within H, but because these signals are out of phase, the interaction produces no flux. We observe also that $\dot{E}_{\rm gw}$ contains a term at order $x^{3/2}$, and such a term has three possible origins. First, it might have originated from an interaction between $H^{[0]}$ and $H^{[3/2]}$, but this produces no flux because these signals also are out of phase. Second, it might have originated from an interaction between $H^{[1]}$ and $H^{[1/2]}$, but this does not contribute for the same reason. The only remaining possibility is an interaction between $H^{[0]}$ and $H^{\rm tail}$; these signals are in phase, and their interaction does indeed contribute to the energy flux. The $4\pi x^{3/2}$ term within

Eq. (7.3.16) has its origin in the tail effect; it is a *wave-propagation correction* to the Newtonian expression that appears outside the large square brackets.

From Eqs. (6.12.6) and (6.12.9) we find that the orbital energy of the two-body system is equal to

$$E_{\text{orbital}} = -\frac{1}{2} m\eta c^2 x \left[1 - \frac{1}{12}(9+\eta)x + O(x^2)\right]. \tag{7.3.19}$$

This, we recall, includes the kinetic and gravitational potential energies, but excludes the rest-mass energy of each body. The orbital energy must decrease according to $\dot{E}_{\text{orbital}} = -\dot{E}_{\text{gw}}$, and inserting Eqs. (7.3.18) and (7.3.19), we obtain a differential equation for the post-Newtonian parameter x:

$$\dot{x} = \frac{64}{5}\frac{c^3}{G}\frac{\eta}{m} x^5 \left[1 - \left(\frac{743}{336} + \frac{11}{4}\eta\right)x + 4\pi x^{3/2} + O(x^2)\right]. \tag{7.3.20}$$

This equation governs how the angular velocity ω increases with time; it describes the reaction of the orbital motion to the emission of gravitational waves. Notice, however, and this was already pointed out in Sec. 6.12.1, that this *radiation reaction* is not incorporated in the 1PN equations of motion that were involved in the derivation of Eqs. (7.3.18) and (7.3.19). This is an effect of higher post-Newtonian order — $\frac{2}{5}$PN order to be precise — whose existence is (plausibly, but not rigourously) inferred on the basis of the statement of energy balance, $\dot{E}_{\text{orbital}} = -\dot{E}_{\text{gw}}$. A calculation at higher order is required to confirm the result of Eq. (7.3.20), and this shall be our focus in the following two sections.

7.4 Radiation-reaction potentials

7.4.1 Introduction

As we have seen, a system of N bodies moving under their mutual gravitational attraction emits gravitational waves, and these waves carry energy away from the system. It is physically imperative that the system respond to this loss of energy, and the equations of motion should contain terms that account for the effect. There should therefore exist a *radiation-reaction force* that does work on each body within the system and dissipates a fraction of its energy; the rate at which these forces do work should be equal to the rate at which the gravitational waves remove energy from the system. Our purpose in this section and the next is to calculate the post-Newtonian radiation-reaction force. Equation (7.3.8) indicates that $\dot{E}_{\text{gw}}$ scales as c^{-5} to leading order in a post-Newtonian expansion, and we expect that the radiation-reaction force also should scale as c^{-5}. This, then, will make a term of $\frac{5}{2}$PN order in the system's equations of motion. Recall that the equations of motion were calculated at 0PN and 1PN order in Chapter 5; there is no term at $\frac{3}{2}$PN order, and we shall bypass a calculation of the 2PN corrections in order to focus on the radiation-reaction term at $\frac{5}{2}$PN order.

It is appropriate that the radiation-reaction force, which causes energy dissipation within the system, would scale as an *odd power* of c^{-1}; this is in contrast with lower-order terms, which are conservative and scale as *even powers* of c^{-1}. This behaviour can be understood as follows.

We have seen that the Einstein field equations can be cast in the form of

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}, \tag{7.4.1}$$

a wave equation for the gravitational potentials $h^{\alpha\beta}$, and that post-Newtonian theory is based on an iterative solution to this equation. It is appropriate to select the

retarded solution to the wave equation, to correctly enforce the notion that cause should precede effect. It is as a result of this choice that the gravitational waves are outgoing, and that they carry energy out to infinity. Mathematically, we see that the wave field depends on retarded-time $t - r/c$, and that $\dot{E}_{\rm gw}$ is a positive quantity that scales as c^{-5}. And finally, we infer that there will be a radiation-reaction force that drives a decrease in the system's energy, so that global energy conservation is maintained; this force also will scale as c^{-5}.

Suppose now that instead of the retarded solution, we incorrectly select the *advanced solution* to the wave equation. We would now find that the waves are incoming instead of outgoing, and that they bring energy to the system instead of taking it away. Mathematically we would see that the wave field depends on advanced-time $t + r/c$, and that $\dot{E}_{\rm gw}$ — the outward flux of gravitational-wave energy — is a negative quantity that scales as c^{-5}. And finally, we would infer that in this situation, the radiation-reaction force should drive an increase in the system's energy, which would match the energy input provided by the waves; the force would still scale as c^{-5}, but it would now come with the opposite sign.

The incorrect solution (advanced potentials, incoming waves, inward flux of gravitational-wave energy, and increase of system's energy) is obtained from the correct solution (retarded potentials, outgoing waves, outward flux of gravitational-wave energy, and decrease of system's energy) simply by reversing the sign of c^{-1}. This reversal must change the sign of the radiation-reaction force, and it follows directly that this force must scale as an *odd power* of c^{-1}. On the other hand, the conservative terms in the equations of motion are not sensitive to the choice of boundary conditions (retarded versus advanced), and they therefore scale as an *even power* of c^{-1}. The radiation-reaction force must therefore be associated with a fractional post-Newtonian order, and as we have seen, it first makes an appearance at $\frac{5}{2}$PN order.

In this section we construct the gravitational potentials that are required in the evaluation of the radiation-reaction force; these necessarily come with an odd power of c^{-1}, and they are easily identified. In the following section we will involve these potentials in a calculation of the equations of motion at $\frac{5}{2}$PN order, skipping 0PN and 1PN orders (which were handled previously in Chapter 5) and bypassing 2PN order (which would require many additional computations).

Before we proceed it is useful briefly to review the situation in flat-spacetime electrodynamics. (We consider the slow-motion limit, and ignore all relativistic effects.) It is well known that the radiation-reaction force acting on a point particle of electric charge q is given by $\boldsymbol{F}_{\rm rr} = kq^2\dot{\boldsymbol{a}}$, where $k^{-1} := 6\pi\epsilon_0 c^3$, and $\boldsymbol{a}$ is the particle's acceleration vector. Notice that as we might expect, the force scales as an odd power of c^{-1}. As the charge moves with velocity $\boldsymbol{v}$, the force does work at a rate $\dot{W} = \boldsymbol{F}_{\rm rr}\cdot\boldsymbol{v} = kq^2\dot{\boldsymbol{a}}\cdot\boldsymbol{v}$. We next write $\dot{\boldsymbol{a}}\cdot\boldsymbol{v} = d(\boldsymbol{a}\cdot\boldsymbol{v})/dt - |\boldsymbol{a}|^2$ and obtain the fine-grained conservation statement

$$\dot{W} + \dot{E}_{\rm waves} = -\frac{d}{dt}E_{\rm bound},$$

where $\dot{E}_{\rm waves} := kq^2|\boldsymbol{a}|^2$ is the rate at which the electromagnetic waves carry energy to infinity, as calculated in the electric dipole approximation. We also have introduced $E_{\rm bound} := -kq^2\boldsymbol{a}\cdot\boldsymbol{v}$ as the piece of the electromagnetic field energy that stays bound to the particle. Averaging over a time interval Δt produces a coarse-grained statement of energy conservation:

$$\langle\dot{W}\rangle + \langle\dot{E}_{\rm waves}\rangle = -\frac{\Delta E_{\rm bound}}{\Delta t},$$

where $\Delta E_{\rm bound}$ is the net change in $E_{\rm bound}$ during the time interval. In situations in which the motion is periodic with period Δt, or when it begins and ends with a

vanishing acceleration, we find that $\Delta E_{\rm bound} = 0$ and the conservation statement becomes

$$\langle \dot{W} \rangle = -\langle \dot{E}_{\rm waves} \rangle.$$

Under these conditions, and in a coarse-grained sense, we have energy balance: The work done by the radiation-reaction force matches the energy taken away by the electromagnetic waves. We recall that coarse-graining was an essential aspect of the Bondi-Sachs derivation of the energy lost by radiating sources, and we should anticipate that coarse-graining will be involved also in a statement of gravitational energy balance (to be written down in Sec. 7.5.8). It is interesting to find that coarse-graining plays an important role even in the relatively mundane context of flat-spacetime electrodynamics.

7.4.2 Post-Newtonian expansion of the potentials in the near zone

We introduce the notation

$$h^{00} := \frac{4}{c^2} V, \qquad h^{0a} := \frac{4}{c^3} V^a, \qquad h^{ab} := \frac{4}{c^4} W^{ab}, \tag{7.4.2}$$

as well as

$$\tau^{00} := \rho c^2, \qquad \tau^{0a} := j^a c, \qquad \tau^{ab} = \tau^{ab}, \tag{7.4.3}$$

and we write the wave equation of Eq. (7.4.1) as the set

$$\Box V = -4\pi G \rho, \qquad \Box V^a = -4\pi G j^a, \qquad \Box W^{ab} = -4\pi G \tau^{ab}. \tag{7.4.4}$$

A method to integrate the wave equation was developed in Chapter 2. The solution in the near zone is written as an integral over the past light cone of the field point x, which is decomposed into contributions from a near-zone domain $\mathscr{N}$ and a complementary wave-zone domain $\mathscr{W}$. It was shown in Sec. 4.2.8 that in the near zone, $h^{\alpha\beta}_{\mathscr{W}}$ first appears at 3PN order, and because our considerations in this section are limited to the $\frac{5}{2}$PN order, it makes no contribution to our near-zone potentials. An expression for $h^{\alpha\beta}_{\mathscr{N}}$ can be found in Sec. 2.4.2, and Eq. (2.4.7) reveals that this takes the form of an expansion in powers of c^{-1}. Explicitly, and to a sufficient degree of accuracy, we have

$$\begin{aligned}
V &= G\biggl[\int \frac{\rho}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x' - \frac{1}{c}\frac{\partial}{\partial t}\int \rho\, d^3x' + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int \rho|\boldsymbol{x}-\boldsymbol{x'}|\, d^3x' \\
&\quad - \frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int \rho|\boldsymbol{x}-\boldsymbol{x'}|^2\, d^3x' + \frac{1}{24c^4}\frac{\partial^4}{\partial t^4}\int \rho|\boldsymbol{x}-\boldsymbol{x'}|^3\, d^3x' \\
&\quad - \frac{1}{120c^5}\frac{\partial^5}{\partial t^5}\int \rho|\boldsymbol{x}-\boldsymbol{x'}|^4\, d^3x' + O(c^{-6})\biggr],
\end{aligned}\tag{7.4.5}$$

$$\begin{aligned}
V^a &= G\biggl[\int \frac{j^a}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x' - \frac{1}{c}\frac{\partial}{\partial t}\int j^a\, d^3x' + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int j^a|\boldsymbol{x}-\boldsymbol{x'}|\, d^3x' \\
&\quad - \frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int j^a|\boldsymbol{x}-\boldsymbol{x'}|^2\, d^3x' + O(c^{-4})\biggr],
\end{aligned}\tag{7.4.6}$$

$$\begin{aligned}
W^{ab} &= G\biggl[\int \frac{\tau^{ab}}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x' - \frac{1}{c}\frac{\partial}{\partial t}\int \tau^{ab}\, d^3x' + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int \tau^{ab}|\boldsymbol{x}-\boldsymbol{x'}|\, d^3x' \\
&\quad - \frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int \tau^{ab}|\boldsymbol{x}-\boldsymbol{x'}|^2\, d^3x' + O(c^{-4})\biggr].
\end{aligned}\tag{7.4.7}$$

In each integral $\tau^{\alpha\beta}$ is expressed as a function of t and $\boldsymbol{x'}$, and the integration is over the near-zone domain $\mathscr{M}$ defined by $r' := |\boldsymbol{x'}| < \mathcal{R}$, where $\mathcal{R}$ is the arbitrary cutoff

radius between the near zone and the wave zone. As usual we shall be interested in the $\mathcal{R}$-independent pieces of the potentials.

The terms that come with an *odd power* of c^{-1} in Eqs. (7.4.5)–(7.4.7) shall be the focus of our attention. We shall have to be careful and keep in mind that while some of the factors of c^{-1} appear explicitly in these equations, some are contained implicitly in the source functions ρ, j^a, and τ^{ab}, which are constructed partly from the potentials. To account for the complete (explicit and implicit) dependence on powers of c^{-1}, we write

$$\begin{aligned} V &= V[0] + c^{-2}V[2] + c^{-4}V[4] + O(c^{-6}) \\ &\quad + c^{-1}V[1] + c^{-3}V[3] + c^{-5}V[5] + O(c^{-7}), \end{aligned} \tag{7.4.8}$$

$$\begin{aligned} V^a &= V^a[0] + c^{-2}V^a[2] + O(c^{-4}) \\ &\quad + c^{-1}V^a[1] + c^{-3}V^a[3] + O(c^{-5}), \end{aligned} \tag{7.4.9}$$

$$\begin{aligned} W^{ab} &= W^{ab}[0] + c^{-2}W^{ab}[2] + O(c^{-4}) \\ &\quad + c^{-1}W^{ab}[1] + c^{-3}W^{ab}[3] + O(c^{-5}). \end{aligned} \tag{7.4.10}$$

Here, for example, $V[5]$ includes a contribution from the last term in Eq. (7.4.5), in which we would substitute $\rho = \rho[0]$, but it includes also a contribution from the first term, in which we would substitute $\rho = c^{-5}\rho[5]$. The dependence of the source terms on powers of c^{-1} will be revealed in due course.

7.4.3 Multipole moments and conservation identities

To help with the evaluation of the potentials we introduce a number of multipole moments and link them with a number of identities. These are a consequence of the conservation equations

$$\partial_t \rho + \partial_a j^a = 0, \qquad \partial_t j^a + \partial_b \tau^{ab} = 0, \tag{7.4.11}$$

which follow directly from $\partial_\beta \tau^{\alpha\beta} = 0$ after involving the definitions of Eq. (7.4.3). The discussion here follows closely the developments of Sec. 3.3.1, except that those applied to the wave zone instead of the near zone.

We define

$$\mathcal{I} := \int \rho \, d^3x, \tag{7.4.12}$$

$$\mathcal{I}^a := \int \rho \, x^a \, d^3x, \tag{7.4.13}$$

$$\mathcal{I}^{ab} := \int \rho \, x^a x^b \, d^3x, \tag{7.4.14}$$

$$\mathcal{I}^{abc} := \int \rho \, x^a x^b x^c \, d^3x, \tag{7.4.15}$$

$$\mathcal{P}^a := \int j^a \, d^3x, \tag{7.4.16}$$

$$\mathcal{P}^{ab} := \int j^a x^b \, d^3x, \tag{7.4.17}$$

$$\mathcal{P}^{abc} := \int j^a x^b x^c \, d^3x, \tag{7.4.18}$$

$$\mathcal{J}^{ab} := \int \big(j^a x^b - j^b x^a\big) \, d^3x, \tag{7.4.19}$$

$$\mathcal{J}^{abc} := \int \big(j^a x^b - j^b x^a\big) x^c \, d^3x, \tag{7.4.20}$$

$$\mathcal{M}^{ab} := \int \tau^{ab} \, d^3x, \tag{7.4.21}$$

$$\mathcal{M}^{abc} := \int \tau^{ab} x^c \, d^3x, \tag{7.4.22}$$

$$\mathcal{M}^{abcd} := \int \tau^{ab} x^c x^d \, d^3x, \tag{7.4.23}$$

in which the sources are expressed as functions of t and $\boldsymbol{x}$, and the integrations are over the domain $\mathcal{M}$ now described by $r := \boldsymbol{x} < \mathcal{R}$; the multipole moments are functions of t only. At the Newtonian order we have that $\rho = \sum_A m_A \delta(\boldsymbol{x} - \boldsymbol{z}_A)$, and $\mathcal{I}$ reduces to the total mass $m := \sum_A m_A$, $\mathcal{I}^a$ reduces $mZ^a := \sum_A m_A \boldsymbol{z}_A^a$, and $\mathcal{I}^{ab}$ reduces to the Newtonian quadrupole moment $I^{ab} := \sum_A m_A z_A^a z_A^b$. Similarly, at the Newtonian order $\mathcal{P}^a$ reduces to the total momentum $P^a := \sum_A m_A v_A^a$ and $\mathcal{J}^{ab}$ reduces to the angular-momentum tensor $J^{ab} := \sum_A m_A (v_A^a z_A^b - z_A^a v_A^a)$.

From the conservation identities of Eq. (1.4.1)–(1.4.4) we deduce that

$$\dot{\mathcal{I}} = -\oint j^a \, dS_a, \tag{7.4.24}$$

$$\dot{\mathcal{I}}^a = \mathcal{P}^a - \oint j^b x^a \, dS_b, \tag{7.4.25}$$

$$\dot{\mathcal{P}}^a = -\oint \tau^{ab} \, dS_b, \tag{7.4.26}$$

$$\dot{\mathcal{J}}^{ab} = -\oint \bigl(\tau^{ac} x^b - \tau^{bc} x^a\bigr) \, dS_c, \tag{7.4.27}$$

$$\mathcal{P}^{ab} = \frac{1}{2}\bigl(\dot{I}^{ab} + J^{ab}\bigr) + \frac{1}{2}\oint j^c x^a x^b \, dS_c, \tag{7.4.28}$$

$$\mathcal{P}^{abc} = \frac{1}{3}\bigl(\dot{I}^{abc} + J^{abc} + J^{acb}\bigr) + \frac{1}{3}\oint j^d x^a x^b x^c \, dS_d, \tag{7.4.29}$$

$$\mathcal{M}^{ab} = \frac{1}{2}\ddot{I}^{ab} + \frac{1}{2}\oint \bigl(\tau^{ac} x^b + \tau^{bc} x^a - \partial_d \tau^{cd} x^a x^b\bigr) \, dS_c, \tag{7.4.30}$$

$$\begin{aligned}\mathcal{M}^{abc} = {} & \frac{1}{6}\ddot{I}^{abc} + \frac{1}{3}\bigl(\dot{J}^{acb} + \dot{J}^{bca}\bigr) + \frac{1}{6}\frac{\partial}{\partial t}\oint j^d x^a x^b x^c \, dS_d \\ & + \frac{1}{2}\oint \bigl(\tau^{ad} x^b x^c + \tau^{bd} x^a x^c - \tau^{cd} x^a x^b\bigr) \, dS_d,\end{aligned} \tag{7.4.31}$$

where the surface integrals are evaluated on $r = \mathcal{R}$, and where an overdot indicates differentiation with respect to t. The derivation of these identities is straightforward, and it follows the general strategy outlined in Sec. 3.3.1. For example, Eq. (7.4.25) follows from $j^a = \partial_t(\rho x^a) + \partial_b(j^b x^a)$, which is a direct consequence of the first of Eqs. (7.4.11).

One major difference with respect to the developments of Sec. 3.3.1 concerns the boundary terms. These were not present in the earlier treatment, because the source functions were constructed entirely from the material energy-momentum tensor, which has its support in a small region deep within the near zone. Here the source functions contain contributions from the potentials, and these do not vanish at $r = \mathcal{R}$. The boundary terms must be carefully evaluated, but we assert that *at all post-Newtonian orders to be considered within this section, the boundary terms contain no $\mathcal{R}$-independent pieces, and they can be safely discarded.* (We shall not prove this assertion, but you may be comforted with the recollection that each boundary integral evaluated in Chapter 6 was shown to make no $\mathcal{R}$-independent contribution to the final result.)

We may set

$$\mathcal{I}^a = 0 = \mathcal{P}^a \tag{7.4.32}$$

by placing the origin of the coordinate system at barycentre, and Eqs. (7.4.25) and (7.4.26) guarantee that these conditions can be imposed at all times.

The conservation identities allow us to simplify the expression of the odd terms that appear in Eqs. (7.4.5)–(7.4.7). First, Eqs. (7.4.24) and (7.4.26) imply immediately that the terms of order c^{-1} vanish in Eqs. (7.4.5) and (7.4.6). Second, expanding $|\boldsymbol{x}-\boldsymbol{x'}|^2$ as $r^2 - 2\boldsymbol{x}\cdot\boldsymbol{x'} + r'^2$ and involving the definition of the multipole moments reveals that

$$\begin{aligned}
\int \rho|\boldsymbol{x}-\boldsymbol{x'}|^2\,d^3x' &= r^2\mathcal{I} - 2x^a\mathcal{I}_a + \mathcal{I}^c{}_c,\\
\int \rho|\boldsymbol{x}-\boldsymbol{x'}|^4\,d^3x' &= r^4\mathcal{I} - 4r^2x^a\mathcal{I}_a + 2r^2\mathcal{I}^c{}_c + 4x^ax^b\mathcal{I}_{ab} - 4x^a\mathcal{I}_a{}^c{}_c + \mathcal{I}^{cd}{}_{cd},\\
\int j^a|\boldsymbol{x}-\boldsymbol{x'}|^2\,d^3x' &= r^2\mathcal{P}^a - 2x^b\mathcal{P}^a{}_b + \mathcal{P}^{ac}{}_c,\\
\int \tau^{ab}\,d^3x' &= \mathcal{M}^{ab},\\
\int \tau^{ab}|\boldsymbol{x}-\boldsymbol{x'}|^2\,d^3x' &= r^2\mathcal{M}^{ab} - 2x^c\mathcal{M}^{ab}{}_c + \mathcal{M}^{abc}{}_c,
\end{aligned}$$

These relations become

$$\begin{aligned}
\int \rho|\boldsymbol{x}-\boldsymbol{x'}|^2\,d^3x' &= \mathcal{I}^c{}_c,\\
\int \rho|\boldsymbol{x}-\boldsymbol{x'}|^4\,d^3x' &= 2\big(r^2\delta^{ab} + 2x^ax^b\big)\mathcal{I}_{ab} - 4x^a\mathcal{I}^{ac}{}_c + \mathcal{I}^{cd}{}_{cd},\\
\int j^a|\boldsymbol{x}-\boldsymbol{x'}|^2\,d^3x' &= -x^b\dot{\mathcal{I}}^a{}_b + \frac{1}{3}\dot{\mathcal{I}}^{ac}{}_c + \frac{2}{3}\mathcal{J}^{ac}{}_c,\\
\int \tau^{ab}\,d^3x' &= \frac{1}{2}\ddot{\mathcal{I}}^{ab},\\
\int \tau^{ab}|\boldsymbol{x}-\boldsymbol{x'}|^2\,d^3x' &= \frac{1}{2}r^2\ddot{\mathcal{I}}^{ab} - \frac{1}{3}x^c\ddot{\mathcal{I}}^{ab}{}_c - \frac{2}{3}x^c\big(\dot{\mathcal{J}}^a{}_c{}^b + \dot{\mathcal{J}}^b{}_c{}^a\big) + \mathcal{M}^{abc}{}_c,
\end{aligned}$$

after involving Eq. (7.4.32) and the conservation identities, and discarding terms that will vanish after differentiation with respect to t.

After taking all this into account, Eqs. (7.4.5)–(7.4.7) become

$$\begin{aligned}
V &= G\bigg\{\int \frac{\rho}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x' + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int \rho|\boldsymbol{x}-\boldsymbol{x'}|\,d^3x' + O(c^{-4})\\
&\quad - \frac{1}{6c^3}\mathcal{I}^{(3)}_{cc} - \frac{1}{120c^5}\Big[2\big(r^2\delta^{ab} + 2x^ax^b\big)\mathcal{I}^{(5)}_{ab} - 4x^a\mathcal{I}^{(5)}_{acc} + \mathcal{I}^{(5)}_{cdcd}\\
&\quad + O(c^{-7})\bigg\}, \qquad (7.4.33)\\
V^a &= G\bigg\{\int \frac{j^a}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x' + O(c^{-2})\\
&\quad + \frac{1}{18c^3}\Big[3x^b\mathcal{I}^{a(4)}_{\ b} - \mathcal{I}^{a(4)}_{cc} - 2\mathcal{J}^{a(3)}_{cc}\Big] + O(c^{-5})\bigg\}, \qquad (7.4.34)\\
W^{ab} &= G\bigg\{\int \frac{\tau^{ab}}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x' + O(c^{-2})\\
&\quad - \frac{1}{2c}\mathcal{I}^{ab(3)} - \frac{1}{36c^3}\Big[3r^2\mathcal{I}^{ab(5)} - 2x^c\mathcal{I}^{ab(5)}_{\ \ c} - 4x^c\big(\mathcal{J}^a{}_c{}^{b(4)} + \mathcal{J}^b{}_c{}^{a(4)}\big)\\
&\quad + 6\mathcal{M}^{ab(3)}_{cc}\Big] + O(c^{-5})\bigg\}, \qquad (7.4.35)
\end{aligned}$$

where a number within brackets indicates the number of differentiations with respect to t.

As a consequence of the fact that $\mathcal{I}$ is a conserved quantity, Eq. (7.4.33) does not contain a term that scales explicitly as c^{-1}. And because ρ cannot contain a c^{-1} term, there is no implicit dependence, and we conclude that

$$V[1] = 0. \tag{7.4.36}$$

Similarly, $\mathcal{P}^a$ is a conserved quantity, and Eq. (7.4.34) does not contain an explicit term at order c^{-1}. Because j^a cannot contain a c^{-1} term, we conclude that

$$V^a[1] = 0. \tag{7.4.37}$$

7.4.4 Odd terms in the effective energy-momentum tensor

To proceed we must identify the orders at which odd powers of c^{-1} appear within the source functions ρ, j^a, and τ^{ab}. We recall from Eq. (1.3.5) that the effective energy-momentum tensor is expressed as

$$\tau^{\alpha\beta} = (-g)\bigl(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL} + t^{\alpha\beta}_{\rm H}\bigr), \tag{7.4.38}$$

in terms of the material energy-momentum tensor $T^{\alpha\beta}$, the Landau-Lifshitz pseudotensor of Eq. (1.1.5), and the harmonic-gauge contribution of Eq. (1.3.6).

We begin with an examination of the material contribution. The energy-momentum tensor of a system of N point masses is given by Eq. (4.1.3),

$$(-g)T^{\alpha\beta} = \sum_A m_A v^\alpha_A v^\beta_A \,\Gamma\, \delta\bigl(\boldsymbol{x} - \boldsymbol{z}_A\bigr), \tag{7.4.39}$$

where $v^\alpha_A = (c, \boldsymbol{v}_A)$ and the relativistic factor Γ is defined by

$$\Gamma := \frac{\sqrt{-g}}{\sqrt{-g_{\mu\nu} v^\mu_A v^\nu_A / c^2}}. \tag{7.4.40}$$

The odd terms will be contained in Γ, and to calculate this we must first obtain the metric from the gravitational potentials. It is sufficient to work at linear order in $h^{\alpha\beta}$, and according to Eqs. (1.6.4) and (1.6.6), we have $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2} h \eta_{\alpha\beta}$ and $\sqrt{-g} = 1 - \frac{1}{2}h$, where $h = \eta_{\alpha\beta} h^{\alpha\beta}$. After involving Eqs. (7.4.2) and (7.4.8)–(7.4.10), we obtain

$$\begin{aligned}
\sqrt{-g} &= 1 + (\text{even}) + \frac{2}{c^5}\bigl(V[3] - W[1]\bigr) + O(c^{-7}), \\
g_{00} &= -1 + (\text{even}) + \frac{2}{c^5}\bigl(V[3] + W[1]\bigr) + O(c^{-7}), \\
g_{ab} &= \delta_{ab} + (\text{even}) + O(c^{-5}),
\end{aligned}$$

where (even) designates terms of order c^{-2}, c^{-4}, and so on, and $W[1] := \delta_{ab} W^{ab}[1]$. To arrive at these results we have set $V[1] = 0$ according to Eq. (7.4.36), and as a consequence of Eq. (7.4.37), we find that the odd terms in g_{0a} first appear at order c^{-6} and can be neglected. We next obtain

$$\Gamma = 1 + (\text{even}) + \frac{1}{c^5}\bigl(3V[3] - W[1]\bigr) + O(c^{-7}), \tag{7.4.41}$$

and we conclude that odd terms first appear at order c^{-5} within the material energy-momentum tensor. (This conclusion will require revision. We shall find that in actual fact, $3V[3] - W[1] = 0$, so that the first odd term in Γ appears at order c^{-7}.)

We next examine the Landau-Lifshitz pseudotensor. We shall not go through a detailed computation here (this is postponed until Sec. 7.5.2), but merely determine

the expected scaling of the leading odd terms. Noticing that $(-g)t_{\rm LL}^{\alpha\beta}$ is at least quadratic in the gravitational potentials, we observe that the leading odd terms in $(h^{00})^2$ and $h^{00}h^{ab}$ scale as c^{-7}, and that other products come with additional powers of c^{-1}; inserting these scalings within Eq. (1.1.5), we find that the leading odd terms in $(-g)t_{\rm LL}^{00}$ and $(-g)t_{\rm LL}^{ab}$ scale as c^{-3}, while the leading odd term in $(-g)t_{\rm LL}^{0a}$ scales as c^{-4}. The conclusion of this simple exercise is that the Landau-Lifshitz pseudotensor is expected to make an odd contribution to ρ at order c^{-5}, to j^a at order c^{-5}, and to τ^{ab} at order c^{-3}. (These conclusions will require revision. A closer examination will reveal that in actual fact, the pseudotensor contributes to ρ and j^a at order c^{-7} only, and to τ^{ab} at order c^{-5} only.)

The expected scaling of the leading odd terms in $(-g)t_{\rm H}^{\alpha\beta}$ is determined in a similar way. Here we find that this pseudotensor contributes to ρ at order c^{-5}, to j^a at order c^{-5}, and to τ^{ab} at order c^{-5}. (These conclusions will not require revision.)

These considerations allow us to write down the following expansions for the source functions:

$$\rho = \rho[0] + c^{-2}\rho[2] + O(c^{-4}) + c^{-5}\rho[5] + O(c^{-7}), \tag{7.4.42}$$
$$j^a = j^a[0] + c^{-2}j^a[2] + O(c^{-4}) + c^{-5}j^a[5] + O(c^{-7}), \tag{7.4.43}$$
$$\tau^{ab} = \tau^{ab}[0] + c^{-2}\tau^{ab}[2] + O(c^{-4}) + c^{-3}\tau^{ab}[3] + c^{-5}\tau^{ab}[5] + O(c^{-7}). \tag{7.4.44}$$

The zeroth-order terms are of course the Newtonian expressions. The terms in c^{-2} are the post-Newtonian corrections, and these were carefully evaluated in Chapter 4; they will not be needed in this section. Expressions for $\rho[5]$ and $\tau^{ab}[3]$ will be obtained below, but $j^a[5]$ and $\tau^{ab}[5]$ will not be required.

7.4.5 Odd terms in the gravitational potentials

We next substitute Eqs. (7.4.42)–(7.4.44) into Eqs. (7.4.33)–(7.4.35) and compare with Eqs. (7.4.8)–(7.4.10). This reveals that the odd terms in the gravitational potentials are given by

$$V[3] = -\frac{1}{6}G\mathcal{I}^{(3)}_{cc}[0], \tag{7.4.45}$$
$$\begin{aligned} V[5] = G\biggl\{ &\int \frac{\rho[5]}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x' - \frac{1}{6}\mathcal{I}^{(3)}_{cc}[2] - \frac{1}{60}\bigl(r^2\delta^{ab} + 2x^ax^b\bigr)\mathcal{I}^{(5)}_{ab}[0] \\ &+ \frac{1}{30}x^a\mathcal{I}^{(5)}_{acc}[0] - \frac{1}{120}\mathcal{I}^{(5)}_{cdcd}[0]\biggr\}, \end{aligned} \tag{7.4.46}$$
$$V^a[3] = G\biggl\{\frac{1}{6}x^b\mathcal{I}^{a(4)}_{\ \ b}[0] - \frac{1}{18}\mathcal{I}^{a(4)}_{\ \ cc}[0] - \frac{1}{9}\mathcal{J}^{a(3)}_{\ \ cc}[0]\biggr\}, \tag{7.4.47}$$
$$W^{ab}[1] = -\frac{1}{2}G\mathcal{I}^{ab(3)}[0], \tag{7.4.48}$$
$$\begin{aligned} W^{ab}[3] = G\biggl\{ &\int \frac{\tau^{ab}[3]}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x' - \frac{1}{2}\mathcal{I}^{ab(3)}[2] - \frac{1}{12}r^2\mathcal{I}^{ab(5)}[0] + \frac{1}{18}x^c\mathcal{I}^{ab(5)}_{\ \ \ c}[0] \\ &+ \frac{1}{9}x^c\bigl(\mathcal{J}^{a\ \,b(4)}_{\ c}[0] + \mathcal{J}^{b\ \,a(4)}_{\ c}[0]\bigr) - \frac{1}{9}\mathcal{M}^{ab(3)}_{\ \ \ cc}[0]\biggr]\biggr\}. \end{aligned} \tag{7.4.49}$$

In these expressions we indicate the order in c^{-1} at which the multipole moments are to be evaluated. For example, $\mathcal{I}^{ab}[0]$ is the zeroth-order term in an expansion of the quadrupole moment in powers of c^{-1}, and $c^{-2}\mathcal{I}^{ab}[2]$ is the second-order term. Said differently, $\mathcal{I}^{ab}[0]$ is the Newtonian quadrupole moment $I^{ab} = \sum_A m_A z_A^a z_A^b$, and $c^{-2}\mathcal{I}^{ab}[2]$ is its post-Newtonian correction.

A number of observations are in order. First, we notice that apart from two exceptions, $V[3]$, $V[5]$, $V^a[3]$, $W^{ab}[1]$, and $W^{ab}[3]$ involve the Newtonian multipole

moments only; the exceptions concern $\mathcal{I}^{ab}[2]$, the 1PN correction to the quadrupole moment. Second, we notice that $V[3]$ and $W^{ab}[1]$ are functions of time only, while $V[5]$, $V^a[3]$, and $W^{ab}[5]$ are also functions of the spatial coordinates. And third, we observe that $3V[3] - W[1] = 0$, which implies that the term of order c^{-5} vanishes in Eq. (7.4.41).

Our results thus far imply that the gravitational potentials admit the following expansions:

$$\begin{aligned} h^{00} &= \frac{4}{c^2}\Big\{V[0] + O(c^{-2}) + c^{-3}V[3] + c^{-5}V[5] + O(c^{-7})\Big\}, \\ h^{0a} &= \frac{4}{c^3}\Big\{V^a[0] + O(c^{-2}) + c^{-3}V^a[3] + O(c^{-5})\Big\}, \\ h^{ab} &= \frac{4}{c^4}\Big\{W^{ab}[0] + O(c^{-2}) + c^{-1}W^{ab}[1] + c^{-3}W^{ab}[3] + O(c^{-5})\Big\}. \end{aligned}$$

Their spatial derivatives are given by

$$\begin{aligned} \partial_c h^{00} &= \frac{4}{c^2}\Big\{\partial_c V[0] + O(c^{-2}) + c^{-5}\partial_c V[5] + O(c^{-7})\Big\}, \\ \partial_c h^{0a} &= \frac{4}{c^3}\Big\{\partial_c V^a[0] + O(c^{-2}) + c^{-3}\partial_c V^a[3] + O(c^{-5})\Big\}, \\ \partial_c h^{ab} &= \frac{4}{c^4}\Big\{\partial_c W^{ab}[0] + O(c^{-2}) + c^{-3}\partial_c W^{ab}[3] + O(c^{-5})\Big\}, \end{aligned}$$

and their time derivatives are

$$\begin{aligned} \partial_0 h^{00} &= \frac{4}{c^3}\Big\{\dot{V}[0] + O(c^{-2}) + c^{-3}\dot{V}[3] + O(c^{-5})\Big\}, \\ \partial_0 h^{0a} &= \frac{4}{c^4}\Big\{\dot{V}^a[0] + O(c^{-2}) + c^{-3}\dot{V}^a[3] + O(c^{-5})\Big\}, \\ \partial_0 h^{ab} &= \frac{4}{c^5}\Big\{\dot{W}^{ab}[0] + O(c^{-2}) + c^{-1}\dot{W}^{ab}[1] + O(c^{-3})\Big\}, \end{aligned}$$

in which an overdot indicates differentiation with respect to $t = x^0/c$. We notice that $V[3]$ and $W^{ab}[1]$ do not appear in our expressions for $\partial_c h^{00}$ and $\partial_c h^{ab}$, because these potentials do not depend on the spatial coordinates.

7.4.6 Computation of $\rho[5]$ and $\tau^{ab}[3]$

The time has come to do some real work and to evaluate the source terms for the radiation-reaction potentials. We must carefully construct $\rho[5]$ and $\tau^{ab}[3]$, which come from $(-g)t^{\alpha\beta}_{\rm LL}$ and $(-g)t^{\alpha\beta}_{\rm H}$; there is no contribution from the material energy-momentum tensor, because as we have seen, the odd terms contained within Γ in Eq. (7.4.39) scale as c^{-7}.

Equation (1.1.5) reveals that a typical term in the Landau-Lifshitz pseudotensor has the form of $gg\partial h\partial h$. (There are also terms of the form $gggg\partial h\partial h$, but they need not be distinguished for the purpose of this argument.) There are two ways of generating terms that contain an odd power of c^{-1}. The first is to let $\partial h\partial h$ be odd in c^{-1}, and to keep the prefactor gg even; the second is to let gg be odd, and to keep $\partial h\partial h$ even.

In the first scenario, we need to multiply an even term in one of the factors ∂h by an odd term in the remaining ∂h. Using the expansions displayed at the end of the preceding subsection, we find that the dominant scaling of such products is c^{-8}, and that it is produced by the set

$$\mathscr{S}_1 = \Big\{\partial_c h^{00}\partial_d h^{0a}, \partial_c h^{00}\partial_0 h^{00}, \partial_c h^{00}\partial_d h^{ab}\Big\}.$$

We also find that the set of products

$$\begin{aligned}\mathscr{S}_2 &= \Big\{\partial_c h^{00}\partial_d h^{00}, \partial_c h^{00}\partial_d h^{ab}, \partial_c h^{00}\partial_0 h^{0a}, \partial_c h^{0a}\partial_d h^{0b},\\ &\quad \partial_c h^{0a}\partial_0 h^{00}, \partial_c h^{0a}\partial_0 h^{bd}, \partial_0 h^{00}\partial_0 h^{00}, \partial_0 h^{00}\partial_0 h^{ab}\Big\}\end{aligned}$$

participates at order c^{-9}.

In the second scenario we let the factors of g supply the odd terms, and we keep $\partial h\partial h$ even. The leading odd terms in g come from h^{00} at order c^{-5}, h^{0a} at order c^{-6}, and h^{ab} at order c^{-5}. The leading even term in $\partial h\partial h$ comes from $\partial_c h^{00}\partial_d h^{00}$ at order c^{-4}. After multiplication we find that the set

$$\mathscr{S}_3 = \Big\{h^{00}\partial_c h^{00}\partial_d h^{00}, h^{ab}\partial_c h^{00}\partial_d h^{00}\Big\}$$

also participates at order c^{-9}.

The next step is to decide how the various terms listed in $\mathscr{S}_1$, $\mathscr{S}_2$, and $\mathscr{S}_3$ enter in the components of the Landau-Lifshitz pseudotensor. A careful examination of Eq. (1.1.5) reveals that $\mathscr{S}_1$ appears only in $(-g)t^{0a}_{\rm LL}$, whose dominant odd term therefore scales as c^{-4}; this produces a contribution to $j^a[5]$. It reveals also that $\mathscr{S}_2$ and $\mathscr{S}_3$ appear in $(-g)t^{00}_{\rm LL}$ and $(-g)t^{ab}_{\rm LL}$, whose dominant odd terms scale as c^{-5}; this produces a contribution to $\rho[7]$ and $\tau^{ab}[5]$. We may conclude from all this that the Landau-Lifshitz pseudotensor makes no contribution to $\rho[5]$ and $\tau^{ab}[3]$, the quantities that concern us in this subsection.

The source functions must therefore originate from $(-g)t^{\alpha\beta}_{\rm H}$, the harmonic-gauge contribution to the effective energy-momentum tensor. This quantity is defined in Eq. (1.3.6),

$$(-g)t^{\alpha\beta}_{\rm H} = \frac{c^4}{16\pi G}\Big(\partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} - h^{\mu\nu}\partial_{\mu\nu}h^{\alpha\beta}\Big),$$

and here odd terms must be produced by the first scenario. Using the expansions displayed at the end of the preceding subsection, we easily find that the leading odd term in $(-g)t^{00}_{\rm H}$ scales as c^{-3} and comes from the product $h^{ab}\partial_{ab}h^{00}$; this makes a contribution to $\rho[5]$. We also find that the leading odd term in $(-g)t^{ab}_{\rm H}$ scales as c^{-5} and therefore makes no contribution to $\tau^{ab}[3]$.

These considerations lead us to the conclusion that only $(-g)t^{00}_{\rm H}$ contributes to $\rho[5]$, and that $\tau^{ab}[3] = 0$. The mass density is produced by

$$-\frac{c^2}{16\pi G}h^{ab}\partial_{ab}h^{00},$$

and from the equations listed at the end of the previous subsection, we find that

$$\rho[5] = -\frac{1}{\pi G}W^{ab}[1]\partial_{ab}V[0].$$

Inserting Eq. (7.4.48) gives

$$\rho[5] = \frac{1}{2\pi}\mathcal{I}^{ab(3)}[0]\partial_{ab}V[0].$$

To put this in its final form, we recall that the [0] label refers to the Newtonian limit. The quadrupole moment is therefore the Newtonian moment I^{ab}, and the potential $V[0]$ is the Newtonian potential, which was denoted U in previous chapters.

What we have obtained, therefore, is

$$\rho[5] = \frac{1}{2\pi}I^{ab(3)}\partial_{ab}U, \tag{7.4.50}$$

and we have also established that

$$\tau^{ab}[3] = 0. \tag{7.4.51}$$

Another important outcome of this subsection is that we have identified the types of terms that contribute to $(-g)t^{00}_{\rm LL}$ at order c^{-5}, to $(-g)t^{0a}_{\rm LL}$ at order c^{-4}, and to $(-g)t^{ab}_{\rm LL}$ at order c^{-5}; this information will be required in Sec. 7.5.2).

7.4.7 Computation of $V[5]$

The only term that remains to be evaluated in Eq. (7.4.46) is the integral

$$\int \frac{\rho[5]}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x',$$

in which $\rho[5]$, given by Eq. (7.4.50), is expressed as a function of t and $\boldsymbol{x'}$. Making the substitution gives

$$\frac{1}{2\pi} I^{ab(3)} \int \frac{\partial_{a'b'}U(t,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x'.$$

The integral is defined over the domain $\mathscr{M}$, and it is evaluated in the following paragraph.

We begin by writing

$$\frac{\partial_{a'b'}U}{|\boldsymbol{x}-\boldsymbol{x'}|} = \partial_{a'}\biggl(\frac{\partial_{b'}U}{|\boldsymbol{x}-\boldsymbol{x'}|}\biggr) - \partial_{b'}U\partial_{a'}\frac{1}{|\boldsymbol{x}-\boldsymbol{x'}|}.$$

Noticing that $\partial_{a'}|\boldsymbol{x}-\boldsymbol{x'}| = -\partial_a|\boldsymbol{x}-\boldsymbol{x'}|$, this is

$$\frac{\partial_{a'b'}U}{|\boldsymbol{x}-\boldsymbol{x'}|} = \partial_{a'}\biggl(\frac{\partial_{b'}U}{|\boldsymbol{x}-\boldsymbol{x'}|}\biggr) + \partial_a\biggl(\frac{\partial_{b'}U}{|\boldsymbol{x}-\boldsymbol{x'}|}\biggr).$$

Applying this trick once more, we obtain

$$\frac{\partial_{a'b'}U}{|\boldsymbol{x}-\boldsymbol{x'}|} = \partial_{a'}\biggl(\frac{\partial_{b'}U}{|\boldsymbol{x}-\boldsymbol{x'}|}\biggr) + \partial_{ab'}\biggl(\frac{U}{|\boldsymbol{x}-\boldsymbol{x'}|}\biggr) + \partial_{ab}\biggl(\frac{U}{|\boldsymbol{x}-\boldsymbol{x'}|}\biggr).$$

Integration over the domain $\mathscr{M}$ yields

$$\int \frac{\partial_{a'b'}U}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x' = \oint \frac{\partial_{b'}U}{|\boldsymbol{x}-\boldsymbol{x'}|}\, dS_a + \partial_a \oint \frac{U}{|\boldsymbol{x}-\boldsymbol{x'}|}\, dS_b + \partial_{ab}\int \frac{U}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x'.$$

Inspection of the surface integrals, which are evaluated on $\partial\mathscr{M}$, reveals that they scale as $\mathcal{R}^{-1}$; they do not give rise to $\mathcal{R}$-independent contributions to the potential. The remaining volume integral is, within the domain $\mathscr{M}$, a solution to $\nabla^2\psi = -4\pi U$. We already know the solution to this equation: According to Eq. (3.2.4), $\nabla^2 X = 2U$, ψ must be equal to $-2\pi X$, where X is the post-Newtonian superpotential. Our final expression for the integral is therefore $-2\pi\partial_{ab}X$.

We have arrived at

$$\int \frac{\rho[5]}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x' = -I^{ab(3)}\partial_{ab}X,$$

and with this established, Eq. (7.4.46) becomes

$$\begin{aligned} V[5] \quad &= \quad G\biggl\{-I^{ab(3)}\partial_{ab}X - \frac{1}{6}\mathcal{I}^{(3)}_{cc}[2] - \frac{1}{60}\bigl(r^2\delta^{ab} + 2x^a x^b\bigr)I^{(5)}_{ab} \\ &\qquad + \frac{1}{30}x^a I^{(5)}_{acc} - \frac{1}{120}I^{(5)}_{cdcd}\biggr\}. \end{aligned} \tag{7.4.52}$$

The potential is expressed in terms of Newtonian multipole moments, the superpotential X, and the post-Newtonian correction $c^{-2}\mathcal{I}^{ab}[2]$ to the Newtonian quadrupole moment.

7.4.8 Summary: Radiation-reaction potentials

Our computation of the radiation-reaction potentials is complete. We have shown that the gravitational potentials can be expanded as

$$h^{00} = \frac{4}{c^2}\Big\{U + O(c^{-2}) + c^{-3}V[3] + c^{-5}V[5] + O(c^{-7})\Big\}, \tag{7.4.53}$$

$$h^{0a} = \frac{4}{c^3}\Big\{U^a + O(c^{-2}) + c^{-3}V^a[3] + O(c^{-5})\Big\}, \tag{7.4.54}$$

$$h^{ab} = \frac{4}{c^4}\Big\{P^{ab} + O(c^{-2}) + c^{-1}W^{ab}[1] + c^{-3}W^{ab}[3] + O(c^{-5})\Big\}, \tag{7.4.55}$$

in which $U := V[0]$, $U^a := V^a[0]$, and $P^{ab} := W^{ab}[0]$ are the leading-order, near-zone potentials listed in Sec. 4.2.10; the Newtonian potential, in particular, is given by

$$U = \sum_A \frac{Gm_A}{|\boldsymbol{x} - \boldsymbol{z}_A|}. \tag{7.4.56}$$

The terms that come with an odd power of c^{-1} are the radiation-reaction potentials, and they are given by

$$V[3] = -\frac{1}{6}GI^{(3)}_{cc}, \tag{7.4.57}$$

$$V[5] = G\Big[-I^{ab(3)}\partial_{ab}X - \frac{1}{6}\mathcal{I}^{(3)}_{cc}[2] - \frac{1}{60}\big(r^2\delta^{ab} + 2x^a x^b\big)I^{(5)}_{ab} + \frac{1}{30}x^a I^{(5)}_{acc} - \frac{1}{120}I^{(5)}_{cdcd}\Big], \tag{7.4.58}$$

$$V^a[3] = G\Big[\frac{1}{6}x^b I^{a(4)}_{\ \ b} - \frac{1}{18}I^{a(4)}_{\ \ cc} - \frac{1}{9}J^{a(3)}_{\ \ cc}\Big], \tag{7.4.59}$$

$$W^{ab}[1] = -\frac{1}{2}GI^{ab(3)}, \tag{7.4.60}$$

$$W^{ab}[3] = G\Big[-\frac{1}{2}\mathcal{I}^{ab(3)}[2] - \frac{1}{12}r^2 I^{ab(5)} + \frac{1}{18}x^c I^{ab(5)}_{\ \ \ c} + \frac{1}{9}x^c\big(J^{a\ b(4)}_{\ c} + J^{b\ a(4)}_{\ c}\big) - \frac{1}{9}M^{ab(3)}_{\ \ \ cc}\Big]. \tag{7.4.61}$$

They are expressed in terms of the Newtonian multipole moments

$$I^{ab} = \sum_A m_A z^a_A z^b_A, \tag{7.4.62}$$

$$I^{abc} = \sum_A m_A z^a_A z^b_A z^c_A, \tag{7.4.63}$$

$$I^{abcd} = \sum_A m_A z^a_A z^b_A z^c_A z^d_A, \tag{7.4.64}$$

$$J^{abc} = \sum_A m_A \big(v^a_A z^b_A - z^a_A v^b_A\big) z^c_A, \tag{7.4.65}$$

$$M^{abcd} = \sum_A m_A v^a_A v^b_A z^c_A z^d_A. \tag{7.4.66}$$

The potentials also depend on $c^{-2}\mathcal{I}^{ab}[2]$, the 1PN correction to the Newtonian quadrupole moment $I^{ab} := \mathcal{I}^{ab}[0]$; an expression for this could be obtained by importing the relevant results from Chapter 4, but this is not necessary, because

$\mathcal{I}^{ab}[2]$ will not be required in future calculations. In addition, $V[5]$ depends on the post-Newtonian superpotential X. According to Eq. (4.2.37), this is given by

$$X = \sum_A Gm_A|\boldsymbol{x} - \boldsymbol{z}_A|. \tag{7.4.67}$$

It is noteworthy that $V[3]$ and $W^{ab}[1]$ are functions of time only. This property ensures that these potentials will have no effect on the equations of motion, because only spatial gradients of these potentials could be involved. This, in turn, ensures that the radiation-reaction force scales as c^{-5}, and not as c^{-3} as might naively be expected. In fact, $V[3]$ and $W^{ab}[1]$ could be eliminated by means of a coordinate transformation; we shall not pursue this here, as the transformation would take us away from the harmonic gauge adopted throughout this work.

7.4.9 Transformation to Burke-Thorne gauge

[TO BE WRITTEN? IF SO REVISE PREVIOUS SENTENCE.]

7.5 Radiation-reaction force and energy balance

7.5.1 Strategy

In this section we calculate the $\frac{5}{2}$PN term in the acceleration vector $\boldsymbol{a}_A$ of each body within the N-body system. Our general strategy is based on the methods of Chapter 5, in which the post-Newtonian equations of motion are derived on the basis of conservations identities that follow from the Einstein field equations. We recall from Sec. 5.1 that the basic law of motion for each body A is

$$M_A\boldsymbol{a}_A = \dot{\boldsymbol{P}}_A - \dot{M}_A\boldsymbol{v}_A - \dot{\boldsymbol{Q}}_A - \ddot{\boldsymbol{D}}_A, \tag{7.5.1}$$

where

$$M_A := \frac{c^2}{16\pi G}\oint_{S_A} \partial_c H^{0c0b}\, dS_b, \tag{7.5.2}$$

is the mass parameter of each body, which changes in time according to

$$\dot{M}_A = -\frac{1}{c}\oint_{S_A} (-g)\biggl(t^{0b}_{\rm LL} - t^{00}_{\rm LL}\frac{v^b_A}{c}\biggr)\, dS_b. \tag{7.5.3}$$

We also have that $\boldsymbol{P}_A$ is the momentum vector of each body, and its rate of change is given by

$$\dot{P}^a_A = -\oint_{S_A} (-g)\biggl(t^{ab}_{\rm LL} - t^{0a}_{\rm LL}\frac{v^b_A}{c}\biggr)\, dS_b. \tag{7.5.4}$$

The law also involves the quantities

$$Q^a_A := \frac{1}{c}\oint_{S_A} (-g)\biggl(t^{0b}_{\rm LL} - t^{00}_{\rm LL}\frac{v^b_A}{c}\biggr)\bigl(x^a - z^a_A\bigr)\, dS_b \tag{7.5.5}$$

and

$$D^a_A = \frac{c^2}{16\pi G}\oint_{S_A} \Bigl[\bigl(\partial_d H^{0c0d}\bigr)\bigl(x^a - z^a_A\bigr) - H^{0a0c}\Bigr]\, dS_c. \tag{7.5.6}$$

Each integral is evaluated on a two-sphere S_A surrounding each body, which is described by the equation $s_A := |\boldsymbol{x} - \boldsymbol{z}_A| =$ constant. And finally, we recall that $H^{\alpha\mu\beta\nu}$ is related to the gravitational potentials through the relation

$$H^{\alpha\mu\beta\nu} = \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu}, \tag{7.5.7}$$

where $\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$ is the gothic inverse metric.

We want to compute the $\frac{5}{2}$PN contributions to M_A, $\dot{M}_A$, $\dot{\boldsymbol{P}}_A$, $\dot{\boldsymbol{Q}}_A$, and $\ddot{\boldsymbol{D}}_A$, and insert them within Eq. (7.5.1) to determine the $\frac{5}{2}$PN contribution to the acceleration vector. (We will also have to make sure that there are no contributions at $\frac{3}{2}$PN order.) According to the equations listed previously, these computations require, in addition to the radiation-reaction potentials obtained in Sec. 7.4, expressions for the odd terms in the Landau-Lifshitz pseudotensor. In particular, we need an expression for $(-g)t^{00}_{\rm LL}$ that is accurate to order c^{-3}, an expression for $(-g)t^{0a}_{\rm LL}$ that is accurate to order c^{-4}, and an expression for $(-g)t^{ab}_{\rm LL}$ that is accurate to order c^{-5}. According to the considerations of Sec. 7.4.6, however, there is no contribution to $(-g)t^{00}_{\rm LL}$ at order c^{-3}, and this component of the pseudotensor plays no role in the radiation reaction. The same considerations revealed that $\mathscr{S}_1$ enters $(-g)t^{0a}_{\rm LL}$ at order c^{-4}, while $\mathscr{S}_2$ and $\mathscr{S}_3$ enter $(-g)t^{ab}_{\rm LL}$ at order c^{-5}; these components will be computed carefully.

Once the relevant components of $H^{\alpha\mu\beta\nu}$ and $(-g)t^{\alpha\beta}_{\rm LL}$ are at hand, it will be a simple matter to follow the methods outlined in Chapter 5 and to calculate the various quantities that appear in Eq. (7.5.1). The end result will be an explicit expression for

$$\boldsymbol{a}_A[\mathrm{rr}] := c^{-5}\boldsymbol{a}_A[5], \tag{7.5.8}$$

the radiation-reaction force (per unit mass) acting on body A. And once this is known, we shall be able to verify whether the work done by all the radiation-reaction forces matches the energy radiated by the source in the form of gravitational waves. This, we recall, is expressed by the quadrupole formula of Eq. (7.3.6),

$$\dot{E}_{\rm gw} = \frac{1}{5}\frac{G}{c^5}\Bigl(I^{ab(3)}I^{(3)}_{ab} - \frac{1}{3}I^{cc(3)}I^{(3)}_{cc}\Bigr). \tag{7.5.9}$$

The answer, of course, will be in the affirmative, but in the same coarse-grained sense that applies to flat-spacetime electrodynamics (as reviewed in Sec. 7.4.1).

7.5.2 Computation of the Landau-Lifshitz pseudotensor

We already have noted that

$$(-g)t^{00}_{\rm LL} = 0 \tag{7.5.10}$$

at order c^{-3}; its leading, odd-order contribution scales as c^{-5}.

To calculate the time-space components at order c^{-4}, we return to the considerations of Sec. 7.4.6, which indicated that this quantity must be constructed from $\mathscr{S}_1$, the first set of products $\partial h \partial h$ that are listed there. A careful expansion of Eq. (1.1.5) next reveals that the answer is given by the c^{-4} piece of

$$(-g)t^{0a}_{\rm LL} = \frac{c^4}{16\pi G}\biggl[\frac{3}{4}\partial_0 h^{00}\partial^a h^{00} + \partial_c h^{00}\bigl(\partial^a h^{0c} - \partial^c h^{0a}\bigr) - \frac{1}{4}\partial_0 h^{cc}\partial^a h^{00}\biggr].$$

This becomes

$$(-g)t^{0a}_{\rm LL} = \frac{1}{\pi G c^4}\biggl[\frac{1}{4}\bigl(3\dot{V}[3] - \dot{W}[1]\bigr)\partial^a U + \bigl(\partial^a V^c[3] - \partial^c V^a[3]\bigr)\partial_c U\biggr]$$

after involving the equations listed near the end of Sec. 7.4.5. Finally, from Eqs. (7.4.57) and (7.4.60) we find that the first group of terms vanishes, while from Eq. (7.4.59) we see that $\partial_a V_b[3] = \frac{1}{6}GI^{(4)}_{ab}$, which implies that the second group vanishes also. We conclude that

$$(-g)t^{0a}_{\rm LL} = 0 \tag{7.5.11}$$

at order c^{-4}.

The spatial components of the pseudotensor require a much more labourious calculation. Returning once more to Sec. 7.4.6, we recognize that the terms that can participate at order c^{-5} are contained in $\mathscr{S}_2$ and $\mathscr{S}_3$. A careful expansion of Eq. (1.1.5) next reveals that the answer is given by the c^{-5} piece of

$$\begin{aligned}(-g)t^{ab}_{\rm LL} &= \frac{c^4}{16\pi G}\biggl\{\frac{1}{4}\partial^a h^{00}\partial^b h^{00} + \frac{1}{4}\partial^a h^{00}\partial^b h^{cc} + \frac{1}{4}\partial^a h^{cc}\partial^b h^{00} \\ &\quad - \bigl(\partial^a h^{0c} - \partial^c h^{0a}\bigr)\bigl(\partial^b h^{0c} - \partial^c h^{0b}\bigr) + \partial_0 h^{0a}\partial^b h^{00} + \partial_0 h^{0b}\partial^a h^{00} \\ &\quad - \partial_0 h^{ac}\partial^b h^{0c} - \partial_0 h^{bc}\partial^a h^{0c} + \delta^{ab}\biggl[-\frac{1}{8}\partial_c h^{00}\partial^c h^{00} - \frac{1}{4}\partial_c h^{00}\partial^c h^{dd} \\ &\quad + \frac{1}{2}\partial_c h^{0d}\bigl(\partial^c h^{0d} - \partial^d h^{0c}\bigr) - \partial_0 h^{0c}\partial_c h^{00} + \partial_0 h^{cd}\partial_d h^{0c} \\ &\quad - \frac{3}{8}\partial_0 h^{00}\partial_0 h^{00} + \frac{1}{4}\partial_0 h^{00}\partial_0 h^{00}\biggr] \\ &\quad + \frac{1}{8}\bigl(2g^{ac}g^{bd} - g^{ab}g^{cd}\bigr)g_{00}g_{00}\partial_c h^{00}\partial_d h^{00}\biggr\}.\end{aligned}$$

This eventually becomes

$$\begin{aligned}(-g)t^{ab}_{\rm LL} &= \frac{1}{4\pi c^5}\biggl[-\bigl(\partial^a U\partial^{bcd}X + \partial^b U\partial^{acd}X - \delta^{ab}\partial_e U\partial^{ecd}X\bigr)I^{(3)}_{cd} \\ &\quad - \frac{1}{5}\bigl(x^a\partial^b U + x^b\partial^a U - \delta^{ab}x^d\partial_d U\bigr)I^{(5)}_{cc} \\ &\quad + \frac{3}{5}x^c\Bigl(\partial^a U I^{b(5)}_{c} + \partial^b U I^{a(5)}_{c} - \delta^{ab}\partial^d U I^{(5)}_{cd}\Bigr) \\ &\quad - \frac{2}{15}\Bigl(\partial^a U I^{b(5)}_{cc} + \partial^b U I^{a(5)}_{cc} - \delta^{ab}\partial^d U I^{(5)}_{dcc}\Bigr) \\ &\quad - \frac{2}{3}\Bigl(\partial^a U J^{b(4)}_{cc} + \partial^b U J^{a(4)}_{cc} - \delta^{ab}\partial^d U J^{(4)}_{dcc}\Bigr) \\ &\quad + 2\Bigl(\partial^a U^c I^{b(4)}_{c} + \partial^b U^c I^{a(4)}_{c} - \delta^{ab}\partial^d U^c I^{(4)}_{cd}\Bigr) \\ &\quad + \frac{4}{3}\partial^a U\partial^b U I^{(3)}_{cc} + 2\partial^a U\partial^c U I^{b(3)}_{c} + 2\partial^b U\partial^c U I^{a(3)}_{c} \\ &\quad - \frac{2}{3}\delta^{ab}\partial_c U\partial^c U I^{(3)}_{dd} - \delta^{ab}\partial^c U\partial^d U I^{(3)}_{cd} - \partial_c U\partial^c U I^{ab(3)}\biggr] \qquad (7.5.12)\end{aligned}$$

after a long computation involving the equations listed near the end of Sec. 7.4.5, as well as the radiation-reaction potentials of Eqs. (7.4.57)–(7.4.61).

7.5.3 Internal and external potentials

To simplify the notation it will be advantageous to proceed as in Sec. 5.2, and to focus our attention on a particular body, the one labeled by $A = 1$. We let $m := m_1$, $\boldsymbol{z} := \boldsymbol{z}_1$, $\boldsymbol{v} := \boldsymbol{v}_1$, and so on. In addition, we introduce the vector $\boldsymbol{s} := \boldsymbol{x} - \boldsymbol{z}$, and decompose it as $\boldsymbol{s} = s\boldsymbol{n}$, in terms of its length $s := |\boldsymbol{s}|$ and the unit vector $\boldsymbol{n} := \boldsymbol{s}/s$. In this notation, the two-sphere S that surrounds the body is described by $s =$ constant, and the surface element on S is $dS_a = s^2 n_a\, d\Omega$, in which $d\Omega$ is the usual element of solid angle.

The potentials U, U^a, and X are decomposed into internal and external pieces according to Eqs. (5.2.10), (5.2.12), (5.2.15), (5.2.18), (5.2.20), and (5.2.22). We have

$$U = \frac{Gm}{s} + U_{\rm ext}, \qquad (7.5.13)$$

$$\begin{aligned} U^a &= \frac{Gmv^a}{s} + U^a_{\text{ext}}, && (7.5.14)\\ X &= Gms + X_{\text{ext}}, && (7.5.15) \end{aligned}$$

with

$$\begin{aligned} U_{\text{ext}} &= \sum_{A\neq 1} \frac{Gm_A}{|\boldsymbol{x}-\boldsymbol{z}_A|}, && (7.5.16)\\ U^a_{\text{ext}} &= \sum_{A\neq 1} \frac{Gm_A v^a_A}{|\boldsymbol{x}-\boldsymbol{z}_A|}, && (7.5.17)\\ X_{\text{ext}} &= \sum_{A\neq 1} Gm_A|\boldsymbol{x}-\boldsymbol{z}_A|. && (7.5.18) \end{aligned}$$

For later convenience we also list the identities

$$\begin{aligned} \partial_a s &= n_a, && (7.5.19)\\ \partial_{ab} s &= \frac{1}{s}\big(\delta_{ab} - n_a n_b\big), && (7.5.20)\\ \partial_{abc} s &= -\frac{1}{s^2}\big(\delta_{ab}n_c + \delta_{ac}n_b + \delta_{bc}n_a - 3n_a n_b n_c\big) && (7.5.21) \end{aligned}$$

involving derivatives of $s := |\boldsymbol{x}-\boldsymbol{z}|$.

7.5.4 Computation of M and $\boldsymbol{D}$

We first compute the odd contributions to $M := M_1$ and $\boldsymbol{D} := \boldsymbol{D}_1$, starting from their definitions in Eqs. (7.5.2) and (7.5.6). Recalling the work carried out in Chapter 5 — and especially the discussion at the beginning for Sec. 5.4.1 — we understand that it is sufficient to calculate all surface integrals to order s^0, and to ignore all contributions at order s and higher.

According to Eq. (7.5.7),

$$H^{0a0b} = -\delta^{ab} - h^{00}\delta^{ab} + h^{ab} + h^{00}h^{ab} - h^{0a}h^{0b},$$

and we wish to evaluate this at orders c^{-5} and c^{-7} in order to calculate the c^{-3} and c^{-5} terms in M and $\boldsymbol{D}$, respectively. With Eqs. (7.4.53)–(7.4.55) we find that $H^{0a0b}[5] = -4\delta^{ab}V[3] + 4W^{ab}[1]$, and inserting Eqs. (7.4.57) and (7.4.60) produces

$$H^{0a0b}[5] = -2G\bigg(I^{ab(3)} - \frac{1}{3}\delta^{ab}I^{(3)}_{cc}\bigg). \qquad (7.5.22)$$

At the next order we get

$$H^{0a0b}[7] = -4\delta^{ab}V[5] + 4W^{ab}[3] - 8GUI^{ab(3)}, \qquad (7.5.23)$$

in which we may substitute the radiation-reaction potentials of Eqs. (7.4.58) and (7.4.61).

The fact that $\partial_a H^{0a0b}[5] = 0$ implies that there is no contribution to M at order c^{-3}. To compute the contribution at order c^{-5} we must substitute Eq. (7.5.23) into Eq. (7.5.2) and evaluate the surface integral to order s^0. After inserting the known expressions for $V[5]$ and $W^{ab}[3]$, we find that the terms within $\partial_a H^{0a0b}[7]$ that are sufficiently singular to produce a finite integral in the limit $s \to 0$ are

$$4GI^{(3)}_{cd}\partial^{bcd}X - 8GI^{b(3)}_{c}\partial^c U;$$

these scale as s^{-2}, while all other contributions are bounded as $s \to 0$. We next decompose U and X into internal and external pieces, as in Eqs. (7.5.13) and

(7.5.15), and we make use of Eqs. (7.5.19) and (7.5.21). The previous expression becomes

$$-\frac{4G^2m}{s^2} I^{(3)}_{cd}\bigl(n^b\delta^{cd} - 3n^bn^cn^d\bigr).$$

Multiplication by $dS_b = s^2 n_b\, d\Omega$ produces

$$\partial_a H^{0a0b}[7] dS_b = -4G^2 m I^{(3)}_{cd}\bigl(\delta^{cd} - 3n^cn^d\bigr),$$

and integration over the sphere gives zero, because according to the results of Sec. 1.8.4, $\langle\!\langle n^an^b\rangle\!\rangle := (4\pi)^{-1}\int n^an^b\, d\Omega = \frac{1}{3}\delta^{ab}$.

We have obtained the statements that

$$M[3] = M[5] = 0, \tag{7.5.24}$$

and we find that M makes no contribution to the radiation-reaction force. On the other hand, inspection of the integrand in Eq. (7.5.6) reveals that it is of order s^{-1}, so that the integral itself is of order s. We therefore write

$$\boldsymbol{D}[3] = \boldsymbol{D}[5] = 0, \tag{7.5.25}$$

and conclude that $\boldsymbol{D}$ also makes no contribution to the radiation-reaction force.

7.5.5 Computation of $\dot{M}$, $\boldsymbol{Q}$, and $\dot{\boldsymbol{P}}$

The computation of $\dot{M} := \dot{M}_1$ and $\boldsymbol{Q} := \boldsymbol{Q}_1$ is exceedingly simple in view of Eqs. (7.5.10) and (7.5.11). Inserting these within Eqs. (7.5.3) and (7.5.5), we immediately obtain

$$\dot{M}[5] = 0 \tag{7.5.26}$$

and

$$\boldsymbol{Q}[5] = 0. \tag{7.5.27}$$

These quantities also do not participate in the radiation-reaction force.

We are left with the computation of $\dot{\boldsymbol{P}} := \dot{\boldsymbol{P}}_1$, which is based on Eq. (7.5.4) and the stress tensor of Eq. (7.5.12). The calculations are very similar to those carried out in Sec. 5.3.3. We write

$$\dot{P}^a[5] = -\bigl\langle\!\bigl\langle s^2\Gamma^{ab}n_b\bigr\rangle\!\bigr\rangle,$$

where $\Gamma^{ab} := 4\pi c^5(-g)t^{ab}_{\rm LL}$ and the angular brackets indicate an average over a two-sphere $s =$ constant. We take each line in turn in Eq. (7.5.12) and substitute the decompositions of Eqs. (7.5.13)–(7.5.15) for the potentials U, U^a, and X. We use Eqs. (7.5.19)–(7.5.21) to differentiate the internal potentials, and we expand each external potential in a Taylor series about $\boldsymbol{x} = \boldsymbol{z}$. Finally, we perform the angular integrations using the rules of Sec. 1.8.4, and discard all terms of order s and higher.

We list some of the intermediate results that are produced in these computations:

$$\begin{aligned}
\bigl\langle\!\bigl\langle s^2\partial^aU\partial^{bcd}Xn_b\bigr\rangle\!\bigr\rangle &= -\frac{1}{3}Gm\partial^{acd}X_{\rm ext} - \frac{2}{3}Gm\delta^{cd}\partial^aU_{\rm ext},\\
\bigl\langle\!\bigl\langle s^2\partial^bU\partial^{acd}Xn_b\bigr\rangle\!\bigr\rangle &= -Gm\partial^{acd}X_{\rm ext}\\
&\quad - \frac{2}{15}Gm\bigl(\delta^{cd}\partial^aU_{\rm ext} + \delta^{ac}\partial^dU_{\rm ext} + \delta^{ad}\partial^cU_{\rm ext}\bigr),\\
\bigl\langle\!\bigl\langle s^2n^a\partial_eU\partial^{ecd}X\bigr\rangle\!\bigr\rangle &= -\frac{1}{3}Gm\partial^{acd}X_{\rm ext}\\
&\quad - \frac{2}{15}Gm\bigl(\delta^{cd}\partial^aU_{\rm ext} + \delta^{ac}\partial^dU_{\rm ext} + \delta^{ad}\partial^cU_{\rm ext}\bigr),
\end{aligned}$$

$$\begin{aligned}
\langle\!\langle s^2 \partial^a U n_b \rangle\!\rangle &= -\frac{1}{3} G m \delta^a{}_b \\
\langle\!\langle s^2 \partial^a U^c n_b \rangle\!\rangle &= -\frac{1}{3} G m \delta^a{}_b v^c \\
\langle\!\langle s^2 \partial^a U \partial^c U n_b \rangle\!\rangle &= -\frac{1}{3} G m \delta^a{}_b \partial^c U_{\rm ext} - \frac{1}{3} G m \delta^c{}_b \partial^a U_{\rm ext}.
\end{aligned}$$

We emphasize that these expressions are valid up to corrections of order s, and that the external potentials are evaluated at $\boldsymbol{x} = \boldsymbol{z}$ after differentiation.

Collecting results, we find that

$$\begin{aligned}
\dot{P}^a[5] &= Gm\biggl(-I^{(3)}_{cd} \partial^{acd} X_{\rm ext} + 2 I^{a(3)}_{\ c} \partial^c U_{\rm ext} + \frac{4}{3} I^{(3)}_{cc} \partial^a U_{\rm ext} \\
&\quad + \frac{3}{5} I^{a(5)}_{\ c} z^c - \frac{1}{5} I^{(5)}_{cc} z^a + 2 I^{a(4)}_{\ c} v^c - \frac{2}{15} I^{a(5)}_{cc} - \frac{2}{3} J^{a(4)}_{cc} \biggr). \qquad (7.5.28)
\end{aligned}$$

7.5.6 Radiation-reaction force: Reference body

We now have everything we need to compute the radiation-reaction force. We return to Eq. (7.5.1),

$$M\boldsymbol{a} = \dot{\boldsymbol{P}} - \dot{M}\boldsymbol{v} - \dot{\boldsymbol{Q}} - \ddot{\boldsymbol{D}},$$

and apply it to our reference body. Equation (5.3.3), together with Eq. (7.5.24), imply that at $\frac{5}{2}$PN order,

$$M = m + c^{-2} M[2] + c^{-4} M[4] + O(c^{-6}),$$

with no odd term making an appearance. Equation (5.3.5), together with Eq. (7.5.26), imply that

$$\dot{M} = c^{-2} \dot{M}[2] + c^{-4} \dot{M}[4] + O(c^{-6}),$$

and this also does not include an odd term. Equation (5.3.9), together with Eq. (7.5.27), imply that

$$\boldsymbol{Q} = c^{-2} \boldsymbol{Q}[2] + c^{-4} \boldsymbol{Q}[4] + O(c^{-6}),$$

and once more we notice the absence of an odd term. We note also that $\boldsymbol{D} = O(s)$, and that it is therefore irrelevant to the equations of motion. This leaves us with $\dot{\boldsymbol{P}}$, and Eqs. (5.3.7) and (7.5.28) imply that

$$\dot{\boldsymbol{P}} = \dot{\boldsymbol{P}}[0] + c^{-2} \dot{\boldsymbol{P}}[2] + c^{-4} \dot{\boldsymbol{P}}[4] + c^{-5} \dot{\boldsymbol{P}}[5] + O(c^{-6}),$$

where $\dot{P}^a[0] = m \partial^a U_{\rm ext}$ is the Newtonian gravitational force, and $\dot{P}^a[5]$ is given explicitly by Eq. (7.5.28). An odd term has finally appeared within the law of motion.

The preceding equations imply that the acceleration vector has an expansion of the form

$$\boldsymbol{a} = \boldsymbol{a}[0] + c^{-2} \boldsymbol{a}[2] + c^{-4} \boldsymbol{a}[4] + c^{-5} \boldsymbol{a}[5] + O(c^{-6}), \qquad (7.5.29)$$

and $\boldsymbol{a}[\mathrm{rr}] := c^{-5} \boldsymbol{a}[5]$ is the radiation-reaction force per unit mass. The Newtonian and post-Newtonian terms were evaluated in Secs. 5.4.1 and 5.4.4, and a calculation of $c^{-4} \boldsymbol{a}[4]$, the 2PN acceleration, was completely bypassed. The $\frac{5}{2}$PN term in the acceleration, however, is given by $\dot{\boldsymbol{P}}[5]/m$, and according to Eq. (7.5.28), this is

$$\begin{aligned}
a^a[\mathrm{rr}] &= \frac{G}{c^5}\biggl(-I^{(3)}_{cd} \partial^{acd} X_{\rm ext} + 2 I^{a(3)}_{\ c} \partial^c U_{\rm ext} + \frac{4}{3} I^{(3)}_{cc} \partial^a U_{\rm ext} \\
&\quad + \frac{3}{5} I^{a(5)}_{\ c} z^c - \frac{1}{5} I^{(5)}_{cc} z^a + 2 I^{a(4)}_{\ c} v^c - \frac{2}{15} I^{a(5)}_{cc} - \frac{2}{3} J^{a(4)}_{cc} \biggr). \qquad (7.5.30)
\end{aligned}$$

This is expressed in terms of Newtonian multipole moments, and in terms of the potentials $U_{\rm ext}$ and $X_{\rm ext}$ that are external to the reference body.

For our final expression we calculate $\partial^a U_{\rm ext}$ and $\partial^{abc} X_{\rm ext}$ with the help of Eqs. (7.5.16), (7.5.18), and we evaluate the results at $\boldsymbol{x} = \boldsymbol{z}$. In terms of the vector $\boldsymbol{z}_{1A} := \boldsymbol{z} - \boldsymbol{z}_A$, its length $z_{1A} := |\boldsymbol{z} - \boldsymbol{z}_A|$, and the unit vector $\boldsymbol{n}_{1A} := \boldsymbol{z}_{1A}/z_{1A}$, we have

$$\partial^a U_{\rm ext} = -\sum_{A\neq 1} \frac{G m_A}{z_{1A}^2} n_{1A}^a$$

and

$$\partial^{abc} X_{\rm ext} = -\sum_{A\neq 1} \frac{G m_A}{z_{1A}^2}\Bigl(\delta^{ab} n_{1A}^c + \delta^{ac} n_{1A}^b + \delta^{bc} n_{1A}^a - 3 n_{1A}^a n_{1A}^b n_{1A}^c\Bigr).$$

After insertion of these expressions within Eq. (7.5.30), we arrive at

$$\begin{aligned} a^a[\mathrm{rr}] &= \frac{G}{c^5}\biggl(-3 I_{bc}^{(3)} \sum_{A\neq 1} \frac{G m_A}{z_{1A}^2} n_{1A}^a n_{1A}^b n_{1A}^c - \frac{1}{3} I_{cc}^{(3)} \sum_{A\neq 1} \frac{G m_A}{z_{1A}^2} n_{1A}^a \\ &\quad + \frac{3}{5} I^{a(5)}_{\ \ b} z^b - \frac{1}{5} I_{cc}^{(5)} z^a + 2 I^{a(4)}_{\ \ b} v^b - \frac{2}{15} I^{a(5)}_{\ \ cc} - \frac{2}{3} J^{a(5)}_{\ \ cc}\biggr). \qquad (7.5.31) \end{aligned}$$

This is the radiation-reaction force (per unit mass) acting on the reference body.

7.5.7 Radiation-reaction force: Final answer

Equation (7.5.31) generalizes easily to any body within the N-body system. We simply replace the label "1" by an arbitrary label "A", and we obtain

$$\begin{aligned} a_A^a[\mathrm{rr}] &= \frac{G}{c^5}\biggl(-3 I_{bc}^{(3)} \sum_{B\neq A} \frac{G m_B}{z_{AB}^2} n_{AB}^a n_{AB}^b n_{AB}^c - \frac{1}{3} I_{cc}^{(3)} \sum_{B\neq A} \frac{G m_B}{z_{AB}^2} n_{AB}^a \\ &\quad + \frac{3}{5} I^{a(5)}_{\ \ b} z_A^b - \frac{1}{5} I_{cc}^{(5)} z_A^a + 2 I^{a(4)}_{\ \ b} v_A^b - \frac{2}{15} I^{a(5)}_{\ \ cc} - \frac{2}{3} J^{a(5)}_{\ \ cc}\biggr). \qquad (7.5.32) \end{aligned}$$

This is our final answer. The radiation-reaction force is expressed in terms of the interbody distance $z_{AB} := |\boldsymbol{z}_A - \boldsymbol{z}_B|$ as well as the unit vector

$$\boldsymbol{n}_{AB} = \frac{\boldsymbol{z}_A - \boldsymbol{z}_B}{|\boldsymbol{z}_A - \boldsymbol{z}_B|},$$

which points from body B to body A. It involves also the Newtonian multipole moments

$$\begin{aligned} I^{ab} &= \sum_A m_A z_A^a z_A^b, & (7.5.33) \\ I^{abc} &= \sum_A m_A z_A^a z_A^b z_A^c, & (7.5.34) \\ J^{abc} &= \sum_A m_A \bigl(v_A^a z_A^b - z_A^a v_A^b\bigr) z_A^c, & (7.5.35) \end{aligned}$$

which are functions of time t; the number within brackets indicates the number of differentiations with respect to t. As a final comment we note that Eq. (7.5.32) gives only the leading-order term in a post-Newtonian expansion of the radiation-reaction force; a more complete calculation would reveal correction terms at order c^{-7}, and so on.

7.5.8 Energy balance

We next calculate the rate at which the radiation-reaction forces do work on the N bodies, and verify that this is equal (in a coarse-grained sense) to the rate at which the system's energy is lost to gravitational waves; this loss is expressed by the quadrupole formula of Eq. (7.5.9).

The rate at which the forces do work is

$$\dot{W} = \sum_A m_A \boldsymbol{a}_A[\mathrm{rr}] \cdot \boldsymbol{v}_A. \tag{7.5.36}$$

We insert Eq. (7.5.32), and we notice that the terms involving $I^a{}_{cc}$ and $J^a{}_{cc}$ disappear, because they each multiply $\boldsymbol{P} := \sum_A m_A \boldsymbol{v}_A$; this is the Newtonian total momentum, and this can be set equal to zero by placing the origin of the coordinate system at barycentre. After rearranging the double sums, we obtain

$$\begin{aligned}\dot{W} &= \frac{G}{c^5}\biggl[-\frac{3}{2} I^{(3)}_{ab} \sum_{AB} \frac{G m_A m_B}{z_{AB}^2} \bigl(\boldsymbol{n}_{AB} \cdot \boldsymbol{v}_{AB}\bigr) n^a_{AB} n^b_{AB} \\ &\quad - \frac{1}{6} I^{(3)}_{cc} \sum_{AB} \frac{G m_A m_B}{z_{AB}^2} \bigl(\boldsymbol{n}_{AB} \cdot \boldsymbol{v}_{AB}\bigr) + \frac{3}{5} I^{(5)}_{ab} \sum_A m_A v^a_A z^b_A \\ &\quad - \frac{1}{5} I^{(5)}_{cc} \sum_A m_A \bigl(\boldsymbol{z}_A \cdot \boldsymbol{v}_A\bigr) + 2 I^{(4)}_{ab} \sum_A m_A v^a_A v^b_A\biggr];\end{aligned} \tag{7.5.37}$$

the double sums exclude the case $A = B$, and $\boldsymbol{v}_{AB} := \boldsymbol{v}_A - \boldsymbol{v}_B$ is the relative velocity between bodies A and B.

To proceed we need to establish a number of helpful results. First, we note that

$$\dot{z}_{AB} = \boldsymbol{n}_{AB} \cdot \boldsymbol{v}_{AB}, \qquad \dot{n}_{AB} = \frac{1}{z_{AB}} \bigl(v^a_{AB} - \dot{z}_{AB} n^a_{AB}\bigr).$$

Second, we work out expressions for the first three derivatives of the quadrupole-moment tensor. We evaluate these with the help of the Newtonian acceleration vector,

$$\boldsymbol{a}_A = -\sum_A \frac{G m_B}{z_{AB}^2} \boldsymbol{n}_{AB} + O(c^{-2}),$$

and after some straightforward computations, we obtain

$$\dot{I}^{ab} = \sum_A m_A \bigl(v^a_A z^b_A + z^a_A v^b_A\bigr), \tag{7.5.38}$$

$$\ddot{I}^{ab} = -\sum_{AB} \frac{G m_A m_B}{z_{AB}} n^a_{AB} n^b_{AB} + 2 \sum_A m_A v^a_A v^b_A, \tag{7.5.39}$$

$$I^{ab(3)} = \sum_{AB} \frac{G m_A m_B}{z_{AB}^2} \Bigl[3 \dot{z}_{AB} n^a_{AB} n^b_{AB} - 2\bigl(v^a_{AB} n^b_{AB} + n^a_{AB} v^b_{AB}\bigr)\Bigr]. \tag{7.5.40}$$

From this last expression we also get

$$I^{(3)}_{cc} = -\sum_{AB} \frac{G m_A m_B}{z_{AB}^2} \dot{z}_{AB}. \tag{7.5.41}$$

Returning to our main development, we notice in Eq. (7.5.37) that the first double sum can be expressed as

$$\frac{1}{3} I^{ab(3)} + \frac{2}{3} \sum_{AB} \frac{G m_A m_B}{z_{AB}^2} \bigl(v^a_{AB} n^b_{AB} + n^a_{AB} v^b_{AB}\bigr).$$

We notice also that the second double sum is equal to $-I^{(3)}_{cc}$. In addition, the third term can be expressed as $\frac{3}{10} I^{(5)}_{ab} \dot{I}^{ab}$, and the fourth term as $-\frac{1}{10} I^{(5)}_{cc} \dot{I}^{cc}$. With all this, Eq. (7.5.37) becomes

$$\begin{aligned} \dot{W} &= \frac{G}{c^5} \biggl[-\frac{1}{2} I^{(3)}_{ab} I^{ab(3)} - I^{(3)}_{ab} \sum_{AB} \frac{G m_A m_B}{z_{AB}^2} \bigl(v^a_{AB} n^b_{AB} + n^a_{AB} v^b_{AB} \bigr) \\ &\quad + \frac{1}{6} I^{(3)}_{cc} I^{cc(3)} + \frac{3}{10} I^{(5)}_{ab} \dot{I}^{ab} - \frac{1}{10} I^{(5)}_{cc} \dot{I}^{cc} + 2 I^{(4)}_{ab} \sum_A m_A v^a_A v^b_A \biggr]. \end{aligned} \tag{7.5.42}$$

In the final sequence of steps we distribute the time derivatives. We write, for example,

$$I^{(5)}_{ab} \dot{I}^{ab} = \frac{d}{dt} \Bigl(I^{(4)}_{ab} \dot{I}^{ab} - I^{(3)}_{ab} \ddot{I}^{ab} \Bigr) + I^{(3)}_{ab} I^{ab(3)}$$

and

$$I^{(4)}_{ab} \sum_A m_A v^a_A v^b_A = \frac{d}{dt} \Bigl(I^{(3)}_{ab} \sum_A m_A v^a_A v^b_A \Bigr) - I^{(3)}_{ab} \frac{d}{dt} \sum_A m_A v^a_A v^b_A.$$

The second term becomes

$$\frac{1}{2} I^{(3)}_{ab} \sum_{AB} \frac{G m_A m_B}{z_{AB}^2} \bigl(v^a_{AB} n^b_{AB} + n^a_{AB} v^b_{AB} \bigr),$$

and we obtain our final expression,

$$\begin{aligned} \dot{W} &= \frac{G}{c^5} \biggl\{ -\frac{1}{5} I^{(3)}_{ab} I^{ab(3)} + \frac{1}{15} I^{(3)}_{cc} I^{cc(3)} + \frac{d}{dt} \biggl[\frac{3}{10} \Bigl(I^{(4)}_{ab} \dot{I}^{ab} - I^{(3)}_{ab} \ddot{I}^{ab} \Bigr) \\ &\quad - \frac{1}{10} \Bigl(I^{(4)}_{cc} \dot{I}^{cc} - I^{(3)}_{cc} \ddot{I}^{cc} \Bigr) + 2 I^{(3)}_{ab} \sum_A m_A v^a_A v^b_A \biggr] \biggr\}. \end{aligned} \tag{7.5.43}$$

This is the desired energy-balance equation.

In view of Eq. (7.5.9), Eq. (7.5.43) can be written as

$$\dot{W} + \dot{E}_{\rm gw} = -\frac{d}{dt} E_{\rm bound}, \tag{7.5.44}$$

where $\dot{E}_{\rm gw}$ is the rate at which the gravitational waves carry energy away, and

$$\begin{aligned} E_{\rm bound} &:= -\frac{G}{c^5} \biggl[\frac{3}{10} \Bigl(I^{(4)}_{ab} \dot{I}^{ab} - I^{(3)}_{ab} \ddot{I}^{ab} \Bigr) - \frac{1}{10} \Bigl(I^{(4)}_{cc} \dot{I}^{cc} - I^{(3)}_{cc} \ddot{I}^{cc} \Bigr) \\ &\quad + 2 I^{(3)}_{ab} \sum_A m_A v^a_A v^b_A \biggr] \end{aligned} \tag{7.5.45}$$

is the piece of the gravitational-field energy that stays bound to the system. Equation (7.5.44) is a fine-grained statement of energy balance. Averaging over an appropriately selected time interval Δt gives rise to the coarse-grained statement

$$\langle \dot{W} \rangle = -\langle \dot{E}_{\rm gw} \rangle, \tag{7.5.46}$$

which says that on the average, the radiation-reaction forces do work at a rate that matches the rate at which energy is removed by radiation. Because $\dot{E}_{\rm gw} > 0$, the forces do negative work, the N-body system loses energy, and the effect occurs at the $\frac{5}{2}$PN order.

Notice that in order to arrive at Eq. (7.5.46), we have to assume that the net change in $E_{\rm bound}$ is zero over the time interval. This would be the case if the Newtonian motion is periodic, or if the system begins and ends in an unaccelerated

state. (Recall from Sec. 7.4.1 that the situation is very similar in the context of flat-spacetime electrodynamics.) Alternatively, we might absorb $E_{\rm bound}$ into a redefinition of the Newtonian energy, $E_{\rm new} = E_{\rm old} + E_{\rm bound}$, and write $\dot{W} = \dot{E}_{\rm new}$. In this language, Eq. (7.5.44) becomes $\dot{E}_{\rm new} = -\dot{E}_{\rm gw}$, and we preserve the fine-grained statement of energy balance. Because the definition of energy is ambiguous at $\frac{5}{2}$PN order, by virtue of the very fact that the dynamics is not conservative, this interpretation of the results is just as valid as the original, coarse-grained interpretation. We confess, however, a marked preference in favour of the original interpretation, because as we have argued back in Sec. 7.2.3, the very notion of a gravitational-wave flux involves an implicit coarse-graining operation. We therefore prefer to keep the coarse-graining explicit in Eqs. (7.5.44) and (7.5.46).

7.5.9 Momentum renormalization

We have seen that the terms involving $I^a_{\ cc}$ and $J^a_{\ cc}$ in Eq. (7.5.32) play no role in the energy balance. Indeed, these terms do not really belong to the radiation-reaction force, and they are best absorbed into a change of momentum variable.

Let us write the equations of motion in the form

$$\begin{aligned} \frac{d\boldsymbol{p}_A}{dt} &= m_A\Big[\boldsymbol{a}_A[0] + c^{-2}\boldsymbol{a}_A[2] + c^{-4}\boldsymbol{a}_A[4] + c^{-5}\bar{\boldsymbol{a}}_A[5] + O(c^{-6})\Big] \\ &\quad - \frac{Gm_A}{c^5}\biggl(\frac{2}{15}I^{a(5)}_{\ cc} + \frac{2}{3}J^{a(5)}_{\ cc}\biggr), \end{aligned}$$

where $\boldsymbol{p}_A := m_A\boldsymbol{v}_A$ is the Newtonian momentum of body A, and where $c^{-5}\bar{\boldsymbol{a}}[5]$ is what remains of the radiation-reaction force after removal of the terms involving $I^a_{\ cc}$ and $J^a_{\ cc}$. Because these are total time derivatives, they can be moved the left-hand side of the equation, which can then be written as

$$\frac{d\bar{\boldsymbol{p}}_A}{dt} = m_A\Big[\boldsymbol{a}_A[0] + c^{-2}\boldsymbol{a}_A[2] + c^{-4}\boldsymbol{a}_A[4] + c^{-5}\bar{\boldsymbol{a}}_A[5] + O(c^{-6})\Big],$$

where

$$\bar{p}^a_A = p^a_A + \frac{Gm_A}{c^5}\biggl(\frac{2}{15}I^{a(4)}_{\ cc} + \frac{2}{3}J^{a(4)}_{\ cc}\biggr)$$

is a new momentum variable. The terms in $I^a_{\ cc}$ and $J^a_{\ cc}$, therefore, are naturally interpreted as a $\frac{5}{2}$PN correction to the Newtonian momentum of each body. Adopting the new definition, these terms can be removed from the expression of Eq. (7.5.32) for the radiation-reaction force.

A consequence of this change is that the expression for the total momentum changes also, according to

$$\bar{P}^a = P^a + \frac{Gm}{c^5}\biggl(\frac{2}{15}I^{a(4)}_{\ cc} + \frac{2}{3}J^{a(4)}_{\ cc}\biggr),$$

where $m := \sum_A m_A$ is the total mass. With $\bar{\boldsymbol{a}}_A[\mathrm{rr}] = c^{-5}\bar{\boldsymbol{a}}_A[5]$ as radiation-reaction forces, $\bar{\boldsymbol{P}}$ is a constant of the motion, and the total momentum can be set equal to zero by placing the origin of the coordinate at the (corrected) barycentre.

7.5.10 Specialization to a two-body system

In the barycentric frame, the motion of each body in a two-body system is completely determined by the relative position vector $\boldsymbol{z} := \boldsymbol{z}_1 - \boldsymbol{z}_2$. At Newtonian order, $\boldsymbol{z}_1 = (m_2/m)\boldsymbol{z}$ and $\boldsymbol{z}_2 = -(m_1/m)\boldsymbol{z}$, where $m := m_1 + m_2$ is the total mass. We introduce also the relative velocity $\boldsymbol{v} := \boldsymbol{v}_1 - \boldsymbol{v}_2$, the relative acceleration

$\boldsymbol{a} = \boldsymbol{a}_1 - \boldsymbol{a}_2$, the dimensionless reduced mass $\eta := m_1 m_2/m^2$, and the unit vector $\boldsymbol{n} := \boldsymbol{z}/z$, where $z := |\boldsymbol{z}|$. The relative Newtonian acceleration is

$$\boldsymbol{a} = -\frac{Gm}{z^2}\boldsymbol{n} + O(c^{-2}), \tag{7.5.47}$$

and from Sec. 6.11.3 we recall the consequences

$$v\dot{v} = -\frac{Gm}{z^2}\dot{z} + O(c^{-2}), \qquad z\ddot{z} = v^2 - \dot{z}^2 - \frac{Gm}{z} + O(c^{-2}), \tag{7.5.48}$$

where $\dot{z} = \boldsymbol{n}\cdot\boldsymbol{v}$ is the radial component of the relative velocity vector.

According to Eq. (7.5.32), the radiation-reaction forces acting on bodies 1 and 2 produce the relative acceleration

$$\begin{aligned} a^a[\mathrm{rr}] &= \frac{G}{c^5}\biggl[-\frac{Gm}{z^2}\biggl(3I^{(3)}_{bc}n^b n^c + \frac{1}{3}I^{(3)}_{cc}\biggr)n^a \\ &\quad + \frac{3}{5}zI^{a(5)}_{\ b}n^b - \frac{1}{5}zI^{(5)}_{cc}n^a + 2I^{a(4)}_{\ c}v^c\biggr]. \end{aligned} \tag{7.5.49}$$

Notice that the terms involving $I^a_{\ cc}$ and $J^a_{\ cc}$ have once more dropped out of sight. The quadrupole-moment tensor is given by

$$I^{ab} = m\eta z^a z^b, \tag{7.5.50}$$

and this must differentiated a number of times in order to turn Eq. (7.5.49) into something fully explicit.

Making repeated use of Eqs. (7.5.47) and (7.4.48), a straightforward computation returns

$$I^{ab(3)} = \frac{2Gm^2\eta}{z^2}\Bigl[3\dot{z}n^a n^b - 2\bigl(v^a n^a + n^a v^b\bigr)\Bigr], \tag{7.5.51}$$

$$\begin{aligned} I^{ab(4)} &= \frac{2Gm^2\eta}{z^3}\biggl[\biggl(3v^2 - 15\dot{z}^2 + \frac{Gm}{z}\biggr)n^a n^b \\ &\quad - 4v^a v^b + 9\dot{z}\bigl(v^a n^a + n^a v^b\bigr)\biggr], \end{aligned} \tag{7.5.52}$$

$$\begin{aligned} I^{ab(5)} &= \frac{2Gm^2\eta}{z^4}\biggl[-15\bigl(3v^2 - 7\dot{z}^2\bigr)\dot{z}n^a n^b + 30\dot{z}v^a v^b \\ &\quad + 4\biggl(3v^2 - 15\dot{z}^2 - \frac{Gm}{z}\biggr)\bigl(v^a n^a + n^a v^b\bigr)\biggr]. \end{aligned} \tag{7.5.53}$$

Inserting these results within Eq. (7.4.49), we eventually arrive at

$$\boldsymbol{a}[\mathrm{rr}] = \frac{8G^2m^2\eta}{5c^5z^3}\biggl[\biggl(3v^2 + \frac{17}{3}\frac{Gm}{z}\biggr)\dot{z}\boldsymbol{n} - \biggl(v^2 + 3\frac{Gm}{z}\biggr)\boldsymbol{v}\biggr]. \tag{7.5.54}$$

This is the radiation-reaction force (per unit mass) acting on the relative orbital motion. This should be added to the right-hand side of Eq. (5.5.18) to account for the dissipative nature of the motion at $\frac{5}{2}$PN order. Its effect on the radius of a circular orbit is described by Eq. (7.3.13).

The radiation-reaction forces do work at a rate $\dot{W} = m_1\boldsymbol{a}_1[\mathrm{rr}]\cdot\boldsymbol{v}_1 + m_2\boldsymbol{a}_2[\mathrm{rr}]\cdot\boldsymbol{v}_2 = m\eta\boldsymbol{a}[\mathrm{rr}]\cdot\boldsymbol{v}$, and according to Eq. (7.5.54), this is

$$\dot{W} = \frac{8G^2m^3\eta^2}{5c^5z^3}\biggl[\biggl(3v^2 + \frac{17}{3}\frac{Gm}{z}\biggr)\dot{z}^2 - \biggl(v^2 + 3\frac{Gm}{z}\biggr)v^2\biggr]. \tag{7.5.55}$$

The rate at which energy is radiated in the form of gravitational waves was computed in Sec. 7.3.1, and according to Eq. (7.3.11), this is

$$\dot{E}_{\rm gw} = \frac{8G^3m^4\eta^2}{15c^5z^4}\bigl(12v^2 - 11\dot{z}^2\bigr). \tag{7.5.56}$$

It is easy to verify that these rates are related by the (fine-grained) energy-balance equation

$$\dot{W} + \dot{E}_{\rm gw} = -\frac{d}{dt}E_{\rm bound}, \tag{7.5.57}$$

where

$$E_{\rm bound} = \frac{8G^2m^3\eta^2}{5c^5}\frac{\dot{z}v^2}{z^2}. \tag{7.5.58}$$

This expression for $E_{\rm bound}$ can be obtained on the basis of Eq. (7.5.45), using the results of Eqs. (7.5.51) and (7.5.52).

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