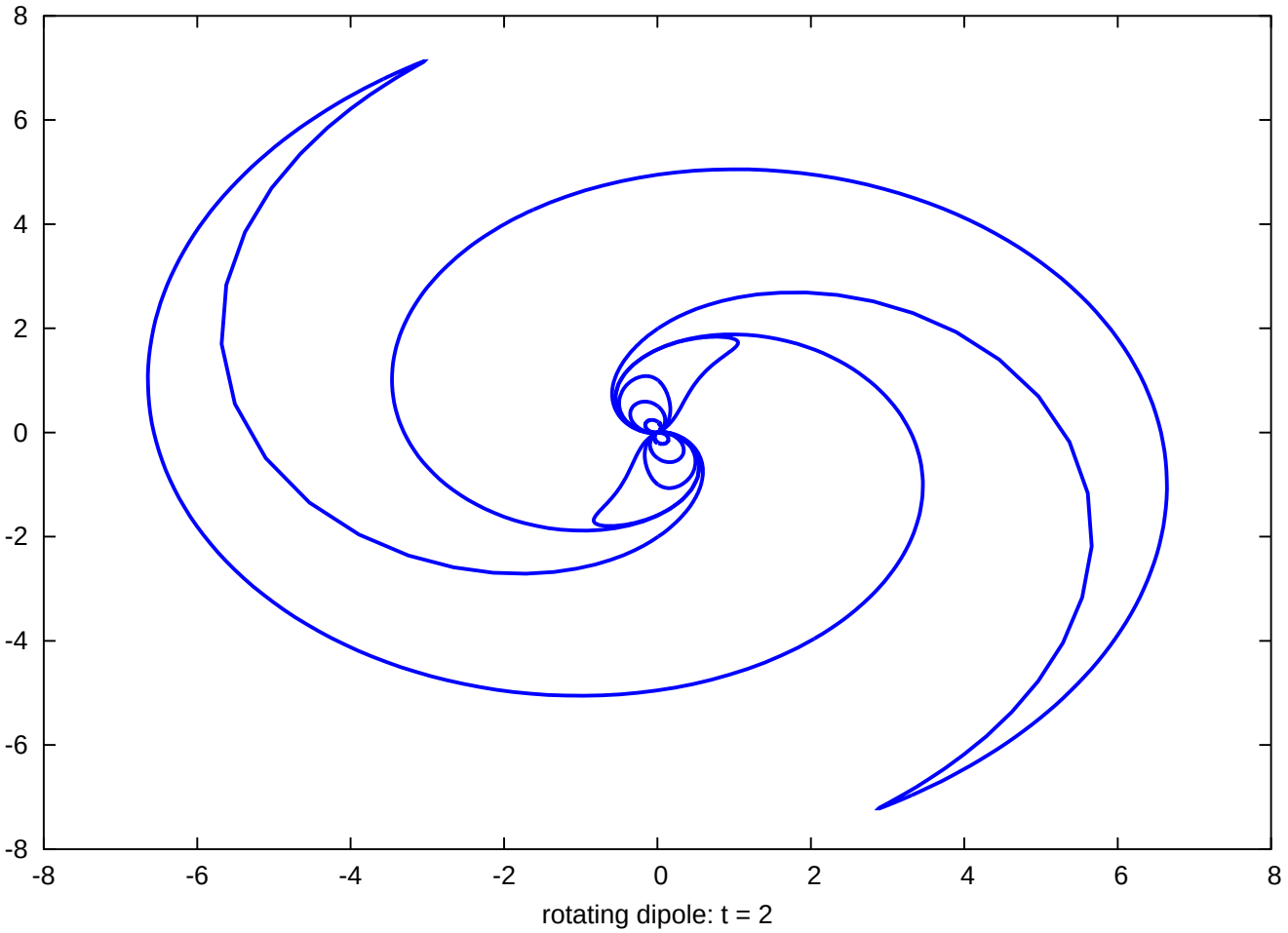
Electromagnetic theory
PHYS*7060

Lecture notes (July 2005)



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Chapter 1
Maxwell's electrodynamics

1.1 Dynamical variables of electromagnetism

The classical theory of the electromagnetic field, as formulated by Maxwell, involves the vector fields

$$\boldsymbol{E}(t,\boldsymbol{x}) \equiv \text{electric field at position } \boldsymbol{x} \text{ and time } t \tag{1.1.1}$$

and

$$\boldsymbol{B}(t,\boldsymbol{x}) \equiv \text{magnetic field at position } \boldsymbol{x} \text{ and time } t. \tag{1.1.2}$$

We have two vectors at each position of space and at each moment of time. The dynamical system is therefore much more complicated than in mechanics, in which there is a *finite number* of degrees of freedom. Here the number of degrees of freedom is infinite.

The electric and magnetic fields are produced by charges and currents. In a classical theory these are best described in terms of a *fluid picture* in which the charge and current distributions are imagined to be continuous (and not made of pointlike charge carriers). Although this is not a true picture of reality, these continuous distributions fit naturally within a classical treatment of electrodynamics. This will be our point of view here, but we shall see that the formalism is robust enough to also allow for a description in terms of point particles.

An *element of charge* is a macroscopically small (but microscopically large) portion of matter that contains a net charge. An element of charge is located at position $\boldsymbol{x}$ and has a volume dV. It moves with a velocity $\boldsymbol{v}$ that depends on time and on position. The volume of a charge element must be sufficiently large that it contains a macroscopic number of elementary charges, but sufficiently small that the density of charge within the volume is uniform to a high degree of accuracy.

In the mathematical description, an element of charge at position $\boldsymbol{x}$ is idealized as the point $\boldsymbol{x}$ itself. The quantities that describe the charge and current distributions are

$$\rho(t,\boldsymbol{x}) \equiv \text{density of charge at position } \boldsymbol{x} \text{ and time } t, \tag{1.1.3}$$

$$\boldsymbol{v}(t,\boldsymbol{x}) \equiv \text{velocity of an element of charge at position } \boldsymbol{x} \text{ and time } t, \tag{1.1.4}$$

$$\boldsymbol{j}(t,\boldsymbol{x}) \equiv \text{current density at position } \boldsymbol{x} \text{ and time } t. \tag{1.1.5}$$

We shall now establish the important relation

$$\boldsymbol{j} = \rho\boldsymbol{v}. \tag{1.1.6}$$

The current density $\boldsymbol{j}$ is defined by the statement

$$\boldsymbol{j}\cdot d\boldsymbol{a} \equiv \text{current crossing an element of area } d\boldsymbol{a}, \tag{1.1.7}$$

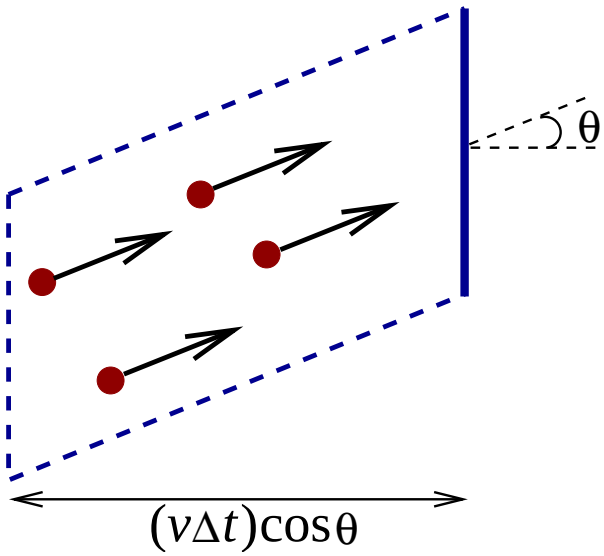

Figure 1.1: Point charges hitting a small surface.

so that $\boldsymbol{j}$ is the current per unit area. The current flowing across a surface S is then

$$\int_S \boldsymbol{j} \cdot d\boldsymbol{a}.$$

On the other hand, current is defined as the quantity of charge crossing a surface per unit time. Let us then calculate the current associated with a distribution of charge with density ρ and velocity $\boldsymbol{v}$.

Figure 1.1 shows a number of charged particles approaching with speed v a small surface of area Δa; there is an angle θ between the direction of the velocity vector $\boldsymbol{v}$ and the normal $\hat{\boldsymbol{n}}$ to the surface. The particles within the dashed box will all hit the surface within a time Δt. The volume of this box is $(v\Delta t \cos\theta)\Delta a = \boldsymbol{v}\cdot\hat{\boldsymbol{n}}\Delta t\Delta a$. The current crossing the surface is then calculated as

$$\begin{aligned}
\text{current} &= (\text{charge crossing the surface})/\Delta t \\
&= (\text{total charge})\left(\frac{\text{volume of box}}{\text{total volume}}\right)\frac{1}{\Delta t} \\
&= (\text{density})(\text{volume of box})/\Delta t \\
&= \rho(\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\Delta t\Delta a)/\Delta t \\
&= \rho\boldsymbol{v}\cdot(\hat{\boldsymbol{n}}\Delta a) \\
&= \rho\boldsymbol{v}\cdot\Delta\boldsymbol{a}.
\end{aligned}$$

Comparing this with the expression given previously, current $= \boldsymbol{j}\cdot\Delta\boldsymbol{x}$, we find that indeed, $\rho\boldsymbol{v}$ is the current density.

As was mentioned previously, the fluid description still allows for the existence of point charges. For these, the density is zero everywhere except at the position of a charge, where it is infinite. This situation can be described by a δ-function. Suppose that we have a point charge q at a position $\boldsymbol{r}$. Its charge density can be written as

$$\rho(\boldsymbol{x}) = q\delta(\boldsymbol{x}-\boldsymbol{r}),$$

where $\delta(\boldsymbol{x}) \equiv \delta(x)\delta(y)\delta(z)$ is a three-dimensional δ-function. If the charge is moving with velocity $\boldsymbol{v}$, then

$$\boldsymbol{j}(\boldsymbol{x}) = q\boldsymbol{v}\delta(\boldsymbol{x}-\boldsymbol{r})$$

is the current density.

More generally, let us have a collection of point charges q_A at positions $\boldsymbol{r}_A(t)$, moving with velocities $\boldsymbol{v}_A(t) = d\boldsymbol{r}_A/dt$. Then the charge and current densities of the charge distribution are given by

$$\rho(t,\boldsymbol{x}) = \sum_A q_A \delta\big(\boldsymbol{x} - \boldsymbol{r}_A(t)\big) \tag{1.1.8}$$

and

$$\boldsymbol{j}(t,\boldsymbol{x}) = \sum_A q_A \boldsymbol{v}_A(t) \delta\big(\boldsymbol{x} - \boldsymbol{r}_A(t)\big). \tag{1.1.9}$$

Each q_A is the integral of $\rho(t,\boldsymbol{x})$ over a volume V_A that encloses this charge but no other:

$$q_A = \int_{V_A} \rho(t,\boldsymbol{x})\, d^3x, \tag{1.1.10}$$

where $d^3x \equiv dxdydz$. The *total charge* of the distribution is obtained by integrating the density over all space:

$$Q \equiv \int \rho(t,\boldsymbol{x})\, d^3x = \sum_A q_A. \tag{1.1.11}$$

1.2 Maxwell's equations and the Lorentz force

The four Maxwell equations determine the electromagnetic field once the charge and current distributions are specified. They are given by

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{1}{\epsilon_0}\rho, \tag{1.2.1}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \tag{1.2.2}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{1.2.3}$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{j} + \epsilon_0\mu_0\frac{\partial \boldsymbol{E}}{\partial t}. \tag{1.2.4}$$

Here, ϵ_0 and μ_0 are constants, and $\boldsymbol{\nabla} = (\nabla_x, \nabla_y, \nabla_z)$ is the gradient operator of vector calculus (with the obvious notation $\nabla_x f = \partial f/\partial x$ for any function f). The Maxwell equations state that the electric field is produced by charges and time-varying magnetic fields, while the magnetic field is produced by currents and time-varying electric fields. Maxwell's equations can also be presented in integral form, by invoking the Gauss and Stokes theorems of vector calculus.

The Lorentz-force law determines the motion of the charges once the electromagnetic field is specified. Let

$$\boldsymbol{f}(t,\boldsymbol{x}) \equiv \text{force density at position } \boldsymbol{x} \text{ and time } t, \tag{1.2.5}$$

where the force density is defined to be the net force acting on an element of charge at $\boldsymbol{x}$ divided by the volume of this charge element. The statement of the Lorentz-force law is then

$$\boldsymbol{f} = \rho\boldsymbol{E} + \boldsymbol{j}\times\boldsymbol{B} = \rho\big(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\big). \tag{1.2.6}$$

The net force $\boldsymbol{F}$ acting on a volume V of the charge distribution is the integral of the force density over this volume:

$$\boldsymbol{F}(t,V) = \int_V \boldsymbol{f}(t,\boldsymbol{x})\, d^3x. \tag{1.2.7}$$

For a single point charge we have $\rho = q\delta(\boldsymbol{x}-\boldsymbol{r})$, $\boldsymbol{j} = q\boldsymbol{v}\delta(\boldsymbol{x}-\boldsymbol{r})$, and the total force becomes

$$\boldsymbol{F} = q\big[\boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{r})\big].$$

Here the fields are evaluated at the charge's position. This is the usual expression for the Lorentz force, but the definition of Eq. (1.2.6) is more general.

Taken together, the Maxwell equations and the Lorentz-force law determine the behaviour of the charge and current distributions, and the evolution of the electric and magnetic fields. Those five equations summarize the *complete theory* of classical electrodynamics. Every conceivable phenomenon involving electromagnetism can be predicted from them.

1.3 Conservation of charge

One of the most fundamental consequences of Maxwell's equations is that *charge is locally conserved*: charge can move around but it cannot be destroyed nor created.

Consider a volume V bounded by a two-dimensional surface S. Charge conservation means that the rate of decrease of charge within V must be equal to the total current flowing across S:

$$-\frac{d}{dt}\int_V \rho\, d^3x = \oint_S \boldsymbol{j}\cdot d\boldsymbol{a}. \tag{1.3.1}$$

Local charge conservation means that this statement must be true for any volume V, however small or large. We may use Gauss' theorem to turn Eq. (1.3.1) into a differential statement. For any smooth vector field $\boldsymbol{u}$ within V we have

$$\int_V \boldsymbol{\nabla}\cdot\boldsymbol{u}\, d^3x = \oint_S \boldsymbol{u}\cdot d\boldsymbol{a}.$$

The right-hand side of Eq. (1.3.1) can thus be written as $\int_V \boldsymbol{\nabla}\cdot\boldsymbol{j}\, d^3x$, while the left-hand side is equal to $-\int_V(\partial\rho/\partial t)\, d^3x$. Equality of the two sides for arbitrary volumes V implies

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0. \tag{1.3.2}$$

This is the differential statement of local charge conservation, and we would like to prove that this comes as a consequence of Maxwell's equations.

To establish this we first differentiate Eq. (1.2.1) with respect to time to obtain

$$\frac{\partial\rho}{\partial t} = \epsilon_0\boldsymbol{\nabla}\cdot\frac{\partial\boldsymbol{E}}{\partial t},$$

where we have exchanged the order with which we take derivatives of the electric field. If we now use Eq. (1.2.4) to eliminate the electric field, we get

$$\frac{\partial\rho}{\partial t} = \frac{1}{\mu_0}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) - \boldsymbol{\nabla}\cdot\boldsymbol{j}.$$

The first term on the right-hand side vanishes identically (the divergence of a curl is always zero) and we arrive at Eq. (1.3.2). We have therefore established that local charge conservation is a consequence of Maxwell's equations.

1.4 Conservation of energy

[The material presented in this section is also covered in Sec. 6.7 of Jackson's text.]

Other conservation statements follow from Maxwell's equations and the Lorentz-force law. In this section we formulate and derive a statement of energy conservation. In the next section we will consider the conservation of linear momentum. It is also possible to prove conservation of angular momentum, but we shall not pursue this here.

Conservation of energy is one of the most fundamental principle of physics, and Maxwell's electrodynamics must be compatible with it. A statement of local energy conservation can be patterned after our previous statement of charge conservation. Let

$$\varepsilon(t,\boldsymbol{x}) \equiv \text{electromagnetic field energy density,} \tag{1.4.1}$$

$$\boldsymbol{S}(t,\boldsymbol{x})\cdot d\boldsymbol{a} \equiv \text{field energy crossing an element of surface } d\boldsymbol{a} \text{ per unit time.} \tag{1.4.2}$$

So ε is analogous to charge density, and $\boldsymbol{S}$ (which is known as *Poynting's vector*), is analogous to current density.

If electromagnetic field energy were locally conserved, we would write the statement

$$-\frac{d}{dt}\int_V \varepsilon\, d^3x = \oint_S \boldsymbol{S}\cdot d\boldsymbol{a},$$

which is analogous to Eq. (1.3.1). But we should *not* expect field energy to be conserved, and this statement is *not correct*. The reason is that the field does work on the charge distribution, and this takes energy away from the field. This energy goes to the charge distribution, and *total energy* is conserved. A correct statement of energy conservation would therefore be (field energy leaving V per unit time) = (field energy crossing S per unit time) + (work done on charges within V per unit time). To figure out what the work term should be, consider an element of charge within V. It moves with a velocity $\boldsymbol{v}$ and the net force acting on it is

$$\boldsymbol{f}\,dV = \rho(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\,dV.$$

While the element undergoes a displacement $d\boldsymbol{x}$, the force does a quantity of work equal to $\boldsymbol{f}\cdot d\boldsymbol{x}\,dV$. The work done per unit time is then $\boldsymbol{f}\cdot\boldsymbol{v}\,dV = \rho\boldsymbol{E}\cdot\boldsymbol{v}\,dV = \boldsymbol{j}\cdot\boldsymbol{E}\,dV$. Integrating this over V gives the total work done on the charges contained in V, per unit time. The correct statement of energy conservation must therefore have the form

$$-\frac{d}{dt}\int_V \varepsilon\, d^3x = \oint_S \boldsymbol{S}\cdot d\boldsymbol{a} + \int_V \boldsymbol{j}\cdot\boldsymbol{E}\, d^3x. \tag{1.4.3}$$

In differential form, this is

$$\frac{\partial\varepsilon}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{S} = -\boldsymbol{j}\cdot\boldsymbol{E}. \tag{1.4.4}$$

This last equation must be derivable from Maxwell's equations, which have not yet been involved. And indeed, the derivation should provide *expressions* for the quantities ε and $\boldsymbol{S}$. What we have at this stage is an educated guess for a correct statement of energy conservation, but Eq. (1.4.4) has not yet been derived nor the the quantities ε and $\boldsymbol{S}$ properly defined in terms of field variables. Now the hard work begins.

We start with the right-hand side of Eq. (1.4.4) and eliminate $\boldsymbol{j}$ in favour of field variables using Eq. (1.2.4). This gives

$$-\boldsymbol{j}\cdot\boldsymbol{E} = -\frac{1}{\mu_0}\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) + \epsilon_0\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}.$$

We use a vector-calculus identity to replace $\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})$ with $\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E})-\boldsymbol{\nabla}\cdot(\boldsymbol{E}\times\boldsymbol{B})$, and in this we replace $\boldsymbol{\nabla}\times\boldsymbol{E}$ with $-\partial\boldsymbol{B}/\partial t$, using Eq. (1.2.3). All of this gives us

$$\begin{aligned} -\boldsymbol{j}\cdot\boldsymbol{E} &= \frac{1}{\mu_0}\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t}+\epsilon_0\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t}+\frac{1}{\mu_0}\boldsymbol{\nabla}\cdot(\boldsymbol{E}\times\boldsymbol{B}) \\ &= \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0\boldsymbol{E}^2+\frac{1}{2\mu_0}\boldsymbol{B}^2\right)+\boldsymbol{\nabla}\cdot\left(\frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}\right). \end{aligned}$$

We have arrived at an equation of the same form as Eq. (1.4.4), and this shows that we do indeed have energy conservation as a consequence of Maxwell's equations and the Lorentz-force law.

The calculation also gives us definitions for the field's energy density,

$$\varepsilon\equiv\frac{1}{2}\epsilon_0\boldsymbol{E}^2+\frac{1}{2\mu_0}\boldsymbol{B}^2, \tag{1.4.5}$$

and for the field's energy flux (Poynting) vector,

$$\boldsymbol{S}\equiv\frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}. \tag{1.4.6}$$

It is an important consequence of Maxwell's theory that the electromagnetic field carries its own energy.

1.5 Conservation of momentum

[The material presented in this section is also covered in Sec. 6.7 of Jackson's text.]

We should expect a statement of momentum conservation to take a form similar to our previous statement of energy conservation. In that case we had a scalar quantity ε representing the density of energy, a vectorial quantity S_a representing the flux of energy, and the conservation statement took the form of

$$\frac{\partial\varepsilon}{\partial t}+\nabla_a S_a=\text{work term}.$$

We have introduced an explicit component notation for vectors, and summation over a repeated index is understood.

For momentum conservation we will need a vectorial quantity ε_a to represent the density of momentum, and a *tensorial quantity* T_{ab} to represent the flux of momentum (one index for the momentum component, another index for the flux direction). So let

$$\varepsilon_a(t,\boldsymbol{x}) \equiv \text{density of } a\text{-component of field momentum}, \tag{1.5.1}$$

$$-T_{ab}(t,\boldsymbol{x})\,da_b \equiv a\text{-component of field momentum crossing an element of surface } d\boldsymbol{a} \text{ per unit time}. \tag{1.5.2}$$

The minus sign in front of T_{ab} is introduced by convention.

The statement of momentum conservation is (field momentum leaving V per unit time) = (field momentum crossing S per unit time) + (rate at which momentum is communicated to the charges). Because a rate of change of momentum is a force, we have the integral statement

$$-\frac{d}{dt}\int_V\varepsilon_a\,d^3x=-\oint_S T_{ab}\,da_b+\int_V f_a\,d^3x, \tag{1.5.3}$$

or the equivalent differential statement

$$-\frac{\partial\varepsilon_a}{\partial t}+\nabla_b T_{ab}=f_a=\rho E_a+(\boldsymbol{j}\times\boldsymbol{B})_a. \tag{1.5.4}$$

We now would like to derive this from Maxwell's equations, and discover the identities of the vector ε_a and the tensor T_{ab}.

We first use Eqs. (1.2.1) and (1.2.4) to eliminate ρ and $\boldsymbol{j}$ from Eq. (1.5.4):

$$\begin{aligned}\boldsymbol{f} &= (\epsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E})\boldsymbol{E}+\left(\frac{1}{\mu_0}\boldsymbol{\nabla}\times\boldsymbol{B}-\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right)\times\boldsymbol{B}\\ &= \epsilon_0(\boldsymbol{\nabla}\cdot\boldsymbol{E})\boldsymbol{E}+\frac{1}{\mu_0}(\boldsymbol{\nabla}\cdot\boldsymbol{B})\boldsymbol{B}+\frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}\\ &\quad -\epsilon_0\frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{B})+\epsilon_0\boldsymbol{E}\times\frac{\partial\boldsymbol{B}}{\partial t},\end{aligned}$$

where to go from the first to the second line we have inserted a term involving $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$ and allowed the time derivative to operate on $\boldsymbol{E}\times\boldsymbol{B}$. The next step is to use Eq. (1.2.3) to replace $\partial\boldsymbol{B}/\partial t$ with $-\boldsymbol{\nabla}\times\boldsymbol{E}$. Collecting terms, this gives

$$\boldsymbol{f}=-\epsilon_0\frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{B})+\epsilon_0\big[(\boldsymbol{\nabla}\cdot\boldsymbol{E})\boldsymbol{E}-\boldsymbol{E}\times(\boldsymbol{\nabla}\times\boldsymbol{E})\big]+\frac{1}{\mu_0}\big[(\boldsymbol{\nabla}\cdot\boldsymbol{B})\boldsymbol{B}-\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{B})\big].$$

To proceed we invoke the vector-calculus identity $\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{u})=(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}+\boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})$, where $\boldsymbol{u}$ stands for any vector field. We use this to clean up the quantities within square brackets. For example,

$$(\boldsymbol{\nabla}\cdot\boldsymbol{E})\boldsymbol{E}-\boldsymbol{E}\times(\boldsymbol{\nabla}\times\boldsymbol{E})=(\boldsymbol{\nabla}\cdot\boldsymbol{E})\boldsymbol{E}+(\boldsymbol{E}\cdot\boldsymbol{\nabla})\boldsymbol{E}-\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{E}\cdot\boldsymbol{E}).$$

In components, the right-hand side reads

$$\begin{aligned}(\nabla_bE_b)E_a+(E_b\nabla_b)E_a-\frac{1}{2}\nabla_a\boldsymbol{E}^2 &= E_a\nabla_bE_b+E_b\nabla_bE_a-\frac{1}{2}\delta_{ab}\nabla_b\boldsymbol{E}^2\\ &= \nabla_b\left(E_aE_b-\frac{1}{2}\delta_{ab}\boldsymbol{E}^2\right).\end{aligned}$$

Doing the same for the bracketed terms involving the magnetic field, we arrive at

$$f_a=-\epsilon_0\frac{\partial}{\partial t}(\boldsymbol{E}\times\boldsymbol{B})_a+\epsilon_0\nabla_b\left(E_aE_b-\frac{1}{2}\delta_{ab}\boldsymbol{E}^2\right)+\frac{1}{\mu_0}\nabla_b\left(B_aB_b-\frac{1}{2}\delta_{ab}\boldsymbol{B}^2\right).$$

This has the form of Eq. (1.5.4) and we conclude that momentum conservation does indeed follow from Maxwell's equations and the Lorentz-force law.

Our calculation also tells us that the field's momentum density is given by

$$\varepsilon_a\equiv\epsilon_0(\boldsymbol{E}\times\boldsymbol{B})_a, \tag{1.5.5}$$

and that the field's momentum flux tensor is

$$T_{ab}=\epsilon_0\left(E_aE_b-\frac{1}{2}\delta_{ab}\boldsymbol{E}^2\right)+\frac{1}{\mu_0}\left(B_aB_b-\frac{1}{2}\delta_{ab}\boldsymbol{B}^2\right). \tag{1.5.6}$$

Notice that the momentum density is proportional to the Poynting vector: $\boldsymbol{\varepsilon}=\epsilon_0\mu_0\boldsymbol{S}$. This means that the field's momentum points in the same direction as the flow of energy, and that these quantities are related by a constant factor of $\epsilon_0\mu_0$.

1.6 Junction conditions

[The material presented in this section is also covered in Sec. I.5 of Jackson's text.]

Maxwell's equations determine how the electric and magnetic fields must be joined at an *interface*. The interface might be just a mathematical boundary (in

which case nothing special should happen), or it might support a surface layer of charge and/or current. Our task in this section is to formulate these junction conditions.

To begin, we keep things simple by supposing that the interface is the x-y plane at $z = 0$ — a nice, flat surface. If the interface supports a charge distribution, then the charge density must have the form

$$\rho(t, \boldsymbol{x}) = \rho_+(t, \boldsymbol{x})\theta(z) + \rho_-(t, \boldsymbol{x})\theta(-z) + \sigma(t, x, y)\delta(z), \tag{1.6.1}$$

where ρ_+ is the charge density above the interface (in the region $z > 0$), ρ_- the charge density below the interface (in the region $z < 0$), and σ is the surface density of charge (charge per unit surface area) supported by the interface.

We have introduced the Heaviside step function $\theta(z)$, which is equal to 1 if $z > 0$ and 0 otherwise, and the Dirac δ-function, which is such that $\int f(z)\delta(z)\,dz = f(0)$ for any smooth function $f(z)$. These distributions (also known as generalized functions) are related by the identities

$$\frac{d}{dz}\theta(z) = \delta(z), \qquad \frac{d}{dz}\theta(-z) = -\delta(z). \tag{1.6.2}$$

As for any distributional identity, these relations can be established by integrating both sides against a test function $f(z)$. (Test functions are required to be smooth and to fall off sufficiently rapidly as $z \to \pm\infty$.) For example,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d\theta(z)}{dz} f(z)\,dz &= -\int_{-\infty}^{\infty} \theta(z)\frac{df(z)}{dz}\,dz \\ &= -\int_0^{\infty} df \\ &= f(0), \end{aligned}$$

and this shows that $d\theta(z)/dz$ is indeed distributionally equal to $\delta(z)$. You can show similarly that $f(z)\delta(z) = f(0)\delta(z)$ and $f(z)\delta'(z) = f(0)\delta'(z) - f'(0)\delta(z)$ are valid distributional identities (a prime indicates differentiation with respect to z).

If the interface at $z = 0$ also supports a current distribution, then the current density must have the form

$$\boldsymbol{j}(t, \boldsymbol{x}) = \boldsymbol{j}_+(t, \boldsymbol{x})\theta(z) + \boldsymbol{j}_-(t, \boldsymbol{x})\theta(-z) + \boldsymbol{K}(t, x, y)\delta(z), \tag{1.6.3}$$

where $\boldsymbol{j}_+$ is the current density above the interface, $\boldsymbol{j}_-$ the current density below the interface, and $\boldsymbol{K}$ is the surface density of current (charge per unit time and unit length) supported by the interface.

We now would like to determine how $\boldsymbol{E}$ and $\boldsymbol{B}$ behave across the interface. We can express the fields as

$$\begin{aligned} \boldsymbol{E}(t, \boldsymbol{x}) &= \boldsymbol{E}_+(t, \boldsymbol{x})\theta(z) + \boldsymbol{E}_-(t, \boldsymbol{x})\theta(-z), \\ \boldsymbol{B}(t, \boldsymbol{x}) &= \boldsymbol{B}_+(t, \boldsymbol{x})\theta(z) + \boldsymbol{B}_-(t, \boldsymbol{x})\theta(-z), \end{aligned} \tag{1.6.4}$$

in an obvious notation; for example, $\boldsymbol{E}_+$ is the electric field above the interface. We will see that these fields are solutions to Maxwell's equations with sources given by Eqs. (1.6.1) and (1.6.3). To prove this we need simply substitute Eq. (1.6.4) into Eqs. (1.2.1)–(1.2.4). Differentiating the coefficients of $\theta(\pm z)$ is straightforward, but we must also differentiate the step functions. For this we use Eq. (1.6.2) and write

$$\boldsymbol{\nabla}\theta(\pm z) = \pm\delta(z)\hat{\boldsymbol{z}},$$

where $\hat{\boldsymbol{z}}$ is a unit vector that points in the z direction.

A straightforward calculation gives

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{E} &= \big(\boldsymbol{\nabla}\cdot\boldsymbol{E}_+\big)\theta(z) + \big(\boldsymbol{\nabla}\cdot\boldsymbol{E}_-\big)\theta(-z) + \hat{\boldsymbol{z}}\cdot\big(\boldsymbol{E}_+ - \boldsymbol{E}_-\big)\delta(z),\\
\boldsymbol{\nabla}\times\boldsymbol{E} &= \big(\boldsymbol{\nabla}\times\boldsymbol{E}_+\big)\theta(z) + \big(\boldsymbol{\nabla}\times\boldsymbol{E}_-\big)\theta(-z) + \hat{\boldsymbol{z}}\times\big(\boldsymbol{E}_+ - \boldsymbol{E}_-\big)\delta(z),\\
\boldsymbol{\nabla}\cdot\boldsymbol{B} &= \big(\boldsymbol{\nabla}\cdot\boldsymbol{B}_+\big)\theta(z) + \big(\boldsymbol{\nabla}\cdot\boldsymbol{B}_-\big)\theta(-z) + \hat{\boldsymbol{z}}\cdot\big(\boldsymbol{B}_+ - \boldsymbol{B}_-\big)\delta(z),\\
\boldsymbol{\nabla}\times\boldsymbol{B} &= \big(\boldsymbol{\nabla}\times\boldsymbol{B}_+\big)\theta(z) + \big(\boldsymbol{\nabla}\times\boldsymbol{B}_-\big)\theta(-z) + \hat{\boldsymbol{z}}\times\big(\boldsymbol{B}_+ - \boldsymbol{B}_-\big)\delta(z).
\end{aligned}$$

Substituting Eq. (1.6.1) and the expression for $\boldsymbol{\nabla}\cdot\boldsymbol{E}$ into the first of Maxwell's equations, $\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho/\epsilon_0$, reveals that $\boldsymbol{E}_\pm$ is produced by $\rho_\pm$ and that the surface charge distribution creates a discontinuity in the normal component of the electric field:

$$\hat{\boldsymbol{z}}\cdot\big(\boldsymbol{E}_+ - \boldsymbol{E}_-\big)\big|_{z=0} = \frac{1}{\epsilon_0}\sigma. \tag{1.6.5}$$

Substituting the expression for $\boldsymbol{\nabla}\cdot\boldsymbol{B}$ into the second of Maxwell's equations, $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, reveals that $\boldsymbol{B}_\pm$ satisfies this equation on both sides of the interface and that the normal component of the magnetic field must be continuous:

$$\hat{\boldsymbol{z}}\cdot\big(\boldsymbol{B}_+ - \boldsymbol{B}_-\big)\big|_{z=0} = 0. \tag{1.6.6}$$

Substituting Eq. (1.6.4) and the expression for $\boldsymbol{\nabla}\times\boldsymbol{E}$ into the third of Maxwell's equations, $\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$, reveals that $\boldsymbol{E}_\pm$ and $\boldsymbol{B}_\pm$ satisfy this equation on both sides of the interface and that the tangential (x and y) components of the electric must also be continuous:

$$\hat{\boldsymbol{z}}\times\big(\boldsymbol{E}_+ - \boldsymbol{E}_-\big)\big|_{z=0} = \boldsymbol{0}. \tag{1.6.7}$$

Finally, substituting Eqs. (1.6.3), (1.6.4), and the expression for $\boldsymbol{\nabla}\times\boldsymbol{B}$ into the fourth of Maxwell's equations, $\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{j} + \epsilon_0\mu_0\partial\boldsymbol{E}/\partial t$, reveals that $\boldsymbol{B}_\pm$ is produced by $\boldsymbol{j}_\pm$ and that the surface current creates a discontinuity in the tangential components of the magnetic field:

$$\hat{\boldsymbol{z}}\times\big(\boldsymbol{B}_+ - \boldsymbol{B}_-\big)\big|_{z=0} = \mu_0\boldsymbol{K}. \tag{1.6.8}$$

Equations (1.6.5)–(1.6.8) tell us how the fields $\boldsymbol{E}_\pm$ and $\boldsymbol{B}_\pm$ are to be joined at a planar interface.

It is not difficult to generalize this discussion to an interface of arbitrary shape. All that is required is to change the argument of the step and δ functions from z to $s(\boldsymbol{x})$, where s is the distance from the interface in the direction normal to the interface; this is positive above the interface and negative below the interface. Then $\boldsymbol{\nabla}s = \hat{\boldsymbol{n}}$, the interface's unit normal, replaces $\hat{\boldsymbol{z}}$ in the preceding equations. If $\boldsymbol{E}_+$, $\boldsymbol{B}_+$ are the fields just above the interface, and $\boldsymbol{E}_-$, $\boldsymbol{B}_-$ the fields just below the interface, then the general junction conditions are

$$\hat{\boldsymbol{n}}\cdot\big(\boldsymbol{E}_+ - \boldsymbol{E}_-\big) = \frac{1}{\epsilon_0}\sigma, \tag{1.6.9}$$

$$\hat{\boldsymbol{n}}\cdot\big(\boldsymbol{B}_+ - \boldsymbol{B}_-\big) = 0, \tag{1.6.10}$$

$$\hat{\boldsymbol{n}}\times\big(\boldsymbol{E}_+ - \boldsymbol{E}_-\big) = \boldsymbol{0}, \tag{1.6.11}$$

$$\hat{\boldsymbol{n}}\times\big(\boldsymbol{B}_+ - \boldsymbol{B}_-\big) = \mu_0\boldsymbol{K}. \tag{1.6.12}$$

Here, σ is the surface charge density on the general interface, and $\boldsymbol{K}$ is the surface current density. It is understood that the normal vector points from the minus side to the plus side of the interface.

1.7 Potentials

[The material presented in this section is also covered in Secs. 6.2 and 6.3 of Jackson's text.]

The Maxwell equation $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$ and the mathematical identity $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})=0$ imply that the magnetic field can be expressed as the curl of a *vector potential* $\boldsymbol{A}(t,\boldsymbol{x})$:

$$\boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A}. \tag{1.7.1}$$

Suppose that $\boldsymbol{B}$ is expressed in this form. Then the other inhomogeneous Maxwell equation, $\boldsymbol{\nabla}\times\boldsymbol{E}+\partial\boldsymbol{B}/\partial t=0$, can be cast in the form

$$\boldsymbol{\nabla}\times\left(\boldsymbol{E}+\frac{\partial\boldsymbol{A}}{\partial t}\right)=0.$$

This, together with the mathematical identity $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\Phi)=\boldsymbol{0}$, imply that $\boldsymbol{E}+\partial\boldsymbol{A}/\partial t$ can be expressed as the divergence of a *scalar potential* $\Phi(t,\boldsymbol{x})$:

$$\boldsymbol{E}=-\frac{\partial\boldsymbol{A}}{\partial t}-\boldsymbol{\nabla}\Phi; \tag{1.7.2}$$

the minus sign in front of $\boldsymbol{\nabla}\Phi$ is conventional.

Introducing the scalar and vector potentials eliminates two of the four Maxwell equations. The remaining two equations will give us equations to be satisfied by the potentials. Solving these is often much simpler than solving the original equations. An issue that arises is whether the scalar and vector potentials are *unique*: While $\boldsymbol{E}$ and $\boldsymbol{B}$ can both be obtained uniquely from Φ and $\boldsymbol{A}$, is the converse true? The answer is in the negative — the potentials are not unique.

Consider the following *gauge transformation* of the potentials:

$$\Phi \;\rightarrow\; \Phi_{\rm new}=\Phi-\frac{\partial f}{\partial t}, \tag{1.7.3}$$

$$\boldsymbol{A} \;\rightarrow\; \boldsymbol{A}_{\rm new}=\boldsymbol{A}+\boldsymbol{\nabla}f, \tag{1.7.4}$$

in which $f(t,\boldsymbol{x})$ is an arbitrary function of space and time. We wish to show that this transformation leaves the fields unchanged:

$$\boldsymbol{E}_{\rm new}=\boldsymbol{E},\qquad \boldsymbol{B}_{\rm new}=\boldsymbol{B}. \tag{1.7.5}$$

This implies that the potentials are *not unique*: they can be redefined at will by a gauge transformation. The new electric field is given by

$$\begin{aligned}\boldsymbol{E}_{\rm new} &= -\frac{\partial\boldsymbol{A}_{\rm new}}{\partial t}-\boldsymbol{\nabla}\Phi_{\rm new}\\ &= -\frac{\partial}{\partial t}\Big(\boldsymbol{A}+\boldsymbol{\nabla}f\Big)-\boldsymbol{\nabla}\left(\Phi-\frac{\partial f}{\partial t}\right)\\ &= \boldsymbol{E}-\frac{\partial}{\partial t}\boldsymbol{\nabla}f+\boldsymbol{\nabla}\frac{\partial f}{\partial t};\end{aligned}$$

the last two terms cancel out and we have established the invariance of the electric field. On the other hand, the new magnetic field is given by

$$\begin{aligned}\boldsymbol{B}_{\rm new} &= \boldsymbol{\nabla}\times\boldsymbol{A}_{\rm new}\\ &= \boldsymbol{\nabla}\times\big(\boldsymbol{A}+\boldsymbol{\nabla}f\big)\\ &= \boldsymbol{B}+\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f);\end{aligned}$$

the last term is identically zero and we have established the invariance of the magnetic field. We will use the *gauge invariance* of the electromagnetic field to simplify the equations to be satisfied by the scalar and vector potentials.

To derive these we start with Eq. (1.2.1) and cast it in the form

$$\frac{1}{\epsilon_0}\rho = \boldsymbol{\nabla}\cdot\boldsymbol{E} = \boldsymbol{\nabla}\cdot\left(-\frac{\partial A}{\partial t} - \boldsymbol{\nabla}\Phi\right) = -\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{A} - \nabla^2\Phi,$$

or

$$\frac{1}{\epsilon_0}\rho = \epsilon_0\mu_0\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi - \frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A} + \epsilon_0\mu_0\frac{\partial\Phi}{\partial t}\right), \tag{1.7.6}$$

where, for reasons that will become clear, we have added and subtracted a term $\epsilon_0\mu_0\partial^2\Phi/\partial t^2$. Similarly, we re-express Eq. (1.2.4) in terms of the potentials:

$$\begin{aligned}
\mu_0\boldsymbol{j} &= \boldsymbol{\nabla}\times\boldsymbol{B} - \epsilon_0\mu_0\frac{\partial\boldsymbol{E}}{\partial t} \\
&= \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) - \epsilon_0\mu_0\frac{\partial}{\partial t}\left(-\frac{\partial\boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\Phi\right) \\
&= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A} + \epsilon_0\mu_0\frac{\partial^2\boldsymbol{A}}{\partial t^2} + \epsilon_0\mu_0\boldsymbol{\nabla}\frac{\partial\Phi}{\partial t},
\end{aligned}$$

where we have used a vector-calculus identity to eliminate $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})$. We have obtained

$$\mu_0\boldsymbol{j} = \epsilon_0\mu_0\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \nabla^2\boldsymbol{A} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A} + \epsilon_0\mu_0\frac{\partial\Phi}{\partial t}\right). \tag{1.7.7}$$

Equations (1.7.6) and (1.7.7) govern the behaviour of the potentials; they are equivalent to the two Maxwell equations that remain after imposing Eqs. (1.7.1) and (1.7.2).

These equations would be much simplified if we could demand that the potentials satisfy the supplementary condition

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} + \epsilon_0\mu_0\frac{\partial\Phi}{\partial t} = 0. \tag{1.7.8}$$

This is known as the *Lorenz gauge condition*, and as we shall see below, it can always be imposed by redefining the potentials according to Eqs. (1.7.3) and (1.7.4) with a specific choice of function $f(t,\boldsymbol{x})$. When Eq. (1.7.8) holds we observe that the equations for Φ and $\boldsymbol{A}$ decouple from one another, and that they both take the form of a wave equation:

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\Phi(t,\boldsymbol{x}) = -\frac{1}{\epsilon_0}\rho(t,\boldsymbol{x}), \tag{1.7.9}$$

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\boldsymbol{A}(t,\boldsymbol{x}) = -\mu_0\boldsymbol{j}(t,\boldsymbol{x}), \tag{1.7.10}$$

where

$$c \equiv \frac{1}{\sqrt{\epsilon_0\mu_0}} \tag{1.7.11}$$

is the speed with which the waves propagate; this is numerically equal to the speed of light in vacuum.

To see that the Lorenz gauge condition can always be imposed, imagine that we are given potentials $\Phi_{\rm old}$ and $\boldsymbol{A}_{\rm old}$ that do not satisfy the gauge condition. We know that we can transform them according to $\Phi_{\rm old} \to \Phi = \Phi_{\rm old} - \partial f/\partial t$ and $\boldsymbol{A}_{\rm old} \to \boldsymbol{A} = \boldsymbol{A}_{\rm old} + \boldsymbol{\nabla}f$, and we ask whether it is possible to find a function $f(t,\boldsymbol{x})$ such that the new potentials Φ and $\boldsymbol{A}$ will satisfy Eq. (1.7.8). The answer is in the affirmative: If Eq. (1.7.8) is true then

$$0 = \boldsymbol{\nabla}\cdot\boldsymbol{A} + \epsilon_0\mu_0\frac{\partial\Phi}{\partial t}$$

$$
\begin{aligned}
&= \boldsymbol{\nabla}\cdot\left(\boldsymbol{A}_{\text{old}}+\boldsymbol{\nabla}f\right)+\epsilon_0\mu_0\frac{\partial}{\partial t}\left(\Phi_{\text{old}}-\frac{\partial f}{\partial t}\right)\\
&= \boldsymbol{\nabla}\cdot\boldsymbol{A}_{\text{old}}+\epsilon_0\mu_0\frac{\partial\Phi_{\text{old}}}{\partial t}-\epsilon_0\mu_0\frac{\partial^2 f}{\partial t^2}+\nabla^2 f,
\end{aligned}
$$

and we see that the Lorenz gauge condition is satisfied if $f(t,\boldsymbol{x})$ is a solution to the wave equation

$$
\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\right)f=-\boldsymbol{\nabla}\cdot\boldsymbol{A}_{\text{old}}-\epsilon_0\mu_0\frac{\partial\Phi_{\text{old}}}{\partial t}.
$$

Such an equation always admits a solution, and we conclude that the Lorenz gauge condition can *always* be imposed. (Notice that we do not need to solve the wave equation for f; all we need to know is that solutions exist.)

To summarize, we have found that the original set of four Maxwell equations for the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ can be reduced to the two wave equations (1.7.9) and (1.7.10) for the potentials Φ and $\boldsymbol{A}$, supplemented by the Lorenz gauge condition of Eq. (1.7.8). Once solutions to these equations have been found, the fields can be constructed with Eqs. (1.7.1) and (1.7.2). Introducing the potentials has therefore dramatically reduced the complexity of the equations, and correspondingly increased the ease of finding solutions.

In time-independent situations, the fields and potentials no longer depend on t, and the foregoing equations reduce to

$$
\nabla^2\Phi(\boldsymbol{x})=-\frac{1}{\epsilon_0}\rho(\boldsymbol{x}), \tag{1.7.12}
$$

$$
\nabla^2\boldsymbol{A}(\boldsymbol{x})=-\mu_0\boldsymbol{j}(\boldsymbol{x}), \tag{1.7.13}
$$

as well as $\boldsymbol{E}=-\boldsymbol{\nabla}\Phi$ and $\boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A}$. The Lorenz gauge condition reduces to $\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$ (also known as the *Coulomb gauge condition*), and the potentials now satisfy Poisson's equation.

1.8 Green's function for Poisson's equation

[The material presented in this section is also covered in Sec. 1.7 of Jackson's text.]

We have seen in the preceding section that the task of solving Maxwell's equations can be reduced to the simpler task of solving two wave equations. For time-independent situations, the wave equation becomes Poisson's equation. In this and the next section we will develop some of the mathematical tools needed to solve these equations.

We begin in this section with the mathematical problem of solving a generic Poisson equation of the form

$$
\nabla^2\psi(\boldsymbol{x})=-4\pi f(\boldsymbol{x}), \tag{1.8.1}
$$

where $\psi(\boldsymbol{x})$ is the potential, $f(\boldsymbol{x})$ a prescribed source function, and where a factor of 4π was inserted for later convenience.

To construct the general solution to this equation we shall first find a *Green's function* $G(\boldsymbol{x},\boldsymbol{x'})$ that satisfies a specialized form of Poisson's equation:

$$
\nabla^2 G(\boldsymbol{x},\boldsymbol{x'})=-4\pi\delta(\boldsymbol{x}-\boldsymbol{x'}), \tag{1.8.2}
$$

where $\delta(\boldsymbol{x}-\boldsymbol{x'})$ is a three-dimensional Dirac δ-function; the source point $\boldsymbol{x'}$ is arbitrary. It is easy to see that if we have such a Green's function at our disposal, then the general solution to Eq. (1.8.1) can be expressed as

$$
\psi(\boldsymbol{x})=\psi_0(\boldsymbol{x})+\int G(\boldsymbol{x},\boldsymbol{x'})f(\boldsymbol{x'})\,d^3x', \tag{1.8.3}
$$

where $\psi_0(\boldsymbol{x})$ is a solution to the inhomogeneous (Laplace) equation,

$$\nabla^2 \psi_0(\boldsymbol{x}) = 0. \tag{1.8.4}$$

This assertion is proved by substituting Eq. (1.8.3) into the left-hand side of Eq. (1.8.1), and using Eqs. (1.8.2) and (1.8.4) to show that the result is indeed equal to $-4\pi f(\boldsymbol{x})$.

In Eq. (1.8.3), the role of the integral is to account for the source term in Eq. (1.8.1). The role of $\psi_0(\boldsymbol{x})$ is to enforce *boundary conditions* that we might wish to impose on the potential $\psi(\boldsymbol{x})$. In the absence of such boundary conditions, ψ_0 can simply be set equal to zero. Methods to solve boundary-value problems will be introduced in the next chapter.

To construct the Green's function we first argue that $G(\boldsymbol{x}, \boldsymbol{x'})$ can depend only on the difference $\boldsymbol{x} - \boldsymbol{x'}$; this follows from the fact that the source term depends only on $\boldsymbol{x} - \boldsymbol{x'}$, and the requirement that the Green's function must be invariant under a translation of the coordinate system. We then express $G(\boldsymbol{x}, \boldsymbol{x'})$ as the Fourier transform of a function $\tilde{G}(\boldsymbol{k})$:

$$G(\boldsymbol{x}, \boldsymbol{x'}) = \frac{1}{(2\pi)^3} \int \tilde{G}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x'})}\, d^3k, \tag{1.8.5}$$

where $\boldsymbol{k}$ is a vector in reciprocal space; this expression incorporates our assumption that $G(\boldsymbol{x}, \boldsymbol{x'})$ depends only on $\boldsymbol{x} - \boldsymbol{x'}$. Recalling that the three-dimensional δ-function can be represented as

$$\delta(\boldsymbol{x} - \boldsymbol{x'}) = \frac{1}{(2\pi)^3} \int e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x'})}\, d^3k, \tag{1.8.6}$$

we see that Eq. (1.8.2) implies $\boldsymbol{k}^2 \tilde{G}(\boldsymbol{k}) = 4\pi$, so that Eq. (1.8.6) becomes

$$G(\boldsymbol{x}, \boldsymbol{x'}) = \frac{4\pi}{(2\pi)^3} \int \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x'})}}{\boldsymbol{k}^2}\, d^3k. \tag{1.8.7}$$

We must now evaluate this integral.

To do this it is convenient to switch to spherical coordinates in reciprocal space, defining a radius k and angles θ and ϕ by the relations $k_x = k\sin\theta\cos\phi$, $k_y = k\sin\theta\sin\phi$, and $k_z = k\cos\theta$. In these coordinates the volume element is $d^3k = k^2 \sin\theta\, dk d\theta d\phi$. For convenience we orient the coordinate system so that the k_z axis points in the direction of $\boldsymbol{R} \equiv \boldsymbol{x} - \boldsymbol{x'}$. We then have $\boldsymbol{k}\cdot\boldsymbol{R} = kR\cos\theta$, where $R \equiv |\boldsymbol{R}|$ is the length of the vector $\boldsymbol{R}$. All this gives

$$G(\boldsymbol{x}, \boldsymbol{x'}) = \frac{1}{2\pi^2} \int_0^\infty dk \int_0^\pi d\theta \int_0^{2\pi} d\phi\, e^{ikR\cos\theta} \sin\theta.$$

Integration over $d\phi$ is trivial, and the θ integral can easily be evaluated by using $\mu = \cos\theta$ as an integration variable:

$$\int_0^\pi e^{ikR\cos\theta} \sin\theta\, d\theta = \int_{-1}^1 e^{ikR\mu}\, d\mu = \frac{2\sin kR}{kR}.$$

We now have

$$G(\boldsymbol{x}, \boldsymbol{x'}) = \frac{2}{\pi} \int_0^\infty \frac{\sin kR}{kR}\, dk = \frac{2}{\pi R} \int_0^\infty \frac{\sin\xi}{\xi}\, d\xi,$$

where we have switched to $\xi = kR$. The remaining integral can easily be evaluated by contour integration (or simply by looking it up in tables of integrals); it is equal to $\pi/2$.

Our final result is therefore

$$G(\boldsymbol{x}, \boldsymbol{x'}) = \frac{1}{|\boldsymbol{x} - \boldsymbol{x'}|}; \tag{1.8.8}$$

the Green's function for Poisson's equation is simply the reciprocal of the distance between $\boldsymbol{x}$ and $\boldsymbol{x'}$. The general solution to Poisson's equation is then

$$\psi(\boldsymbol{x}) = \psi_0(\boldsymbol{x}) + \int \frac{f(\boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|}\, d^3x'. \tag{1.8.9}$$

The meaning of this equation is that apart from the term ψ_0, the potential at $\boldsymbol{x}$ is built by summing over source contributions from all relevant points $\boldsymbol{x'}$ and dividing by the distance between $\boldsymbol{x}$ and $\boldsymbol{x'}$. Notice that in Eq. (1.8.8) we have recovered a familiar result: the potential of a point charge at $\boldsymbol{x'}$ goes like $1/R$.

1.9 Green's function for the wave equation

[The material presented in this section is also covered in Sec. 6.4 of Jackson's text.]

We now turn to the mathematical problem of solving the wave equation,

$$\Box\psi(t, \boldsymbol{x}) = -4\pi f(t, \boldsymbol{x}), \tag{1.9.1}$$

for a time-dependent potential ψ produced by a prescribed source f; we have introduced the wave, or d'Alembertian, differential operator

$$\Box \equiv -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2. \tag{1.9.2}$$

For this purpose we seek a Green's function $G(t, \boldsymbol{x}; t', \boldsymbol{x'})$ that satisfies

$$\Box G(t, \boldsymbol{x}; t', \boldsymbol{x'}) = -4\pi\delta(t - t')\delta(\boldsymbol{x} - \boldsymbol{x'}). \tag{1.9.3}$$

In terms of this the general solution to Eq. (1.9.2) can be expressed as

$$\psi(t, \boldsymbol{x}) = \psi_0(t, \boldsymbol{x}) + \int\int G(t, \boldsymbol{x}; t', \boldsymbol{x'}) f(t', \boldsymbol{x'})\, dt' d^3x', \tag{1.9.4}$$

where $\psi_0(t, \boldsymbol{x})$ is a solution to the homogeneous wave equation,

$$\Box\psi_0(t, \boldsymbol{x}) = 0. \tag{1.9.5}$$

That the potential of Eq. (1.9.4) does indeed solve Eq. (1.9.1) can be verified by direct substitution.

To construct the Green's function we Fourier transform it with respect to time,

$$G(t, \boldsymbol{x}; t', \boldsymbol{x'}) = \frac{1}{2\pi}\int \tilde{G}(\omega; \boldsymbol{x}, \boldsymbol{x'}) e^{-i\omega(t-t')}\, d\omega, \tag{1.9.6}$$

and we represent the time δ-function as

$$\delta(t - t') = \frac{1}{2\pi}\int e^{-i\omega(t-t')}\, d\omega. \tag{1.9.7}$$

Substituting these expressions into Eq. (1.9.3) yields

$$\left[\nabla^2 + (\omega/c)^2\right]\tilde{G}(\omega; \boldsymbol{x}, \boldsymbol{x'}) = -4\pi\delta(\boldsymbol{x} - \boldsymbol{x'}), \tag{1.9.8}$$

which is a generalized form of Eq. (1.8.2). From this comparison we learn that $\tilde{G}(0; \boldsymbol{x}, \boldsymbol{x'}) = 1/|\boldsymbol{x} - \boldsymbol{x'}|$.

At this stage we might proceed as in Sec. 1.8 and Fourier transform $\tilde{G}(\omega; \boldsymbol{x}, \boldsymbol{x'})$ with respect to the spatial variables. This would eventually lead to

$$\tilde{G}(\omega; \boldsymbol{x}, \boldsymbol{x'}) = \frac{4\pi}{(2\pi)^3} \int \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x'})}}{\boldsymbol{k}^2 - (\omega/c)^2}\, d^3k,$$

which is a generalized form of Eq. (1.8.7). This integral, however, is not defined, because of the singularity at $\boldsymbol{k}^2 = (\omega/c)^2$. We shall therefore reject this method of solution. (There are ways of regularizing the integral so as to obtain a meaningful answer. One method involves deforming the contour of the ω integration so as to avoid the poles. This is described in Sec. 12.11 of Jackson's text.)

We can anticipate that for $\omega \neq 0$, $\tilde{G}$ will be of the form

$$\tilde{G}(\omega; \boldsymbol{x}, \boldsymbol{x'}) = \frac{g(\omega, |\boldsymbol{x} - \boldsymbol{x'}|)}{|\boldsymbol{x} - \boldsymbol{x'}|}, \tag{1.9.9}$$

with g representing a function that stays nonsingular when the second argument, $R \equiv |\boldsymbol{x} - \boldsymbol{x'}|$, approaches zero. That $\tilde{G}$ should depend on the spatial variables through R only can be motivated on the grounds that three-dimensional space is both homogeneous (so that $\tilde{G}$ can only depend on the vector $\boldsymbol{R} \equiv \boldsymbol{x} - \boldsymbol{x'}$) and isotropic (so that only the length of the vector matters, and not its direction). That $\tilde{G}$ should behave as $1/R$ when R is small is justified by the following discussion.

Take Eq. (1.9.8) and integrate both sides over a sphere of small radius ε centered at $\boldsymbol{x'}$. Since $\nabla^2\tilde{G} = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\tilde{G}$, we can use Gauss' theorem to get

$$\oint_{R=\varepsilon} (\boldsymbol{\nabla}\tilde{G}\cdot\hat{\boldsymbol{R}})da + (\omega/c)^2 \int_{R<\varepsilon} \tilde{G}\, d^3x = -4\pi,$$

where $\hat{\boldsymbol{R}} = \boldsymbol{R}/R$. In this equation, the volume integral is of order $\tilde{G}\varepsilon^3$ and it contributes nothing in the limit $\varepsilon \to 0$, unless $\tilde{G}$ happens to be as singular as $1/\varepsilon^3$. The surface integral, on the other hand, is equal to

$$4\pi\varepsilon^2 \frac{d\tilde{G}}{dR}\bigg|_{R=\varepsilon}.$$

If $\tilde{G}$ were to behave as $1/\varepsilon^3$, then $d\tilde{G}/dR$ would be of order $1/\varepsilon^4$, the surface integral would contribute a term of order $1/\varepsilon^2$, and the left-hand side would never give rise to the required -4π. So we conclude that $\tilde{G}$ cannot be so singular, and that the left-hand side is dominated by the surface integral. This implies that $\tilde{G}$ must be of order $1/\varepsilon$, as was expressed by Eq. (1.9.9). Setting $\tilde{G} = g/R$ returns $-4\pi g(\omega, \varepsilon) + O(\varepsilon)$ for the surface integral, and this gives us the condition

$$g(\omega, 0) = 1. \tag{1.9.10}$$

We also recall that $g(0, R) = 1$.

We may now safely take $R \neq 0$ and substitute Eq. (1.9.9) into Eq. (1.9.8), taking its right-hand side to be zero. Since $\tilde{G}$ depends on $\boldsymbol{x}$ only through R, the Laplacian operator becomes

$$\nabla^2 \to \frac{1}{R^2}\frac{d}{dR}R^2\frac{d}{dR}.$$

Acting with this on $\tilde{G} = g/R$ yields g''/R and Eq. (1.9.8) becomes

$$g'' + (\omega/c)^2 g = 0, \tag{1.9.11}$$

with a prime indicating differentiation with respect to R. With the boundary condition of Eq. (1.9.10), two possible solutions to this equation are

$$g_\pm(\omega, R) = e^{\pm i(\omega/c)R}. \tag{1.9.12}$$

Substituting this into Eq. (1.9.9), and that into Eq. (1.9.6), we obtain

$$G_\pm(t,\boldsymbol{x};t',\boldsymbol{x'}) = \frac{1}{2\pi}\int \frac{e^{\pm i(\omega/c)R}}{R} e^{-i\omega(t-t')}\,d\omega = \frac{1}{2\pi R}\int e^{-i\omega(t-t'\mp R/c)}\,d\omega,$$

or

$$G_\pm(t,\boldsymbol{x};t',\boldsymbol{x'}) = \frac{\delta\big(t-t'\mp|\boldsymbol{x}-\boldsymbol{x'}|/c\big)}{|\boldsymbol{x}-\boldsymbol{x'}|}. \tag{1.9.13}$$

These are the two fundamental solutions to Eq. (1.9.3). The function $G_+(t,\boldsymbol{x};t',\boldsymbol{x'})$, which is nonzero when $t-t'=+R/c$, is known as the *retarded Green's function*; the function $G_-(t,\boldsymbol{x};t',\boldsymbol{x'})$, which is nonzero when $t-t'=-R/c$, is known as the *advanced Green's function*.

Substituting the Green's functions into Eq. (1.9.4) gives

$$\psi_\pm(t,\boldsymbol{x}) = \psi_0(t,\boldsymbol{x}) + \int\int \frac{\delta\big(t-t'\mp|\boldsymbol{x}-\boldsymbol{x'}|/c\big)}{|\boldsymbol{x}-\boldsymbol{x'}|} f(t',\boldsymbol{x'})\,dt'd^3x'.$$

The time integration can be immediately carried out, and we obtain

$$\psi_\pm(t,\boldsymbol{x}) = \psi_0(t,\boldsymbol{x}) + \int \frac{f\big(t\mp|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{x'}\big)}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x' \tag{1.9.14}$$

for the fundamental solutions to the wave equation. Except for the shifted time dependence, Eq. (1.9.14) looks very similar to Eq. (1.8.9), which gives the general solution to Poisson's equation. The time translation, however, is very important. It means that the potential at time t depends on the conditions at the source at a shifted time

$$t' = t\mp|\boldsymbol{x}-\boldsymbol{x'}|/c.$$

The second term on the right-hand side is the time required by light to travel the distance between the source point $\boldsymbol{x'}$ and the field point $\boldsymbol{x}$; it encodes the property that information about the source travels to $\boldsymbol{x}$ at the speed of light. The "+" solution, $\psi_+(t,\boldsymbol{x})$, depends on the behaviour of the source at an *earlier time* $t-R/c$ — there is a delay between the time the information leaves the source and the time it reaches the observer. This solution to the wave equation is known as the *retarded solution*, and it properly enforces causality: the cause (source) precedes the effect (potential). On the other hand, $\psi_-(t,\boldsymbol{x})$ depends on the behaviour of the source at a *later time* $t+R/c$; here the behaviour is anti-causal, and this unphysical solution to the wave equation is known as the *advanced solution*.

Although both solutions to the wave equation are mathematically acceptable, causality clearly dictates that we should adopt the retarded solution $\psi_+(t,\boldsymbol{x})$ as the only physically acceptable solution. So we take

$$\psi(t,\boldsymbol{x}) = \psi_0(t,\boldsymbol{x}) + \int \frac{f\big(t-|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{x'}\big)}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x' \tag{1.9.15}$$

to be the solution to the wave equation (1.9.1). Recall that $\psi_0(t,\boldsymbol{x})$ is a solution to the homogeneous equation, $\Box\psi_0=0$.

1.10 Summary

The scalar potential Φ and the vector potential $\boldsymbol{A}$ satisfy the wave equations

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\right)\Phi(t,\boldsymbol{x}) = -\frac{1}{\epsilon_0}\rho(t,\boldsymbol{x}) \tag{1.10.1}$$

and

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\right)\boldsymbol{A}(t,\boldsymbol{x})=-\mu_0\boldsymbol{j}(t,\boldsymbol{x}) \tag{1.10.2}$$

when the Lorenz gauge condition

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}+\frac{1}{c^2}\frac{\partial\Phi}{\partial t}=0 \tag{1.10.3}$$

is imposed. The speed of propagation of the waves, $c=1/\sqrt{\epsilon_0\mu_0}$, is numerically equal to the speed of light in vacuum.

The retarded solutions to the wave equations are

$$\Phi(t,\boldsymbol{x})=\Phi_0(t,\boldsymbol{x})+\frac{1}{4\pi\epsilon_0}\int\frac{\rho\big(t-|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{x'}\big)}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x' \tag{1.10.4}$$

and

$$\boldsymbol{A}(t,\boldsymbol{x})=\boldsymbol{A}_0(t,\boldsymbol{x})+\frac{\mu_0}{4\pi}\int\frac{\boldsymbol{j}\big(t-|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{x'}\big)}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'. \tag{1.10.5}$$

In static situations, the time dependence of the sources and potentials can be dropped.

Once the scalar and vector potentials have been obtained, the electric and magnetic fields are recovered by straightforward differential operations:

$$\boldsymbol{E}=-\frac{\partial\boldsymbol{A}}{\partial t}-\boldsymbol{\nabla}\Phi,\qquad \boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A}. \tag{1.10.6}$$

This efficient reformulation of the equations of electrodynamics is completely equivalent to the original presentation of Maxwell's equations. It will be the starting point of most of our subsequent investigations.

1.11 Problems

1. We have seen that the charge density of a point charge q located at $\boldsymbol{r}(t)$ is given by

   $$\rho(\boldsymbol{x})=q\delta\big(\boldsymbol{x}-\boldsymbol{r}(t)\big),$$

   and that its current density is

   $$\boldsymbol{j}(\boldsymbol{x})=q\boldsymbol{v}(t)\delta\big(\boldsymbol{x}-\boldsymbol{r}(t)\big),$$

   where $\boldsymbol{v}=d\boldsymbol{r}/dt$ is the charge's velocity. Prove that ρ and $\boldsymbol{j}$ satisfy the statement of local charge conservation,

   $$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{j}=0.$$

2. In this problem we explore the formulation of electrodynamics in the *Coulomb gauge*, an alternative to the Lorenz gauge adopted in the text. The Coulomb gauge is especially useful in the formulation of a quantum theory of electrodynamics.

   a) Prove that the Coulomb gauge condition, $\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$, can always be imposed on the vector potential. Then show that in this gauge, the scalar potential satisfies Poisson's equation, $\nabla^2\Phi=-\rho/\epsilon_0$, so that it can be expressed as

   $$\Phi(t,\boldsymbol{x})=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(t,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'.$$

   Show also that the vector potential satisfies the wave equation

   $$\Box\boldsymbol{A}=-\mu_0\boldsymbol{j}+\epsilon_0\mu_0\boldsymbol{\nabla}\frac{\partial\Phi}{\partial t}.$$

b) Define the "longitudinal current" by

$$\boldsymbol{j}_\mathrm{l}(t,\boldsymbol{x}) = -\frac{1}{4\pi}\boldsymbol{\nabla}\int \frac{\boldsymbol{\nabla'}\cdot\boldsymbol{j}(t,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x',$$

and show that this satisfies $\boldsymbol{\nabla}\times\boldsymbol{j}_\mathrm{l} = 0$. Then prove that the wave equation for the vector potential can be rewritten as $\Box\boldsymbol{A} = -\mu_0\boldsymbol{j}_\mathrm{t}$, where $\boldsymbol{j}_\mathrm{t} \equiv \boldsymbol{j} - \boldsymbol{j}_\mathrm{l}$ is the "transverse current".

c) Prove that the transverse current can also be defined by

$$\boldsymbol{j}_\mathrm{t}(t,\boldsymbol{x}) = \frac{1}{4\pi}\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\int\frac{\boldsymbol{j}(t,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'\right),$$

and that it satisfies $\boldsymbol{\nabla}\cdot\boldsymbol{j}_\mathrm{t} = 0$.

d) The labels "longitudinal" and "transverse" can be made more meaningful by formulating the results of part b) and c) in reciprocal space instead of ordinary space. For this purpose, introduce the Fourier transform $\boldsymbol{j}(t,\boldsymbol{k})$ of the current density, such that

$$\boldsymbol{j}(t,\boldsymbol{x}) = \frac{1}{(2\pi)^3}\int \boldsymbol{j}(t,\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,d^3k.$$

Then prove that the Fourier transform of the longitudinal current is

$$\boldsymbol{j}_\mathrm{l}(t,\boldsymbol{k}) = \big[\hat{\boldsymbol{k}}\cdot\boldsymbol{j}(t,\boldsymbol{k})\big]\hat{\boldsymbol{k}},$$

where $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$ is a unit vector aligned with the reciprocal position vector $\boldsymbol{k}$; thus, the longitudinal current has a component in the direction of $\boldsymbol{k}$ only. Similarly, prove that the Fourier transform of the transverse current is

$$\boldsymbol{j}_\mathrm{t}(t,\boldsymbol{k}) = \big[\hat{\boldsymbol{k}}\times\boldsymbol{j}(t,\boldsymbol{k})\big]\times\hat{\boldsymbol{k}};$$

thus, the transverse current has components in the directions orthogonal to $\boldsymbol{k}$ only. For a concrete illustration, assume that $\boldsymbol{k}$ points in the direction of the z axis. Then show that $\boldsymbol{j}_\mathrm{l} = j_z\hat{\boldsymbol{z}}$ and $\boldsymbol{j}_\mathrm{t} = j_x\hat{\boldsymbol{x}} + j_y\hat{\boldsymbol{y}}$.

3. Prove that

$$G(x,x') = \frac{1}{2ik}\,e^{ik|x-x'|}$$

is a solution to

$$\left(\frac{d^2}{dx^2}+k^2\right)G(x,x') = \delta(x-x'),$$

where k is a constant. This shows that $G(x,x')$ is a Green's function for the inhomogeneous Helmholtz equation in one dimension, $(d^2/dx^2 + k^2)\psi(x) = f(x)$, where ψ is the potential and f the source.

4. The differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = f(t)$$

governs the motion of a simple harmonic oscillator of unit mass and natural frequency ω driven by an arbitrary external force $f(t)$. It is supposed that the force starts acting at $t = 0$, and that prior to $t = 0$ the oscillator was at rest, so that $x(0) = \dot{x}(0) = 0$, with an overdot indicating differentiation with respect to t.

Find the retarded Green's function $G(t, t')$ for this differential equation, which must be a solution to

$$\frac{d^2G}{dt^2} + \omega^2 G = \delta(t - t').$$

Then find the solution $x(t)$ to the differential equation (in the form of an integral) that satisfies the stated initial conditions.

Check your results by verifying that if $f(t) = \theta(t)\cos(\omega t)$, so that the oscillator is driven at resonance, then $x(t) = (2\omega)^{-1} t \sin \omega t$.

5. In this problem (adapted from Jackson's problem 6.1) we construct solutions to the inhomogeneous wave equation $\Box\psi = -4\pi f$ in a few simple situations. A particular solution to this equation is

$$\psi(t, \boldsymbol{x}) = \int G(t, \boldsymbol{x}; t', \boldsymbol{x'}) f(t', \boldsymbol{x'})\, dt' d^3x',$$

where

$$G(t, \boldsymbol{x}; t', \boldsymbol{x'}) = \frac{\delta(t - t' - |\boldsymbol{x} - \boldsymbol{x'}|/c)}{|\boldsymbol{x} - \boldsymbol{x'}|}$$

is the retarded Green's function.

a) Take the source of the wave to be a momentary point source, described by $f(t', \boldsymbol{x'}) = \delta(x')\delta(y')\delta(z')\delta(t')$. Show that the solution to the wave equation in this case is given by

$$\psi(t, x, y, z) = \frac{\delta(t - r/c)}{r},$$

where $r \equiv \sqrt{x^2 + y^2 + z^2}$. The wave is therefore a concentrated wavefront expanding outward at the speed of light.

b) Now take the source of the wave to be a line source described by $f(t', \boldsymbol{x'}) = \delta(x')\delta(y')\delta(t')$. (Notice that this also describes a point source in a space of two dimensions.) Show that the solution to the wave equation in this case is given by

$$\psi(t, x, y) = \frac{2c\theta(t - \rho/c)}{\sqrt{(ct)^2 - \rho^2}},$$

where $\rho \equiv \sqrt{x^2 + y^2}$ and $\theta(s)$ is the Heaviside step function. Notice that the wave is no longer well localized, but that the wavefront still travels outward at the speed of light.

c) Finally, take the source to be a sheet source described by $f(t', \boldsymbol{x'}) = \delta(x')\delta(t')$. (Notice that this also describes a point source in a space of a single dimension.) Show that the solution to the wave equation in this case is given by

$$\psi(t, x) = 2\pi c\theta(t - |x|/c).$$

Notice that apart from a sharp cutoff at $|x| = ct$, the wave is now completely uniform.

6. In the quantum version of Maxwell's theory, the electromagnetic interaction is mediated by a massless photon. In this problem we consider a modified

classical theory of electromagnetism that would lead, upon quantization, to a *massive* photon. It is based on the following set of field equations:

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{E}+\kappa^2\Phi &= \frac{1}{\epsilon_0}\rho, \\ \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0, \\ \boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{\partial\boldsymbol{B}}{\partial t}, \\ \boldsymbol{\nabla}\times\boldsymbol{B}+\kappa^2\boldsymbol{A} &= \mu_0\boldsymbol{j}+\epsilon_0\mu_0\frac{\partial\boldsymbol{E}}{\partial t}, \end{aligned}$$

where κ is a new constant related to the photon mass m by $\kappa = mc/\hbar$, and where the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are related in the usual way to the potentials Φ and $\boldsymbol{A}$. The field equations are supplemented by the same Lorentz-force law,

$$\boldsymbol{f} = \rho\boldsymbol{E}+\boldsymbol{j}\times\boldsymbol{B},$$

as in the original theory.

a) Prove that the modified theory enforces charge conservation *provided* that the potentials are linked by

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}+\epsilon_0\mu_0\frac{\partial\Phi}{\partial t}=0.$$

The Lorenz gauge condition must therefore always be imposed in the modified theory.

b) Find the modified wave equations that are satisfied by the potentials Φ and $\boldsymbol{A}$.

c) Verify that in modified electrostatics, the scalar potential outside a point charge q at $\boldsymbol{x}=0$ is given by the Yukawa form

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\kappa r}}{r}.$$

d) Prove that the modified theory enforces energy conservation by deriving an equation of the form

$$\frac{\partial\varepsilon}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{S}=-\boldsymbol{j}\cdot\boldsymbol{E}.$$

Find the new expressions for ε and $\boldsymbol{S}$; they should reduce to the old expressions when $\kappa\to 0$. In the original theory the potentials have no direct physical meaning; is this true also in the modified theory?

Chapter 2
Electrostatics

2.1 Equations of electrostatics

For time-independent situations, the equations of electromagnetism decouple into a set of equations for the electric field alone — electrostatics — and another set for the magnetic field alone — magnetostatics. The equations of magnetostatics will be considered in Chapter 3. The topic of this chapter is electrostatics.

The equations of electrostatics are Poisson's equation for the scalar potential,

$$\nabla^2\Phi(\boldsymbol{x}) = -\frac{1}{\epsilon_0}\rho(\boldsymbol{x}), \tag{2.1.1}$$

and the relation between potential and field,

$$\boldsymbol{E} = -\boldsymbol{\nabla}\Phi. \tag{2.1.2}$$

The general solution to Poisson's equation is

$$\Phi(\boldsymbol{x}) = \Phi_0(\boldsymbol{x}) + \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x', \tag{2.1.3}$$

where $\Phi_0(\boldsymbol{x})$ satisfies Laplace's equation,

$$\nabla^2\Phi_0(\boldsymbol{x}) = 0. \tag{2.1.4}$$

The role of $\Phi_0(\boldsymbol{x})$ is to enforce boundary conditions that we might wish to impose on the scalar potential.

2.2 Point charge

The simplest situation involves a point charge q located at a fixed point $\boldsymbol{b}$. The charge density is

$$\rho(\boldsymbol{x}) = q\delta(\boldsymbol{x}-\boldsymbol{b}), \tag{2.2.1}$$

and in the absence of boundaries, Eq. (2.1.3) gives

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\boldsymbol{x}-\boldsymbol{b}|}. \tag{2.2.2}$$

To compute the electric field we first calculate the gradient of $R \equiv |\boldsymbol{x}-\boldsymbol{b}|$, the distance between the field point $\boldsymbol{x}$ and the charge. We have $R^2 = (x-b_x)^2 + (y-b_y)^2 + (z-b_z)^2$ and differentiating both sides with respect to, say, x gives $2R\partial R/\partial x = 2(x-b_x)$, or $\partial R/\partial x = (x-b_x)/R$. Derivatives of R with respect

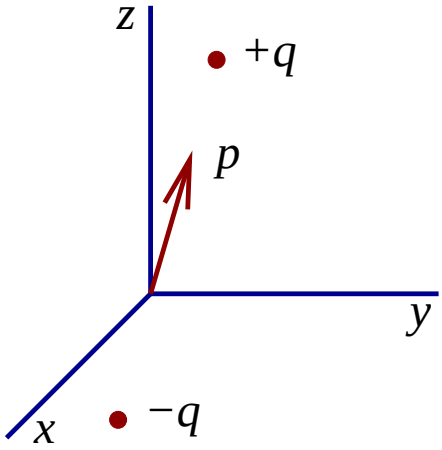

Figure 2.1: Geometry of a dipole.

to y and z can be computed in a similar way, and we have established the useful vectorial relation

$$\boldsymbol{\nabla}|\boldsymbol{x}-\boldsymbol{b}| = \frac{\boldsymbol{x}-\boldsymbol{b}}{|\boldsymbol{x}-\boldsymbol{b}|}. \tag{2.2.3}$$

Notice that $\boldsymbol{\nabla}R$ is a unit vector that points in the direction of $\boldsymbol{R} \equiv \boldsymbol{x}-\boldsymbol{b}$.

The calculation of the electric field involves $\boldsymbol{\nabla}R^{-1} = -R^{-2}\boldsymbol{\nabla}R$, or

$$\boldsymbol{\nabla}\frac{1}{|\boldsymbol{x}-\boldsymbol{b}|} = -\frac{\boldsymbol{x}-\boldsymbol{b}}{|\boldsymbol{x}-\boldsymbol{b}|^3}. \tag{2.2.4}$$

This gives

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{x}-\boldsymbol{b}}{|\boldsymbol{x}-\boldsymbol{b}|^3}. \tag{2.2.5}$$

The electric field goes as q/R^2 and points in the direction of the vector $\boldsymbol{R} = \boldsymbol{x}-\boldsymbol{b}$. This well-known result is known as *Coulomb's law*.

2.3 Dipole

Another elementary situation is that of two equal charges, one positive, the other negative, separated by a distance $d \equiv 2\varepsilon$. We align the charges along the direction of the unit vector $\hat{\boldsymbol{p}}$, and place the origin of the coordinate system at the middle point; see Fig. 2.1.

The charge density of this distribution is given by

$$\rho(\boldsymbol{x}) = q\delta(\boldsymbol{x}-\varepsilon\hat{\boldsymbol{p}}) - q\delta(\boldsymbol{x}+\varepsilon\hat{\boldsymbol{p}}), \tag{2.3.1}$$

and the potential is

$$\Phi(\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\boldsymbol{x}-\varepsilon\hat{\boldsymbol{p}}|} - \frac{1}{|\boldsymbol{x}+\varepsilon\hat{\boldsymbol{p}}|}\right). \tag{2.3.2}$$

This is an exact expression. We can simplify it if we take the observation point $\boldsymbol{x}$ to be at a large distance away from the dipole,

$$r \equiv |\boldsymbol{x}| \gg \varepsilon. \tag{2.3.3}$$

We can then approximate

$$\begin{aligned} |\boldsymbol{x} \mp \varepsilon\hat{\boldsymbol{p}}| &= \sqrt{(\boldsymbol{x} \mp \varepsilon\hat{\boldsymbol{p}}) \cdot (\boldsymbol{x} \mp \varepsilon\hat{\boldsymbol{p}})} \\ &= \sqrt{r^2 \mp 2\varepsilon\hat{\boldsymbol{p}} \cdot \boldsymbol{x} + \varepsilon^2} \\ &= r\Big[1 \mp \varepsilon\hat{\boldsymbol{p}} \cdot \boldsymbol{x}/r^2 + O(\varepsilon^2/r^2)\Big], \end{aligned}$$

or

$$\frac{1}{|\boldsymbol{x} \mp \varepsilon\hat{\boldsymbol{p}}|} = \frac{1}{r}\Big[1 \pm \varepsilon\hat{\boldsymbol{p}} \cdot \boldsymbol{x}/r^2 + O(\varepsilon^2/r^2)\Big].$$

Keeping terms of order ε only, the potential becomes

$$\Phi = \frac{1}{4\pi\epsilon_0} \frac{(2q\varepsilon\hat{\boldsymbol{p}}) \cdot \boldsymbol{x}}{r^3}.$$

The vector $\boldsymbol{p} \equiv 2q\varepsilon\hat{\boldsymbol{p}}$ is the *dipole moment* of the charge distribution. In general, for a collection of several charges, this is defined as

$$\boldsymbol{p} = \sum_A q_A \boldsymbol{x}_A, \tag{2.3.4}$$

where $\boldsymbol{x}_A$ is the position vector of the charge q_A. In the present situation the definition implies $\boldsymbol{p} = (+q)(\varepsilon\hat{\boldsymbol{p}}) + (-q)(-\varepsilon\hat{\boldsymbol{p}}) = 2q\varepsilon\hat{\boldsymbol{p}}$, as was stated previously. The potential of a dipole is then

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0} \frac{\boldsymbol{p} \cdot \boldsymbol{x}}{r^3}, \tag{2.3.5}$$

when r satisfies Eq. (2.3.3). It should be noticed that here, the potential falls off as $1/r^2$, faster than in the case of a single point charge. This has to do with the fact that here, the charge distribution has a vanishing total charge.

It is instructive to rederive this result by directly expanding the charge density in powers of ε. For any function f of a vector $\boldsymbol{x}$ we have $f(\boldsymbol{x} + \varepsilon\hat{\boldsymbol{p}}) = f(\boldsymbol{x}) + \varepsilon\hat{\boldsymbol{p}} \cdot \boldsymbol{\nabla} f(\boldsymbol{x}) + O(\varepsilon^2)$, and we can extend this result to Dirac's distribution:

$$\delta(\boldsymbol{x} \pm \varepsilon\hat{\boldsymbol{p}}) = \delta(\boldsymbol{x}) \pm \varepsilon\hat{\boldsymbol{p}} \cdot \boldsymbol{\nabla}\delta(\boldsymbol{x}) + O(\varepsilon^2).$$

The charge density of Eq. (2.3.1) then becomes $\rho(\boldsymbol{x}) = -2q\varepsilon\hat{\boldsymbol{p}} \cdot \boldsymbol{\nabla}\delta(\boldsymbol{x}) + O(\varepsilon^2)$, or

$$\rho(\boldsymbol{x}) = -\boldsymbol{p} \cdot \boldsymbol{\nabla}\delta(\boldsymbol{x}). \tag{2.3.6}$$

This is the charge density of a point dipole.

Substituting this into Eq. (2.1.3) and setting $\Phi_0 = 0$ yields

$$\Phi(\boldsymbol{x}) = -\frac{p_a}{4\pi\epsilon_0} \int \frac{\nabla'_a \delta(\boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|}\, d^3x',$$

where summation over the vectorial index a is understood. The standard procedure when dealing with derivatives of a δ-function is to integrate by parts. The integral is then

$$-\int \delta(\boldsymbol{x'})\nabla'_a \frac{1}{|\boldsymbol{x} - \boldsymbol{x'}|}\, d^3x' = -\int \delta(\boldsymbol{x'}) \frac{(\boldsymbol{x} - \boldsymbol{x'})_a}{|\boldsymbol{x} - \boldsymbol{x'}|^3}\, d^3x' = -\frac{x_a}{r^3},$$

and we have recovered Eq. (2.3.5). Notice that in these manipulations we have inserted a result analogous to Eq. (2.2.4); the minus sign does not appear because we are now differentiating with respect to the primed variables.

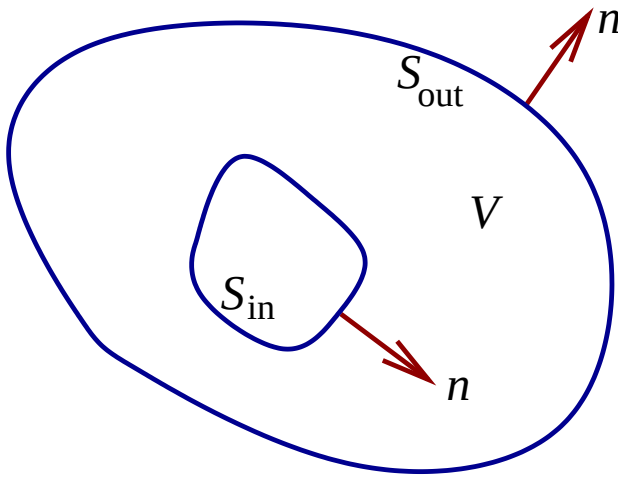

Figure 2.2: An inner boundary S_{in} on which Φ is specified, an outer boundary S_{out} on which Φ is specified, and the region V in between that contains a charge distribution ρ. The outward unit normal to both boundaries is denoted $\hat{\boldsymbol{n}}$.

From the potential of Eq. (2.3.5) we can calculate the electric field of a point dipole. Taking a derivative with respect to x gives

$$\nabla_x \Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{\nabla_x(\boldsymbol{p}\cdot\boldsymbol{x})}{r^3} - \frac{3(\boldsymbol{p}\cdot\boldsymbol{x})}{r^4}\nabla_x r\right].$$

We have $\nabla_x(\boldsymbol{p}\cdot\boldsymbol{x}) = p_x$ and according to Eq. (2.2.3), $\nabla_x r = x/r$. So

$$\nabla_x \Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{p_x}{r^3} - \frac{3(\boldsymbol{p}\cdot\boldsymbol{x})x}{r^5}\right],$$

and a similar calculation can be carried out for the y and z components of $\boldsymbol{\nabla}\Phi$. Introducing the unit vector

$$\hat{\boldsymbol{r}} = \boldsymbol{x}/r = (x/r, y/r, z/r), \tag{2.3.7}$$

we find that the electric field is given by

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\frac{3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}}{r^3}. \tag{2.3.8}$$

This is an *exact* expression for the electric field of a point dipole, for which the charge density is given by Eq. (2.3.6). For a physical dipole of finite size, this expression is only an *approximation* of the actual electric field; the approximation is good for $r \gg \varepsilon$.

2.4 Boundary-value problems: Green's theorem

[The material presented in this section is also covered in Secs. 1.8 and 1.10 of Jackson's text.]

After the warmup exercise of the preceding two sections we are ready to face the more serious challenge of solving boundary-value problems. A typical situation in electrostatics features a charge distribution $\rho(\boldsymbol{x})$ between two boundaries on which the potential $\Phi(\boldsymbol{x})$ is specified (see Fig. 2.2) — the potential is then said to satisfy *Dirichlet boundary conditions*. A concrete situation might involve an inner

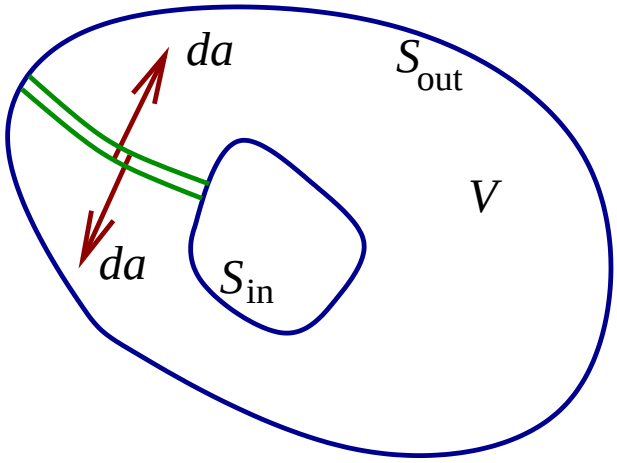

Figure 2.3: The union of the inner boundary S_{in}, the outer boundary S_{out}, and the narrow "channel" between them forms a closed surface S that encloses the region V.

boundary S_{in} that is also a grounded conducting surface (on which $\Phi = 0$) and an outer boundary S_{out} that is pushed to infinity (so that $\Phi = 0$ there also). We would like to derive an expression for $\Phi(\boldsymbol{x})$ in the region between the boundaries; this should account for both the charge density and the boundary data. To arrive at this expression we will need to introduce a new type of Green's function, which we will call a *Dirichlet Green's function* and denote $G_{\text{D}}(\boldsymbol{x}, \boldsymbol{x'})$; its properties will be identified along the way. To get there we will make use of *Green's identity*, which is essentially an application of Gauss' theorem.

Gauss' theorem states that for a volume V enclosed by a surface S,

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{b}\, d^3x = \oint_S \boldsymbol{b} \cdot d\boldsymbol{a},$$

where $\boldsymbol{b}$ is any vector field defined within V and $d\boldsymbol{a}$ is an *outward* surface element on S. The theorem, as stated, says nothing about a region V bounded by two boundaries S_{in} and S_{out}, the situation that interests us. But we can make our situation fit the formulation of the theorem by digging a narrow "channel" from S_{out} to S_{in}, as depicted in Fig. 2.3. The union of S_{in}, S_{out}, and the channel is a closed surface, and integrating $\boldsymbol{b}$ over both sides of the channel produces a zero result because $d\boldsymbol{a}$ points in opposite directions. We therefore have

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{b}\, d^3x = \int_{S_{\text{out}}} \boldsymbol{b} \cdot d\boldsymbol{a} + \int_{S_{\text{in}}} \boldsymbol{b} \cdot d\boldsymbol{a},$$

and noting the opposite orientations of $d\boldsymbol{a}$ and $\hat{\boldsymbol{n}}$ on S_{in} (see Fig. 2.3), we write this as

$$\begin{aligned} \int_V \boldsymbol{\nabla} \cdot \boldsymbol{b}\, d^3x &= \int_{S_{\text{out}}} \boldsymbol{b} \cdot \hat{\boldsymbol{n}} da - \int_{S_{\text{in}}} \boldsymbol{b} \cdot \hat{\boldsymbol{n}} da \\ &\equiv \int_{S_{\text{out}} - S_{\text{in}}} \boldsymbol{b} \cdot \hat{\boldsymbol{n}} da. \end{aligned}$$

It is important to notice that the unit vector $\hat{\boldsymbol{n}}$ points *out* of both S_{in} and S_{out}.

We now choose the vector $\boldsymbol{b}$ to be

$$\boldsymbol{b} = \phi \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \phi,$$

where $\phi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$ are two arbitrary functions. We have $\boldsymbol{\nabla}\cdot\boldsymbol{b} = \phi\nabla^2\psi - \psi\nabla^2\phi$ and Gauss' theorem gives

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) d^3x = \int_{S_{\rm out}-S_{\rm in}} \left(\phi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\phi\right)\cdot\hat{\boldsymbol{n}}da. \tag{2.4.1}$$

This is *Green's identity*. For convenience we write it in the equivalent form

$$\int_V \left(\phi\nabla'^2\psi - \psi\nabla'^2\phi\right) d^3x' = \int_{S_{\rm out}-S_{\rm in}} \left(\phi\boldsymbol{\nabla'}\psi - \psi\boldsymbol{\nabla'}\phi\right)\cdot\hat{\boldsymbol{n}}da',$$

where all fields are now expressed in terms of the new variables $\boldsymbol{x'}$.

We now pick

$$\phi(\boldsymbol{x'}) \equiv \Phi(\boldsymbol{x'}) \equiv \text{scalar potential}$$

and

$$\psi(\boldsymbol{x'}) \equiv G_{\rm D}(\boldsymbol{x},\boldsymbol{x'}) \equiv \text{Dirichlet Green's function},$$

where the properties of the "Dirichlet Green's function" will be described in detail below. For now we assume that it satisfies the equation

$$\nabla'^2 G_{\rm D}(\boldsymbol{x},\boldsymbol{x'}) = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x'}), \tag{2.4.2}$$

while the scalar potential satisfies Poisson's equation, $\nabla'^2\Phi(\boldsymbol{x'}) = -\rho(\boldsymbol{x'})/\epsilon_0$. Making these substitutions in the equation following Eq. (2.4.1) leads to

$$\begin{aligned} -4\pi\Phi(\boldsymbol{x}) + \frac{1}{\epsilon_0}\int_V G_{\rm D}(\boldsymbol{x},\boldsymbol{x'})\rho(\boldsymbol{x'})\,d^3x' &= \int_{S_{\rm out}-S_{\rm in}} \Big[\Phi(\boldsymbol{x'})\boldsymbol{\nabla'}G_{\rm D}(\boldsymbol{x},\boldsymbol{x'}) \\ &\quad - G_{\rm D}(\boldsymbol{x},\boldsymbol{x'})\boldsymbol{\nabla'}\Phi(\boldsymbol{x'})\Big]\cdot\hat{\boldsymbol{n}}da', \end{aligned}$$

or

$$\begin{aligned} \Phi(\boldsymbol{x}) &= \frac{1}{4\pi\epsilon_0}\int_V G_{\rm D}(\boldsymbol{x},\boldsymbol{x'})\rho(\boldsymbol{x'})\,d^3x' \\ &\quad - \frac{1}{4\pi}\int_{S_{\rm out}-S_{\rm in}} \Big[\Phi(\boldsymbol{x'})\boldsymbol{\nabla'}G_{\rm D}(\boldsymbol{x},\boldsymbol{x'}) - G_{\rm D}(\boldsymbol{x},\boldsymbol{x'})\boldsymbol{\nabla'}\Phi(\boldsymbol{x'})\Big]\cdot\hat{\boldsymbol{n}}'da'. \end{aligned}$$

Apart from a remaining simplification, this is the kind of expression we were seeking. The volume integral takes care of the charge distribution within V, and the two surface integrals take care of the boundary conditions. But there is one problem: While the value of the scalar potential is specified on the boundaries, so that the functions $\Phi(\boldsymbol{x'})$ are known inside the surface integrals, we are given no information about $\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla'}\Phi(\boldsymbol{x'})$, its normal derivative, which also appears within the surface integrals. It is not clear, therefore, how we might go about evaluating these integrals: the problem does not seem to be well posed.

To sidestep this problem we demand that the Dirichlet Green's function satisfy the condition

$$G_{\rm D}(\boldsymbol{x},\boldsymbol{x'}) = 0 \qquad \text{when } \boldsymbol{x'} \text{ is on the boundaries.} \tag{2.4.3}$$

Then the boundary terms involving $\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla'}\Phi(\boldsymbol{x'})$ simply go away, because they are multiplied by $G_{\rm D}(\boldsymbol{x},\boldsymbol{x'})$ which vanishes on the boundaries. With this property we arrive at

$$\begin{aligned} \Phi(\boldsymbol{x}) &= \frac{1}{4\pi\epsilon_0}\int_V G_{\rm D}(\boldsymbol{x},\boldsymbol{x'})\rho(\boldsymbol{x'})\,d^3x' \\ &\quad - \frac{1}{4\pi}\int_{S_{\rm out}-S_{\rm in}} \Phi(\boldsymbol{x'})\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla'}G_{\rm D}(\boldsymbol{x},\boldsymbol{x'})da'. \end{aligned} \tag{2.4.4}$$

This is our final expression for the scalar potential in the region V. The problem of finding this potential has been reduced to that of finding a Dirichlet Green's function that satisfies Eqs. (2.4.2) and (2.4.3). Notice that this is a specialized form of our original problem: $G_{\rm D}(\boldsymbol{x},\boldsymbol{x'})$ is the potential produced by a point charge of strength $4\pi\epsilon_0$ located at $\boldsymbol{x}$, and its boundary values on $S_{\rm in}$ and $S_{\rm out}$ are zero. If we can solve this problem and obtain the Dirichlet Green's function, then Eq. (2.4.4) gives us the means to calculate $\Phi(\boldsymbol{x})$ in very general circumstances.

We already know that $G_{\rm D}(\boldsymbol{x},\boldsymbol{x'})$ satisfies Eq. (2.4.2) and vanishes when $\boldsymbol{x'}$ lies on $S_{\rm in}$ and $S_{\rm out}$. We now use Green's identity to show that the Dirichlet Green's function is symmetric in its arguments:

$$G_{\rm D}(\boldsymbol{x'},\boldsymbol{x}) = G_{\rm D}(\boldsymbol{x},\boldsymbol{x'}). \tag{2.4.5}$$

This implies that it is also a solution to the standard equation satisfied by a Green's function,

$$\nabla^2 G_{\rm D}(\boldsymbol{x},\boldsymbol{x'}) = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x'}), \tag{2.4.6}$$

which is identical to Eq. (1.8.2). To establish Eq. (2.4.5) we take $\boldsymbol{y}$ to be the integration variables in Green's identity,

$$\int_V \left(\phi\nabla_y^2\psi - \psi\nabla_y^2\phi\right) d^3y = \int_{S_{\rm out}-S_{\rm in}} \left(\phi\boldsymbol{\nabla}_y\psi - \psi\boldsymbol{\nabla}_y\phi\right)\cdot\hat{\boldsymbol{n}}da_y,$$

and we pick $\phi(\boldsymbol{y}) \equiv G_{\rm D}(\boldsymbol{x},\boldsymbol{y})$ and $\psi(\boldsymbol{y}) \equiv G_{\rm D}(\boldsymbol{x'},\boldsymbol{y})$. Then according to Eq. (2.4.2) we have $\nabla_y^2\phi = -4\pi\delta(\boldsymbol{x}-\boldsymbol{y})$ and $\nabla_y^2\psi = -4\pi\delta(\boldsymbol{x'}-\boldsymbol{y})$. The left-hand side of Green's identity gives $-4\pi[G_{\rm D}(\boldsymbol{x},\boldsymbol{x'}) - G_{\rm D}(\boldsymbol{x'},\boldsymbol{x})]$ after integration over d^3y. The right-hand side, on the other hand, gives zero because both ϕ and ψ are zero when $\boldsymbol{y}$ is on $S_{\rm in}$ and $S_{\rm out}$. This produces Eq. (2.4.5).

To summarize, we have found that the Dirichlet Green's function satisfies Eq. (2.4.6),

$$\nabla^2 G_{\rm D}(\boldsymbol{x},\boldsymbol{x'}) = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x'}),$$

possesses the symmetry property of Eq. (2.4.5),

$$G_{\rm D}(\boldsymbol{x'},\boldsymbol{x}) = G_{\rm D}(\boldsymbol{x},\boldsymbol{x'}),$$

and satisfies the boundary conditions of Eq. (2.4.3),

$$G_{\rm D}(\boldsymbol{x},\boldsymbol{x'}) = 0 \qquad \text{when } \boldsymbol{x} \text{ or } \boldsymbol{x'} \text{ is on the boundaries.}$$

Notice that the boundary conditions have been generalized in accordance to Eq. (2.4.5): the Dirichlet Green's function vanishes whenever $\boldsymbol{x}$ *or* $\boldsymbol{x'}$ happens to lie on the boundaries. Once the Dirichlet Green's function has been found, the potential produced by a charge distribution $\rho(\boldsymbol{x})$ in a region V between two boundaries $S_{\rm in}$ and $S_{\rm out}$ on which $\Phi(\boldsymbol{x})$ is specified can be obtained by evaluating the integrals of Eq. (2.4.4). The task of finding a suitable Dirichlet Green's function can only be completed once the boundaries $S_{\rm in}$ and $S_{\rm out}$ are fully specified; the form of the Green's function will depend on the shapes and locations of these boundaries. In the absence of boundaries, $G_{\rm D}(\boldsymbol{x},\boldsymbol{x'})$ reduces to the Green's function constructed in Sec. 1.8, $G(\boldsymbol{x},\boldsymbol{x'}) = |\boldsymbol{x}-\boldsymbol{x'}|^{-1}$.

In the following sections we will consider a *restricted class* of boundary-value problems, for which

- there is only an inner boundary $S_{\rm in}$ ($S_{\rm out}$ is pushed to infinity and does not need to be considered);
- the inner boundary is spherical ($S_{\rm in}$ is described by the statement $|\boldsymbol{x'}| = R$, where R is the surface's radius);

- the inner boundary is the surface of a grounded conductor (Φ vanishes on $S_{\rm in}$).

With these restrictions (introduced only for simplicity), Eq. (2.4.4) reduces to

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int_{|\boldsymbol{x'}|>R} G_{\rm D}(\boldsymbol{x},\boldsymbol{x'})\rho(\boldsymbol{x'})\,d^3x'. \tag{2.4.7}$$

In these situations, the electric field vanishes inside the conductor and there is an induced distribution of charge on the surface. The surface charge density is given by Eq. (1.6.9),

$$\sigma = \epsilon_0 \boldsymbol{E}\cdot\hat{\boldsymbol{n}}, \tag{2.4.8}$$

where $\boldsymbol{E}$ is the electric field just above $S_{\rm in}$. This is given by $\boldsymbol{E} = -\boldsymbol{\nabla}\Phi(|\boldsymbol{x}| = R)$. To proceed we will need to find a concrete expression for the Dirichlet Green's function. For the spherical boundary considered here, this is done by solving Eq. (2.4.6) in spherical coordinates.

2.5 Laplace's equation in spherical coordinates

[The material presented in this section is also covered in Secs. 3.1, 3.2, and 3.5 of Jackson's text.]

When formulated in spherical coordinates (r,θ,ϕ), Laplace's equation takes the form

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0. \tag{2.5.1}$$

To solve this equation we separate the variables according to

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi), \tag{2.5.2}$$

and we obtain the decoupled equations

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \ell(\ell+1)R \tag{2.5.3}$$

and

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -\ell(\ell+1)Y, \tag{2.5.4}$$

where $\ell(\ell+1)$ is a separation constant.

The solutions to the angular equation (2.5.4) are the spherical harmonics $Y_{\ell m}(\theta,\phi)$, which are labeled by the two integers ℓ and m. While ℓ ranges from zero to infinity, m is limited to the values $-\ell, -\ell+1, \cdots, \ell-1, \ell$. For $m \geq 0$,

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\phi}, \tag{2.5.5}$$

where $P_\ell^m(\cos\theta)$ are the associated Legendre polynomials. For $m<0$ we use

$$Y_{\ell m} = (-1)^m Y^*_{\ell,-m}, \tag{2.5.6}$$

where an asterisk indicates complex conjugation. Particular spherical harmonics are

$$Y_{00} \;=\; \frac{1}{\sqrt{4\pi}},$$

$$\begin{aligned} Y_{11} &= -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\phi}, \\ Y_{10} &= \sqrt{\frac{3}{4\pi}}\,\cos\theta, \\ Y_{22} &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\, e^{2i\phi}, \\ Y_{21} &= -\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{i\phi}, \\ Y_{20} &= \frac{1}{2}\sqrt{\frac{5}{4\pi}}\,(3\cos^2\theta - 1). \end{aligned}$$

Two fundamental properties of the spherical harmonics are that they are *orthonormal functions*, and that they form a *complete set* of functions of the angles θ and ϕ. The statement of orthonormality is

$$\int Y^*_{\ell' m'}(\theta,\phi) Y_{\ell m}(\theta,\phi)\, d\Omega = \delta_{\ell\ell'}\delta_{mm'}, \tag{2.5.7}$$

where $d\Omega = \sin\theta\, d\theta \mathrm{d}\phi$ is an element of solid angle. The statement of completeness is that *any* function $f(\theta,\phi)$ can be represented as a sum over spherical harmonics:

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell m}(\theta,\phi) \tag{2.5.8}$$

for some coefficients $f_{\ell m}$. By virtue of Eq. (2.5.7), these can in fact be calculated as

$$f_{\ell m} = \int f(\theta,\phi) Y^*_{\ell m}(\theta,\phi)\, d\Omega. \tag{2.5.9}$$

Equation (2.5.8) means that the spherical harmonics form a complete set of *basis functions* on the sphere.

It is interesting to see what happens when Eq. (2.5.9) is substituted into Eq. (2.5.8). To avoid confusion we change the variables of integration to θ' and ϕ':

$$\begin{aligned} f(\theta,\phi) &= \sum_\ell \sum_m Y_{\ell m}(\theta,\phi) \int f(\theta',\phi') Y^*_{\ell m}(\theta',\phi')\, d\Omega' \\ &= \int f(\theta',\phi') \left[\sum_\ell \sum_m Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi) \right] d\Omega'. \end{aligned}$$

The quantity within the large square brackets is such that when it is multiplied by $f(\theta',\phi')$ and integrated over the primed angles, it returns $f(\theta,\phi)$. This must therefore be a product of two δ-functions, one for θ and the other for ϕ. More precisely stated,

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi) = \frac{\delta(\theta-\theta')\delta(\phi-\phi')}{\sin\theta}, \tag{2.5.10}$$

where the factor of $1/\sin\theta$ was inserted to compensate for the factor of $\sin\theta'$ in $d\Omega'$ (the δ-function is enforcing the condition $\theta' = \theta$). Equation (2.5.10) is known as the *completeness relation* for the spherical harmonics. This is analogous to a well-known identity,

$$\int \left(\frac{1}{\sqrt{2\pi}} e^{ikx'}\right)^* \left(\frac{1}{\sqrt{2\pi}} e^{ikx}\right) dk = \delta(x - x'),$$

in which the integral over dk replaces the discrete summation over ℓ and m; the basis functions $(2\pi)^{-1/2}e^{ikx}$ are then analogous to the spherical harmonics.

The solutions to the radial equation (2.5.3) are power laws, $R \propto r^\ell$ or $R \propto r^{-(\ell+1)}$. The general solution is

$$R_{\ell m}(r) = a_{\ell m} r^\ell + b_{\ell m} r^{-(\ell+1)}, \tag{2.5.11}$$

where $a_{\ell m}$ and $b_{\ell m}$ are constants.

The general solution to Laplace's equation is obtained by combining Eqs. (2.5.2), (2.5.5), (2.5.11) and summing over all possible values of ℓ and m:

$$\psi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Big(a_{\ell m} r^\ell + b_{\ell m} r^{-(\ell+1)} \Big) Y_{\ell m}(\theta,\phi). \tag{2.5.12}$$

The coefficients $a_{\ell m}$ and $b_{\ell m}$ are determined by the boundary conditions imposed on $\psi(r,\theta,\phi)$. If, for example, ψ is to be well behaved at $r = 0$, then $b_{\ell m} \equiv 0$. If, on the other hand, ψ is to vanish at $r = \infty$, then $a_{\ell m} \equiv 0$. From these observations we deduce that the only solution to Laplace's equation that is well behaved at the origin and vanishes at infinity is the trivial solution $\psi = 0$.

2.6 Green's function in the absence of boundaries

[The material presented in this section is also covered in Secs. 3.6, and 3.9 of Jackson's text.]

Keeping in mind that our ultimate goal is to obtain the Dirichlet Green's function $G_{\rm D}(\boldsymbol{x},\boldsymbol{x'})$ for a spherical inner boundary, in this section we attempt something simpler and solve

$$\nabla^2 G(\boldsymbol{x},\boldsymbol{x'}) = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x'}) \tag{2.6.1}$$

in the absence of boundaries. We already know the answer: In Sec. 1.8 we found that

$$G(\boldsymbol{x},\boldsymbol{x'}) = \frac{1}{|\boldsymbol{x}-\boldsymbol{x'}|}. \tag{2.6.2}$$

We will obtain an alternative expression for this, one that can be generalized to account for the presence of spherical boundaries.

The δ-function on the right-hand side of Eq. (2.6.1) can be represented as

$$\begin{aligned} \delta(\boldsymbol{x}-\boldsymbol{x'}) &= \frac{\delta(r-r')}{r^2}\frac{\delta(\theta-\theta')\delta(\phi-\phi')}{\sin\theta} \\ &= \frac{\delta(r-r')}{r^2}\sum_{\ell m} Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi), \end{aligned} \tag{2.6.3}$$

where we have involved Eq. (2.5.10). The factor of $1/(r^2\sin\theta)$ was inserted because in spherical coordinates, the volume element is $d^3x = r^2\sin\theta\, dr d\theta d\phi$, and the Jacobian factor of $r^2\sin\theta$ must be compensated for. We then have, for example,

$$\begin{aligned} f(\boldsymbol{x'}) &= \int f(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{x'})\, d^3x \\ &= \int f(r,\theta,\phi)\frac{\delta(r-r')}{r^2}\frac{\delta(\theta-\theta')\delta(\phi-\phi')}{\sin\theta}\, r^2\sin\theta\, dr d\theta d\phi \\ &= \int f(r,\theta,\phi)\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')\, dr d\theta d\phi \\ &= f(r',\theta',\phi'), \end{aligned}$$

as we should.

Inspired by Eq. (2.6.3) we expand the Green's function as

$$G(\boldsymbol{x},\boldsymbol{x'}) = \sum_{\ell m} g_{\ell m}(r,r') Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi), \tag{2.6.4}$$

using the spherical harmonics $Y_{\ell m}(\theta,\phi)$ for angular functions and $g_{\ell m}(r,r')Y^*_{\ell m}(\theta',\phi')$ for radial functions. For convenience we have made the "radial function" depend on $\boldsymbol{x'}$ through the variables r', θ', and ϕ'; these are treated as constant parameters when substituting $G(\boldsymbol{x},\boldsymbol{x'})$ into Eq. (2.6.1). Applying the Laplacian operator of Eq. (2.5.1) on the Green's function yields

$$\nabla^2 G(\boldsymbol{x},\boldsymbol{x'}) = \sum_{\ell m}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dg_{\ell m}}{dr}\right) - \frac{\ell(\ell+1)}{r^2}g_{\ell m}\right] Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi),$$

and setting this equal to $-4\pi\delta(\boldsymbol{x}-\boldsymbol{x'})$ as expressed in Eq. (2.6.3) produces an ordinary differential equation for $g_{\ell m}(r,r')$:

$$\frac{d}{dr}\left(r^2\frac{dg_{\ell m}}{dr}\right) - \ell(\ell+1)g_{\ell m} = -4\pi\delta(r-r'). \tag{2.6.5}$$

This equation implies that the radial function depends only on ℓ, and not on m: $g_{\ell m}(r,r') \equiv g_\ell(r,r')$. To simplify the notation we will now omit the label "ℓ" on the radial function.

To solve Eq. (2.6.5) we first observe that when $r \neq r'$, the right-hand side of the equation is zero and g is a solution to Eq. (2.5.3). We let $g_<(r,r')$ be the solution to the left of r' (for $r<r'$) and $g_>(r,r')$ be the solution to the right of r' (for $r>r'$). We then assume that $g(r,r')$ can be obtained by joining the solutions at $r=r'$:

$$g(r,r') = g_<(r,r')\theta(r'-r) + g_>(r,r')\theta(r-r'), \tag{2.6.6}$$

where $\theta(r'-r)$ and $\theta(r-r')$ are step functions. Differentiating this equation with respect to r gives

$$g'(r,r') = g'_<(r,r')\theta(r'-r) + g'_>(r,r')\theta(r-r') + \big[g_>(r',r') - g_<(r',r')\big]\delta(r-r'),$$

where a prime on g indicates differentiation. We have involved the distributional identities $\theta'(r-r') = \delta(r-r')$, $\theta'(r'-r) = -\delta(r-r')$, and $f(r)\delta(r-r') = f(r')\delta(r-r')$; these were first encountered in Sec. 1.6. Multiplying by r^2 yields

$$\begin{aligned} r^2 g'(r,r') &= r^2 g'_<(r,r')\theta(r'-r) + r^2 g'_>(r,r')\theta(r-r') \\ &\quad + r'^2\big[g_>(r',r') - g_<(r',r')\big]\delta(r-r') \end{aligned}$$

and taking another derivative brings

$$\begin{aligned} \big[r^2 g'(r,r')\big]' &= \big[r^2 g'_<(r,r')\big]'\theta(r'-r) + \big[r^2 g'_>(r,r')\big]'\theta(r-r') \\ &\quad + r'^2\big[g'_>(r',r') - g'_<(r',r')\big]\delta(r-r') \\ &\quad + r'^2\big[g_>(r',r') - g_<(r',r')\big]\delta'(r-r'). \end{aligned}$$

Substituting this and Eq. (2.6.6) back into Eq. (2.6.5), we obtain

$$\begin{aligned} -4\pi\delta(r-r') &= \left[\frac{d}{dr}\left(r^2\frac{dg_<}{dr}\right) - \ell(\ell+1)g_<\right]\theta(r'-r) \\ &\quad + \left[\frac{d}{dr}\left(r^2\frac{dg_>}{dr}\right) - \ell(\ell+1)g_>\right]\theta(r-r') \\ &\quad + r'^2\left[\frac{dg_>}{dr}(r',r') - \frac{dg_<}{dr}(r',r')\right]\delta(r-r') \\ &\quad + r'^2\Big[g_>(r',r') - g_<(r',r')\Big]\frac{d}{dr}\delta(r-r'). \end{aligned}$$

For the right-hand side to match the left-hand side we must demand that $g_<(r,r')$ and $g_>(r,r')$ be solutions to the homogeneous version of Eq. (2.6.5), as was already understood, and that these solutions be matched according to

$$g_>(r',r') - g_<(r',r') = 0 \tag{2.6.7}$$

and

$$\frac{dg_>}{dr}(r',r') - \frac{dg_<}{dr}(r',r') = -\frac{4\pi}{r'^2}. \tag{2.6.8}$$

The function $g(r,r')$ is therefore continuous at $r = r'$, but its first derivative must be discontinuous to account for the δ-function on the right-hand side of Eq. (2.6.5).

Because $g_<$ is a solution to Eq. (2.5.3), according to Eq. (2.5.11) it must be a linear combination of terms proportional to r^ℓ and $1/r^{\ell+1}$. But only the first term is well behaved at $r = 0$, and we set

$$g_<(r,r') = a(r')r^\ell.$$

For $g_>$ we select instead

$$g_>(r,r') = b(r')/r^{\ell+1},$$

which is well behaved at $r = \infty$. It does not matter that $g_<$ diverges at $r = \infty$ because this function is restricted to the domain $r < r' < \infty$; similarly, it does not matter that $g_>$ diverges at $r = 0$ because it is restricted to the domain $r > r' > 0$. To determine the "constants" a and b we use the matching conditions. First, Eq. (2.6.7) gives $b = ar'^{2\ell+1}$. Second, Eq. (2.6.8) gives $a = 4\pi r'^{-(\ell+1)}/(2\ell+1)$. So

$$g_<(r,r') = \frac{4\pi}{2\ell+1}\frac{r^\ell}{r'^{\ell+1}} \tag{2.6.9}$$

and

$$g_>(r,r') = \frac{4\pi}{2\ell+1}\frac{r'^\ell}{r^{\ell+1}}. \tag{2.6.10}$$

We observe a nice symmetry between these two results. Combining Eqs. (2.6.6), (2.6.9), and (2.6.10), we find that the radial function can be expressed in the compact form

$$g_\ell(r,r') = \frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}, \tag{2.6.11}$$

where $r_<$ denotes the lesser of r and r', while $r_>$ is the greater of r and r'. Thus, if $r < r'$ then $r_< = r$, $r_> = r'$, and Eq. (2.6.11) gives $g(r,r') = g_<(r,r')$, as it should. Similarly, when $r > r'$ we have $r_< = r'$, $r_> = r$, and Eq. (2.6.11) gives $g(r,r') = g_>(r,r')$.

Substituting Eq. (2.6.11) into Eq. (2.6.4) and taking into account Eq. (2.6.2), we arrive at

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x'}|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi). \tag{2.6.12}$$

This very useful identity is known as the *addition theorem* for spherical harmonics.

At first sight the usefulness of this identity might seem doubtful. After all, we have turned something simple like an inverse distance into something horrible involving special functions and an infinite double sum. But we will see that when $1/|\boldsymbol{x}-\boldsymbol{x'}|$ appears inside an integral, the representation of Eq. (2.6.12) can be very useful indeed. Not relying on this identity would mean having to evaluate an integral that involves

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x'}|} = \frac{1}{\sqrt{r^2 - 2rr'[\sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta'] + r'^2}},$$

and such integrals typically cannot be evaluated directly: they are just too complicated.

Let us consider an example to illustrate the power of the addition theorem. We want to calculate the electrostatic potential both inside and outside a spherical distribution of charge with density $\rho(\boldsymbol{x'}) = \rho(r')$. There are no boundaries in this problem, but the distribution is confined to a sphere of radius R. After substituting Eq. (2.6.12) into Eq. (2.1.3) and writing $d^3x' = r'^2 dr' d\Omega'$, we obtain

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell m} \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta,\phi) \int_0^R \frac{r_<^\ell}{r_>^{\ell+1}} \rho(r') r'^2 \, dr' \int Y^*_{\ell m}(\theta',\phi') \, d\Omega'.$$

To evaluate the angular integral we multiply and divide by $(4\pi)^{-1/2} \equiv Y_{00}(\theta',\phi')$:

$$\int Y^*_{\ell m}(\theta',\phi') \, d\Omega' = \sqrt{4\pi} \int Y^*_{\ell m}(\theta',\phi') Y_{00}(\theta',\phi') \, d\Omega' = \sqrt{4\pi}\delta_{\ell,0}\delta_{m,0},$$

where we have involved Eq. (2.5.7). At this stage we happily realize that the infinite sum over ℓ and m involves but a single term. We therefore have

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0} 4\pi\sqrt{4\pi} Y_{00}(\theta,\phi) \int_0^R \frac{r'^2}{r_>} \rho(r') \, dr' = \frac{1}{4\pi\epsilon_0} 4\pi \int_0^R \frac{r'^2}{r_>} \rho(r') \, dr',$$

and as we might have expected from the spherical symmetry of the problem, the potential depends on r only.

To evaluate the radial integral we must be careful with the meaning of $r_> \equiv \max(r, r')$. Suppose first that $\boldsymbol{x}$ is *outside* the charge distribution, so that $r > R$. Then $r > r'$, $r_> = r$, and we have

$$4\pi \int_0^R \frac{r'^2}{r_>} \rho(r') \, dr' = \frac{1}{r} 4\pi \int_0^R \rho(r') r'^2 \, dr' = \frac{Q}{r},$$

where $Q \equiv \int \rho(\boldsymbol{x'}) \, d^3x' = 4\pi \int_0^R \rho(r') r'^2 \, dr'$ is the *total charge* of the distribution. In this case we have

$$\Phi_{\text{out}}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}, \tag{2.6.13}$$

the same result as in Eq. (2.2.2). Suppose next that $\boldsymbol{x}$ is *inside* the charge distribution, so that $r < R$. Then part of the integration covers the interval $0 \le r' < r$ and the remaining part covers $r \le r' \le R$. In the first part r' is smaller than r and $r_> = r$; in the second part r' is larger than r and $r_> = r'$. So

$$4\pi \int_0^R \frac{r'^2}{r_>} \rho(r') \, dr' = \frac{1}{r} 4\pi \int_0^r \rho(r') r'^2 \, dr' + 4\pi \int_r^R \rho(r') r' \, dr'.$$

In the first term we recognize

$$q(r) \equiv \int_{|\boldsymbol{x'}| \le r} \rho(\boldsymbol{x'}) \, d^3x' = 4\pi \int_0^r \rho(r') r'^2 \, dr', \tag{2.6.14}$$

the charge enclosed by a sphere of radius r; if $r > R$ then $q(r) = Q$. The internal potential can thus be expressed as

$$\Phi_{\text{in}}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0} \left[\frac{q(r)}{r} + 4\pi \int_r^R \rho(r') r' \, dr' \right]. \tag{2.6.15}$$

While the expression (2.6.15) for the internal potential depends on the details of the charge distribution — the particular form of the function $\rho(r')$ — a nicer

conclusion applies to the electric field. The field is obtained by differentiating $\Phi_{\rm in}$ with respect to r, and

$$\frac{d\Phi_{\rm in}}{dr} = \frac{1}{4\pi\epsilon_0}\left[-\frac{q}{r^2} + \frac{1}{r}\frac{dq}{dr} - 4\pi\rho r\right].$$

But since $dq/dr = 4\pi\rho r^2$ according to Eq. (2.6.14), we see that the last two terms cancel out. The electric field is then

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\frac{q(r)}{r^2}\hat{\boldsymbol{r}}, \tag{2.6.16}$$

both inside and outside the charge distribution; recall that when $r > R$, $q(r)$ becomes Q, the total charge. We have recovered the well-known result that for any spherical distribution of charge, the electric field at a radius r depends only on the charge $q(r)$ enclosed by a sphere of that radius; and the field is the same as if all this charge were concentrated in a point at $r = 0$.

2.7 Dirichlet Green's function for a spherical inner boundary

[The material presented in this section is also covered in Sec. 3.9 of Jackson's text.]

We now return to the task of finding the Dirichlet Green's function $G_{\rm D}(\boldsymbol{x}, \boldsymbol{x'})$ for the specific situation described near the end of Sec. 2.4, in which we have a spherical inner boundary at $|\boldsymbol{x}| = R$ and no outer boundary. It will be a simple matter to modify the treatment presented in Sec. 2.6 to account for the inner boundary.

As in Eq. (2.6.4), the Dirichlet Green's function can be expanded as

$$G_{\rm D}(\boldsymbol{x}, \boldsymbol{x'}) = \sum_{\ell m} g_\ell(r, r') Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi). \tag{2.7.1}$$

As in Eq. (2.6.6), the radial function can be expressed as

$$g_\ell(r, r') = g_<(r, r')\theta(r' - r) + g_>(r, r')\theta(r - r'), \tag{2.7.2}$$

where $g_<$ and $g_>$ are solutions to Eq. (2.5.3) that must also satisfy the matching conditions of Eqs. (2.6.7) and (2.6.8),

$$g_>(r', r') - g_<(r', r') = 0 \tag{2.7.3}$$

and

$$\frac{dg_>}{dr}(r', r') - \frac{dg_<}{dr}(r', r') = -\frac{4\pi}{r'^2}. \tag{2.7.4}$$

The only difference with respect to what was done in Sec. 2.6 is that now, according to Eq. (2.4.3), the Green's function must vanish on the inner boundary. This gives the additional conditions $g_\ell(r = R, r') = g_\ell(r, r' = R) = 0$, or

$$g_<(r = R, r') = 0, \qquad g_>(r, r' = R) = 0; \tag{2.7.5}$$

recall that R is now the smallest possible value of both r and r'.

Since $g_<(r, r')$ is no longer required to be well behaved at $r = 0$, we must set it equal to

$$g_<(r, r') = a(r', R)r^\ell + \alpha(r', R)/r^{\ell+1},$$

the general solution to Eq. (2.5.3). For $g_>(r, r')$ we choose instead

$$g_>(r, r') = b(r', R)/r^{\ell+1},$$

which is well behaved at $r = \infty$. Solving Eqs. (2.7.3)–(2.7.5) for the "constants" produces $a = 4\pi r'^{-(\ell+1)}/(2\ell+1)$, $\alpha = -aR^{2\ell+1}$, and $b = a(r'^{2\ell+1} - R^{2\ell+1})$. Collecting these results gives

$$g_<(r,r') = \frac{4\pi}{2\ell+1}\frac{1}{r'^{\ell+1}}\left(r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}}\right) \tag{2.7.6}$$

and

$$g_>(r,r') = \frac{4\pi}{2\ell+1}\frac{1}{r^{\ell+1}}\left(r'^\ell - \frac{R^{2\ell+1}}{r'^{\ell+1}}\right). \tag{2.7.7}$$

Combining Eqs. (2.7.2), (2.7.6), and (2.7.7), we find that the radial function can be expressed in the compact form

$$g_\ell(r,r') = \frac{4\pi}{2\ell+1}\frac{1}{r_>^{\ell+1}}\left(r_<^\ell - \frac{R^{2\ell+1}}{r_<^{\ell+1}}\right) = \frac{4\pi}{2\ell+1}\frac{r_<^{2\ell+1} - R^{2\ell+1}}{(rr')^{\ell+1}}, \tag{2.7.8}$$

where $r_<$ denotes the lesser of r and r', while $r_>$ is the greater of r and r' (notice that $r_<r_> \equiv rr'$). We see that $g_\ell(r,r')$ vanishes on the inner boundary: When $r = R$ it must be that $r < r'$, so that $r_< = r = R$, and the terms within the brackets vanish; on the other hand, when $r' = R$ it must be that $r' < r$, so that $r_< = r' = R$, and the same conclusion applies.

The Dirichlet Green's function is obtained by substituting Eq. (2.7.8) into Eq. (2.7.1):

$$G_{\rm D}(\boldsymbol{x},\boldsymbol{x'}) = \sum_{\ell m}\frac{4\pi}{2\ell+1}\frac{1}{r_>^{\ell+1}}\left(r_<^\ell - \frac{R^{2\ell+1}}{r_<^{\ell+1}}\right)Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi). \tag{2.7.9}$$

This is slightly more complicated than Eq. (2.6.12), but still manageable. Notice that the Dirichlet Green's function reduces to $1/|\boldsymbol{x} - \boldsymbol{x'}|$ when $R = 0$. It would be straightforward to generalize this derivation to account also for the presence of an outer boundary; we shall not pursue this here.

2.8 Point charge outside a grounded, spherical conductor

[The material presented in this section is also covered in Sec. 2.2 of Jackson's text.]

The electrostatic potential outside a grounded, spherical conductor of radius R is given by Eq. (2.4.7),

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int_{|\boldsymbol{x'}|>R} G_{\rm D}(\boldsymbol{x},\boldsymbol{x'})\rho(\boldsymbol{x'})\,d^3x', \tag{2.8.1}$$

and the Dirichlet Green's function was obtained as

$$G_{\rm D}(\boldsymbol{x},\boldsymbol{x'}) = \sum_{\ell m}\frac{4\pi}{2\ell+1}\frac{1}{r_>^{\ell+1}}\left(r_<^\ell - \frac{R^{2\ell+1}}{r_<^{\ell+1}}\right)Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi) \tag{2.8.2}$$

in the preceding section. The charge density induced on the surface of the conductor is given by Eq. (2.4.8), $\sigma = \epsilon_0\boldsymbol{E}\cdot\hat{\boldsymbol{n}}$, where $\boldsymbol{E}$ is the electric field just above the conductor and $\hat{\boldsymbol{n}} \equiv \hat{\boldsymbol{r}}$ is the surface's unit normal. Because $\boldsymbol{E} = -\boldsymbol{\nabla}\Phi$, this works out to be

$$\sigma = -\epsilon_0\frac{\partial\Phi}{\partial r}\bigg|_{r=R}. \tag{2.8.3}$$

We place a point charge q on the z axis, at a distance $z_0 > R$ from the centre of the conducting sphere. The charge's position vector is $\boldsymbol{z}_0 = z_0\hat{\boldsymbol{z}}$ and its spherical coordinates are $r_0 = z_0$, $\theta_0 = 0$, and ϕ_0 is undetermined. The charge density is

$$\rho(\boldsymbol{x'}) = q\delta(\boldsymbol{x'} - \boldsymbol{z}_0). \tag{2.8.4}$$

Substituting this into Eq. (2.8.1) gives

$$\begin{aligned}\Phi(\boldsymbol{x}) &= \frac{q}{4\pi\epsilon_0} G_{\rm D}(\boldsymbol{x},\boldsymbol{z}_0) \\ &= \frac{q}{4\pi\epsilon_0}\sum_{\ell m}\frac{4\pi}{2\ell+1}\left(\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{R^{2\ell+1}}{r_<^{\ell+1}r_>^{\ell+1}}\right) Y^*_{\ell m}(0,\phi_0)Y_{\ell m}(\theta,\phi),\end{aligned}$$

where $r_<$ now stands for the lesser of r and z_0, while $r_>$ stands for the greater of the two. It is a property of the spherical harmonics that

$$Y_{\ell m}(0,\phi_0) = \sqrt{\frac{2\ell+1}{4\pi}}\delta_{m,0},$$

so that the potential is actually independent of ϕ_0 and involves a single sum over ℓ. It is another property of the spherical harmonics that for $m=0$, they reduce to ordinary Legendre polynomials,

$$Y_{\ell,0}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta).$$

After cleaning up the algebra, the final expression for the potential is

$$\Phi(\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\left(\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{R^{2\ell+1}/z_0^{\ell+1}}{r^{\ell+1}}\right)P_\ell(\cos\theta). \tag{2.8.5}$$

We see that the potential depends on r and θ only; the fact that it does not depend on ϕ reflects the axial symmetry of the problem.

Before moving on we pause and recall the main properties of the Legendre polynomials. These are ordinary polynomials of order ℓ; if ℓ is even the function $P_\ell(x)$ contains only even terms, while if ℓ is odd it contains only odd terms. For example,

$$\begin{aligned}P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= \frac{1}{2}(3x^2-1), \\ P_3(x) &= \frac{1}{2}(5x^3-3x), \\ P_4(x) &= \frac{1}{8}(35x^4-30x^2+3).\end{aligned}$$

The Legendre polynomials are orthogonal functions,

$$\int_{-1}^{1} P_{\ell'}(x)P_\ell(x)\,dx = \frac{2}{2\ell+1}\delta_{\ell'\ell}, \tag{2.8.6}$$

and they possess the special values

$$P_{2n}(0) = (-1)^n\frac{(2n-1)!!}{(2n)!!}, \qquad P_{2n+1}(0) = 0, \qquad P_\ell(1) = 1, \tag{2.8.7}$$

where the double factorial notation means $(2n)!! = (2n)(2n-2)(2n-4)\cdots(2)$ or $(2n-1)!! = (2n-1)(2n-3)(2n-5)\cdots(1)$.

The potential of Eq. (2.8.5) is expressed as an infinite sum over ℓ. We will evaluate this sum in a moment, but for now we calculate the charge density $\sigma(\theta)$

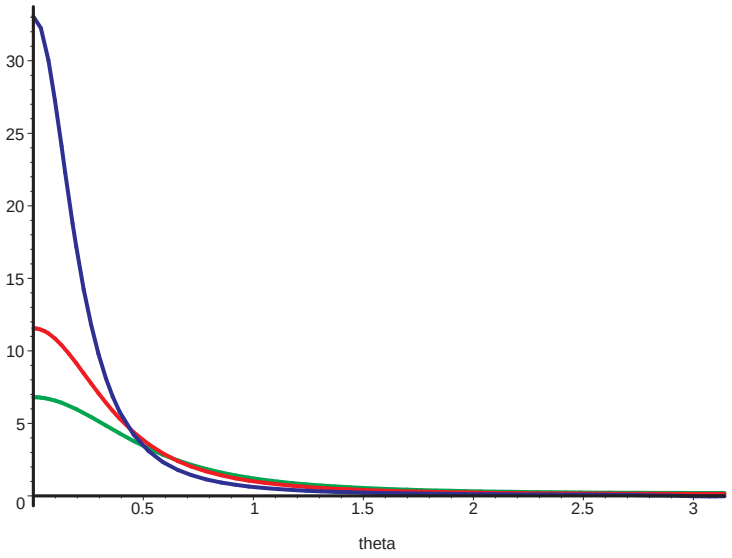

Figure 2.4: Surface charge induced by a point charge above a spherical conductor. Plotted is $4\pi R^2\sigma/q'$ as a function of θ for $z_0/R = 1.3$ (upper curve), $z_0/R = 1.6$ (middle curve), and $z_0/R = 1.9$ (lower curve). The charge density is peaked at $\theta = 0$ and its width increases with increasing values of z_0/R: the closer the charge is to the surface the narrower is the charge distribution.

on the surface of the conductor. For r close to R we have $r < z_0$, $r_< = r$, $r_> = z_0$, and Eq. (2.8.5) becomes

$$\Phi(\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\sum_\ell \frac{1}{z_0^{\ell+1}}\left(r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}}\right)P_\ell(\cos\theta).$$

Let $f(r) = r^\ell - R^{2\ell+1}/r^{\ell+1}$ be the function of r that appears inside the brackets. Its derivative is $f'(r) = \ell r^{\ell-1} + (\ell+1)R^{2\ell+1}/r^{\ell+2}$ and evaluating this at $r = R$ gives $f'(R) = (2\ell+1)R^{\ell-1}$. From Eq. (2.8.3) we then obtain

$$\sigma = -\frac{q}{4\pi}\sum_\ell (2\ell+1)\frac{R^{\ell-1}}{z_0^{\ell+1}}P_\ell(\cos\theta),$$

or

$$\sigma(\theta) = \frac{q'}{4\pi R^2}\sum_{\ell=0}^{\infty}(2\ell+1)(R/z_0)^\ell P_\ell(\cos\theta), \tag{2.8.8}$$

where

$$q' \equiv -q(R/z_0) \tag{2.8.9}$$

will be shown presently to be the *total induced charge* on the surface of the conductor. Plots of the charge density are presented in Fig. 2.4 for selected values of R/z_0.

That q' is truly the total surface charge can be shown by evaluating the integral of the charge density over the surface of the conductor; we should recover $q' = \oint \sigma\, da$. To evaluate this we let $da = R^2\, d\Omega = R^2 \sin\theta\, d\theta d\phi$, which reduces to $da = 2\pi R^2 \sin\theta\, d\theta$ after integration over ϕ. Then

$$\oint \sigma\, da = \frac{1}{2}q'\sum_{\ell=0}^{\infty}(2\ell+1)(R/z_0)^\ell \int_0^\pi P_\ell(\cos\theta)\sin\theta\, d\theta.$$

The integral over θ is equal to $\int_{-1}^{1} P_\ell(x)P_0(x)\,dx = 2\delta_{\ell,0}$ by virtue of Eq. (2.8.6), and we obtain $\oint \sigma\,da = q'$, as required.

The potential of Eq. (2.8.5) can be decomposed as

$$\Phi(\boldsymbol{x}) = \Phi_q(\boldsymbol{x}) + \Phi_{q'}(\boldsymbol{x}), \tag{2.8.10}$$

where

$$\Phi_q(\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\theta) \tag{2.8.11}$$

is the potential that would be obtained in the absence of a boundary, and

$$\Phi_{q'}(\boldsymbol{x}) = \frac{q'}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{(z_0')^\ell}{r^{\ell+1}}P_\ell(\cos\theta) \tag{2.8.12}$$

is the modification introduced by the boundary condition $\Phi(r = R) = 0$; here $q' = -q(R/z_0)$ was defined in Eq. (2.8.9) and

$$z_0' \equiv R^2/z_0. \tag{2.8.13}$$

Notice that $z_0'/R = R/z_0 < 1$, so that $z_0' < R \le r$.

The sums over ℓ can now be evaluated. For Φ_q we already know that Eq. (2.8.11) is equivalent to

$$\Phi_q(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\boldsymbol{x} - \boldsymbol{z}_0|}, \tag{2.8.14}$$

because Φ_q is the potential produced by a charge q at a position $\boldsymbol{z}_0 = z_0\hat{\boldsymbol{z}}$ in the absence of boundaries. By analogy, because $z_0' < r$ we have that Eq. (2.8.12) is equivalent to

$$\Phi_{q'}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\frac{q'}{|\boldsymbol{x} - \boldsymbol{z}_0'|}, \tag{2.8.15}$$

where $\boldsymbol{z}_0' = z_0'\hat{\boldsymbol{z}}$. This is the potential of a fictitious *image charge* q' at a position $\boldsymbol{z}_0'$ inside the conductor. While in reality the charge q' is distributed on the surface of the conductor, the potential $\Phi_{q'}$ is the same as if all this charge were concentrated at $\boldsymbol{z}_0'$.

This observation is the basis for the *method of images*, which consists of finding solutions for the potential in the presence of conducting surfaces by guessing the correct positions of the correct number of image charges. It is a powerful, but tricky, method.

2.9 Ring of charge outside a grounded, spherical conductor

In this section we consider a uniformly charged ring of radius $a > R$. If the ring is placed in the x-y plane, its charge density is described by

$$\rho(\boldsymbol{x'}) = \frac{q}{2\pi a^2}\delta(r' - a)\delta(\theta' - \tfrac{\pi}{2}), \tag{2.9.1}$$

where $q = \int \rho(\boldsymbol{x'})\,d^3x'$ is the ring's total charge. The δ-function in r' indicates that the ring is infinitely thin, while the δ-function in θ' tells us that it is placed in the x-y plane, as was previously stated.

Substituting Eq. (2.9.1) into Eq. (2.8.1) along with Eq. (2.8.2) gives

$$\begin{aligned}\Phi(\boldsymbol{x}) \;=\; & \frac{1}{4\pi\epsilon_0}\frac{q}{2\pi a^2}\sum_{\ell m}\frac{4\pi}{2\ell+1}Y_{\ell m}(\theta,\phi)\int_0^\infty\left(\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{R^{2\ell+1}}{r_<^{\ell+1}r_>^{\ell+1}}\right)\delta(r' - a)r'^2\,dr' \\ & \times\int Y_{\ell m}^*(\theta',\phi')\delta(\theta' - \tfrac{\pi}{2})\,d\Omega',\end{aligned}$$

where $d\Omega' = \sin\theta'\, d\theta' d\phi$, $r_< = \min(r, r')$, and $r_> = \max(r, r')$. To perform the ϕ' integration we recall that $Y^*_{\ell m} \propto e^{-im\phi'}$, so that integrating gives zero unless $m = 0$. So

$$\int_0^{2\pi} Y^*_{\ell m}(\theta', \phi')\, d\phi' = 2\pi Y_{\ell,0}(\theta')\delta_{m,0}.$$

Integration over θ' then gives

$$2\pi Y_{\ell,0}(\tfrac{\pi}{2})\delta_{m,0} = 2\pi\sqrt{\frac{2\ell+1}{4\pi}}P_\ell(0)\delta_{m,0}.$$

On the other hand, the radial integration gives

$$a^2\left(\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{R^{2\ell+1}}{r_<^{\ell+1}r_>^{\ell+1}}\right)$$

where $r_<$ now stands for the lesser of r and a, while $r_>$ stands for the greater of the two. With these results the potential becomes

$$\Phi(\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\sum_\ell \frac{4\pi}{2\ell+1}Y_{\ell,0}(\theta)\left(\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{R^{2\ell+1}}{r_<^{\ell+1}r_>^{\ell+1}}\right)\sqrt{\frac{2\ell+1}{4\pi}}P_\ell(0).$$

Invoking once more the relation between $Y_{\ell,0}(\theta)$ and $P_\ell(\cos\theta)$, we arrive at

$$\Phi(\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty} P_\ell(0)\left(\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{R^{2\ell+1}/a^{\ell+1}}{r^{\ell+1}}\right)P_\ell(\cos\theta). \tag{2.9.2}$$

Once more the potential is independent of ϕ, and is expressed as an infinite sum over ℓ. Once again we will see that the sum over ℓ can be evaluated.

We first calculate the charge density on the surface of the conductor. For r close to R and therefore smaller than a, we have $r_< = r$, $r_> = a$, and Eq. (2.9.2) becomes

$$\Phi(\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty} P_\ell(0)\frac{f(r)}{a^{\ell+1}}P_\ell(\cos\theta),$$

where the function $f(r) = r^\ell - R^{2\ell+1}/r^{\ell+1}$ was previously encountered in Sec. 2.8. We recall that $f'(R) = (2\ell+1)R^{\ell-1}$, and Eq. (2.8.3) gives

$$\sigma = -\frac{q}{4\pi}\sum_\ell (2\ell+1)P_\ell(0)\frac{R^{\ell-1}}{a^{\ell+1}}P_\ell(\cos\theta),$$

or

$$\sigma(\theta) = \frac{q'}{4\pi R^2}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(0)(R/a)^\ell P_\ell(\cos\theta), \tag{2.9.3}$$

where

$$q' \equiv -q(R/a) \tag{2.9.4}$$

is the total charge on the conducting surface. Plots of the surface charge density are presented in Fig. 2.5 for selected values of R/a.

As in the previous section the potential of Eq. (2.9.2) can be decomposed as

$$\Phi(\boldsymbol{x}) = \Phi_q(\boldsymbol{x}) + \Phi_{q'}(\boldsymbol{x}), \tag{2.9.5}$$

where Φ_q is the potential of a ring of total charge q and radius a, while $\Phi_{q'}$ is the potential of a fictitious image ring of total charge q' and radius $a' = R^2/a$. (You should work through the details and make sure that these statements are true.)

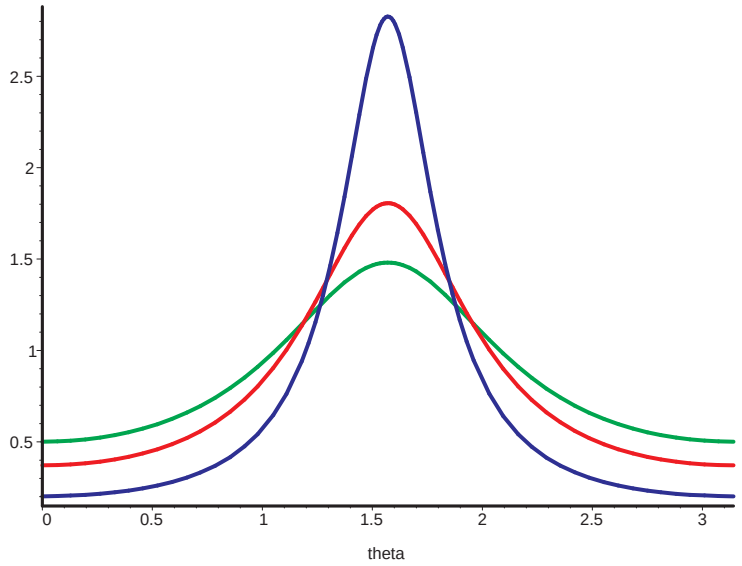

Figure 2.5: Surface charge induced by a ring of charge outside a spherical conductor. Plotted is $4\pi R^2\sigma/q'$ as a function of θ for $a/R = 1.3$ (upper curve), $a/R = 1.6$ (middle curve), and $a/R = 1.9$ (lower curve). The charge density is peaked at $\theta = \frac{\pi}{2}$ and its width increases with increasing values of a/R: the closer the ring is to the surface the narrower is the charge distribution.

2.10 Multipole expansion of the electric field

[The material presented in this section is also covered in Sec. 4.1 of Jackson's text.]

We now leave boundary-value problems behind, and derive the fact that in the absence of boundaries, the electrostatic potential outside a bounded distribution of charge can be characterized by a (potentially infinite) number of *multipole moments*. Among them are

$$Q \equiv \int \rho(\boldsymbol{x})\, d^3x \equiv \text{total charge} \equiv \text{monopole moment}, \tag{2.10.1}$$

$$p_a \equiv \int \rho(\boldsymbol{x}) x_a\, d^3x \equiv \text{dipole moment vector}, \tag{2.10.2}$$

$$Q_{ab} \equiv \int \rho(\boldsymbol{x})\big(3x_a x_b - r^2\delta_{ab}\big)\, d^3x \equiv \text{quadrupole moment tensor}. \tag{2.10.3}$$

Additional multipole moments can be constructed in a similar way, up to an infinite number of tensorial indices. But to proceed along this road quickly becomes cumbersome, and we shall find a better way of packaging the components of these multipole-moment tensors. Notice that the quadrupole moment is a symmetric and tracefree tensor: $Q_{ba} = Q_{ab}$ and $Q_{aa} = 0$ (summation over the repeated index is understood); it therefore possesses 5 independent components.

Consider the complex quantities

$$q_{\ell m} \equiv \int \rho(\boldsymbol{x}) r^\ell Y^*_{\ell m}(\theta,\phi)\, d^3x. \tag{2.10.4}$$

By virtue of Eq. (2.5.6), they satisfy the identity

$$q_{\ell,-m} = (-1)^m q^*_{\ell m}. \tag{2.10.5}$$

Because m ranges from $-\ell$ to ℓ, for a given ℓ there are $2\ell+1$ independent real quantities within the $q_{\ell m}$s. For $\ell=0$ we have only one, and we will see that q_{00} is proportional to the total charge. For $\ell=1$ we have three independent quantities, and we will see that the q_{1m}s give the components of the dipole moment vector. For $\ell=2$ we have five independent quantities, and we will see that the q_{2m}s give the components of the quadrupole moment tensor. Thus, the quantities of Eq. (2.10.4) give a convenient packaging of the multipole moments, to all orders.

To evaluate Eq. (2.10.4) for $\ell=0$ we recall that $Y_{00}^* = (4\pi)^{-1/2}$, so that

$$q_{00} = \frac{1}{\sqrt{4\pi}} Q. \tag{2.10.6}$$

We see that indeed, q_{00} is directly related to the total charge. For $\ell=1$ we have $rY_{11}^* = -(3/8\pi)^{1/2}(r\sin\theta)e^{-i\phi} = -(3/8\pi)^{1/2}(x-iy)$ and $rY_{10}^* = (3/4\pi)^{1/2} r\cos\theta = (3/4\pi)^{1/2} z$. This gives

$$q_{11} = -\sqrt{\frac{3}{8\pi}}\big(p_x - ip_y\big), \qquad q_{10} = \sqrt{\frac{3}{4\pi}}\, p_z, \tag{2.10.7}$$

and we see that indeed, the q_{1m}s are directly related to the dipole moment vector. For $\ell=2$ we have $r^2 Y_{22}^* = (15/32\pi)^{1/2}(r\sin\theta e^{-i\phi})^2 = (15/32\pi)^{1/2}(x-iy)^2$, $r^2 Y_{21}^* = -(15/8\pi)^{1/2}(r\sin\theta e^{-i\phi})(r\cos\theta) = -(15/8\pi)^{1/2}(x-iy)z$, and $r^2 Y_{20}^* = (5/16\pi)^{1/2} r^2(3\cos^2\theta - 1) = (5/16\pi)^{1/2}(3z^2 - r^2)$. This gives

$$\begin{aligned} q_{22} &= \frac{1}{12}\sqrt{\frac{15}{2\pi}}\big(Q_{xx} - Q_{yy} - 2iQ_{xy}\big), \\ q_{21} &= -\frac{1}{6}\sqrt{\frac{15}{2\pi}}\big(Q_{xz} - iQ_{yz}\big), \\ q_{20} &= \frac{1}{2}\sqrt{\frac{5}{4\pi}}\, Q_{zz}, \end{aligned} \tag{2.10.8}$$

and we see that indeed, the q_{2m}s are directly related to the quadrupole moment tensor. For higher values of ℓ it is best to stick with the definition of Eq. (2.10.4).

How are the multipole moments $q_{\ell m}$ related to the potential? The answer comes from Eq. (2.1.3) in which we set $\Phi_0(\boldsymbol{x}) = 0$ (because we are no longer considering boundaries) and substitute Eq. (2.6.12):

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell m} \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta,\phi) \int \rho(\boldsymbol{x'}) \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta',\phi')\, d^3x'.$$

Because we are looking at the potential *outside* the charge distribution, r is larger than r' and we can set $r_< = r'$, $r_> = r$. This gives

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell m} \frac{4\pi}{2\ell+1} \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} \int \rho(\boldsymbol{x'}) r'^\ell Y_{\ell m}^*(\theta',\phi')\, d^3x'.$$

We recognize the integral over the primed variables as the definition of the $q_{\ell m}$s, and we arrive at

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} q_{\ell m} \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}}. \tag{2.10.9}$$

This is an expansion of the potential in inverse powers of r, and the multipole moments play the role of expansion coefficients.

The leading term in Eq. (2.10.9) comes from the monopole moment q_{00}, or total charge Q:

$$\Phi_{\text{monopole}}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}. \tag{2.10.10}$$

The next term comes from the dipole moment. Using Eq. (2.10.7) and recalling the explicit expressions for the spherical harmonics of degree $\ell = 1$, we have

$$\sum_{m=-1}^{1} q_{1,m}Y_{1,m} = \frac{3}{4\pi}\left(p_x x + p_y y + p_z z\right)$$

and

$$\Phi_{\text{dipole}}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p}\cdot\boldsymbol{x}}{r^3}, \tag{2.10.11}$$

as in Eq. (2.3.5). The next term in the expansion of the electrostatic potential comes from the quadrupole moment. Using Eq. (2.10.8) and the explicit expressions for the spherical harmonics of degree $\ell = 2$, it is easy to show that

$$\Phi_{\text{quadrupole}}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\frac{1}{2}Q_{ab}\frac{x_a x_b}{r^5}. \tag{2.10.12}$$

In a situation in which $Q \neq 0$, the electrostatic potential far outside the charge distribution is well approximated by Φ_{monopole}; in this case the potential is characterized by a single quantity, the total charge Q. When the total charge vanishes, however, the potential is approximated by Φ_{dipole}, and in this case we need three quantities, the components of $\boldsymbol{p}$, the dipole moment vector. When this vanishes also, the potential is more complicated and is approximated by $\Phi_{\text{quadrupole}}$; in this case the characterization of the potential involves the five independent components of the quadrupole moment tensor.

It is important to be aware of the fact that the multipole moments $q_{\ell m}$ depend on the choice of origin for the coordinate system. For example, a point charge q at $\boldsymbol{x} = \boldsymbol{0}$ has a charge density described by $\rho(\boldsymbol{x}) = q\delta(\boldsymbol{x})$ and multipole moments given by

$$q_{\ell m} = \begin{cases} q/\sqrt{4\pi} & \ell = 0 \\ 0 & \text{otherwise} \end{cases}.$$

As expected, only the monopole moment is nonzero. But the same point charge placed at another position $\boldsymbol{x}_0$ possesses a very different set of multipole moments. In this case the density is $\rho(\boldsymbol{x}) = q\delta(\boldsymbol{x} - \boldsymbol{x}_0)$ and Eq. (2.10.4) gives

$$q_{\ell m} = q{r_0}^{\ell}Y^*_{\ell m}(\theta_0, \phi_0),$$

where (r_0, θ_0, ϕ_0) are the spherical coordinates of the point $\boldsymbol{x}_0$. We have agreement only for q_{00}, the lowest nonvanishing multipole moment. This is a special case of a general result: *Only the lowest nonvanishing multipole moments have values that are independent of the choice of origin for the coordinate system; all other moments change under a translation of the coordinates.* We shall not provide a proof of this statement.

2.11 Multipolar fields

To conclude this chapter we construct plots of electric field lines that would be produced by a pure multipole of order ℓ. We shall consider only axisymmetric situations and set $m = 0$.

The potential for such a multipole is obtained from Eq. (2.10.9), in which we set $[4\pi/(2\ell+1)]q_{\ell 0}Y_{\ell 0}(\theta,\phi) = q_\ell P_\ell(\cos\theta)$. This gives

$$\Phi(r,\theta) = \frac{q_\ell}{4\pi\epsilon_0}\frac{P_\ell(\cos\theta)}{r^{\ell+1}}, \tag{2.11.1}$$

and the corresponding electric field is

$$\boldsymbol{E}(r,\theta) = \frac{q_\ell}{4\pi\epsilon_0}\frac{1}{r^{\ell+2}}\Big[(\ell+1)P_\ell(\cos\theta)\hat{\boldsymbol{r}} + \sin\theta P'_\ell(\cos\theta)\hat{\boldsymbol{\theta}}\Big], \tag{2.11.2}$$

in which a prime indicates differentiation with respect to the argument.

The electric field lines are represented by parametric curves $\boldsymbol{x}(\lambda)$ for which λ is an arbitrary parameter. These curves are integral curves of the vector field $\boldsymbol{E}(\boldsymbol{x})$, which means that the tangent vector to the curves is defined to be everywhere equal to $\boldsymbol{E}$. Thus, the field lines are determined by solving the differential equations

$$\frac{d\boldsymbol{x}}{d\lambda} = \boldsymbol{E}(\boldsymbol{x}). \tag{2.11.3}$$

In the case of an axisymmetric field the curves can depend on r and θ only, and we have

$$\frac{d\boldsymbol{x}}{d\lambda} = \frac{\partial\boldsymbol{x}}{\partial r}\frac{dr}{d\lambda} + \frac{\partial\boldsymbol{x}}{\partial\theta}\frac{d\theta}{d\lambda} = \frac{dr}{d\lambda}\hat{\boldsymbol{r}} + r\frac{d\theta}{d\lambda}\hat{\boldsymbol{\theta}}.$$

This shows that $dr/d\lambda = E_r$ and $rd\theta/d\lambda = E_\theta$, and using Eq. (2.11.2) produces $dr \propto (\ell+1)P_\ell$ and $rd\theta \propto \sin\theta P'_\ell$, or

$$\frac{dr}{r} = \frac{(\ell+1)P_\ell(\cos\theta)}{\sin\theta P'_\ell(\cos\theta)}\,d\theta = -\frac{(\ell+1)P_\ell(\mu)}{(1-\mu^2)P'_\ell(\mu)}\,d\mu, \tag{2.11.4}$$

where $\mu = \cos\theta$. The field lines are therefore determined by integrating Eq. (2.11.4).

To accomplish this it is useful to recall the differential equation satisfied by the Legendre polynomials,

$$\frac{d}{d\mu}(1-\mu^2)\frac{dP_\ell}{d\mu} + \ell(\ell+1)P_\ell = 0.$$

We solve this algebraically for P_ℓ and substitute into Eq. (2.11.4) to get

$$\ell\frac{dr}{r} = \frac{\big[(1-\mu^2)P'_\ell\big]'}{(1-\mu^2)P'_\ell(\mu)}\,d\mu.$$

This can be integrated at once, and we obtain

$$\ell\ln r = \ln\big[(1-\mu^2)P'_\ell\big] + \text{constant},$$

or

$$r(\theta) = r_0\big|\sin^2\theta P'_\ell(\cos\theta)\big|^{1/\ell}, \tag{2.11.5}$$

where r_0 is constant on each curve, but varies from one curve to the next. Equation (2.11.5) gives the mathematical description of the electric field lines of an axisymmetric multipole of degree ℓ. The field lines are plotted for selected values of ℓ in Fig. 2.6.

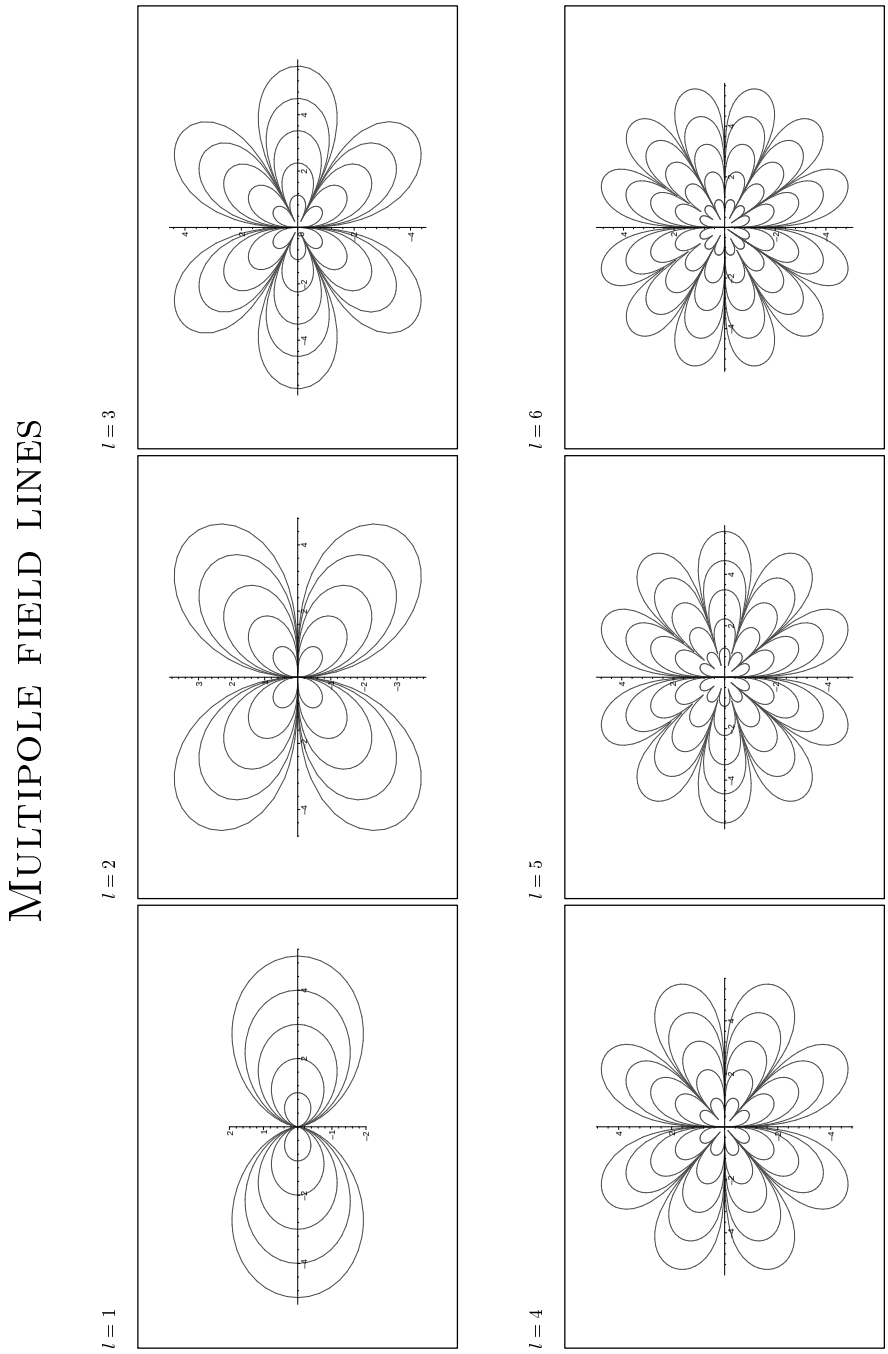

Figure 2.6: Electric field lines of an axisymmetric multipole, for selected values of the multipole order ℓ.

2.12 Problems

1. Calculate the electrostatic potential $\Phi(\boldsymbol{x})$ inside a charge distribution whose density is given by

$$\rho(\boldsymbol{x}) = \frac{3}{8\pi}\rho_0 (r/r_0)^2 \sin^2\theta \qquad \text{for } r < r_0,$$

where ρ_0 and r_0 are constants. For $r > r_0$ the density is zero. There are no boundary surfaces in this problem.

2. (Jackson's Problem 1.12) Prove Green's reciprocation theorem: If Φ_1 is the potential due to a volume charge density ρ_1 within a volume V and a surface charge density σ_1 on the conducting surface S bounding the volume V, while Φ_2 is the potential due to another charge distribution ρ_2 and σ_2, then

$$\int_V \rho_1 \Phi_2 \, d^3x + \oint_S \sigma_1 \Phi_2 \, da = \int_V \rho_2 \Phi_1 \, d^3x + \oint_S \sigma_2 \Phi_1 \, da.$$

3. A point charge q is brought to a distance $z_0 > R$ measured from the centre of a conducting sphere of radius R (as in Sec. 2.8). The conductor, however, is not grounded, but kept in careful isolation; as it initially supports a zero net charge on its surface, its final net surface charge must also vanish. The surface of the conductor is therefore at a constant potential $V \neq 0$. Find V as a function of q, z_0, and R.

4. Suppose that at a spherical boundary surface S of radius R, the electrostatic potential is assigned the values $\Phi(R, \theta, \phi) = V \sin\theta \cos\theta \cos\phi$, where V is a constant. Suppose that there are no charges outside this surface, and that S is the only boundary present in the problem. Find the potential for $r > R$.

5. In this problem we consider the electrostatic potential due to a uniformly charged disk located outside a grounded, conducting sphere. The conductor has a radius R and the potential over its surface is zero. The disk has an inner radius $a > R$ and an outer radius $b > a$. It is convenient to place the disk in the equatorial plane of the spherical coordinate system, so that its charge density is described by

$$\rho(\boldsymbol{x}) = \frac{3q}{2\pi} \frac{\delta(\theta - \frac{\pi}{2})}{b^3 - a^3}, \qquad a < r < b,$$

with q denoting the disk's total charge.

a) Verify that q is indeed the disk's total charge.

b) Calculate the potential in the region $R < r < a$.

c) Calculate the surface charge density $\sigma(\theta)$ on the surface of the conducting sphere, and provide a plot of this function. (Use the values $q = 1$, $R = 1$, $a = 1.2$, and $b = 2$; make sure to include a sufficient number of terms in the infinite sum over ℓ.)

d) Show that the total charge on the surface of the conductor is given by

$$q' = -\frac{3q}{2} \frac{(a+b)R}{a^2 + ab + b^2}.$$

6. We generalize the situation considered in Sec. 2.8 by placing an additional charge $-q$ at position $-\boldsymbol{z}_0 = -z_0\hat{\boldsymbol{z}}$ below the conducting sphere. The first charge $+q$ is still at $+\boldsymbol{z}_0 = z_0\hat{\boldsymbol{z}}$, the conducting sphere still has a radius R, and the sphere is still grounded.

Calculate the vector $\boldsymbol{p}$, the total electric dipole moment of this charge distribution. Express your result in terms of q, z_0, and R.

7. Suppose that the upper hemisphere of a conducting sphere is maintained at a potential $+V$, while its lower hemisphere is maintained at a potential $-V$. (The two hemispheres are separated by a thin layer of insulating material, but there is no need to include this into the mathematical model.) The conducting surface has a radius R, and there are no charges outside the conductor.

   Show that the potential outside the conducting sphere can be expressed as

   $$\Phi(r,\theta) = V \sum_{\ell=1}^{\infty} a_\ell (R/r)^{\ell+1} P_\ell(\cos\theta),$$

   where the numerical coefficients a_ℓ are zero if ℓ is even, and equal to $(2\ell + 1)\int_0^1 P_\ell(x)\,dx$ if ℓ is odd. Calculate a_ℓ explicitly for $\ell = (1, 3, 5)$.

8. A conducting sphere of radius R is maintained at a potential V. It is surrounded by a thin spherical shell of radius $a > R$ on which there is a surface charge density proportional to $\cos\theta$. The volume density of charge is given by

   $$\rho(r,\theta) = \frac{3p}{4\pi a^3}\,\cos\theta\,\delta(r-a),$$

   where p is a constant — it is the dipole moment of the charge distribution on the thin shell.

   a) Verify that the shell's total charge is zero, and that $\boldsymbol{p} = p\hat{\boldsymbol{z}}$ is the dipole moment vector of this charge distribution.

   b) Calculate the potential in the region $R < r < a$.

   c) Calculate the potential in the region $r > a$.

   d) What is the total charge of this configuration? What is the total dipole moment vector? (The *total* charge configuration includes the free charge on the spherical shell and the induced charge on the surface of the conductor.)

   e) Calculate the density of charge $\sigma(\theta)$ on the surface of the conductor. What is the total charge on the conductor?

9. A dielectric sphere of radius R (the inner boundary) is maintained at a potential $\Phi = V_0\cos\theta$. It is surrounded by a thin spherical shell of radius $a > R$ with a density of charge given by

   $$\rho = \frac{3p}{4\pi a^3}\,\cos\theta\,\delta(r-a),$$

   where p is a constant.

   Find the value of V_0 which makes the scalar potential $\Phi(\boldsymbol{x})$ vanish for $r > a$.

10. Calculate the electrostatic potential $\Phi(\boldsymbol{x})$ outside a thin, hollow sphere of radius R on which there is a surface density of charge proportional to $1 + 3\cos(2\theta)$. You may assume that there are no boundaries, and that the volume charge density is given by

    $$\rho(r,\theta) = \frac{5\lambda}{8\pi R^4}\Big[1 + 3\cos(2\theta)\Big]\delta(r-R),$$

    where λ is a constant.

    What are the components of the quadrupole moment tensor for this charge distribution?

Chapter 3
Magnetostatics

3.1 Equations of magnetostatics

For time-independent situations, the part of Maxwell's equations that involve the magnetic field are decoupled from those involving the electric field, and they can be dealt with separately. The equations of magnetostatics are Poisson's equation for the vector potential,

$$\nabla^2 \boldsymbol{A}(\boldsymbol{x}) = -\mu_0 \boldsymbol{j}(\boldsymbol{x}), \tag{3.1.1}$$

the Lorenz (or Coulomb) gauge condition

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0, \tag{3.1.2}$$

and the relation between potential and field,

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{3.1.3}$$

We recall that the current density must satisfy the statement of charge conservation, $\boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$.

Because Eq. (3.1.1) is very similar to the electrostatic equation (2.1.1), the tools introduced in Chapter 2 to solve Poisson's equation can be taken over directly to magnetostatics. And because we already have gained much experience solving such equations, it will suffice to keep our discussion of magnetostatics quite brief. In particular, it will be sufficient to solve Eq. (3.1.1) in the absence of boundaries. We therefore write the solution as

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}(\boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|}\, d^3x', \tag{3.1.4}$$

recalling the results of Sec. 1.8.

3.2 Circular current loop

[The material presented in this section is also covered in Sec. 5.5 of Jackson's text.]

As our first application we calculate the magnetic field produced by a current I flowing inside a circular loop of radius R. We place the loop in the x-y plane, at $z = 0$. To evaluate the integral of Eq. (3.1.4) for this situation we will use spherical coordinates and express the Green's function as in Eq. (2.6.12),

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{x'}|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi), \tag{3.2.1}$$

where $r_<$ is the lesser of r and r', while $r_>$ is the greater of the two.

The density of the negative charge carriers inside the wire is given by

$$\rho(\boldsymbol{x}) = -\frac{\lambda}{R}\delta(r-R)\delta(\theta-\tfrac{\pi}{2}), \tag{3.2.2}$$

where λ is the magnitude of the linear charge density (minus the wire's total charge per unit length); the total charge is given by $\int \rho(\boldsymbol{x})\,d^3x = -2\pi R\lambda$. The charge carriers move in the negative ϕ direction so as to produce a positive current. Their velocity vector is

$$\boldsymbol{v} = -v\hat{\boldsymbol{\phi}} = -v\big(-\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}\big), \tag{3.2.3}$$

where $\hat{\boldsymbol{\phi}}$ is a unit vector that points in the direction of increasing ϕ. Its expression in terms of the constant vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ is obtained by noting that the position vector of a point in the plane is given by $\boldsymbol{r} = r\cos\phi\hat{\boldsymbol{x}} + r\sin\phi\hat{\boldsymbol{y}}$, so that a vector pointing in the ϕ direction is $\partial\boldsymbol{r}/\partial\phi = -r\sin\phi\hat{\boldsymbol{x}} + r\cos\phi\hat{\boldsymbol{y}}$; dividing by r produces the normalized vector $\hat{\boldsymbol{\phi}}$. From Eqs. (3.2.2) and (3.2.3) we obtain the current density,

$$\boldsymbol{j}(\boldsymbol{x}) = \rho\boldsymbol{v} = \frac{I}{R}\hat{\boldsymbol{\phi}}\,\delta(r-R)\delta(\theta-\tfrac{\pi}{2}), \tag{3.2.4}$$

where $I \equiv \lambda v$ is the current.

To obtain the vector potential we substitute Eqs. (3.2.1) and (3.2.4) into Eq. (3.1.4). For concreteness we take $\boldsymbol{x}$ to be outside the loop, so that $r > r'$, $r_< = r'$, and $r_> = r$. We must pay attention to the fact that $\hat{\boldsymbol{\phi}}$ is not a constant vector: its expression in terms of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ must be substituted inside the integral before integration is attempted. It is convenient to consider the complex combination $A_y - iA_x$, which is generated by

$$j_y - ij_x = \frac{I}{R}(\cos\phi' + i\sin\phi')\delta(r'-R)\delta(\theta'-\tfrac{\pi}{2}) = \frac{I}{R}e^{i\phi'}\delta(r'-R)\delta(\theta'-\tfrac{\pi}{2});$$

we have now expressed this in terms of the primed variables. We notice that the ϕ' dependence of the current density is contained entirely in the exponential factor $e^{i\phi'}$. After evaluating the r' and θ' integrals the vector potential becomes

$$A_y - iA_x = \frac{\mu_0}{4\pi}\frac{I}{R}\sum_{\ell m}\frac{4\pi}{2\ell+1}\frac{R^{\ell+2}}{r^{\ell+1}}Y_{\ell m}(\theta,\phi)\int_0^{2\pi}Y^*_{\ell m}(\tfrac{\pi}{2},\phi')e^{i\phi'}\,d\phi'.$$

The ϕ' integral evaluates to $Y_{\ell m}(\frac{\pi}{2},0)\int_0^{2\pi}e^{-im\phi'}e^{i\phi'}\,d\phi' = 2\pi Y_{\ell m}(\frac{\pi}{2},0)\delta_{m,1}$, and the potential simplifies to

$$A_y - iA_x = \frac{\mu_0}{4\pi}I\sum_{\ell=1}^{\infty}\frac{8\pi^2}{2\ell+1}Y_{\ell,1}(\tfrac{\pi}{2},0)\left(\frac{R}{r}\right)^{\ell+1}Y_{\ell,1}(\theta,\phi);$$

the sum begins at $\ell = 1$ because there is no spherical harmonic Y_{01}. The spherical-harmonic functions can then be expressed as $Y_{\ell,1}(\theta,\phi) = Y_{\ell,1}(\theta,0)e^{i\phi}$, and this allows us to extract the real and imaginary parts of the right-hand side. We obtain

$$A_x = -\frac{\mu_0}{4\pi}I\sum_{\ell=1}^{\infty}\frac{8\pi^2}{2\ell+1}Y_{\ell,1}(\tfrac{\pi}{2},0)\left(\frac{R}{r}\right)^{\ell+1}Y_{\ell,1}(\theta,0)\sin\phi$$

and

$$A_y = \frac{\mu_0}{4\pi}I\sum_{\ell=1}^{\infty}\frac{8\pi^2}{2\ell+1}Y_{\ell,1}(\tfrac{\pi}{2},0)\left(\frac{R}{r}\right)^{\ell+1}Y_{\ell,1}(\theta,0)\cos\phi,$$

and these combine to form the vector

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}I\hat{\boldsymbol{\phi}}\sum_{\ell=1}^{\infty}\frac{8\pi^2}{2\ell+1}Y_{\ell,1}(\tfrac{\pi}{2},0)\left(\frac{R}{r}\right)^{\ell+1}Y_{\ell,1}(\theta,0). \tag{3.2.5}$$

We see that the vector potential is expressed as an expansion in powers of R/r.

When $r \gg R$ the vector potential is well approximated by the leading term of the expansion, given by $\ell = 1$. In this case we have $Y_{11}(\theta, 0) = -\sqrt{3/(8\pi)}\sin\theta$ and Eq. (3.2.5) becomes

$$\boldsymbol{A}(\boldsymbol{x}) \simeq \frac{\mu_0}{4\pi}(\pi R^2 I)\frac{1}{r^2}\big(\sin\theta\hat{\boldsymbol{\phi}}\big). \tag{3.2.6}$$

We will write this in a different form. First we define the vector

$$\boldsymbol{m} = (\pi R^2 I)\hat{\boldsymbol{z}}, \tag{3.2.7}$$

which points in the z direction and whose magnitude is the product of the current I and the area πR^2 enclosed by the loop; this vector is known as the *magnetic moment* of the current distribution. Then we notice that $\hat{\boldsymbol{z}} \times \boldsymbol{x}/r = -(y/r)\hat{\boldsymbol{x}} + (x/r)\hat{\boldsymbol{y}} = -\sin\theta\sin\phi\hat{\boldsymbol{x}} + \sin\theta\cos\phi\hat{\boldsymbol{y}} = \sin\theta\hat{\boldsymbol{\phi}}$. So Eq. (3.2.6) can be written as

$$\boldsymbol{A}(\boldsymbol{x}) \simeq \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{x}}{r^3}, \tag{3.2.8}$$

and this shall be our final expression.

The curl of $\boldsymbol{A}$ gives the magnetic field. Apart from a common factor of $(\mu_0/4\pi)|\boldsymbol{m}|$, its x component is equal to $-\nabla_z(x/r^3) = 3xz/r^5$, its y component is equal to $-\nabla_z(y/r^3) = 3yz/r^5$, and finally, its z component is equal to $\nabla_x(x/r^3)+\nabla_y(y/r^3) = 2/r^3 - 3x^5/r^5 - 3y^2/r^5 = -1/r^3 + 3z^2/r^5$. These results combine to form the vector $3z\boldsymbol{x}/r^5 - \hat{\boldsymbol{z}}/r^3$, and multiplying by $|\boldsymbol{m}|$ gives $3(\boldsymbol{m}\cdot\boldsymbol{x})\boldsymbol{x}/r^5 - \boldsymbol{m}/r^3$. The magnetic field is therefore

$$\boldsymbol{B}(\boldsymbol{x}) \simeq \frac{\mu_0}{4\pi}\frac{3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}}{r^3}, \tag{3.2.9}$$

where $\hat{\boldsymbol{r}} = \boldsymbol{x}/r$ is a unit vector that points in the radial direction. Notice that this expression for the magnetic field is very similar to what was obtained in Sec. 2.3 for the electric field of a dipole:

$$\boldsymbol{E}_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\frac{3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}}{r^3}.$$

So the magnetic field far outside a current loop has a dipolar behaviour, with $\boldsymbol{m}$ playing the role of dipole moment.

3.3 Spinning charged sphere

As a second application we calculate the magnetic field produced by a hollow sphere of radius R, uniformly charged, that is rotating with an angular velocity $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{z}}$.

Here the charge density is given by

$$\rho(\boldsymbol{x}) = \frac{q}{4\pi R^2}\delta(r - R), \tag{3.3.1}$$

where q is the sphere's total charge. To calculate the velocity vector $\boldsymbol{v}$ of a point $\boldsymbol{x}$ on the surface of the sphere, we consider a narrow strip at an angle θ with respect to the z axis (see Fig. 3.1). A point on this strip moves in the ϕ direction; in a time dt the angle increases by $d\phi = \Omega\, dt$. Since this point is at a distance $R\sin\theta$ from the z axis, the length traveled is $dl = (R\sin\theta)\, d\phi$. The speed is therefore $v = dl/dt = (R\sin\theta)\Omega$, and the velocity vector is $\boldsymbol{v} = v\hat{\boldsymbol{\phi}} = \Omega R\sin\theta\hat{\boldsymbol{\phi}}$, or

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\Omega}\times\boldsymbol{x}. \tag{3.3.2}$$

This is the velocity of the charge distribution, and the current density is given by

$$\boldsymbol{j}(\boldsymbol{x}) = \frac{q\Omega}{4\pi R}\sin\theta\hat{\boldsymbol{\phi}}\delta(r - R). \tag{3.3.3}$$

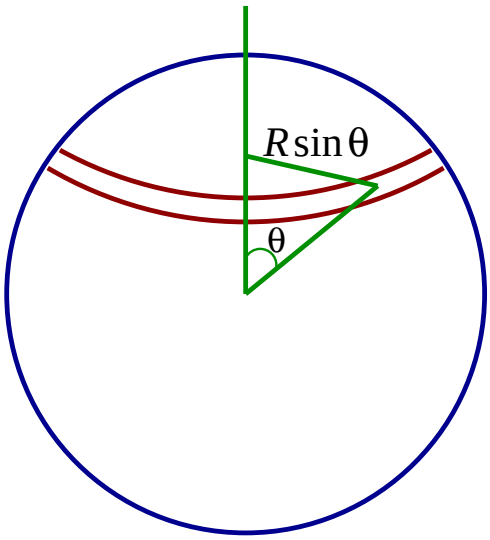

Figure 3.1: A hollow sphere rotating at an angular velocity Ω around the z axis.

We recall that $\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}$.

Again it is convenient to consider the complex combination $A_y - iA_x$, which is generated by

$$j_y - ij_x = \frac{q\Omega}{4\pi R}\sin\theta' e^{i\phi'}\delta(r' - R).$$

The angular function happens to be equal to $-\sqrt{8\pi/3}Y_{11}(\theta', \phi')$. Substituting this and Eq. (3.2.1) into the expression for $A_y - iA_x$ that comes from Eq. (3.1.4), we find that the angular integration is nonzero if and only if $\ell = m = 1$; the double sum over ℓ and m therefore contains a single term. The radial integration reduces to

$$\int_0^\infty \frac{r_<}{r_>^2}\delta(r' - R)r'^2\,dr' = \begin{cases} r & r < R \\ R^3/r^2 & r > R \end{cases},$$

and we obtain

$$A_y - iA_x = \frac{\mu_0}{4\pi}\frac{q\Omega}{4\pi R}\frac{4\pi}{3}\left[-\sqrt{\frac{8\pi}{3}}Y_{11}(\theta, \phi)\right]\begin{Bmatrix} r \\ R^3/r^2 \end{Bmatrix}.$$

The angular function within square brackets is $\sin\theta e^{i\phi}$, and the vector potential is

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\frac{q\Omega}{3R}\big(-\sin\theta\sin\phi\hat{\boldsymbol{x}} + \sin\theta\cos\phi\hat{\boldsymbol{y}}\big)\begin{Bmatrix} r \\ R^3/r^2 \end{Bmatrix}.$$

After multiplication by r the vector within brackets becomes $-y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}} = \hat{\boldsymbol{z}} \times \boldsymbol{x}$.

If we let

$$\boldsymbol{m} \equiv \frac{1}{3}q\Omega R^2\hat{\boldsymbol{z}} = \frac{1}{3}qR^2\boldsymbol{\Omega}, \tag{3.3.4}$$

we find that the vector potential inside the sphere is given by

$$\boldsymbol{A}_{\text{in}}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m} \times \boldsymbol{x}}{R^3} \qquad (r < R), \tag{3.3.5}$$

while the potential outside the sphere is

$$\boldsymbol{A}_{\text{out}}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m} \times \boldsymbol{x}}{r^3} \qquad (r > R). \tag{3.3.6}$$

Notice that these are *exact results* — no approximations were involved in the calculation of the vector potential. Notice also that $\boldsymbol{A}_{\rm out}$ takes the same form as in Eq. (3.2.8), except for the fact that here, the magnetic moment $\boldsymbol{m}$ is a different vector. From this we conclude that the magnetic field $\boldsymbol{B}_{\rm out}$ outside the sphere has a dipolar behaviour, and that it takes the form of Eq. (3.2.9).

To calculate the magnetic field inside the sphere we express the vector potential as $\boldsymbol{A}_{\rm in} = (\mu_0/4\pi)(m/R^3)(-y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}})$ and take its curl. Its x and y components are both zero, and its z component is proportional to $\nabla_x x + \nabla_y y = 2$. We arrive at

$$\boldsymbol{B}_{\rm in}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{2\boldsymbol{m}}{R^3}. \tag{3.3.7}$$

This states that inside the sphere, the magnetic field is constant and points in the same direction as the magnetic moment (which corresponds to the rotation axis).

3.4 Multipole expansion of the magnetic field

[The material presented in this section is also covered in Sec. 5.6 of Jackson's text.]

For both the current loop (Sec. 3.2) and the rotating sphere (Sec. 3.3) we found that the vector potential at a large distance from the current distribution was given by

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{x}}{r^3}, \tag{3.4.1}$$

where the vector $\boldsymbol{m}$, known as the magnetic moment, characterizes the distribution of current. And in both cases we found the magnetic field to be described by

$$\boldsymbol{B}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}}{r^3}, \tag{3.4.2}$$

where $\hat{\boldsymbol{r}} = \boldsymbol{x}/r$; this is a dipolar field. Here we will show that these expressions are in fact very general: they are the leading terms in an expansion of the vector potential and the magnetic field in powers of $1/r$. This *multipole expansion* is valid for points $\boldsymbol{x}$ that lie outside of the current distribution.

Let the distribution of current be confined to a volume V bounded by a surface S. We take $\boldsymbol{x}$ to be outside of V, while $\boldsymbol{x'}$ is necessarily inside. We examine the vector potential at a large distance $r \gg r'$, starting from the exact expression

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{j}(\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'.$$

We recall that the current density satisfies $\boldsymbol{\nabla}\cdot\boldsymbol{j} = 0$, and we recognize that there is no current flowing across S; this implies that $\boldsymbol{j}\cdot d\boldsymbol{a} = 0$, where $d\boldsymbol{a}$ is the surface element on S. As we shall show below, these equations imply the integral statements

$$\int_V \boldsymbol{j}(\boldsymbol{x'})\,d^3x' = 0 \tag{3.4.3}$$

and

$$\int_V (\hat{\boldsymbol{r}}\cdot\boldsymbol{x'})\boldsymbol{j}(\boldsymbol{x'})\,d^3x' = -\frac{1}{2}\hat{\boldsymbol{r}}\times\int_V \big[\boldsymbol{x'}\times\boldsymbol{j}(\boldsymbol{x'})\big]\,d^3x'. \tag{3.4.4}$$

To simplify the vector potential we expand $1/|\boldsymbol{x}-\boldsymbol{x'}|$ in powers of $1/r$. We could do this very systematically by involving the addition theorem for spherical harmonics (Sec. 2.6), but it will be good enough to keep only a small number of terms. For this we proceed directly:

$$\begin{aligned}
|\boldsymbol{x}-\boldsymbol{x'}| &= \sqrt{(\boldsymbol{x}-\boldsymbol{x'})\cdot(\boldsymbol{x}-\boldsymbol{x'})} \\
&= \sqrt{r^2 - 2\boldsymbol{x}\cdot\boldsymbol{x'} - r'^2} \\
&= r\sqrt{1 - 2\hat{\boldsymbol{r}}\cdot\boldsymbol{x'}/r + \cdots} \\
&= r\big(1 - \hat{\boldsymbol{r}}\cdot\boldsymbol{x'}/r + \cdots\big),
\end{aligned}$$

so that

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x'}|} = \frac{1}{r}\Big(1 + \hat{\boldsymbol{r}}\cdot\boldsymbol{x'}/r + \cdots\Big).$$

Substituting this into the exact expression for $\boldsymbol{A}(\boldsymbol{x})$ gives

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\left[\frac{1}{r}\int_V \boldsymbol{j}(\boldsymbol{x'})\,d^3x' + \frac{1}{r^2}\int_V (\hat{\boldsymbol{r}}\cdot\boldsymbol{x'})\boldsymbol{j}(\boldsymbol{x'})\,d^3x' + \cdots\right].$$

The first term on the right-hand side vanishes by virtue of Eq. (3.4.3), and the second term is altered by Eq. (3.4.4). We have

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{1}{r^2}\left\{-\frac{1}{2}\hat{\boldsymbol{r}}\times\int_V \big[\boldsymbol{x'}\times\boldsymbol{j}(\boldsymbol{x'})\big]\,d^3x' + \cdots\right\},$$

and if we *define* the magnetic moment of the current distribution by

$$\boldsymbol{m} = \frac{1}{2}\int_V \boldsymbol{x'}\times\boldsymbol{j}(\boldsymbol{x'})\,d^3x', \tag{3.4.5}$$

then we recover Eq. (3.4.1). We see that indeed,

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\hat{\boldsymbol{r}}}{r^2} + O\left(\frac{1}{r^3}\right)$$

is a very general expression, and that the magnetic moment of an arbitrary distribution of current is given by Eq. (3.4.5). It is a straightforward exercise to show that for the current loop of Sec. 3.2, Eq. (3.4.5) reproduces Eq. (3.2.7). Similarly, Eq. (3.4.5) reduces to Eq. (3.3.4) for the rotating sphere of Sec. 3.3.

We still have to establish the identities of Eqs. (3.4.3) and (3.4.4). We begin with the z component of Eq. (3.4.3), in which we drop the primes on the integration variables. Consider

$$\boldsymbol{\nabla}\cdot(z\boldsymbol{j}) = z(\boldsymbol{\nabla}\cdot\boldsymbol{j}) + \boldsymbol{j}\cdot\boldsymbol{\nabla}z = j_z,$$

which follows because $\boldsymbol{\nabla}\cdot\boldsymbol{j} = 0$ and $\boldsymbol{\nabla}z = \hat{\boldsymbol{z}}$. Integrating this equation over V gives

$$\int_V j_z\,d^3x = \int_V \boldsymbol{\nabla}\cdot(z\boldsymbol{j})\,d^3x = \oint_S z\boldsymbol{j}\cdot d\boldsymbol{a},$$

where we have invoked Gauss' theorem. But the right-hand side is zero because no current crosses the surface S, and we have $\int_V j_z\,d^3x = 0$. Proceeding similarly with the x and y components, we arrive at Eq. (3.4.3).

To establish Eq. (3.4.4) requires a bit more work, but the idea is the same. We consider the z component of this equation, in which we drop the primes on the integration variables. (At the same time we replace the vector $\hat{\boldsymbol{r}}$, which is constant with respect to the primed variables, by an arbitrary constant vector $\boldsymbol{e}$.) We have

$$\int_V (\boldsymbol{e}\cdot\boldsymbol{x})j_z\,d^3x = e_x\int_V xj_z\,d^3x + e_y\int_V yj_z\,d^3x + e_z\int_V zj_z\,d^3x$$

for the left-hand side of the identity. Consider now

$$\boldsymbol{\nabla}\cdot(xz\boldsymbol{j}) = xz(\boldsymbol{\nabla}\cdot\boldsymbol{j}) + \boldsymbol{j}\cdot\boldsymbol{\nabla}(xz) = xj_z + zj_x,$$

which integrates to

$$\int_V (xj_z + zj_x)\,d^3x = \oint_S xz\boldsymbol{j}\cdot d\boldsymbol{a} = 0,$$

giving rise to the integral identity

$$\int_V x j_z \, d^3x = -\int_V z j_x \, d^3x.$$

Similarly,

$$\int_V y j_z \, d^3x = -\int_V z j_y \, d^3x.$$

On the other hand,

$$\boldsymbol{\nabla} \cdot (z^2 \boldsymbol{j}) = z^2 (\boldsymbol{\nabla} \cdot \boldsymbol{j}) + \boldsymbol{j} \cdot \boldsymbol{\nabla} z^2 = 2 z j_z,$$

and this yields

$$\int_V z j_z \, d^3x = 0.$$

Using these results, we go back to the left-hand side of Eq. (3.4.4) and write

$$\begin{aligned} \int_V (\boldsymbol{e} \cdot \boldsymbol{x}) j_z \, d^3x &= \frac{1}{2} e_x \left(\int_V x j_z \, d^3x - \int_V z j_x \, d^3x \right) \\ &\quad + \frac{1}{2} e_y \left(\int_V y j_z \, d^3x - \int_V z j_y \, d^3x \right) \\ &= -\frac{1}{2} e_x \int_V (z j_x - x j_z) \, d^3x + \frac{1}{2} e_y \int_V (y j_z - z j_y) \, d^3x. \end{aligned}$$

Inside the first integral we recognize the y component of the vector $\boldsymbol{x} \times \boldsymbol{j}$, while the second integral contains its x component. So

$$\begin{aligned} \int_V (\boldsymbol{e} \cdot \boldsymbol{x}) j_z \, d^3x &= -\frac{1}{2} \left(e_x \int_V (\boldsymbol{x} \times \boldsymbol{j})_y \, d^3x - e_y \int_V (\boldsymbol{x} \times \boldsymbol{j})_x \, d^3x \right) \\ &= -\frac{1}{2} \left(\boldsymbol{e} \times \int_V (\boldsymbol{x} \times \boldsymbol{j}) \, d^3x \right)_z, \end{aligned}$$

and we have arrived at the right-hand side of Eq. (3.4.4).

3.5 Problems

1. Show that for the current loop of Sec. 3.2, Eq. (3.4.5) reproduces Eq. (3.2.7). And show that Eq. (3.4.5) reduces to Eq. (3.3.4) for the rotating sphere of Sec. 3.3.

2. A uniformly charged sphere of radius R is rotating around the z axis with an angular velocity Ω. Take the density of charge inside the sphere to be $3q/(4\pi R^3)$ (for $r < R$), where q is the sphere's total charge. Calculate $\boldsymbol{m}$, the sphere's magnetic moment vector.

3. Calculate the vector potential $\boldsymbol{A}(\boldsymbol{x})$ inside the spinning sphere of the preceding problem. Express it in terms of the sphere's magnetic moment vector $\boldsymbol{m}$.

4. Prove that for a current loop of arbitrary shape that lies in a plane, Eq. (3.4.5) gives

   $$|\boldsymbol{m}| = (\text{current})(\text{area enclosed by loop}),$$

   and that the direction of $\boldsymbol{m}$ is normal to the plane.

Chapter 4
Electromagnetic waves in matter

4.1 Macroscopic form of Maxwell's equations

[The material presented in this section is also covered in Sec. 6.6 of Jackson's text.]

Our goal in this chapter is to describe how electromagnetic waves propagate in dielectric materials. Our first task will be to reformulate Maxwell's equations in a way that conveniently incorporates the electromagnetic response of the medium.

4.1.1 Microscopic and macroscopic quantities

Maxwell's equations, as they were written down in Chapter 1, are the fundamental equations of classical electrodynamics; in principle they require no modification even in the presence of a nontrivial medium. But it is convenient to write them in a different form, in which the charges that are free to move within the medium are distinguished from the charges that are tied to the medium's molecules, and in which the microscopic fluctuations of all the variables involved have been smoothed out. The resulting form of Maxwell's equations involves only *macroscopic* densities of *free* charges and currents, and *macroscopic* electromagnetic fields.

To distinguish them from macroscopic quantities, the fundamental, microscopic variables will be assigned different symbols in this chapter. The *microscopic* density of charge (including both free and bound charges) is denoted η, while the *microscopic* density of current (including both free and bound currents) is denoted $\boldsymbol{J}$. The *microscopic* electric field is denoted $\boldsymbol{e}$, and the *microscopic* magnetic field is denoted $\boldsymbol{b}$. These quantities are related by the *microscopic* Maxwell equations,

$$\boldsymbol{\nabla}\cdot\boldsymbol{e} = \frac{1}{\epsilon_0}\eta, \tag{4.1.1}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{b} = 0, \tag{4.1.2}$$

$$\boldsymbol{\nabla}\times\boldsymbol{e} + \frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{0}, \tag{4.1.3}$$

$$\boldsymbol{\nabla}\times\boldsymbol{b} - \epsilon_0\mu_0\frac{\partial \boldsymbol{e}}{\partial t} = \mu_0\boldsymbol{J}, \tag{4.1.4}$$

which are the same as in Chapter 1. The *macroscopic* density of *free* charge will be denoted ρ, while the *macroscopic* density of *free* current will be denoted $\boldsymbol{j}$. These are obtained from the microscopic quantities by removing the contribution from the bound charges (those that are tied to the molecules and become part of the medium's electromagnetic response) and by macroscopic smoothing; the precise relationships will be given below. The *macroscopic* fields will be denoted $\boldsymbol{E}$, $\boldsymbol{D}$,

$\boldsymbol{B}$, and $\boldsymbol{H}$. The macroscopic electric field $\boldsymbol{E}$ is obtained from $\boldsymbol{e}$ by macroscopic smoothing, and $\boldsymbol{D}$ accounts for the electric polarizability of the medium; precise relationships will be given below. Similarly, the macroscopic magnetic field $\boldsymbol{B}$ is obtained from $\boldsymbol{b}$ by macroscopic smoothing, and $\boldsymbol{H}$ accounts for the magnetization of the medium; precise relationships will also be given below.

4.1.2 Macroscopic smoothing

Microscopic quantities fluctuate widely over atomic distances, and we would like to smooth out these fluctuations to define corresponding macroscopic quantities. For example, the charge density η switches from zero to infinity whenever a point charge is encountered, and we would like to replace this by a quantity that varies smoothly to reflect only macroscopic changes of density.

One simple-minded way of smoothing out a microscopic quantity f is to average it over a volume $V(\boldsymbol{x})$ centered at some position $\boldsymbol{x}$; this volume is imagined to be microscopically large (so that all microscopic fluctuations will be averaged over) but macroscopically small (so that variations over relevant macroscopic scales are not accidentally discarded). The averaging operation is an integration of f over the volume $V(\boldsymbol{x})$: $\langle f(\boldsymbol{x})\rangle = V^{-1}\int f(\boldsymbol{x'})\,d^3x'$, where the domain of integration consists of all points $\boldsymbol{x'}$ that lie within $V(\boldsymbol{x})$. If the volume is a sphere of radius R, then the condition is $|\boldsymbol{x}-\boldsymbol{x'}|<R$, and $\langle f(\boldsymbol{x})\rangle$ can be expressed as

$$\frac{1}{V}\int f(\boldsymbol{x'})\theta(R-|\boldsymbol{x}-\boldsymbol{x'}|)\,d^3x',$$

with a now infinite domain of integration.

In this averaging operation it is convenient to replace the sharply defined function $V^{-1}\theta(R-|\boldsymbol{x}-\boldsymbol{x'}|)$ by a better-behaved function $w(\boldsymbol{x}-\boldsymbol{x'})$ that smoothly goes from V^{-1} when $\boldsymbol{x'}=\boldsymbol{x}$ to zero when $\boldsymbol{x'}$ is far away from $\boldsymbol{x}$. Such a function, whose detailed specification is not required, will be called a *smoothing function*; a specific choice would be a Gaussian function of width R centered at $\boldsymbol{x}-\boldsymbol{x'}=\boldsymbol{0}$. The smoothing function is required to be normalized, in the sense that $\int w(\boldsymbol{x}-\boldsymbol{x'})\,d^3x'=1$.

We define the *macroscopic average* of a microscopic quantity $f(t,\boldsymbol{x})$ to be

$$F(t,\boldsymbol{x})\equiv\big\langle f(t,\boldsymbol{x})\big\rangle\equiv\int f(t,\boldsymbol{x'})w(\boldsymbol{x}-\boldsymbol{x'})\,d^3x', \tag{4.1.5}$$

where w is a suitable smoothing function (assumed to be normalized) and the integration is over all space. The property that w does not have sharp boundaries implies that the operations of averaging and differentiation commute. For example,

$$\nabla_a\big\langle f(t,\boldsymbol{x})\big\rangle=\big\langle\nabla_a f(t,\boldsymbol{x})\big\rangle, \tag{4.1.6}$$

and the same is true of time derivatives. This important result follows by straightforward manipulations. Consider, for example,

$$\begin{aligned}\frac{\partial\langle f\rangle}{\partial x} &= \int f(t,\boldsymbol{x'})\frac{\partial}{\partial x}w(\boldsymbol{x}-\boldsymbol{x'})\,d^3x'\\ &= -\int f(t,\boldsymbol{x'})\frac{\partial}{\partial x'}w(\boldsymbol{x}-\boldsymbol{x'})\,d^3x'\\ &= \int\frac{\partial f}{\partial x'}\,w(\boldsymbol{x}-\boldsymbol{x'})\,d^3x'\\ &= \left\langle\frac{\partial f}{\partial x}\right\rangle.\end{aligned}$$

In the second line we use the fact that w depends on the difference $\boldsymbol{x}-\boldsymbol{x'}$, so that a derivative with respect to x is minus a derivative with respect to x'. In the third line

we integrate by parts, noting that the boundary terms at infinity do not contribute because w vanishes there; the result is by definition the average of the microscopic quantity $\partial f/\partial x$.

Following the usage introduced in Eq. (4.1.5), we define the *macroscopic electric field* to be

$$\boldsymbol{E}(t,\boldsymbol{x}) = \big\langle \boldsymbol{e}(t,\boldsymbol{x})\big\rangle, \tag{4.1.7}$$

and the *macroscopic magnetic field* to be

$$\boldsymbol{B}(t,\boldsymbol{x}) = \big\langle \boldsymbol{b}(t,\boldsymbol{x})\big\rangle. \tag{4.1.8}$$

Because differentiation and averaging are commuting operations, the macroscopic form of Maxwell's equations are

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{1}{\epsilon_0}\langle\eta\rangle, \tag{4.1.9}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \tag{4.1.10}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} + \frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{0}, \tag{4.1.11}$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} - \epsilon_0\mu_0\frac{\partial\boldsymbol{E}}{\partial t} = \mu_0\langle\boldsymbol{J}\rangle. \tag{4.1.12}$$

The homogeneous equations (4.1.10) and (4.1.11) are the same as their microscopic version, but the inhomogeneous equations (4.1.9) and (4.1.12) involve a macroscopic averaging of the source terms.

4.1.3 Macroscopic averaging of the charge density

We distinguish between the charges that can move freely within the medium — the *free charges* — and the charges that are tied to the molecules — the *bound charges*. We will consider the bound charges to be part of the medium and remove them from the right-hand side of Maxwell's equations, which will then involve the free charges only. The influence of the bound charges will thus be taken to the left-hand side of the equations; it will be described by new macroscopic quantities, $\boldsymbol{P}$ (electric polarizability of the medium) and $\boldsymbol{M}$ (magnetization of the medium). From these we will define the new fields $\boldsymbol{D}\equiv\epsilon_0\boldsymbol{E}+\boldsymbol{P}$ and $\boldsymbol{H}\equiv\boldsymbol{B}/\mu_0-\boldsymbol{M}$.

In Fig. 4.1 we have a picture of the charge distribution in the medium. Let $\boldsymbol{x}_A(t)$ be the position vector of the free charge q_A labeled by "A", and let $\boldsymbol{v}_A(t)=d\boldsymbol{x}_A/dt$ be its velocity vector. Let $\boldsymbol{x}_n$ be the position vector of the centre of mass of the molecule labeled by "n"; for simplicity we assume that the molecules do not move within the medium. Finally, let $\boldsymbol{x}_B(t)$ be the position vector of the bound charge q_B labeled by "B" within a given molecule, relative to the centre of mass of this molecule, and let $\boldsymbol{v}_B(t)=d\boldsymbol{x}_B/dt$ be its velocity vector.

The density of free charge is

$$\eta_{\text{free}}(t,\boldsymbol{x}) = \sum_A q_A\delta\big(\boldsymbol{x}-\boldsymbol{x}_A(t)\big). \tag{4.1.13}$$

The density of bound charge within the molecule labeled by "n" is

$$\eta_n(t,\boldsymbol{x}) = \sum_B q_B\delta\big(\boldsymbol{x}-\boldsymbol{x}_n-\boldsymbol{x}_B(t)\big). \tag{4.1.14}$$

We assume that each molecule is electrically neutral, so that $\sum_B q_B = 0$. The density of bound charge is then

$$\eta_{\text{bound}}(t,\boldsymbol{x}) = \sum_n \eta_n(t,\boldsymbol{x}), \tag{4.1.15}$$

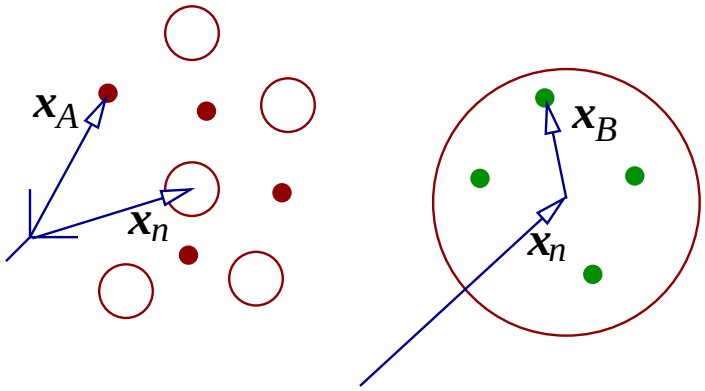

Figure 4.1: The left panel shows the free charges (small solid disks) and molecules (large open circles) in the medium. The right panel shows the distribution of bound charges within a given molecule.

and the total density of charge is $\eta = \eta_{\text{free}} + \eta_{\text{bound}}$.

We first calculate the macroscopic average of the molecular charge density. We have

$$\begin{aligned}\langle \eta_n \rangle &= \int \eta_n(t, \boldsymbol{x'}) w(\boldsymbol{x} - \boldsymbol{x'})\, d^3x' \\ &= \sum_B q_B \int \delta(\boldsymbol{x'} - \boldsymbol{x}_n - \boldsymbol{x}_B) w(\boldsymbol{x} - \boldsymbol{x'})\, d^3x' \\ &= \sum_B q_B w(\boldsymbol{x} - \boldsymbol{x}_n - \boldsymbol{x}_B).\end{aligned}$$

The function $w(\boldsymbol{x} - \boldsymbol{x}_n - \boldsymbol{x}_B)$ is peaked at $\boldsymbol{x} = \boldsymbol{x}_n + \boldsymbol{x}_B$. Because the scale R over which w varies is very large compared with the intra-molecular displacement $\boldsymbol{x}_B$, we can approximate w by Taylor-expanding about $\boldsymbol{x} - \boldsymbol{x}_n$:

$$w(\boldsymbol{x} - \boldsymbol{x}_n - \boldsymbol{x}_B) = w(\boldsymbol{x} - \boldsymbol{x}_n) - x_B^a \nabla_a w(\boldsymbol{x} - \boldsymbol{x}_n) + \frac{1}{2} x_B^a x_B^b \nabla_a \nabla_b w(\boldsymbol{x} - \boldsymbol{x'}) + \cdots,$$

where to avoid confusing the notation we make the vectorial index on $\boldsymbol{x}_B$ a superscript instead of a subscript. (Recalling that repeated indices are summed over, we have that $x_B^a \nabla_a$ stands for $\boldsymbol{x}_B \cdot \boldsymbol{\nabla}$.) Substituting this into the previous expression yields

$$\langle \eta_n \rangle = \sum_B q_B w - \sum_B q_B x_B^a \nabla_a w + \frac{1}{2} \sum_B q_B x_B^a x_B^b \nabla_a \nabla_b w + \cdots,$$

where w now stands for $w(\boldsymbol{x} - \boldsymbol{x}_n)$. We simplify this equation by recalling that each molecule is electrically neutral, so that the first sum vanishes. In the second term we recognize the expression for the molecular dipole moment, $\boldsymbol{p}_n = \sum_B q_B \boldsymbol{x}_B$, or

$$p_n^a(t) = \sum_B q_B x_B^a(t), \tag{4.1.16}$$

and in the third term we recognize the expression for the molecular quadrupole moment,

$$Q_n^{ab}(t) = 3 \sum_B q_B x_B^a(t) x_B^b(t). \tag{4.1.17}$$

Notice that this actually differs from the definition made in Sec. 2.10: here our quadrupole moment is not defined as a tracefree tensor. Combining these results

gives

$$\big\langle \eta_n(t,\boldsymbol{x})\big\rangle = -p_n^a(t)\nabla_a w(\boldsymbol{x}-\boldsymbol{x}_n) + \frac{1}{6}Q_n^{ab}(t)\nabla_a\nabla_b w(\boldsymbol{x}-\boldsymbol{x}_n) + \cdots \tag{4.1.18}$$

for the molecular charge density.

The macroscopic average of the density of bound charge is obtained by summing $\langle\eta_n\rangle$ over all molecules. This gives

$$\begin{aligned}\langle\eta_{\text{bound}}\rangle &= -\sum_n p_n^a\nabla_a w + \frac{1}{6}\sum_n Q_n^{ab}\nabla_a\nabla_b w + \cdots \\ &= -\nabla_a\sum_n p_n^a w + \frac{1}{6}\nabla_a\nabla_b\sum_n Q_n^{ab} w + \cdots \\ &= -\nabla_a\Big\langle\sum_n p_n^a\delta(\boldsymbol{x}-\boldsymbol{x}_n)\Big\rangle + \frac{1}{6}\nabla_a\nabla_b\Big\langle\sum_n Q_n^{ab}\delta(\boldsymbol{x}-\boldsymbol{x}_n)\Big\rangle + \cdots,\end{aligned}$$

because $w \equiv w(\boldsymbol{x}-\boldsymbol{x}_n)$ is the macroscopic average of $\delta(\boldsymbol{x}-\boldsymbol{x}_n)$. The quantities

$$P_a(t,\boldsymbol{x}) \equiv \Big\langle\sum_n p_n^a(t)\delta(\boldsymbol{x}-\boldsymbol{x}_n)\Big\rangle \equiv \text{macroscopic polarization} \tag{4.1.19}$$

and

$$Q_{ab}(t,\boldsymbol{x}) \equiv \Big\langle\sum_n Q_n^{ab}(t)\delta(\boldsymbol{x}-\boldsymbol{x}_n)\Big\rangle \equiv \text{macroscopic quadrupole density} \tag{4.1.20}$$

have a straightforward physical interpretation. Consider, for example, Eq. (4.1.19). The quantity within the averaging brackets is $\sum_n p_n^a\delta(\boldsymbol{x}-\boldsymbol{x}_n)$, the sum over all molecules of the product of each molecule's dipole moment with a δ-function centered at the molecule. This is clearly the microscopic density of molecular dipole moments, and P_a is its macroscopic average. So the vector $\boldsymbol{P}(t,\boldsymbol{x})$, called the macroscopic polarization, is nothing but the macroscopic average of the dipole-moment density of the medium's molecules. Similarly, $Q_{ab}(t,\boldsymbol{x})$ is the macroscopic average of the molecular quadrupole-moment density. In terms of these we have

$$\big\langle\eta_{\text{bound}}(t,\boldsymbol{x})\big\rangle = -\nabla_a P_a(t,\boldsymbol{x}) + \frac{1}{6}\nabla_a\nabla_b Q_{ab}(t,\boldsymbol{x}) + \cdots \tag{4.1.21}$$

for the macroscopic average of the density of bound charge.

The average of the free-charge density is simply

$$\rho(t,\boldsymbol{x}) \equiv \big\langle\eta_{\text{free}}(t,\boldsymbol{x})\big\rangle \equiv \Big\langle\sum_A q_A\delta(\boldsymbol{x}-\boldsymbol{x}_A)\Big\rangle, \tag{4.1.22}$$

and the macroscopic average of the total charge density is therefore

$$\langle\eta\rangle = \rho - \boldsymbol{\nabla}\cdot\boldsymbol{P} + \frac{1}{6}\nabla_a\nabla_b Q_{ab} + \cdots. \tag{4.1.23}$$

Substituting this into Eq. (4.1.9) yields

$$\nabla_a\Big(\epsilon_0 E_a + P_a - \frac{1}{6}\nabla_b Q_{ab} + \cdots\Big) = \rho.$$

The vector within the large brackets is the macroscopic field $\boldsymbol{D}(t,\boldsymbol{x})$, and we arrive at

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho, \tag{4.1.24}$$

with the definition

$$D_a = \epsilon_0 E_a + P_a - \frac{1}{6}\nabla_b Q_{ab} + \cdots. \tag{4.1.25}$$

Notice that this macroscopic Maxwell equation involves only the density of free charges on the right-hand side, and that apart from a factor of ϵ_0, the effective electric field $\boldsymbol{D}$ is the electric field $\boldsymbol{E}$ augmented by the electrical response of the medium, which is described by the molecular dipole-moment and quadrupole-moment densities. In practice the term involving Q_{ab} in Eq. (4.1.25) is almost always negligible, and we can write $\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}$.

4.1.4 Macroscopic averaging of the current density

Following the definitions of Eqs. (4.1.13)–(4.1.15) we introduce the current densities

$$\boldsymbol{J}_{\rm free}(t,\boldsymbol{x}) = \sum_A q_a \boldsymbol{v}_A(t)\delta\big(\boldsymbol{x} - \boldsymbol{x}_A(t)\big), \tag{4.1.26}$$

$$\boldsymbol{J}_n(t,\boldsymbol{x}) = \sum_B q_B \boldsymbol{v}_B(t)\delta\big(\boldsymbol{x} - \boldsymbol{x}_n - \boldsymbol{x}_B(t)\big), \tag{4.1.27}$$

$$\boldsymbol{J}_{\rm bound}(t,\boldsymbol{x}) = \sum_n \boldsymbol{J}_n(t,\boldsymbol{x}), \tag{4.1.28}$$

and the total microscopic current density is $\boldsymbol{J} = \boldsymbol{J}_{\rm free} + \boldsymbol{J}_{\rm bound}$.

The macroscopic average of the molecular current density is

$$\begin{aligned}\big\langle J_n^a\big\rangle &= \sum_B v_B^a w(\boldsymbol{x} - \boldsymbol{x}_n - \boldsymbol{x}_B)\\ &= \sum_B q_B v_B^a w - \sum_B q_B v_B^a x_B^b \nabla_b w + \cdots,\end{aligned}$$

where w stands for $w(\boldsymbol{x} - \boldsymbol{x}_n)$ in the second line. In the first term we recognize the time derivative of the molecular dipole moment, $dp_n^a/dt = (d/dt)\sum_B q_B x_B^a = \sum_B q_B v_B^a$. To put the second term in a recognizable form we define the molecular magnetic moment by

$$\boldsymbol{m}_n(t) = \frac{1}{2}\sum_B q_B(\boldsymbol{x}_B \times \boldsymbol{v}_B) \tag{4.1.29}$$

and we note that the vector $\boldsymbol{\nabla} w \times \boldsymbol{m}_n$ has components

$$\begin{aligned}\big(\boldsymbol{\nabla} w \times \boldsymbol{m}_n\big)^a &= \frac{1}{2}\sum_B q_B\big(x_B^a v_B^b - v_B^a x_B^b\big)\nabla_b w\\ &= \frac{1}{2}\sum_B q_B\big(x_B^a v_B^b + v_B^a x_B^b\big)\nabla_b w - \sum_B q_B v_B^a x_B^b \nabla_b w\\ &= \frac{1}{6}\left(\frac{d}{dt}Q_n^{ab}\right)\nabla_b w - \sum_B q_B v_B^a x_B^b \nabla_b w,\end{aligned}$$

recalling the definition $Q_n^{ab} = 3\sum_B q_B x_B^a x_B^b$ for the molecular quadrupole moment. Combining these results gives

$$\big\langle J_n^a\big\rangle = \frac{\partial}{\partial t}\left(p_n^a w - \frac{1}{6}Q_n^{ab}\nabla_b w\right) + \big(\boldsymbol{\nabla} w \times \boldsymbol{m}_n\big)^a + \cdots, \tag{4.1.30}$$

and summing over n yields

$$\big\langle J_{\rm bound}^a\big\rangle = \frac{\partial}{\partial t}\left(\sum_n p_n^a w - \frac{1}{6}\nabla_b \sum_n Q_n^{ab} w\right) + \left(\boldsymbol{\nabla} \times \sum_n \boldsymbol{m}_n w\right)^a + \cdots,$$

$$= \frac{\partial}{\partial t}\Big(\Big\langle\sum_n p_n^a\delta(\boldsymbol{x}-\boldsymbol{x}_n)\Big\rangle - \frac{1}{6}\nabla_b\Big\langle\sum_n Q_n^{ab}\delta(\boldsymbol{x}-\boldsymbol{x}_n)\Big\rangle\Big) + \Big[\boldsymbol{\nabla}\times\Big\langle\sum_n \boldsymbol{m}_n\delta(\boldsymbol{x}-\boldsymbol{x}_n)\Big\rangle\Big]^a + \cdots.$$

Introducing

$$\boldsymbol{M}(t,\boldsymbol{x}) \equiv \Big\langle\sum_n \boldsymbol{m}_n(t)\delta(\boldsymbol{x}-\boldsymbol{x}_n)\Big\rangle \equiv \text{macroscopic magnetization} \tag{4.1.31}$$

as the macroscopic average of the magnetic-moment density of the medium's molecules, we finally arrive at the expression

$$\langle J^a_{\text{bound}}\rangle = \frac{\partial}{\partial t}\Big(P_a - \frac{1}{6}Q_{ab}\Big) + (\boldsymbol{\nabla}\times\boldsymbol{M})_a + \cdots \tag{4.1.32}$$

for the macroscopic average of the density of bound current.

The average of the free-current density is simply

$$\boldsymbol{j}(t,\boldsymbol{x}) \equiv \langle \boldsymbol{J}_{\text{free}}(t,\boldsymbol{x})\rangle \equiv \Big\langle\sum_A q_A\boldsymbol{v}_A\delta(\boldsymbol{x}-\boldsymbol{x}_A)\Big\rangle, \tag{4.1.33}$$

and the macroscopic average of the total current density is therefore

$$\langle \boldsymbol{J}\rangle = \boldsymbol{j} + \frac{\partial}{\partial t}\Big(\boldsymbol{D}-\epsilon_0\boldsymbol{E}\Big) + \boldsymbol{\nabla}\times\boldsymbol{M} + \cdots, \tag{4.1.34}$$

having used Eq. (4.1.25). Substituting Eq. (4.1.34) into Eq. (4.1.12) gives

$$\boldsymbol{\nabla}\times\big(\boldsymbol{B}/\mu_0 - \boldsymbol{M}\big) - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{j}.$$

The vector within brackets is the macroscopic field $\boldsymbol{H}(t,\boldsymbol{x})$, and we arrive at

$$\boldsymbol{\nabla}\times\boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{j} \tag{4.1.35}$$

with the definition

$$\boldsymbol{H} = \boldsymbol{B}/\mu_0 - \boldsymbol{M}. \tag{4.1.36}$$

Notice that this macroscopic Maxwell equation involves only the density of free currents on the right-hand side, and that apart from a factor of μ_0, the effective magnetic field $\boldsymbol{H}$ is the magnetic field $\boldsymbol{B}$ diminished by the magnetic response of the medium, which is described by the molecular magnetic-moment density.

4.1.5 Summary — macroscopic Maxwell equations

The macroscopic Maxwell equations are

$$\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho \tag{4.1.37}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \tag{4.1.38}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0}, \tag{4.1.39}$$

$$\boldsymbol{\nabla}\times\boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{j}. \tag{4.1.40}$$

Here ρ is the macroscopic free-charge density, and $\boldsymbol{j}$ is the macroscopic free-current density. The fields $\boldsymbol{D}$ and $\boldsymbol{H}$ are defined by

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} \tag{4.1.41}$$

and

$$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}, \tag{4.1.42}$$

where

$$\boldsymbol{P} = \Big\langle \sum_n \boldsymbol{p}_n(t)\delta(\boldsymbol{x} - \boldsymbol{x}_n) \Big\rangle \tag{4.1.43}$$

is the macroscopic polarization ($\boldsymbol{p}_n = \sum_B q_B \boldsymbol{x}_B$ is the molecular dipole moment) and

$$\boldsymbol{M} = \Big\langle \sum_n \boldsymbol{m}_n(t)\delta(\boldsymbol{x} - \boldsymbol{x}_n) \Big\rangle \tag{4.1.44}$$

the macroscopic magnetization ($\boldsymbol{m}_n = \frac{1}{2}\sum_B q_B \boldsymbol{x}_B \times \boldsymbol{v}_B$ is the molecular magnetic moment).

4.2 Maxwell's equations in the frequency domain

[The material presented in this section is also covered in Sec. 7.1 of Jackson's text.]

We now begin our discussion of electromagnetic waves propagating in a dielectric medium. We suppose that the medium is not magnetized, so that we can set $\boldsymbol{M} = \boldsymbol{0}$ is our equations. We further suppose that the waves are propagating in the absence of sources, so that we can also set $\rho = 0$ and $\boldsymbol{j} = \boldsymbol{0}$. Maxwell's equations therefore reduce to

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{D} &= 0 && (4.2.1)\\
\boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0, && (4.2.2)\\
\boldsymbol{\nabla}\times\boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} &= \boldsymbol{0}, && (4.2.3)\\
\boldsymbol{\nabla}\times\boldsymbol{B} - \mu_0\frac{\partial \boldsymbol{D}}{\partial t} &= \boldsymbol{0}, && (4.2.4)
\end{aligned}$$

where $\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}$.

We want to form plane-wave solutions to these equations, and examine how the waves interact with the medium. We shall assume that the medium is uniform, isotropic, and unbounded, and leave unexplored issues of reflection and transmission at boundaries. To begin we will look for monochromatic waves oscillating with an angular frequency ω. We therefore set

$$\begin{aligned}
\boldsymbol{E}(t,\boldsymbol{x}) &= \tilde{\boldsymbol{E}}(\boldsymbol{x})e^{-i\omega t}, && (4.2.5)\\
\boldsymbol{D}(t,\boldsymbol{x}) &= \tilde{\boldsymbol{D}}(\boldsymbol{x})e^{-i\omega t}, && (4.2.6)\\
\boldsymbol{P}(t,\boldsymbol{x}) &= \tilde{\boldsymbol{P}}(\boldsymbol{x})e^{-i\omega t}, && (4.2.7)\\
\boldsymbol{B}(t,\boldsymbol{x}) &= \tilde{\boldsymbol{B}}(\boldsymbol{x})e^{-i\omega t}, && (4.2.8)
\end{aligned}$$

with the understanding that only the *real parts* of these complex fields have physical meaning.

The restriction to monochromatic waves is not too severe. Because the field equations are linear (assuming, as we shall verify below, that the relation between $\boldsymbol{D}$ and $\boldsymbol{E}$ is itself linear), solutions with different frequencies can be added to form wave-packet solutions. For example, we can express the electric field as

$$\boldsymbol{E}(t,\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\boldsymbol{E}}(\omega,\boldsymbol{x})e^{-i\omega t}\,d\omega, \tag{4.2.9}$$

in terms of the frequency-domain field $\tilde{\boldsymbol{E}}(\omega,\boldsymbol{x})$; this is a *Fourier representation* of the electric field. The inverse transformation is

$$\tilde{\boldsymbol{E}}(\omega,\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\boldsymbol{E}(t,\boldsymbol{x})e^{i\omega t}\,dt, \tag{4.2.10}$$

and frequency-domain fields $\tilde{\boldsymbol{D}}(\omega,\boldsymbol{x})$, $\tilde{\boldsymbol{P}}(\omega,\boldsymbol{x})$, $\tilde{\boldsymbol{B}}(\omega,\boldsymbol{x})$ can be defined a similar way.

The frequency-domain fields satisfy the equations

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{D}} &= 0 && (4.2.11)\\
\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{B}} &= 0, && (4.2.12)\\
\boldsymbol{\nabla}\times\tilde{\boldsymbol{E}} - i\omega\tilde{\boldsymbol{B}} &= \boldsymbol{0}, && (4.2.13)\\
\boldsymbol{\nabla}\times\tilde{\boldsymbol{B}} + i\omega\mu_0\tilde{\boldsymbol{D}} &= \boldsymbol{0}, && (4.2.14)
\end{aligned}$$

and we have the relation

$$\tilde{\boldsymbol{D}} = \epsilon_0\tilde{\boldsymbol{E}} + \tilde{\boldsymbol{P}}. \qquad (4.2.15)$$

These are the sourceless Maxwell equations in the frequency domain. They can only be solved once an explicit relationship (called a constitutive relation) is introduced between the polarization $\tilde{\boldsymbol{P}}$ and the electric field $\tilde{\boldsymbol{E}}$. The constitutive relation describes the medium's response to an applied electric field, and its derivation is based on complicated molecular processes. While quantum mechanics is required for a proper treatment, in the next section we shall consider a simple classical model that nevertheless captures the essential features.

4.3 Dielectric constant

[The material presented in this section is also covered in Sec. 7.5 of Jackson's text.]

We suppose that the medium is sufficiently dilute that the electric field felt by any given molecule is just the applied field $\tilde{\boldsymbol{E}}$ itself, the field exerted by other molecules being negligible. We assume that in the absence of an applied field, the molecular dipole moment is zero, at least on average. When an electric field is applied, however, some of the molecular charges (the electrons) move away from their equilibrium positions, and a dipole moment develops.

The displacement of charge q_B from its equilibrium position is denoted $\boldsymbol{\xi}_B(t)$. This charge undergoes a motion that is governed by the applied electric field (which tends to drive ξ_B away from zero) and the intra-molecular forces (which tend to drive the charge back to its equilibrium position). We model these forces as harmonic forces, all sharing the same natural frequency ω_0. What we have, therefore, is a system of simple harmonic oscillators of natural frequency ω_0 that are driven at a frequency ω by an applied force. Since the motion of the charges will produce electromagnetic radiation, energy will gradually be removed from the oscillators (to be carried off by the radiation), and the oscillations will be damped; we incorporate this effect through a phenomenological damping parameter $\gamma > 0$. The equations of motion for the charge q_B are therefore

$$m_B\big(\ddot{\boldsymbol{\xi}}_B + \gamma\dot{\boldsymbol{\xi}}_B + \omega_0^2\boldsymbol{\xi}\big) = q_B\tilde{\boldsymbol{E}}(\omega,\boldsymbol{x}_n)e^{-i\omega t}, \qquad (4.3.1)$$

where overdots indicate differentiation with respect to t. Notice that the applied electric field, which is macroscopic and varies slowly over intra-molecular distances, is evaluated at the molecule's centre of mass. To solve the equations of motion in the steady-state regime we write $\boldsymbol{\xi}_B(t) = \tilde{\boldsymbol{\xi}}_B e^{-i\omega t}$ and we substitute this into Eq. (4.3.1). This yields

$$\tilde{\boldsymbol{\xi}}_B(\omega) = \frac{q_B/m_B}{\omega_0^2 - \omega^2 - i\gamma\omega}\tilde{\boldsymbol{E}}(\omega,\boldsymbol{x}_n) \qquad (4.3.2)$$

for the frequency-domain displacement.

The molecule's dipole moment is $\boldsymbol{p}_n = \sum_B q_B(\boldsymbol{x}_B + \boldsymbol{\xi}_B)$, where $\boldsymbol{x}_B$ is the equilibrium position of charge q_B. Because the dipole moment vanishes at equilibrium, this reduces to $\boldsymbol{p}_n = \sum_B q_B \boldsymbol{\xi}_B$. Substituting Eq. (4.3.2) then gives

$$\tilde{\boldsymbol{p}}_n = \sum_B \frac{q_B^2/m_B}{\omega_0^2 - \omega^2 - i\gamma\omega} \tilde{\boldsymbol{E}}(\omega, \boldsymbol{x}_n).$$

Since all the contributing charges q_B are electrons, we can write $\sum_B q_B^2/m_B = Ze^2/m$, where Z is the number of electrons per molecule, e the electronic charge, and m the electron's mass. We therefore arrive at

$$\tilde{\boldsymbol{p}}_n(\omega) = \frac{Ze^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \tilde{\boldsymbol{E}}(\omega, \boldsymbol{x}_n) \tag{4.3.3}$$

for the molecular dipole moment in the frequency domain.

The polarization vector is then

$$\begin{aligned} \tilde{\boldsymbol{P}} &= \Big\langle \sum_n \tilde{\boldsymbol{p}}_n \delta(\boldsymbol{x} - \boldsymbol{x}_n) \Big\rangle \\ &= \frac{Ze^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \Big\langle \sum_n \tilde{\boldsymbol{E}}(\omega, \boldsymbol{x}_n) \delta(\boldsymbol{x} - \boldsymbol{x}_n) \Big\rangle \\ &= \frac{Ze^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \tilde{\boldsymbol{E}}(\omega, \boldsymbol{x}) \Big\langle \sum_n \delta(\boldsymbol{x} - \boldsymbol{x}_n) \Big\rangle. \end{aligned}$$

The quantity within the averaging brackets is the microscopic molecular density. Its macroscopic average is therefore the medium's density N (the number of molecules per macroscopic unit volume), which we take to be uniform. We have arrived at

$$\tilde{\boldsymbol{P}}(\omega, \boldsymbol{x}) = \frac{Ze^2 N}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \tilde{\boldsymbol{E}}(\omega, \boldsymbol{x}) \tag{4.3.4}$$

for the frequency-domain polarization. We recall that Z is the number of electrons per molecule, that N is the molecular number density, and that e and m are the electron's charge and mass, respectively. We notice that the relationship between $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{E}}$ is *linear*, as was anticipated in Sec. 4.2.

The relationship between $\tilde{\boldsymbol{D}} \equiv \epsilon_0 \tilde{\boldsymbol{E}} + \tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{E}}$ also is linear. We express it as

$$\tilde{\boldsymbol{D}} = \epsilon_0 \left(1 + \frac{Ze^2 N}{\epsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \right) \tilde{\boldsymbol{E}},$$

or as

$$\tilde{\boldsymbol{D}}(\omega, \boldsymbol{x}) = \epsilon(\omega) \tilde{\boldsymbol{E}}(\omega, \boldsymbol{x}), \tag{4.3.5}$$

having introduced the (complex, and frequency-dependent) *dielectric constant*

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} \tag{4.3.6}$$

and the medium's *plasma frequency*

$$\omega_p = \left(\frac{Ze^2 N}{\epsilon_0 m} \right)^{1/2}. \tag{4.3.7}$$

Equation (4.3.6) is the product of a simplistic classical model. But a proper quantum-mechanical treatment would produce a similar result, except for the fact

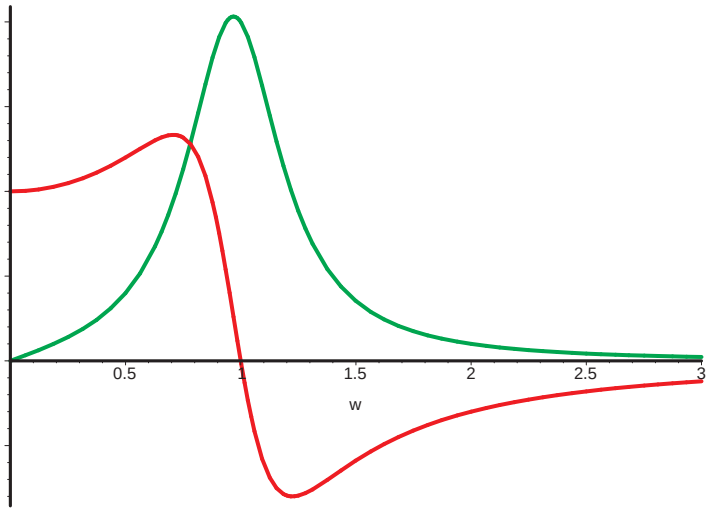

Figure 4.2: Real and imaginary parts of the dielectric constant, as functions of ω/ω_0. To fit both curves on the same graph we removed 1 from $\mathrm{Re}(\epsilon/\epsilon_0)$. The real part of the dielectric constant goes through 1 at resonance. The imaginary part of the dielectric constant achieves its maximum at resonance.

that the electrons actually oscillate with a discrete spectrum of natural frequencies ω_j and damping coefficients γ_j. A more realistic expression for the dielectric constant is then

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \omega_p^2 \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

where f_j is the fraction of electrons that share the same natural frequency ω_j. In the rest of this chapter we will continue to deal with the simple expression of Eq. (4.3.6). It is good to keep in mind, however, that the model is a bit crude.

In Fig. 4.2 we plot the real and imaginary parts of the dielectric constant. You will notice that for most frequencies, when ω is not close to ω_0, $\mathrm{Re}(\epsilon/\epsilon_0)$ increases with increasing ω, and $\mathrm{Im}(\epsilon/\epsilon_0)$ is very small; this typical behaviour is associated with *normal dispersion*. When ω is very close to ω_0, however, the electrons are driven at almost their natural frequency, and resonance occurs. Near resonance, then, $\mathrm{Re}(\epsilon/\epsilon_0)$ decreases with increasing ω, and $\mathrm{Im}(\epsilon/\epsilon_0)$ is no longer small; this behaviour is associated with *anomalous dispersion*. In realistic models the number of resonances is larger than one: there is one resonance for each natural frequency ω_j.

At low frequencies the dielectric constant behaves as $\epsilon/\epsilon_0 \simeq 1 + (\omega_p/\omega_0)^2$, and is therefore a constant slightly larger than unity. At high frequencies it behaves as $\epsilon/\epsilon_0 \simeq 1 - (\omega_p/\omega)^2$, and therefore approaches unity from below.

4.4 Propagation of plane, monochromatic waves

[The material presented in this section is also covered in Sec. 7.1 of Jackson's text.]

After making the substitution of Eq. (4.3.5), the frequency-domain Maxwell equations (4.2.11)–(4.2.14) become

$$\boldsymbol{\nabla} \cdot \tilde{\boldsymbol{E}} = 0 \qquad (4.4.1)$$

$$\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{B}} = 0, \tag{4.4.2}$$
$$\boldsymbol{\nabla}\times\tilde{\boldsymbol{E}} - i\omega\tilde{\boldsymbol{B}} = \boldsymbol{0}, \tag{4.4.3}$$
$$\boldsymbol{\nabla}\times\tilde{\boldsymbol{B}} + i\omega\epsilon(\omega)\mu_0\tilde{\boldsymbol{E}} = \boldsymbol{0}. \tag{4.4.4}$$

We recall that the time-domain fields are reconstructed, for example, as $\boldsymbol{E} = \tilde{\boldsymbol{E}}e^{-i\omega t}$ if the wave is monochromatic, or as in Eq. (4.2.9) if the wave possesses a spectrum of frequencies.

It is easy to show that as a consequence of Eqs. (4.4.1)–(4.4.4), both $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{B}}$ satisfy a frequency-domain wave equation. For example, taking the curl of Eq. (4.4.3) gives

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\tilde{\boldsymbol{E}}) - i\omega\boldsymbol{\nabla}\times\tilde{\boldsymbol{B}} = \boldsymbol{0};$$

substituting Eq. (4.4.4) and making use of the vectorial identity $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\tilde{\boldsymbol{E}}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{E}}) - \nabla^2\tilde{\boldsymbol{E}}$ puts this in the form

$$\big(\nabla^2 + k^2\big)\tilde{\boldsymbol{E}}(\omega,\boldsymbol{x}) = \boldsymbol{0}, \tag{4.4.5}$$

where

$$k^2 \equiv \frac{\omega^2}{c^2}\frac{\epsilon(\omega)}{\epsilon_0}. \tag{4.4.6}$$

We have introduced a complex wave number $k(\omega)$, and we recall that $c \equiv (\epsilon_0\mu_0)^{-1/2}$ is the speed of light in vacuum. Similar manipulations return

$$\big(\nabla^2 + k^2\big)\tilde{\boldsymbol{B}}(\omega,\boldsymbol{x}) = \boldsymbol{0}, \tag{4.4.7}$$

the statement that the magnetic field also satisfies a wave equation.

To see how monochromatic electromagnetic waves propagate in a dielectric medium, consider any component of $\tilde{\boldsymbol{E}}$ or $\tilde{\boldsymbol{B}}$, and call it $\tilde{\psi}$. Then suppose that the wave is a plane wave that propagates in the z direction, so that $\tilde{\psi} = \tilde{\psi}(z)$. The wave equation simplifies to

$$\left(\frac{d^2}{dz^2} + k^2\right)\tilde{\psi}(\omega,z) = 0. \tag{4.4.8}$$

A right-moving wave is described by the particular solution $\tilde{\psi} = e^{ikz}$. To help identify the physical properties of this wave we substitute Eq. (4.3.6) into Eq. (4.4.6) and get

$$k^2 = \frac{\omega^2}{c^2}\left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}\right). \tag{4.4.9}$$

To be concrete we suppose that the second term within the large brackets is small. Taking the square root gives

$$k = \frac{\omega}{c}\left[1 + \frac{\omega_p^2}{2\big(\omega_0^2 - \omega^2 - i\gamma\omega\big)}\right],$$

or

$$k = k_R + ik_I, \tag{4.4.10}$$

where

$$k_R = \frac{\omega}{c}\left\{1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{2\big[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\big]}\right\}$$

is the real part of the complex wave number, while

$$k_I = \frac{\omega}{c}\frac{\omega_c^2\gamma\omega}{2\big[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\big]}$$

is its imaginary part.

The right-moving wave is therefore described by

$$\psi(t,z) \equiv \tilde{\psi}(\omega,z)e^{-i\omega t} = e^{-k_I z}e^{-i(\omega t - k_R z)}. \tag{4.4.11}$$

From this expression we recognize that the wave is indeed traveling in the positive z direction, with a speed

$$v_p = \frac{\omega}{k_R(\omega)} \equiv \frac{c}{n(\omega)} \tag{4.4.12}$$

known as the *phase velocity*; $n \equiv ck_R/\omega$ is the medium's *index of refraction*. From our previous expression for k_R we obtain

$$n = 1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{2\big[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\big]},$$

and we see that $n \simeq 1 + \frac{1}{2}(\omega_p/\omega_0)^2 > 1$ for low frequencies, so that $v_p < c$. On the other hand, $n \simeq 1 - \frac{1}{2}(\omega_p/\omega)^2 < 1$ for high frequencies, so that $v_p > c$. The phase velocity of the wave can therefore exceed the speed of light in vacuum! This strange fact does not violate any cherished relativistic notion, because a monochromatic wave does not carry information. To form a signal one must modulate the wave and superpose solutions with different frequencies. As we shall see, in such situations the wave always travels with a speed that does not exceed c.

Another feature of Eq. (4.4.10) is that the wave is an exponentially decreasing function of z provided that k_I is positive, which is guaranteed to be true if $\gamma > 0$. A positive damping constant means that some of the wave's energy is dissipated by the oscillating electrons (which emit their own electromagnetic radiation), and the wave must therefore decrease in amplitude as it travels through the medium. It is interesting to note that in the case of lasers and masers, the electronic oscillations actually reinforce the wave and lead to an exponential *amplification* instead of an attenuation; in such media the damping constant is negative.

Having constructed solutions to Eqs. (4.4.5) and (4.4.7), we should examine what constraints the full set of equations (4.4.1)–(4.4.4) place on these solutions. We write

$$\tilde{\boldsymbol{E}}(\omega,z) = \hat{\boldsymbol{e}}E_0(\omega)e^{ik(\omega)z} \tag{4.4.13}$$

and

$$\tilde{\boldsymbol{B}}(\omega,z) = \hat{\boldsymbol{b}}B_0(\omega)e^{ik(\omega)z}, \tag{4.4.14}$$

where $\hat{\boldsymbol{e}}$ and $\hat{\boldsymbol{b}}$ are real unit vectors, and where E_0 and B_0 are complex amplitudes. Substituting these expressions into Eqs. (4.4.1) and (4.4.2) informs us that the unit vectors are both orthogonal to $\hat{\boldsymbol{z}}$,

$$\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{z}} = \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{z}} = 0; \tag{4.4.15}$$

the fields are therefore transverse to the direction in which the wave propagates. Equation (4.4.3) then produces $B_0\hat{\boldsymbol{b}} = (k/\omega)E_0(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{e}})$, which implies

$$B_0(\omega) = \frac{k(\omega)}{\omega}E_0(\omega) \tag{4.4.16}$$

and

$$\hat{\boldsymbol{b}} = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{e}}. \tag{4.4.17}$$

These equations state that the magnetic field is orthogonal to the electric field, and that its amplitude is linked to that of the electric field. Finally, Eq. (4.4.4) gives us $B_0(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{b}}) = -(k/\omega)E_0\hat{\boldsymbol{e}}$, which merely confirms Eqs. (4.4.16) and (4.4.17).

4.5 Propagation of wave packets

[The material presented in this section is also covered in Secs. 7.8 and 7.9 of Jackson's text.]

4.5.1 Description of wave packets

In this section we shall assume, for simplicity, that $\epsilon(\omega)$ is *real*: we work away from resonances and ignore dissipative effects. We let $k(\omega)$ be the positive solution to Eq. (4.4.6).

A wave packet is obtained by superposing solutions to Eq. (4.4.8) with different frequencies ω. Including both left- and right-moving waves, we write

$$\psi(t,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big[a_+(\omega) e^{ik(\omega)z} + a_-(\omega) e^{-ik(\omega)z} \Big] e^{-i\omega t}\, d\omega, \tag{4.5.1}$$

where $a_\pm(\omega)$ are complex amplitudes that determine the shape of the wave packet.

Alternatively, and more conveniently for our purposes, we can instead superpose solutions with different wave numbers k. Defining $\omega(k)$ to be the positive solution to Eq. (4.4.6), we write

$$\psi(t,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big[A_+(k) e^{i\omega(k)t} + A_-(k) e^{-i\omega(k)t} \Big] e^{ikz}\, dk, \tag{4.5.2}$$

and it is not too difficult to show that the decomposition of Eq. (4.5.2) is equivalent to that of Eq. (4.5.1).

The complex amplitudes $A_\pm(k)$ are determined by the initial conditions we wish to impose on $\psi(t,z)$ and its time derivative. It follows from Eq. (4.5.2) that

$$\psi(0,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \big[A_+(k) + A_-(k) \big] e^{ikz}\, dk,$$

and Fourier's inversion theorem gives

$$A_+(k) + A_-(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(0,z) e^{-ikz}\, dz.$$

On the other hand, differentiating Eq. (4.5.2) with respect to t produces

$$\dot{\psi}(0,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\omega(k) \big[A_+(k) - A_-(k) \big] e^{ikz}\, dk,$$

and inversion gives

$$i\omega(k) \big[A_+(k) - A_-(k) \big] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \dot{\psi}(0,z) e^{-ikz}\, dz.$$

Solving for $A_\pm(k)$, we finally obtain

$$A_\pm(k) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\psi(0,z) \pm \frac{1}{i\omega(k)} \dot{\psi}(0,z) \right] e^{-ikz}\, dz. \tag{4.5.3}$$

The procedure to follow to construct a wave packet is therefore to choose an initial configuration by specifying $\psi(0,z)$ and $\dot{\psi}(0,z)$, then calculate $A_\pm(k)$ using Eq. (4.5.3), and finally, obtain $\psi(t,z)$ by evaluating the integral of Eq. (4.5.2).

In the following discussion we will choose the initial configuration to be *time-symmetric*, in the sense that

$$\dot{\psi}(0,z) \equiv 0. \tag{4.5.4}$$

Equation (4.5.3) then simplifies to

$$A(k) \equiv A_\pm(k) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(0,z) e^{-ikz}\, dz, \tag{4.5.5}$$

and we have an equal superposition of left- and right-moving waves.

4.5.2 Propagation without dispersion

Let us first see what the formalism of the preceding subsection gives us when there is no dispersion, that is, when $n \equiv c|k|/\omega$ is a constant independent of k. We then have

$$\omega(k) = v_p|k|, \tag{4.5.6}$$

where $v_p = c/n$ is the wave's phase velocity.

For time-symmetric initial data, Eq. (4.5.2) reduces to

$$\psi(t,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\Big[e^{i|k|v_p t} + e^{-i|k|v_p t}\Big] e^{ikz}\, dk.$$

The factor within the square brackets is $2\cos(|k|v_p t) = 2\cos(kv_p t)$, and we can therefore replace the previous expression with

$$\psi(t,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\Big[e^{ik(z+v_p t)} + e^{ik(z-v_p t)}\Big]\, dk.$$

But from Eq. (4.5.5) we have

$$A(k) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(0,z') e^{-ikz'}\, dz',$$

and substituting this gives

$$\begin{aligned} \psi(t,z) &= \frac{1}{2}\int_{-\infty}^{\infty} dz' \psi(0,z') \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big[e^{ik(z+v_p t-z')} + e^{ik(z-v_p t-z')}\Big]\, dk \\ &= \frac{1}{2}\int_{-\infty}^{\infty} dz' \psi(0,z')\Big[\delta(z+v_p t-z') + \delta(z-v_p t-z')\Big], \end{aligned}$$

or

$$\psi(t,z) = \frac{1}{2}\psi(0,z+v_p t) + \frac{1}{2}\psi(0,z-v_p t). \tag{4.5.7}$$

The first term on the right-hand side is (half) the initial wave packet, $\psi(0,z)$, translated in the z direction by $-v_p t$; it represents that part of the wave packet that travels undisturbed in the negative z direction with a speed v_p. The second term, on the other hand, is the remaining half of the wave packet, which travels toward the positive z direction with the same speed v_p.

The main features of wave propagation without dispersion are that the wave packet travels with the phase velocity $v_p = \omega/k$ and that its shape stays unchanged during propagation.

4.5.3 Propagation with dispersion

Let us now return to the general case of propagation with dispersion. Here ω is a nonlinear function of k, and for time-symmetric initial data we have

$$\psi(t,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\Big[e^{i\omega(k)t} + e^{-i\omega(k)t}\Big] e^{ikz}\, dk.$$

To be concrete, suppose that $A(k)$ is sharply peaked around $k = k_0$, with k_0 some arbitrary wave number. [A proper treatment would have $A(k)$ be a symmetric function peaked about both $k = k_0$ and $k = -k_0$. Unlike what we shall find below, this would produce a *real* wave function $\psi(t,z)$. It is not difficult to make this change, but we shall not do so here.] Suppose further that $\omega(k)$ varies slowly near $k = k_0$, so that it can be approximated by

$$\omega(k) \simeq \omega_0 + \omega_0'(k-k_0) = (\omega_0 - \omega_0' k_0) + \omega_0' k, \tag{4.5.8}$$

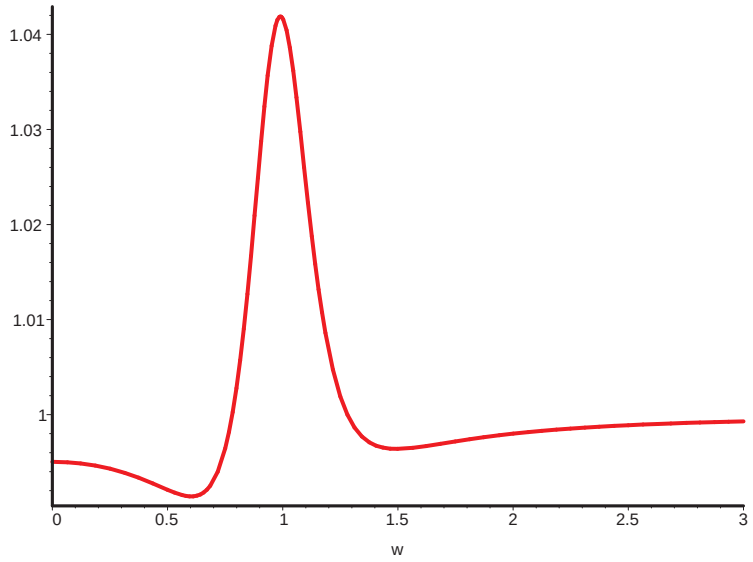

Figure 4.3: Group velocity v_g in a dielectric medium, in units of c, as a function of ω/ω_0.

where $\omega_0 \equiv \omega(k_0)$ and $\omega_0' \equiv (d\omega/dk)(k_0)$. Then the wave packet can be expressed as

$$\begin{aligned}\psi &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\left[e^{ik(z+\omega_0' t)}e^{i(\omega_0-\omega_0' k_0)t} + e^{ik(z-\omega_0' t)}e^{-i(\omega_0-\omega_0' k_0)t}\right] dk \\ &= e^{i(\omega_0-\omega_0' k_0)t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ik(z+\omega_0' t)}\, dk \\ &\quad + e^{-i(\omega_0-\omega_0' k_0)t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ik(z-\omega_0' t)}\, dk,\end{aligned}$$

or

$$\psi(t,z) = e^{i(\omega_0-\omega_0' k_0)t}\psi(0, z+v_g t) + e^{-i(\omega_0-\omega_0' k_0)t}\psi(0, z-v_g t), \tag{4.5.9}$$

where

$$v_g = \frac{d\omega}{dk}(k_0) \tag{4.5.10}$$

is known as the *group velocity*. From Eq. (4.5.9) we learn that in the presence of dispersion, the wave packet travels with the group velocity v_g instead of the phase velocity $v_p = k_0/\omega_0$. And the phase factors indicate that the shape of the wave suffers a distortion during propagation. These are the main features of wave propagation in a dispersive medium.

If the dispersion relation is expressed as $k = n(\omega)\omega/c$, in terms of the index of refraction $n(\omega)$, then $dk/d\omega = c^{-1}[n + \omega(dn/d\omega)]$ and

$$\frac{v_g}{c} = \frac{1}{n + \omega(dn/d\omega)}. \tag{4.5.11}$$

For the dielectric medium described in Sec. 4.3,

$$n^2 = 1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2},$$

and the group velocity is plotted in Fig. 4.3. You will notice that v_g is smaller than c for most frequencies, but that it seems to exceed c near resonance. This

phenomenon is entirely artificial: when ω is close to ω_0 it is no longer true that ω is a slowly varying function of k, and the approximations involved in defining the group velocity are no longer valid. Furthermore, the imaginary part of ω, together with the dissipative effects it represents, cannot be neglected near resonance. The bottom line is that the very notion of group velocity is no longer meaningful near a resonance.

4.5.4 Gaussian wave packet

As a concrete example of wave propagation with dispersion we consider an initial configuration given by

$$\psi(0,z) = e^{-z^2/(2L^2)}\cos(k_0 z); \tag{4.5.12}$$

this is an oscillating wave (with wave number k_0) modulated by a Gaussian envelope of width L. Once more we take the initial data to be time-symmetric, and we impose $\dot{\psi}(0,z) = 0$. In order to be able to carry out the calculations we shall take the dispersion relation to be

$$\omega(k) = \nu\left(1 + \frac{1}{2}\alpha^2 k^2\right), \tag{4.5.13}$$

where ν is a minimum frequency and α an arbitrary length scale. This dispersion relation is not particularly realistic, but its quadratic form will allow us to evaluate the integrals that will appear in the course of our analysis. These will have the form of Gaussian integrals, and we quote the standard result

$$\int_{-\infty}^{\infty} e^{-az^2+bz+c}\,dz = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a} + c\right). \tag{4.5.14}$$

Here the constants a, b, and c can be complex, but $\mathrm{Re}(a)$ must be positive for the integral to converge.

We first calculate the Fourier amplitudes associated with Eq. (4.5.12). Equation (4.5.5) gives

$$\begin{aligned} A(k) &= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(0,z)e^{-ikz}\,dz \\ &= \frac{1}{4}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/(2L^2)}\left[e^{-i(k-k_0)z} + e^{-i(k+k_0)z}\right]dz. \end{aligned}$$

For the first integral we have $a = 1/(2L^2)$, $b = -i(k-k_0)$, $c = 0$, and $b^2/(4a) = -\frac{1}{2}(k-k_0)^2L^2$; for the second integral we simply make the substitution $k_0 \to -k_0$ in this result. Equation (4.5.14) then gives

$$A(k) = \frac{L}{4}\left[e^{-\frac{1}{2}(k-k_0)^2L^2} + e^{-\frac{1}{2}(k+k_0)^2L^2}\right]. \tag{4.5.15}$$

We see that $A(k)$ is a symmetric function of k, and that it is peaked at both $k = k_0$ and $k = -k_0$. The width of the distribution around both peaks is equal to $1/L$.

The wave packet at times $t \neq 0$ is given by Eq. (4.5.2),

$$\psi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{i[kz-\omega(k)t]}\,dk + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{i[kz+\omega(k)t]}\,dk,$$

which we may also express as

$$\psi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{i[kz-\omega(k)t]}\,dk + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(-k)e^{-i[kz-\omega(-k)t]}\,dk,$$

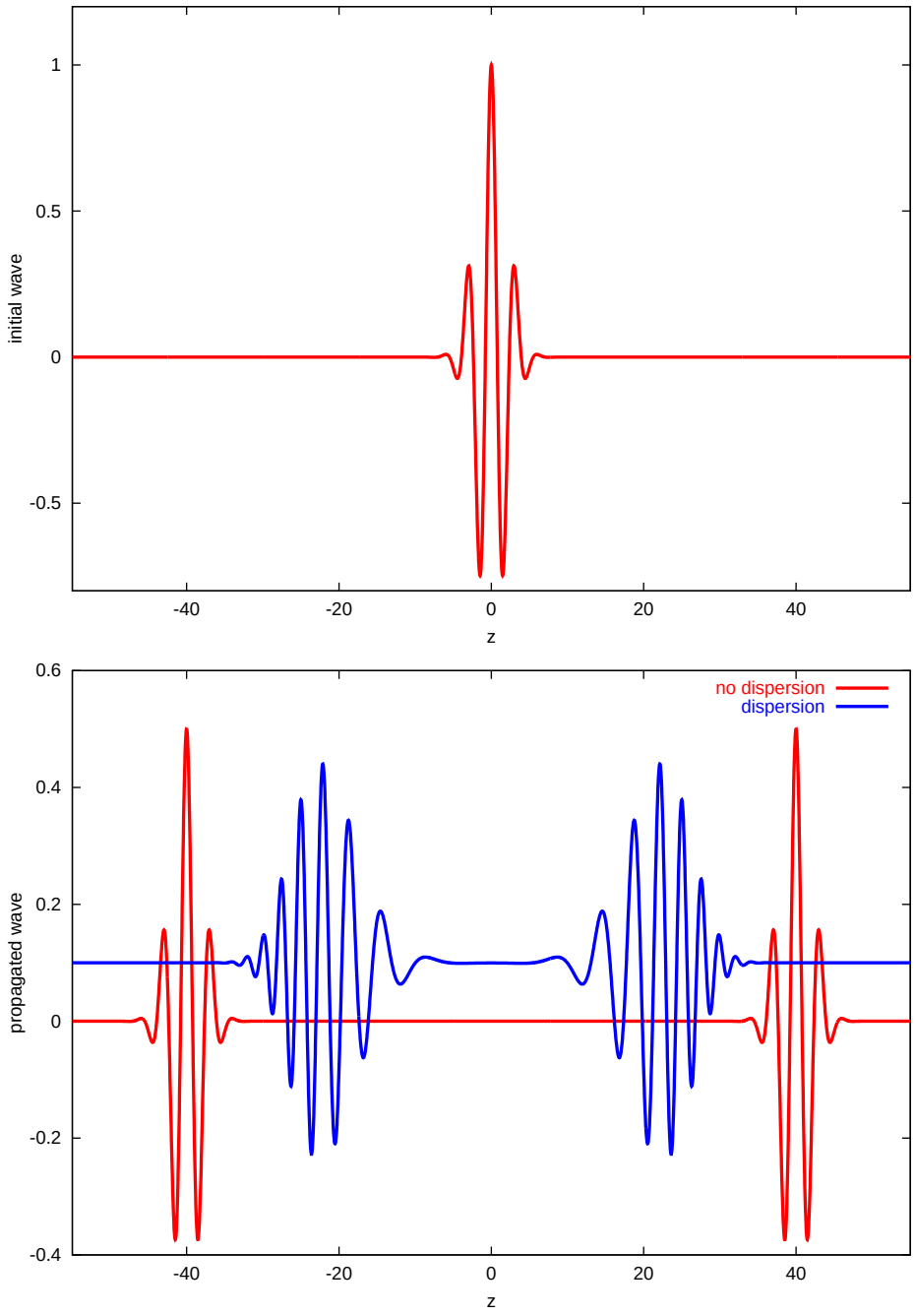

Figure 4.4: Wave propagation with a high-frequency dispersion relation $\omega(k) = (\omega_p^2 + c^2k^2)^{1/2}$, calculated numerically with the help of a Fast Fourier Transform routine. The upper panel shows the initial wave profile, described by Eq. (4.5.12). The lower panel shows two versions of the propagated wave. The first (lower curve) is obtained without dispersion by setting $\omega_p = 0$; this wave propagates with speed c without altering its shape. The second (higher curve — the wave function is translated upward for ease of visualization) is obtained with dispersion; this wave spreads out as it travels with speed $v_g < c$. Notice the drop in amplitude as the initial wave packet splits into an equal superposition of left- and right-moving waves.

having changed the variable of integration to $-k$ in the second integral. Because $A(-k) = A(k)$ and $\omega(-k) = \omega(k)$, we have

$$\begin{aligned}\psi &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{i[kz-\omega(k)t]}\,dk + \text{complex conjugate},\\ &= \frac{2}{\sqrt{2\pi}}\text{Re}\int_{-\infty}^{\infty} A(k)e^{i[kz-\omega(k)t]}\,dk.\end{aligned}$$

Substituting Eq. (4.5.15) then gives

$$\psi = \frac{L}{2\sqrt{2\pi}}\text{Re}\left\{\int_{-\infty}^{\infty} e^{-\frac{1}{2}(k-k_0)^2L^2}e^{i[kz-\omega(k)t]}\,dk + (k_0 \to -k_0)\right\}.$$

Taking into account Eq. (4.5.13), we see that the argument of the exponential function is a second-order polynomial in k, which we write as $-ak^2 + bk + c$, with $a = \frac{1}{2}(L^2 + i\nu t\alpha^2)$, $b = k_0L^2 + iz$, and $c = -(i\nu t + \frac{1}{2}k_0^2L^2)$. With this we have

$$\frac{b^2}{4a} = -\frac{(z - ik_0L^2)^2}{2L^2(1 + i\nu tf^2)},$$

where $f \equiv \alpha/L$, and Eq. (4.5.14) yields

$$\psi = \frac{1}{2}\text{Re}\left\{(1 + i\nu tf^2)^{-1/2}\exp\left[-\frac{(z - ik_0L^2)^2}{2L^2(1 + i\nu tf^2)}\right]e^{-i\nu t - \frac{1}{2}k_0^2L^2} + (k_0 \to -k_0)\right\}.$$

It is then a matter of straightforward algebra to bring this to its final form,

$$\begin{aligned}\psi(t,z) &= \frac{1}{2}\text{Re}\left\{(1 + i\nu tf^2)^{-1/2}\exp\left[-\frac{(z - v_gt)^2}{2L^2(1 + i\nu tf^2)}\right]e^{i[k_0z-\omega(k_0)t]}\right.\\ &\quad \left. + (k_0 \to -k_0)\right\},\end{aligned} \tag{4.5.16}$$

where $f = \alpha/L$ and

$$v_g = \nu\alpha^2k_0 = \frac{d\omega}{dk}(k_0) \tag{4.5.17}$$

is the group velocity at $k = k_0$ for the dispersion relation of Eq. (4.5.13).

Equation (4.5.16) informs us that the wave packet retains its Gaussian shape as it propagates through the medium, but that its width increases with time:

$$L(t) = L\sqrt{1 + (f^2\nu t)^2}. \tag{4.5.18}$$

The wave therefore spreads out, and this is a typical feature of propagation with dispersion. In addition, we see that while the oscillations travel at the phase velocity $v_p = \omega(k_0)/k_0$, the centre of the Gaussian peak travels at the group velocity v_g; this result was expected after the discussion of the preceding subsection.

A more realistic example of wave dispersion (calculated numerically) is displayed in Fig. 4.4. Notice that the features discussed in the preceding paragraphs are present in this example as well: the wave packet spreads out as it propagates with a group velocity v_g that is smaller than c.

4.6 Problems

1. (This problem is adapted from Jackson's problem 7.19.) A wave packet in one dimension has an initial shape given by $\psi(0, z) = f(z)e^{ik_0z}$, where $f(z)$ is the modulation envelope; the wave packet is assumed to be time-symmetric. For each of the forms $f(z)$ given below, calculate the wave-number spectrum $|A(k)|^2$ of the wave packet, sketch $|\psi(0, z)|^2$ and $|A(k)|^2$, calculate the root-mean-square deviations Δz and Δk from the means (defined in terms of $|\psi(0, z)|^2$ and $|A(k)|^2$, respectively), and test the inequality $\Delta z\Delta k \geq \frac{1}{2}$.

a) $f(z) = e^{-\alpha|z|/2}$;

b) $f(z) = e^{-\alpha^2 z^2/4}$;

c) $f(z) = 1 - \alpha|z|$ for $\alpha|z| < 1$ and $f(z) = 0$ for $\alpha|z| > 1$;

d) $f(z) = 1$ for $\alpha|z| < 1$ and $f(z) = 0$ for $\alpha|z| > 1$.

2. We have seen that the relationship between the frequency-domain fields $\tilde{\boldsymbol{D}}$ and $\tilde{\boldsymbol{E}}$ is

$$\tilde{\boldsymbol{D}}(\omega, \boldsymbol{x}) = \epsilon(\omega)\tilde{\boldsymbol{E}}(\omega, \boldsymbol{x}),$$

where

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}.$$

Prove that as a consequence of this, the relationship between the time-domain fields $\boldsymbol{D}$ and $\boldsymbol{E}$ is

$$\boldsymbol{D}(t, \boldsymbol{x}) = \epsilon_0 \left[\boldsymbol{E}(t, \boldsymbol{x}) + \frac{\omega_p^2}{\nu} \int_0^\infty e^{-\gamma\tau/2} \sin(\nu\tau)\, \boldsymbol{E}(t - \tau, \boldsymbol{x})\, d\tau \right],$$

where $\nu \equiv \sqrt{\omega_0^2 - \frac{1}{4}\gamma^2}$. This reveals that $\boldsymbol{D}$ at time t depends on the electric field at *all previous times*: the relationship is nonlocal in time. The exponential factor, however, indicates that delays longer than $1/\gamma$ do not matter very much: only the electric field's recent history is important.

Chapter 5
Electromagnetic radiation from slowly moving sources

5.1 Equations of electrodynamics

In this chapter and the next we will explore some of the mechanisms by which electromagnetic radiation is produced. While Chapter 6 will be concerned with relativistic situations, here we shall restrict our attention to charge and current distributions whose internal velocities are small compared with the speed of light.

We recall the main equations of electrodynamics, as we have formulated them in Chapter 1. We have wave equations for the scalar and vector potentials,

$$\Box\Phi(t,\boldsymbol{x}) = -\frac{1}{\epsilon_0}\rho(t,\boldsymbol{x}) \tag{5.1.1}$$

and

$$\Box\boldsymbol{A}(t,\boldsymbol{x}) = -\mu_0\boldsymbol{j}(t,\boldsymbol{x}), \tag{5.1.2}$$

where

$$\Box \equiv -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \tag{5.1.3}$$

is the wave operator and $c \equiv (\epsilon_0\mu_0)^{-1/2}$ is the speed of light. For Eqs. (5.1.1) and (5.1.2) to be equivalent to Maxwell's equations, the potentials must satisfy the Lorenz gauge condition,

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0. \tag{5.1.4}$$

Maxwell's equations imply that the charge and current densities satisfy the continuity equation,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0, \tag{5.1.5}$$

which states that charge is locally conserved.

In the absence of boundaries, the wave equations have solutions

$$\Phi(t,\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(t-|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x' \tag{5.1.6}$$

and

$$\boldsymbol{A}(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int\frac{\boldsymbol{j}(t-|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'; \tag{5.1.7}$$

these are *retarded solutions*, which incorporate the physically appropriate causal relationship between sources and fields.

From the potentials we obtain the fields via the relations

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\Phi \tag{5.1.8}$$

and

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{5.1.9}$$

And finally, we recall from Sec. 1.4 that the fields carry energy. The electromagnetic field energy density is given by

$$\varepsilon(t, \boldsymbol{x}) = \frac{1}{2}\epsilon_0 \boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2 \tag{5.1.10}$$

and the energy flux vector (Poynting vector) is

$$\boldsymbol{S}(t, \boldsymbol{x}) = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B}. \tag{5.1.11}$$

5.2 Plane waves

The simplest situation featuring the production of electromagnetic waves involves monochromatic plane waves. These are generated by a planar distribution of charges, all oscillating in the plane with the same frequency ω.

We take the charge distribution to be uniform in the x-y plane, so that the charge density is given by

$$\rho(t, \boldsymbol{x}) = \sigma\delta(z), \tag{5.2.1}$$

where σ is a constant surface density. Notice that the charge density is actually time independent. The resulting scalar potential will also not depend on time, and the electromagnetic waves will be described entirely by the vector potential.

We take each charge to be oscillating in the x direction with a velocity

$$\boldsymbol{v}(t, \boldsymbol{x}) = v_0 \hat{\boldsymbol{x}} \cos\omega t. \tag{5.2.2}$$

Notice that the velocity vector is actually independent of position in the plane, and that $\hat{\boldsymbol{x}}$ is actually an arbitrary direction within the plane; we could pick another direction by rotating the x-y coordinates around the z axis. The current density is

$$\boldsymbol{j}(t, \boldsymbol{x}) = \sigma v_0 \delta(z) \hat{\boldsymbol{x}} \cos\omega t, \tag{5.2.3}$$

and we will calculate the associated vector potential without approximations.

In principle we might proceed by substituting Eq. (5.2.3) into Eq. (5.1.7), but it is easier to start instead with the equivalent form

$$\boldsymbol{A}(t, \boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \boldsymbol{j}(t', \boldsymbol{x'})\frac{\delta(t - t' - |\boldsymbol{x} - \boldsymbol{x'}|/c)}{|\boldsymbol{x} - \boldsymbol{x'}|}\, dt' d^3x',$$

in which we substitute Eq. (5.2.3) expressed in terms of primed variables. We have

$$\boldsymbol{A}(t, \boldsymbol{x}) = \frac{\mu_0}{4\pi}\sigma v_0 \hat{\boldsymbol{x}} \int dt'\, \Gamma(t - t', \boldsymbol{x}) \cos\omega t',$$

where

$$\Gamma(t - t', \boldsymbol{x}) = \int \delta(z')\frac{\delta(t - t' - |\boldsymbol{x} - \boldsymbol{x'}|/c)}{|\boldsymbol{x} - \boldsymbol{x'}|}\, d^3x'$$

is an integral that we must now evaluate.

Integration over dz' is immediate:

$$\Gamma = \int \frac{\delta(t - t' - \sqrt{(x-x')^2 + (y-y')^2 + z^2}/c)}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}}\, dx'dy',$$

or

$$\Gamma = \int \frac{\delta(t - t' - \sqrt{x'^2 + y'^2 + z^2}/c)}{\sqrt{x'^2 + y'^2 + z^2}}\, dx'dy'$$

after translating the origin of the (x', y') coordinates to the point (x, y); this is allowed, because the domain of integration is the whole x'-y' plane. To integrate this we make the change of variables $x' = \ell\cos\theta$, $y' = \ell\sin\theta$. This gives, after writing $dx'dy' = \ell\, d\ell d\theta$ and integrating over $d\theta$,

$$\Gamma = 2\pi \int_0^\infty \frac{\delta(t - t' - \sqrt{\ell^2 + z^2}/c)}{\sqrt{\ell^2 + z^2}}\, \ell\, d\ell.$$

The easiest way to integrate over the δ-function is to make its argument be the integration variable. So let $s = \sqrt{\ell^2 + z^2}/c - \Delta t$, where $\Delta t \equiv t - t'$. Because ℓ is nonnegative, we have that s is bounded below by $|z|/c - \Delta t$. We also have

$$ds = \frac{\ell\, d\ell}{c\sqrt{\ell^2 + z^2}} = \frac{\ell\, d\ell}{c^2(s + \Delta t)},$$

and the integral becomes

$$\Gamma = 2\pi c \int_{|z|/c - \Delta t}^\infty \delta(s)\, ds,$$

or

$$\Gamma(t - t', \boldsymbol{x}) = 2\pi c\theta(t - t' - |z|/c),$$

where $\theta(s)$ is the step function, which is unity when $s > 0$ and zero otherwise.

Substituting this result into our previous expression for the vector potential, we obtain

$$\begin{aligned} \boldsymbol{A}(t, \boldsymbol{x}) &= \frac{\mu_0}{4\pi}(\sigma v_0)(2\pi c)\hat{\boldsymbol{x}} \int_{-\infty}^\infty dt'\, \theta(t - |z|/c - t')\cos\omega t' \\ &= \frac{\mu_0}{4\pi} 2\pi\sigma v_0 c\, \hat{\boldsymbol{x}} \int_{-\infty}^{t - |z|/c} \cos\omega t'\, dt'. \end{aligned}$$

The dt' integral evaluates to $\omega^{-1}\sin\omega(t - |z|/c)$, apart from an ambiguous constant associated with the integral's infinite lower bound. Since this constant will never appear in expressions for the electric and magnetic fields, we simply set it to zero. Our final expression for the vector potential is therefore

$$\boldsymbol{A}(t, z) = \frac{E_0}{\omega}\sin\omega(t - |z|/c)\hat{\boldsymbol{x}}, \tag{5.2.4}$$

where

$$E_0 = \frac{1}{2}\mu_0\sigma v_0 c \tag{5.2.5}$$

has the dimensions of an electric field. For $z > 0$ we have that $\sin\omega(t - |z|/c) = \sin\omega(t - z/c) = -\sin k(z - ct)$, where $k = \omega/c$ is the wave number, and we see that the wave travels in the positive z direction with speed c. For $z < 0$ we have instead $\sin\omega(t - |z|/c) = \sin\omega(t + z/c) = \sin k(z + ct)$, and the wave now travels in the negative z direction. Thus, Eq. (5.2.4) describes a plane wave that propagates away from $z = 0$, the place where it is generated; the wave has a frequency ω, a wave number $k = \omega/c$, and it travels with the speed of light.

The time-dependent part of the electric field is obtained from Eqs. (5.1.8) and (5.2.4):

$$\boldsymbol{E}(t,z) = -E_0 \cos\omega(t-|z|/c)\hat{\boldsymbol{x}}. \tag{5.2.6}$$

We see that the electric field behaves as a plane wave that travels away from $z=0$ with the speed of light, and that it points in the x direction, which is perpendicular to the direction of propagation. The magnetic field, on the other hand, is obtained by taking the curl of Eq. (5.2.4). This involves differentiating $|z|$ with respect to z, which is $+1$ for $z>0$ and -1 if $z<0$. Denoting this function by $\epsilon(z) \equiv \theta(z)-\theta(-z)$, we have

$$\boldsymbol{B}(t,z) = \frac{E_0}{c}\epsilon(z)\cos\omega(t-|z|/c)\hat{\boldsymbol{y}}. \tag{5.2.7}$$

The magnetic field also is a plane wave that travels away from $z=0$ at the speed of light, and it points in the y direction, which is perpendicular to both the direction of propagation and the direction of the electric field. Notice that $\boldsymbol{E}$ and $\boldsymbol{B}$ are both proportional to $\cos\omega(t-|z|/c)$: the fields are in phase. Notice also that $|\boldsymbol{B}|$ is smaller than $|\boldsymbol{E}|$ by a factor $1/c$.

To finish off our discussion of plane electromagnetic waves, we calculate the field's energy density and flux vector. From Eqs. (5.1.10), (5.2.6), and (5.6.7) we obtain

$$\varepsilon(t,z) = \epsilon_0 E_0^2 \cos^2\omega(t-|z|/c), \tag{5.2.8}$$

and from Eq. (5.1.11) we get

$$\boldsymbol{S}(t,z) = c\epsilon(z)\varepsilon(t,z)\hat{\boldsymbol{z}}. \tag{5.2.9}$$

This shows that the electromagnetic field energy travels with the wave, in the positive or negative z direction, away from the source, and with the speed of light.

Let us summarize the main points:

- The plane wave travels in the (positive or negative) z direction with speed c.
- The electric and magnetic fields are perpendicular to $\hat{\boldsymbol{z}}$ and to each other.
- The fields are in phase.
- Their magnitudes are related by $|\boldsymbol{B}|/|\boldsymbol{E}| = 1/c$.
- The electromagnetic field energy travels with the wave.

These properties are generic: they are shared by all electromagnetic waves. The only caveat is that in general, the direction of propagation is not uniform: the constant vector $\hat{\boldsymbol{z}}$ goes into a vector $\hat{\boldsymbol{n}}$ that may change from point to point.

5.3 Spherical waves — the oscillating dipole

[The material presented in this section is also covered in Secs. 9.1 and 9.2 of Jackson's text.]

5.3.1 Plane versus spherical waves

Any function of the form

$$\psi(t,z) = f(t \mp z/c), \tag{5.3.1}$$

where $f(\xi)$ is an arbitrary function of $\xi = t \mp z/c$, is a solution to the wave equation $\Box\psi = 0$. If $\xi = t - z/c$, the plane wave travels in the positive z direction with speed c; if instead $\xi = t + z/c$, the wave travels in the negative z direction. To see that Eq. (5.3.1) does indeed form a valid solution to the wave equation, we note that

$\partial\psi/\partial t = f'(\xi)$ while $\partial\psi/\partial z = \mp f'(\xi)/c$, where a prime indicates differentiation with respect to ξ; substitution into the wave operator yields $\Box\psi = -c^{-2}f'' + c^{-2}f'' = 0$, as required.

Plane waves are not a very good approximation to realistic waves, which originate in a localized region of space and propagate outward with a decreasing amplitude. A better approximation that incorporates these features is the spherical wave

$$\psi(t,r) = \frac{1}{r}f(t - r/c), \tag{5.3.2}$$

where $f(u)$ is an arbitrary function of $u = t - r/c$. This is also a solution to $\Box\psi = 0$. Here the wavefronts are spherical instead of planar, and the wave propagates outward in the radial direction; its amplitude decays as $1/r$, giving rise to an intensity that goes as $1/r^2$. That Eq. (5.3.2) forms a valid solution to the wave equation follows by direct substitution: In the absence of an angular dependence the wave operator reduces to

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}.$$

Differentiating with respect to t yields $\partial^2\psi/\partial t^2 = r^{-1}\ddot{f}$, while we have $r^2\partial\psi/\partial r = -r\dot{f}/c - f$ and

$$\frac{\partial}{\partial r}r^2\frac{\partial\psi}{\partial r} = -\frac{1}{c}\dot{f} + \frac{1}{c^2}r\ddot{f} + \frac{1}{c}\dot{f} = \frac{1}{c^2}r\ddot{f},$$

where an overdot indicates differentiation with respect to u. All this gives $\Box\psi = 0$, as required. Note that a similar calculation would reveal that $\psi = r^{-1}f(t + r/c)$ also is a solution to the wave equation; this wave travels inward instead of outward.

5.3.2 Potentials of an oscillating dipole

The simplest electromagnetic spherical wave is produced by an *oscillating dipole* located at the origin of the coordinate system. Here and in the following subsections we will calculate the potentials and fields produced by this dipole, and examine the nature of the corresponding electromagnetic radiation.

In Sec. 2.3 we saw that the charge density of a point dipole could be expressed as $\rho(\boldsymbol{x}) = -\boldsymbol{p}\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x})$, where $\boldsymbol{p}$ was a constant dipole moment vector. Here we introduce a time dependence by making $\boldsymbol{p}$ oscillate with angular frequency ω: $\boldsymbol{p} \to \boldsymbol{p}\cos\omega t$. The charge density of an oscillating dipole is then

$$\rho(t,\boldsymbol{x}) = -\boldsymbol{p}\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x})\,\cos\omega t. \tag{5.3.3}$$

Notice that in this equation, $\boldsymbol{p}$ is still a constant vector, which we align with the z axis: $\boldsymbol{p} = p\hat{\boldsymbol{z}}$.

A time-varying charge density must be associated with a current density $\boldsymbol{j}$, because

$$\boldsymbol{\nabla}\cdot\boldsymbol{j} = -\frac{\partial\rho}{\partial t} = -\omega\boldsymbol{p}\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x})\,\sin\omega t = \boldsymbol{\nabla}\cdot\big[-\omega\boldsymbol{p}\delta(\boldsymbol{x})\,\sin\omega t\big].$$

The simplest solution to this equation is

$$\boldsymbol{j}(t,\boldsymbol{x}) = -\omega\boldsymbol{p}\delta(\boldsymbol{x})\,\sin\omega t, \tag{5.3.4}$$

and this is the current density of an oscillating dipole.

We first compute the vector potential produced by the current density of Eq. (5.3.4). Once more it is convenient to start with the expression

$$\boldsymbol{A}(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \boldsymbol{j}(t',\boldsymbol{x'})\frac{\delta(t - t' - |\boldsymbol{x} - \boldsymbol{x'}|/c)}{|\boldsymbol{x} - \boldsymbol{x'}|}\,dt'd^3x'.$$

After substituting Eq. (5.3.5) we obtain

$$\begin{aligned}\boldsymbol{A}(t,\boldsymbol{x}) &= -\frac{\mu_0}{4\pi}\omega\boldsymbol{p}\int dt'\,\sin\omega t'\int\delta(\boldsymbol{x'})\frac{\delta(t-t'-|\boldsymbol{x}-\boldsymbol{x'}|/c)}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x' \\ &= -\frac{\mu_0}{4\pi}\omega\boldsymbol{p}\int dt'\,\sin\omega t'\,\frac{\delta(t-t'-r/c)}{r},\end{aligned}$$

or

$$\boldsymbol{A}(t,\boldsymbol{x}) = -\frac{\mu_0}{4\pi}\omega\boldsymbol{p}\,\frac{\sin\omega(t-r/c)}{r}. \tag{5.3.5}$$

This has the structure of a spherical wave, with its expected dependence on $u = t - r/c$ and its amplitude decaying as $1/r$.

We may now compute the scalar potential. This calculation is a bit more laborious, and we start with

$$\Phi(t,\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\int\rho(t',\boldsymbol{x'})\frac{\delta(t-t'-|\boldsymbol{x}-\boldsymbol{x'}|/c)}{|\boldsymbol{x}-\boldsymbol{x'}|}\,dt'd^3x'.$$

After substituting Eq. (5.3.3) we get

$$\Phi(t,\boldsymbol{x}) = -\frac{1}{4\pi\epsilon_0}\boldsymbol{p}\cdot\int dt'\boldsymbol{\Gamma}(t-t',\boldsymbol{x})\,\cos\omega t',$$

where

$$\boldsymbol{\Gamma}(t-t',\boldsymbol{x}) = \int\left[\boldsymbol{\nabla'}\delta(\boldsymbol{x'})\right]\frac{\delta(t-t'-|\boldsymbol{x}-\boldsymbol{x'}|/c)}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'$$

is a vectorial integral that we must now evaluate.

Integration by parts allows us to write

$$\boldsymbol{\Gamma} = -\int\delta(\boldsymbol{x'})\boldsymbol{\nabla'}\left[\frac{\delta(t-t'-|\boldsymbol{x}-\boldsymbol{x'}|/c)}{|\boldsymbol{x}-\boldsymbol{x'}|}\right]d^3x'.$$

Because the dependence on $\boldsymbol{x'}$ is contained entirely in $|\boldsymbol{x}-\boldsymbol{x'}|$, and because $\boldsymbol{\nabla'}|\boldsymbol{x}-\boldsymbol{x'}| = -(\boldsymbol{x}-\boldsymbol{x'})/|\boldsymbol{x}-\boldsymbol{x'}|$, evaluating the gradient gives

$$\begin{aligned}\boldsymbol{\Gamma} &= -\int\delta(\boldsymbol{x'})\left[\frac{\delta(t-t'-|\boldsymbol{x}-\boldsymbol{x'}|/c)}{|\boldsymbol{x}-\boldsymbol{x'}|^2}+\frac{1}{c}\frac{\delta'(t-t'-|\boldsymbol{x}-\boldsymbol{x'}|/c)}{|\boldsymbol{x}-\boldsymbol{x'}|}\right]\frac{\boldsymbol{x}-\boldsymbol{x'}}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x' \\ &= -\left[\frac{\delta(t-t'-r/c)}{r^2}+\frac{1}{c}\frac{\delta'(t-t'-r/c)}{r}\right]\frac{\boldsymbol{x}}{r},\end{aligned}$$

or

$$\boldsymbol{\Gamma}(t-t',\boldsymbol{x}) = -\hat{\boldsymbol{r}}\left[\frac{\delta(t-t'-r/c)}{r^2}+\frac{1}{c}\frac{\delta'(t-t'-r/c)}{r}\right],$$

where $\hat{\boldsymbol{r}} = \boldsymbol{x}/r$.

Substituting this result into our previous expression for the scalar potential, we obtain

$$\Phi(t,\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\boldsymbol{p}\cdot\hat{\boldsymbol{r}}\int dt'\left[\frac{\delta(t-t'-r/c)}{r^2}+\frac{1}{c}\frac{\delta'(t-t'-r/c)}{r}\right]\cos\omega t',$$

and integrating against the δ-functions gives

$$\Phi(t,\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\boldsymbol{p}\cdot\hat{\boldsymbol{r}}\left[\frac{\cos\omega(t-r/c)}{r^2}-\frac{\omega}{c}\frac{\sin\omega(t-r/c)}{r}\right].$$

One must be careful with the second term: a minus sign comes from the integration by part needed to convert $\delta'(t-t'-r/c)$ into an actual δ-function, but this minus

sign is canceled out because the prime on the δ-function indicates differentiation with respect to $t - t' - r/c$ instead of t'. We express our final result as

$$\Phi = \Phi_{\text{near}} + \Phi_{\text{wave}}, \tag{5.3.6}$$

where

$$\Phi_{\text{near}}(t, \boldsymbol{x}) = \frac{1}{4\pi\epsilon_0} \frac{\boldsymbol{p} \cdot \hat{\boldsymbol{r}}}{r^2} \cos(\omega u) \tag{5.3.7}$$

and

$$\Phi_{\text{wave}}(t, \boldsymbol{x}) = -\frac{1}{4\pi\epsilon_0} \frac{\omega}{c} \boldsymbol{p} \cdot \hat{\boldsymbol{r}} \frac{\sin(\omega u)}{r}, \tag{5.3.8}$$

and where $u = t - r/c$ is retarded time.

The scalar potential contains two terms. Of these, only Φ_{wave} has the mathematical structure of a spherical wave. The other term, Φ_{near}, has the structure of a static dipole field,

$$\Phi_{\text{dipole}}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0} \frac{\boldsymbol{p} \cdot \hat{\boldsymbol{r}}}{r^2},$$

to which a time dependence $\cos(\omega u)$ is attached. Unlike Φ_{wave}, this term does not represent electromagnetic radiation. Instead, Φ_{near} describes the field of a point dipole, and this field follows the dipole's time dependence with the usual delay: the $\cos \omega t$ dependence of the dipole is replaced by $\cos \omega(t - r/c)$.

As the name indicates, Φ_{near} dominates when r is small, while Φ_{wave} dominates when r is large. The scalar potential becomes purely radiative when r is sufficiently large that Φ_{near} can be neglected; the region of space for which this is true is called the *wave zone*. On the other hand, the region of space for which r is sufficiently small that Φ_{wave} can be neglected in front of Φ_{near} is called the *near zone*.

In the near zone we must have $1/r^2 \gg \omega/(cr)$, or

$$r \ll \frac{c}{\omega} = \frac{1}{k} = \frac{\lambda}{2\pi} \qquad \text{(near zone)}, \tag{5.3.9}$$

where k is the wave number and λ the wavelength. In the near zone, $u = t - r/c \simeq t$ and the delay contained in the retarded time is hardly noticeable — the near-zone field responds virtually instantaneously to changes in the source. In the wave zone we have instead $1/r^2 \ll \omega/(cr)$, or

$$r \gg \frac{c}{\omega} = \frac{1}{k} = \frac{\lambda}{2\pi} \qquad \text{(wave zone)}. \tag{5.3.10}$$

Here the time delay is important, and the scalar potential behaves as a true spherical wave. In the region of space where r is comparable to c/ω, between the near zone and the wave zone (a region that is sometimes called the induction, or intermediate, zone), Φ_{near} and Φ_{wave} are comparable, and the scalar potential behaves neither as an instantaneously-changing dipole field nor as a spherical wave.

5.3.3 Wave-zone fields of an oscillating dipole

Let us calculate the electric and magnetic fields in the wave zone. We use the expressions

$$\boldsymbol{A}(t, \boldsymbol{x}) = -\frac{\mu_0}{4\pi} \omega \boldsymbol{p} \frac{\sin(\omega u)}{r} \tag{5.3.11}$$

and

$$\Phi(t, \boldsymbol{x}) = -\frac{1}{4\pi\epsilon_0} \frac{\omega}{c} \boldsymbol{p} \cdot \hat{\boldsymbol{r}} \frac{\sin(\omega u)}{r}, \tag{5.3.12}$$

where $u = t - r/c$ is retarded time. Notice that Eq. (5.3.11) is exact, while Eq. (5.3.12) is valid in the wave zone only.

To do this calculation we will need to compute the gradient of

$$\psi(t,r) \equiv \frac{\sin(\omega u)}{r}.$$

We have $\boldsymbol{\nabla}\psi = (\partial\psi/\partial r)\boldsymbol{\nabla}r = (\partial\psi/\partial r)\hat{\boldsymbol{r}}$ and

$$\frac{\partial\psi}{\partial r} = -\frac{\sin(\omega u)}{r^2} - \frac{\omega}{c}\frac{\cos(\omega u)}{r},$$

because $\partial u/\partial r = 1/c$. But since we are working in the wave zone, the first term can be neglected and we obtain the useful approximation

$$\boldsymbol{\nabla}\frac{\sin(\omega u)}{r} = -\frac{\omega}{c}\frac{\cos(\omega u)}{r}\hat{\boldsymbol{r}}. \tag{5.3.13}$$

We might also need the gradient of $\boldsymbol{p}\cdot\hat{\boldsymbol{r}}$. To see what this is, consider its x component:

$$\nabla_x\left(\boldsymbol{p}\cdot\frac{\boldsymbol{x}}{r}\right) = -\frac{\boldsymbol{p}\cdot\boldsymbol{x}}{r^2}\frac{x}{r} + \frac{p_x}{r},$$

so that

$$\boldsymbol{\nabla}(\boldsymbol{p}\cdot\hat{\boldsymbol{r}}) = \frac{\boldsymbol{p} - (\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}}{r}. \tag{5.3.14}$$

The fact that this is of order $1/r$ means that the gradient of $\boldsymbol{p}\cdot\hat{\boldsymbol{r}}$ will be negligible in our forthcoming calculations.

The electric field is related to the potentials by Eq. (5.1.8),

$$\boldsymbol{E} = -\frac{\partial\boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\Phi.$$

Substituting Eqs. (5.3.11) and (5.3.12), we obtain

$$\boldsymbol{E} = \frac{\mu_0}{4\pi}\omega^2\boldsymbol{p}\frac{\cos(\omega u)}{r} + \frac{1}{4\pi\epsilon_0}\frac{\omega}{c}\left[\frac{\sin(\omega u)}{r}\boldsymbol{\nabla}(\boldsymbol{p}\cdot\hat{\boldsymbol{r}}) + (\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\boldsymbol{\nabla}\frac{\sin(\omega u)}{r}\right].$$

Using Eq. (5.3.14) shows that the first term within the square brackets is of order $1/r^2$ and can be neglected in the wave zone, while using Eq. (5.3.13) gives

$$\boldsymbol{E} = \frac{\mu_0}{4\pi}\omega^2\boldsymbol{p}\frac{\cos(\omega u)}{r} - \frac{1}{4\pi\epsilon_0}\frac{\omega^2}{c^2}(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\frac{\cos(\omega u)}{r}\hat{\boldsymbol{r}}.$$

Collecting the terms and recalling that $\epsilon_0\mu_0 = 1/c^2$ brings this to its final form

$$\boldsymbol{E}_{\rm wave}(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi}\omega^2\left[\boldsymbol{p} - (\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}\right]\frac{\cos(\omega u)}{r}. \tag{5.3.15}$$

This has the structure of a spherical wave, and we notice that $\boldsymbol{E}_{\rm wave}$ is *transverse to the direction in which the wave propagates*:

$$\boldsymbol{E}_{\rm wave}\cdot\hat{\boldsymbol{r}} = 0. \tag{5.3.16}$$

If, for concreteness, we let $\boldsymbol{p}$ point in the z direction, then $\boldsymbol{p} = p\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{r}} = \sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}}$, $\boldsymbol{p}\cdot\hat{\boldsymbol{r}} = p\cos\theta$, and $\boldsymbol{p} - (\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} = -p\sin\theta(\cos\theta\cos\phi\hat{\boldsymbol{x}} + \cos\theta\sin\phi\hat{\boldsymbol{y}} - \sin\theta\hat{\boldsymbol{z}}) = -p\sin\theta\hat{\boldsymbol{\theta}}$, where $\hat{\boldsymbol{\theta}}$ is a unit vector that points in the direction of increasing θ. This yields

$$\boldsymbol{E}_{\rm wave}(t,\boldsymbol{x}) = -\frac{\mu_0}{4\pi}\omega^2 p\sin\theta\frac{\cos(\omega u)}{r}\hat{\boldsymbol{\theta}} \tag{5.3.17}$$

for the electric field.

The magnetic field is obtained by taking the curl of the vector potential. We have

$$\boldsymbol{B}(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi}\omega\boldsymbol{p}\times\boldsymbol{\nabla}\frac{\sin(\omega u)}{r},$$

and Eq. (5.3.13) gives us

$$\boldsymbol{B}_{\rm wave}(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{\omega^2}{c}(\hat{\boldsymbol{r}}\times\boldsymbol{p})\frac{\cos(\omega u)}{r}. \tag{5.3.18}$$

The magnetic field also is a spherical wave, and like the electric field it is transverse to the direction of propagation:

$$\boldsymbol{B}_{\rm wave}\cdot\hat{\boldsymbol{r}} = 0. \tag{5.3.19}$$

Furthermore, we see from Eqs. (5.3.15) and (5.3.18) that the fields are mutually orthogonal, because $[\boldsymbol{p}-(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}]\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{p}) = \boldsymbol{p}\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{p}) - (\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{p}) = \boldsymbol{0}$, since the vector $\hat{\boldsymbol{r}}\times\boldsymbol{p}$ is perpendicular to both $\boldsymbol{p}$ and $\hat{\boldsymbol{r}}$. So the fields satisfy

$$\boldsymbol{E}_{\rm wave}\cdot\boldsymbol{B}_{\rm wave} = 0 \tag{5.3.20}$$

in the wave zone. For $\boldsymbol{p} = p\hat{\boldsymbol{z}}$ we have $\hat{\boldsymbol{r}}\times\boldsymbol{p} = -p\sin\theta(-\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}) = -p\sin\theta\hat{\boldsymbol{\phi}}$, and Eq. (5.3.18) becomes

$$\boldsymbol{B}_{\rm wave}(t,\boldsymbol{x}) = -\frac{\mu_0}{4\pi}\frac{\omega^2}{c}p\sin\theta\frac{\cos(\omega u)}{r}\,\hat{\boldsymbol{\phi}}. \tag{5.3.21}$$

5.3.4 Poynting vector of an oscillating dipole

We now calculate $\boldsymbol{S} = \mu_0^{-1}\boldsymbol{E}\times\boldsymbol{B}$, the energy flux vector associated with the electromagnetic radiation produced by the oscillating dipole. It is easy to check that in the wave zone, the electric and magnetic fields are related by

$$\boldsymbol{E}_{\rm wave} = c\boldsymbol{B}_{\rm wave}\times\hat{\boldsymbol{r}}. \tag{5.3.22}$$

This follows because the magnetic field is proportional to $\hat{\boldsymbol{r}}\times\boldsymbol{p}$, and $(\hat{\boldsymbol{r}}\times\boldsymbol{p})\times\hat{\boldsymbol{r}} = (\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}})\boldsymbol{p} - (\hat{\boldsymbol{r}}\cdot\boldsymbol{p})\hat{\boldsymbol{r}} = \boldsymbol{p} - (\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}$, which gives the direction of the electric field. Using once more the same vectorial identity, the Poynting vector is

$$\boldsymbol{S} = \frac{c}{\mu_0}\big(\boldsymbol{B}\times\hat{\boldsymbol{r}}\big)\times\boldsymbol{B} = \frac{c}{\mu_0}\Big[(\boldsymbol{B}\cdot\boldsymbol{B})\hat{\boldsymbol{r}} - (\boldsymbol{B}\cdot\hat{\boldsymbol{r}})\boldsymbol{B}\Big].$$

The second term vanishes because $\boldsymbol{B}$ is orthogonal to $\hat{\boldsymbol{r}}$, and we arrive at

$$\boldsymbol{S}_{\rm wave}(t,\boldsymbol{x}) = \frac{c}{\mu_0}\big|\boldsymbol{B}_{\rm wave}(t,\boldsymbol{x})\big|^2\hat{\boldsymbol{r}}. \tag{5.3.23}$$

This shows that the energy is transported in the radial direction, the same direction in which the wave propagates.

For $\boldsymbol{p} = p\hat{\boldsymbol{z}}$ we can substitute Eq. (5.3.21) into Eq. (5.3.23) and obtain

$$\begin{aligned}\boldsymbol{S} &= \frac{c}{\mu_0}\left(\frac{\mu_0}{4\pi}\right)^2\frac{\omega^4p^2\sin^2\theta}{c^2}\frac{\cos^2(\omega u)}{r^2}\,\hat{\boldsymbol{r}}\\ &= \frac{\mu_0p^2\omega^4}{16\pi^2c}\frac{\sin^2\theta}{r^2}\cos^2(\omega u)\,\hat{\boldsymbol{r}}.\end{aligned}$$

The energy flux oscillates in time, and after averaging over a complete wave cycle we get

$$\langle\boldsymbol{S}_{\rm wave}\rangle = \frac{\mu_0p^2\omega^4}{32\pi^2c}\frac{\sin^2\theta}{r^2}\,\hat{\boldsymbol{r}}. \tag{5.3.24}$$

From the angular dependence of this expression we see that most of the energy is emitted near the equatorial plane ($\theta = \frac{\pi}{2}$), and that none of the energy propagates along the z axis ($\theta = 0$ or $\theta = \pi$). The average of any function f of retarded time u over a complete wave cycle is defined by $\langle f(u)\rangle \equiv T^{-1}\int_0^T f(u)\,du$, where $T = 2\pi/\omega$ is the oscillation period. Using this rule we get $\langle \cos^2(\omega u)\rangle = \frac{1}{2}$, and this gives Eq. (5.3.24).

The total power radiated (energy per unit time) is obtained by integrating $\boldsymbol{S}$ (energy per unit time per unit area) over any closed surface S. The averaged power is then

$$\langle P\rangle = \oint_S \langle \boldsymbol{S}\rangle \cdot d\boldsymbol{a}. \tag{5.3.25}$$

It is simplest to take S to be a sphere of constant r. Then $d\boldsymbol{a} = \hat{\boldsymbol{r}} r^2 \sin\theta\, d\theta d\phi$, and Eq. (5.3.25) becomes

$$\langle P\rangle = \frac{\mu_0 p^2 \omega^4}{32\pi^2 c} 2\pi \int_0^\pi \sin^3\theta\, d\theta.$$

The integral evaluates to 4/3, and we obtain

$$\langle P\rangle = \frac{\mu_0}{12\pi c} p^2 \omega^4. \tag{5.3.26}$$

5.3.5 Summary — oscillating dipole

We have seen that in the wave zone ($r \gg c/\omega$), the fields of an oscillating dipole are given by

$$\boldsymbol{E}(t, r, \theta, \phi) = -\frac{\mu_0}{4\pi}\omega^2 p \sin\theta\, \frac{\cos\omega(t - r/c)}{r}\, \hat{\boldsymbol{\theta}}$$

and

$$\boldsymbol{B}(t, r, \theta, \phi) = -\frac{\mu_0}{4\pi}\frac{\omega^2}{c} p \sin\theta\, \frac{\cos\omega(t - r/c)}{r}\, \hat{\boldsymbol{\phi}}.$$

These have the structure of a spherical wave, which travels outward at the speed of light. They are produced by a charge distribution whose dipole moment vector is given by $\boldsymbol{p}(t) = p\cos(\omega t)\hat{\boldsymbol{z}}$. The field amplitudes are related by $|\boldsymbol{B}|/|\boldsymbol{E}| = 1/c$, and both fields are in phase. The electric field is directed along

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\, \hat{\boldsymbol{x}} + \cos\theta\sin\phi\, \hat{\boldsymbol{y}} - \sin\theta\, \hat{\boldsymbol{z}}$$

while the magnetic field is directed along

$$\hat{\boldsymbol{\phi}} = -\sin\phi\, \hat{\boldsymbol{x}} + \cos\phi\, \hat{\boldsymbol{y}}.$$

The fields are mutually orthogonal, and they are both transverse to

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\, \hat{\boldsymbol{x}} + \sin\theta\sin\phi\, \hat{\boldsymbol{y}} + \cos\theta\, \hat{\boldsymbol{z}},$$

the direction in which the wave is traveling. This is also the direction in which the field energy is transported. The angular profile of the energy-flux (Poynting) vector is described by $\sin^2\theta$, so that most of the energy is emitted in the directions orthogonal to the dipole.

5.4 Electric dipole radiation

[The material presented in this section is also covered in Sec. 9.2 of Jackson's text.]

We now consider a much more general situation, in which electromagnetic radiation is produced by an *arbitrary distribution of charges and currents*, with an *arbitrary time dependence* (not necessarily oscillating with a single frequency ω). Our only restrictions are that

- the source is confined to a bounded region V of space;
- the charges are moving slowly.

These conditions will allow us to formulate useful approximations for the behaviour of the electric and magnetic fields. Although the present context is very general, we will see that the results to be derived in this section are very close to those obtained in the preceding section.

5.4.1 Slow-motion approximation; near and wave zones

To make the slow-motion approximation precise, and to define near and wave zones in the general case, we introduce the following scaling quantities:

$$\begin{aligned} r_c &\equiv \text{characteristic length scale of the charge and current distribution,} \\ t_c &\equiv \text{characteristic time scale over which the distribution changes,} \\ v_c &\equiv \frac{r_c}{t_c} \equiv \text{characteristic velocity of the source,} \\ \omega_c &\equiv \frac{2\pi}{t_c} \equiv \text{characteristic frequency of the source,} \\ \lambda_c &\equiv \frac{2\pi c}{\omega_c} \equiv ct_c \equiv \text{characteristic wavelength of the radiation.} \end{aligned}$$

The characteristic length scale is defined such that if $\rho(t, \boldsymbol{x'})$ and $\boldsymbol{j}(t, \boldsymbol{x'})$ are the charge and current densities, respectively, then $|\boldsymbol{x'}|$ is at most of order r_c throughout the source. This implies that the distribution of charge and current is localized within a region whose volume is of the order of r_c^3. The characteristic time scale is defined such that $\partial\rho/\partial t$ is of order ρ/t_c throughout the source. If, for example, ρ oscillates with a frequency Ω, then $t_c \sim 1/\Omega$ and $\omega_c \sim \Omega$.

The slow-motion approximation means that $v_c = r_c/t_c$ is much smaller than the speed of light:

$$v_c \ll c. \tag{5.4.1}$$

This condition gives us $r_c = v_c t_c \ll ct_c = \lambda_c$, or

$$r_c \ll \lambda_c. \tag{5.4.2}$$

The source is therefore confined to a region that is *much smaller* than a typical wavelength of the radiation.

We can now define the near and wave zones:

$$\text{near zone:} \qquad |\boldsymbol{x} - \boldsymbol{x'}| \ll \frac{\lambda_c}{2\pi} = \frac{c}{\omega_c}, \tag{5.4.3}$$

$$\text{wave zone:} \qquad |\boldsymbol{x} - \boldsymbol{x'}| \gg \frac{\lambda_c}{2\pi} = \frac{c}{\omega_c}. \tag{5.4.4}$$

Because $r_c \ll \lambda_c$, the wave zone can also be defined by the condition $r \gg \lambda_c/(2\pi)$. We will compute the potentials and fields in these zones, using Eqs. (5.4.1)–(5.4.4) to simplify our expressions.

5.4.2 Scalar potential

We begin by calculating the scalar potential,

$$\Phi(t, \boldsymbol{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(t - |\boldsymbol{x} - \boldsymbol{x'}|/c, \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|}\, d^3x',$$

where V is the region of space occupied by the source. In the *near zone* we can treat $|\boldsymbol{x}-\boldsymbol{x'}|/c$ as a small quantity and Taylor-expand the charge density about the current time t. Because $\rho(t-\varepsilon)=\rho(t)-\dot\rho(t)\varepsilon+\frac{1}{2}\ddot\rho(t)\varepsilon^2+\cdots$ we have

$$\rho(t-|\boldsymbol{x}-\boldsymbol{x'}|/c)=\rho(t)-\frac{1}{c}\dot\rho(t)|\boldsymbol{x}-\boldsymbol{x'}|+\cdots,$$

where an overdot indicates differentiation with respect to t. Relative to the first term, the second term is of order $|\boldsymbol{x}-\boldsymbol{x'}|/(ct_c)=|\boldsymbol{x}-\boldsymbol{x'}|/\lambda_c$, and by virtue of Eq. (5.4.3), this is small in the near zone. The third term would be smaller still, and we neglect it. We then have

$$\begin{aligned}\Phi(t,\boldsymbol{x}) &= \frac{1}{4\pi\epsilon_0}\left[\int_V\frac{\rho(t,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'-\frac{1}{c}\int\dot\rho(t,\boldsymbol{x'})\,d^3x'+\cdots\right]\\ &= \frac{1}{4\pi\epsilon_0}\left[\int_V\frac{\rho(t,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'-\frac{1}{c}\frac{d}{dt}\int_V\rho(t,\boldsymbol{x'})\,d^3x'+\cdots\right].\end{aligned}$$

The second term vanishes, because it involves the time derivative of the total charge $\int\rho(t,\boldsymbol{x'})\,d^3x'$, which is conserved. We have obtained

$$\Phi_{\rm near}(t,\boldsymbol{x})=\frac{1}{4\pi\epsilon_0}\int_V\frac{\rho(t,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\left[1+O\left(\frac{|\boldsymbol{x}-\boldsymbol{x'}|^2}{\lambda_c^2}\right)\right]d^3x'. \tag{5.4.5}$$

We see that the near-zone potential takes its usual static expression, except for the small correction of order $|\boldsymbol{x}-\boldsymbol{x'}|^2/\lambda_c^2$ and the fact that charge density depends on time. The time delay between the source and the potential has disappeared, and what we have is a potential that adjusts instantaneously to the changes within the distribution. The electric field it produces is then a "time-changing electrostatic field". This near-zone field does not behave as radiation.

To witness radiative effects we must go to the *wave zone*. Here $|\boldsymbol{x}-\boldsymbol{x'}|$ is large and we can no longer Taylor-expand the density as we did previously. Instead we must introduce another approximation technique. We use the fact that in the wave zone, r is much larger than r', so that

$$\begin{aligned}|\boldsymbol{x}-\boldsymbol{x'}| &= \sqrt{(\boldsymbol{x}-\boldsymbol{x'})\cdot(\boldsymbol{x}-\boldsymbol{x'})}\\ &= \sqrt{r^2-2\boldsymbol{x}\cdot\boldsymbol{x'}+r'^2}\\ &= r\Big[1-2\hat{\boldsymbol{r}}\cdot\boldsymbol{x'}/r+O(r_c^2/r^2)\Big]^{1/2}\\ &= r\Big[1-\hat{\boldsymbol{r}}\cdot\boldsymbol{x'}/r+O(r_c^2/r^2)\Big]\\ &= r-\hat{\boldsymbol{r}}\cdot\boldsymbol{x'}+O(r_c^2/r).\end{aligned}$$

This gives

$$\begin{aligned}\rho(t-|\boldsymbol{x}-\boldsymbol{x'}|/c) &= \rho(t-r/c+\hat{\boldsymbol{r}}\cdot\boldsymbol{x'}/c+\epsilon/c)\\ &= \rho(u+\hat{\boldsymbol{r}}\cdot\boldsymbol{x'}/c+\epsilon/c),\end{aligned}$$

where

$$u=t-r/c \tag{5.4.6}$$

is *retarded time* and ϵ is of order r_c^2/r. Let us now Taylor-expand the charge density about the retarded time u instead of the current time t. We have

$$\rho(t-|\boldsymbol{x}-\boldsymbol{x'}|/c)=\rho(u)+\frac{1}{c}\dot\rho(u)\big(\hat{\boldsymbol{r}}\cdot\boldsymbol{x'}+\epsilon\big)+\cdots,$$

where an overdot now indicates differentiation with respect to u. We see that relative to $\hat{\boldsymbol{r}}\cdot\boldsymbol{x'}$, the ϵ term is of order $r_c/r\ll1$ and can be neglected. Now relative

to $\rho(u)$, the time-derivative term is of order $r_c/(ct_c) = v_c/c$, and this is small by virtue of Eq. (5.4.1). We therefore have

$$\rho(t - |\boldsymbol{x} - \boldsymbol{x'}|/c) = \rho(u) + \frac{\hat{\boldsymbol{r}} \cdot \boldsymbol{x'}}{c}\dot{\rho}(u) + O\big(v_c^2/c^2\big) \tag{5.4.7}$$

in the wave zone.

Inside the integral for Φ we also have

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{x'}|} = \frac{1}{r}\Big[1 + \hat{\boldsymbol{r}} \cdot \boldsymbol{x'}/r + O(r_c^2/r^2)\Big] = \frac{1}{r}\Big[1 + O(r_c/r)\Big],$$

and it is sufficient to keep the leading term only. The wave-zone potential is then

$$\begin{aligned}\Phi(t, \boldsymbol{x}) &= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int_V \left[\rho(u) + \frac{\hat{\boldsymbol{r}} \cdot \boldsymbol{x'}}{c}\dot{\rho}(u) + \cdots\right] d^3x' \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\left[\int_V \rho(u, \boldsymbol{x'})\, d^3x' + \frac{\hat{\boldsymbol{r}}}{c} \cdot \frac{\partial}{\partial u}\int_V \rho(u, \boldsymbol{x'})\boldsymbol{x'}\, d^3x' + \cdots\right].\end{aligned}$$

In the first integral we recognize Q, the total charge of the distribution; this is actually independent of u by virtue of charge conservation. In the second integral we recognize $\boldsymbol{p}(u)$, the dipole moment vector of the charge distribution; this does depend on retarded time u. Our final expression for the potential is therefore

$$\Phi_{\text{wave}}(t, \boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} + \frac{1}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{r}} \cdot \dot{\boldsymbol{p}}(u)}{cr} + \cdots, \tag{5.4.8}$$

where

$$Q = \int_V \rho(u, \boldsymbol{x'})\, d^3x' \tag{5.4.9}$$

is the total charge and

$$\boldsymbol{p}(u) = \int_V \rho(u, \boldsymbol{x'})\boldsymbol{x'}\, d^3x' \tag{5.4.10}$$

the dipole moment. The first term on the right-hand side of Eq. (5.4.8) is the static, monopole potential associated with the total charge Q. This term does not depend on time and is not associated with the propagation of radiation; we shall simply omit it in later calculations. The second term, on the other hand, is radiative: it depends on retarded time $u = t - r/c$ and decays as $1/r$. We see that the radiative part of the scalar potential is produced by a time-changing dipole moment of the charge distribution; it is nonzero whenever $d\boldsymbol{p}/dt$ is nonzero. You should check that

$$\Phi_{\text{wave}}(t, \boldsymbol{x}) \simeq \frac{1}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{r}} \cdot \dot{\boldsymbol{p}}(u)}{cr}$$

reduces to Eq. (5.3.8) when the dipole moment is given by $\boldsymbol{p}(t) = \boldsymbol{p}_0 \cos(\omega t)$, where $\boldsymbol{p}_0$ is a constant vector.

5.4.3 Vector potential

The exact expression for the vector potential is

$$\boldsymbol{A}(t, \boldsymbol{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{j}(t - |\boldsymbol{x} - \boldsymbol{x'}|/c, \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|}\, d^3x'.$$

In the *near zone* we approximate the current density as

$$\boldsymbol{j}(t - |\boldsymbol{x} - \boldsymbol{x'}|/c) = \boldsymbol{j}(t)\left[1 + O\left(\frac{|\boldsymbol{x} - \boldsymbol{x'}|}{\lambda_c}\right)\right]$$

and we obtain

$$\boldsymbol{A}_{\rm near}(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{j}(t,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\left[1+O\left(\frac{|\boldsymbol{x}-\boldsymbol{x'}|}{\lambda_c}\right)\right] d^3x'. \tag{5.4.11}$$

We see that here also, the vector potential takes its static form, except for the correction of order $|\boldsymbol{x}-\boldsymbol{x'}|/\lambda_c$ and the fact that the current density depends on time. The potential responds virtually instantaneously to changes in the distribution, and there are no radiative effects in the near zone.

In the *wave zone* we have instead

$$\boldsymbol{j}(t-|\boldsymbol{x}-\boldsymbol{x'}|/c) = \boldsymbol{j}\big(u+\hat{\boldsymbol{r}}\cdot\boldsymbol{x'}/c\big) = \boldsymbol{j}(u)\big[1+O(v_c/c)\big]$$

and

$$\boldsymbol{A}_{\rm wave}(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{1}{r}\int_V \boldsymbol{j}(u,\boldsymbol{x'})\big[1+O(v_c/c)\big]\, d^3x'.$$

In static situations, the volume integral of $\boldsymbol{j}$ vanishes — recall Eq. (3.4.3). But here the current density depends on time, and as we shall prove presently, we have instead

$$\int_V \boldsymbol{j}(u,\boldsymbol{x'})\, d^3x' = \dot{\boldsymbol{p}}(u). \tag{5.4.12}$$

The vector potential is therefore

$$\boldsymbol{A}_{\rm wave}(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{\dot{\boldsymbol{p}}(u)}{r} + \cdots \tag{5.4.13}$$

in the wave zone. This has the structure of a spherical wave, and we see that the radiative part of the vector potential is produced by a time-changing dipole moment. You may verify that Eq. (5.4.13) reduces to reduces to Eq. (5.3.5) when the dipole moment is given by $\boldsymbol{p}(t) = \boldsymbol{p}_0\cos(\omega t)$, where $\boldsymbol{p}_0$ is a constant vector.

To establish the z component of Eq. (5.4.12) we start with the identity $\boldsymbol{\nabla}\cdot(z\boldsymbol{j}) = z(\boldsymbol{\nabla}\cdot\boldsymbol{j}) + \boldsymbol{j}\cdot(\boldsymbol{\nabla}z)$, in which we substitute the statement of charge conservation, $\boldsymbol{\nabla}\cdot\boldsymbol{j} = -\partial\rho/\partial t$. After integration over the source we have

$$\int_V j_z\, d^3x - \frac{d}{dt}\int_V \rho z\, d^3x = \int_V \boldsymbol{\nabla}\cdot(z\boldsymbol{j})\, d^3x = \oint_S z\boldsymbol{j}\cdot d\boldsymbol{a},$$

where S is the two-dimensional surface bounding V. Because no current is crossing this surface, the right-hand side vanishes and we have

$$\int_V j_z\, d^3x = \frac{d}{dt}\int_V \rho z\, d^3x = \frac{dp_z}{dt}.$$

This is the same thing as the z component of Eq. (5.4.12), except for the fact that ρ and $\boldsymbol{j}$ are expressed in terms of the variables $(t,\boldsymbol{x})$ instead of $(u,\boldsymbol{x'})$. The other components of the equation are established with similar manipulations.

5.4.4 Wave-zone fields

The potentials

$$\Phi_{\rm wave}(t,\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{r}}\cdot\dot{\boldsymbol{p}}(u)}{cr}\Big[1+O(v_c/c)\Big] \tag{5.4.14}$$

and

$$\boldsymbol{A}_{\rm wave}(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{\dot{\boldsymbol{p}}(u)}{r}\Big[1+O(v_c/c)\Big], \tag{5.4.15}$$

where $u = t - r/c$ and $\hat{\boldsymbol{r}} = \boldsymbol{x}/r$, are generated by time variations of the dipole moment vector $\boldsymbol{p}$ of the charge and current distribution. They therefore give rise

to *electric-dipole radiation*, the leading-order contribution, in our slow-motion approximation, to the radiation emitted by an arbitrary source. We now compute the electric and magnetic fields in this approximation.

To get the electric field we keep only those terms that decay as $1/r$, and neglect terms that decay faster. For example, when computing the gradient of the scalar potential we can neglect $\boldsymbol{\nabla} r^{-1} = -\hat{\boldsymbol{r}}/r^2$; we only need $\boldsymbol{\nabla}(\hat{\boldsymbol{r}}\cdot\dot{\boldsymbol{p}})$, which we calculate as

$$\begin{aligned}
\nabla_x(\hat{\boldsymbol{r}}\cdot\dot{\boldsymbol{p}}) &= \nabla_x\left(\frac{x}{r}\dot{p}_x + \frac{y}{r}\dot{p}_y + \frac{z}{r}\dot{p}_z\right) \\
&= \frac{x}{r}\ddot{p}_x\left(-\frac{1}{c}\nabla_x r\right) + \frac{y}{r}\ddot{p}_y\left(-\frac{1}{c}\nabla_x r\right) + \frac{z}{r}\ddot{p}_z\left(-\frac{1}{c}\nabla_x r\right) + O\left(\frac{1}{r}\right) \\
&= -\frac{1}{c}\big(\hat{\boldsymbol{r}}\cdot\ddot{\boldsymbol{p}}\big)\frac{x}{r} + O\left(\frac{1}{r}\right),
\end{aligned}$$

so that

$$\boldsymbol{\nabla}\big(\hat{\boldsymbol{r}}\cdot\dot{\boldsymbol{p}}\big) = -\frac{1}{c}\big(\hat{\boldsymbol{r}}\cdot\ddot{\boldsymbol{p}}\big)\hat{\boldsymbol{r}} + O\left(\frac{1}{r}\right). \tag{5.4.16}$$

To obtain this result we made use of the fact that $\dot{\boldsymbol{p}}$ depends on $\boldsymbol{x}$ through $u = t - r/c$, so that acting with $\boldsymbol{\nabla}$ amounts to taking a derivative with respect to u and multiplying by $-c^{-1}\boldsymbol{\nabla} r = -c^{-1}\hat{\boldsymbol{r}}$. The electric field is then

$$\begin{aligned}
\boldsymbol{E} &= -\frac{\partial A}{\partial t} - \boldsymbol{\nabla}\Phi \\
&= -\frac{\mu_0}{4\pi}\frac{\ddot{\boldsymbol{p}}}{r} + \frac{1}{4\pi\epsilon_0}\frac{(\hat{\boldsymbol{r}}\cdot\ddot{\boldsymbol{p}})\hat{\boldsymbol{r}}}{c^2 r},
\end{aligned}$$

or

$$\boldsymbol{E}_{\text{wave}}(t,\boldsymbol{x}) = -\frac{\mu_0}{4\pi}\frac{\ddot{\boldsymbol{p}}(u) - \big[\hat{\boldsymbol{r}}\cdot\ddot{\boldsymbol{p}}(u)\big]\hat{\boldsymbol{r}}}{r} = -\frac{\mu_0}{4\pi}\frac{\big[\hat{\boldsymbol{r}}\times\ddot{\boldsymbol{p}}(u)\big]\times\hat{\boldsymbol{r}}}{r}. \tag{5.4.17}$$

Notice that the wave-zone electric field behaves as a spherical wave, and that it is transverse to $\hat{\boldsymbol{r}}$, the direction in which the wave propagates.

To get the magnetic field we need to compute $\boldsymbol{\nabla}\times\dot{\boldsymbol{p}}(u)$. To see what this is we examine its x component:

$$\begin{aligned}
\big(\boldsymbol{\nabla}\times\dot{\boldsymbol{p}}\big)_x &= \nabla_y\dot{p}_z - \nabla_z\dot{p}_y \\
&= -\frac{1}{c}\ddot{p}_z\frac{y}{r} + \frac{1}{c}\ddot{p}_y\frac{z}{r} \\
&= -\frac{1}{c}\big(\hat{\boldsymbol{r}}\times\ddot{\boldsymbol{p}}\big)_x,
\end{aligned}$$

so that

$$\boldsymbol{\nabla}\times\dot{\boldsymbol{p}}(u) = -\frac{1}{c}\hat{\boldsymbol{r}}\times\ddot{\boldsymbol{p}}(u). \tag{5.4.18}$$

The magnetic field is then

$$\boldsymbol{B}_{\text{wave}}(t,\boldsymbol{x}) = -\frac{\mu_0}{4\pi}\frac{\hat{\boldsymbol{r}}\times\ddot{\boldsymbol{p}}(u)}{cr}. \tag{5.4.19}$$

Notice that the wave-zone magnetic field behaves as a spherical wave, and that it is orthogonal to both $\hat{\boldsymbol{r}}$ and the electric field. Notice finally that the fields are in phase — they both depend on $\ddot{\boldsymbol{p}}(u)$ — and that their magnitudes are related by $|\boldsymbol{B}|/|\boldsymbol{E}| = 1/c$. The properties listed in Sec. 5.3.5 are therefore shared by all electric-dipole radiation fields.

5.4.5 Energy radiated

The Poynting vector is $\boldsymbol{S} = \mu_0^{-1}\boldsymbol{E}\times\boldsymbol{B}$, and since $\boldsymbol{E} = c\boldsymbol{B}\times\hat{\boldsymbol{r}}$ in the wave zone — as follows from Eqs. (5.4.17) and (5.4.19) — we have $\boldsymbol{S} = (c/\mu_0)(\boldsymbol{B}\times\hat{\boldsymbol{r}})\times\boldsymbol{B} = (c/\mu_0)[|\boldsymbol{B}|^2\hat{\boldsymbol{r}} - (\boldsymbol{B}\cdot\hat{\boldsymbol{r}})\boldsymbol{B}]$. The last term vanishes by virtue of Eq. (5.4.19), and we obtain

$$\boldsymbol{S}_{\text{wave}} = \frac{c}{\mu_0}\big|\boldsymbol{B}_{\text{wave}}\big|^2\hat{\boldsymbol{r}}, \tag{5.4.20}$$

the same result as in Eq. (5.3.23). The fact that the Poynting vector is directed along $\hat{\boldsymbol{r}}$ shows that the electromagnetic field energy travels along with the wave.

The energy crossing a sphere of radius r per unit time is given by $P = \oint \boldsymbol{S}\cdot d\boldsymbol{a}$, where $d\boldsymbol{a} = \hat{\boldsymbol{r}}r^2\,d\Omega$ and $d\Omega = \sin\theta\,d\theta d\phi$. Substituting Eqs. (5.4.19) and (5.4.20) yields

$$P = \frac{\mu_0}{(4\pi)^2 c}\oint\big|\hat{\boldsymbol{r}}\times\ddot{\boldsymbol{p}}(u)\big|^2\,d\Omega.$$

To evaluate the integral we use the trick of momentarily aligning the z axis with the instantaneous direction of $\ddot{\boldsymbol{p}}(u)$ — we must do this for each particular value of u. Then $\hat{\boldsymbol{r}}\times\ddot{\boldsymbol{p}}(u) = |\ddot{\boldsymbol{p}}(u)|\sin\theta$ and

$$P = \frac{\mu_0}{(4\pi)^2 c}\big|\ddot{\boldsymbol{p}}(u)\big|^2\oint\sin^2\theta\,d\Omega = \frac{\mu_0}{(4\pi)^2 c}\big|\ddot{\boldsymbol{p}}(u)\big|^2\int_0^{2\pi}d\phi\int_0^{\pi}\sin^3\theta\,d\theta.$$

The angular integration gives $(2\pi)(4/3)$ and we arrive at

$$P = \frac{\mu_0}{6\pi c}\big|\ddot{\boldsymbol{p}}(u)\big|^2. \tag{5.4.21}$$

This is the total power radiated by a slowly-moving distribution of charge and current. The power's angular distribution is described by

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c}\big|\ddot{\boldsymbol{p}}(u)\big|^2\sin^2\Theta(u), \tag{5.4.22}$$

where $\Theta(u)$ is the angle between the vectors $\hat{\boldsymbol{r}}$ and $\ddot{\boldsymbol{p}}(u)$.

5.4.6 Summary — electric dipole radiation

To leading order in a slow-motion approximation, the wave-zone fields of an arbitrary distribution of charges and currents are given by

$$\boldsymbol{B}_{\text{wave}}(t,\boldsymbol{x}) = -\frac{\mu_0}{4\pi}\frac{\hat{\boldsymbol{r}}\times\ddot{\boldsymbol{p}}(t-r/c)}{cr}\Big[1+O(v_c/c)\Big]$$

and

$$\boldsymbol{E}_{\text{wave}} = c\boldsymbol{B}_{\text{wave}}\times\hat{\boldsymbol{r}}.$$

The fields are generated by time variations of the source's dipole moment vector,

$$\boldsymbol{p}(t) = \int_V \rho(t,\boldsymbol{x})\boldsymbol{x}\,d^3x.$$

The fields behave as spherical waves, they are each transverse to $\hat{\boldsymbol{r}}$, and they are mutually orthogonal. The field energy flux vector is

$$\boldsymbol{S}_{\text{wave}} = \frac{c}{\mu_0}\big|\boldsymbol{B}_{\text{wave}}\big|^2\hat{\boldsymbol{r}},$$

and the total power radiated is given by

$$P = \oint\boldsymbol{S}\cdot d\boldsymbol{a} = \frac{\mu_0}{6\pi c}\big|\ddot{\boldsymbol{p}}(u)\big|^2.$$

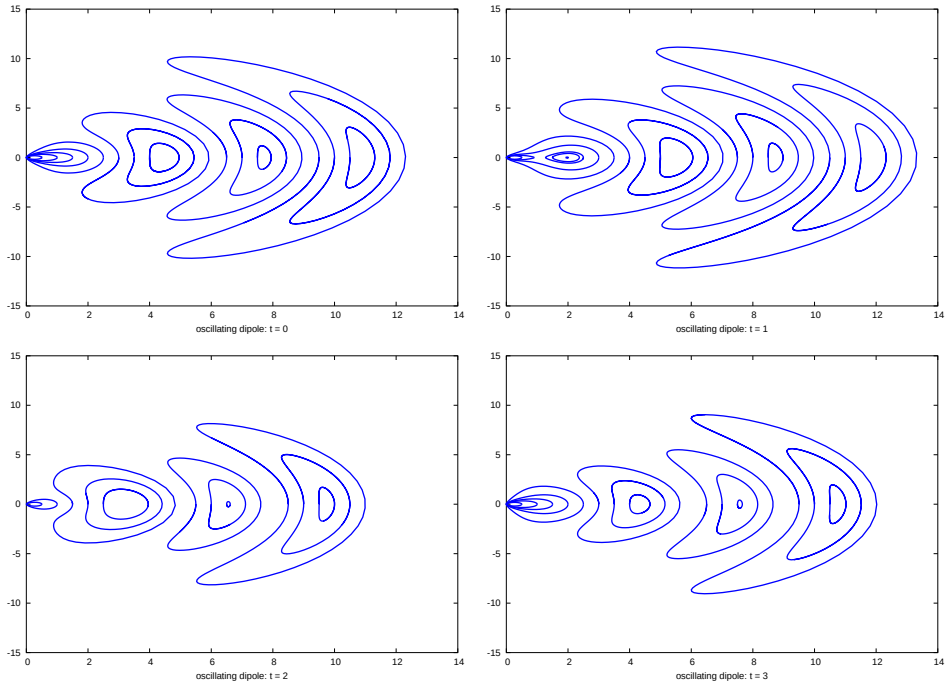

Figure 5.1: Electric field lines of an oscillating dipole, $\boldsymbol{p}(t) = p_0 \cos(\omega t)\, \hat{\boldsymbol{z}}$, at selected moments of time, for $\omega = 1$. Witness the transition between near-zone behaviour and wave-zone behaviour which occurs near $r = \lambda = 2\pi c/\omega \equiv 1$. The curves were obtained by following the method detailed in Sec. 2.11.

The power's angular profile is proportional to $\sin^2\Theta(u)$, where $\Theta(u)$ is the angle between the vectors $\hat{\boldsymbol{r}}$ and $\ddot{\boldsymbol{p}}(u)$.

The electric field lines of an oscillating dipole, $\boldsymbol{p}(t) = p_0 \cos(\omega t)\, \hat{\boldsymbol{z}}$, are represented in Fig. 5.1. The field lines of a rotating dipole, $\boldsymbol{p}(t) = p_0[\cos(\omega t)\, \hat{\boldsymbol{x}} + \sin(\omega t)\, \hat{\boldsymbol{y}}]$, are represented in Fig. 5.2.

5.5 Centre-fed linear antenna

[The material presented in this section is also covered in Sec. 9.4 of Jackson's text.]

As an example of a radiating system we consider a thin wire of total length 2ℓ which is fed an oscillating current through a small gap at its midpoint. The wire runs along the z axis, from $z = -\ell$ to $z = \ell$, and the gap is located at $z = 0$ (see Fig. 5.3.)

For such antennas, the current typically oscillates both in time and in space, and it is usually represented by

$$\boldsymbol{j}(t, \boldsymbol{x}) = I \sin\big[k(\ell - |z|)\big]\, \delta(x)\delta(y)\hat{\boldsymbol{z}} \cos\omega t, \tag{5.5.1}$$

where

$$k = \omega/c. \tag{5.5.2}$$

The current is an even function of z (it is the same in both arms of the antenna) and its goes to zero at both ends (at $z = \pm\ell$). The current at the gap (at $z = 0$) is $I \sin k\ell$, and I is the current's peak value.

We want to calculate the total power radiated by this antenna, using the electric-dipole approximation. To be consistent we must be sure that $v_c \ll c$ for this

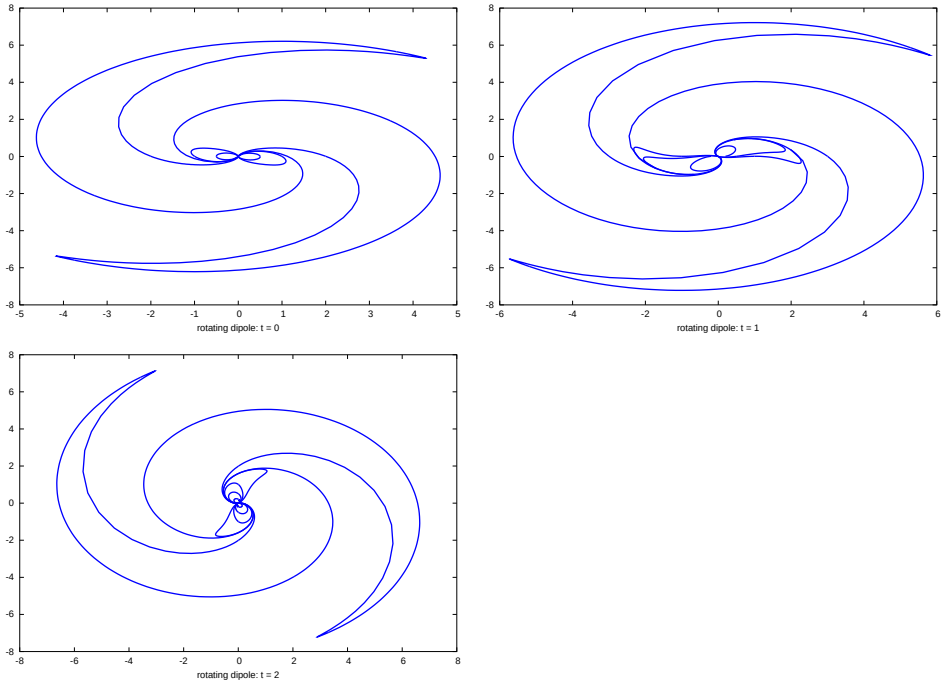

Figure 5.2: Electric field lines of a rotating dipole, $\boldsymbol{p}(t) = p_0[\cos(\omega t)\,\hat{\boldsymbol{x}} + \sin(\omega t)\,\hat{\boldsymbol{y}}]$, at selected moments of time, for $\omega = 1$. Witness the transition between near-zone behaviour and wave-zone behaviour which occurs near $r = \lambda = 2\pi c/\omega \equiv 1$.

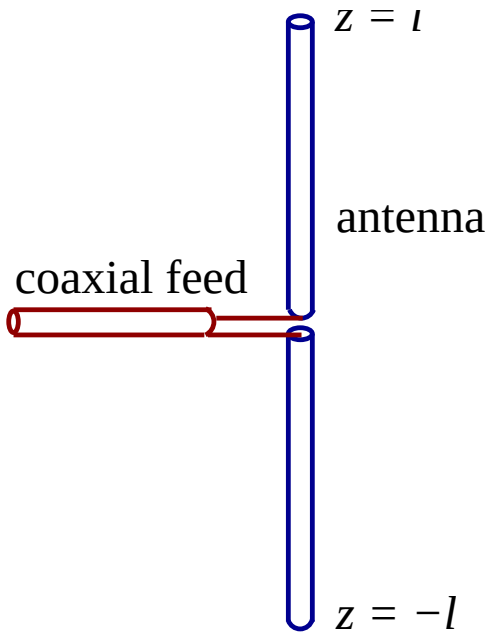

Figure 5.3: Centre-fed linear antenna. The oscillating current is provided by a coaxial feed.

distribution of current, or equivalently, that $\ell \equiv r_c \ll \lambda_c \equiv 2\pi/k$ — refer back to Eq. (5.4.2). In other words, we must demand that

$$k\ell \ll 1,$$

which means that $k|z|$ is small throughout the antenna. We can therefore approximate $\sin[k(\ell - |z|)]$ by $k(\ell - |z|)$, and Eq. (5.5.1) becomes

$$\boldsymbol{j}(t, \boldsymbol{x}) = I_0\big(1 - |z|/\ell\big)\delta(x)\delta(y)\hat{\boldsymbol{z}}\cos\omega t, \tag{5.5.3}$$

where

$$I_0 = Ik\ell \tag{5.5.4}$$

is the value of the current at the gap. In this approximation the current no longer oscillates in space: it simply goes from its peak value I_0 at the gap to zero at the two ends of the wire. In Sec. 9.4 of his book, Jackson gives an exact treatment of the antenna of Eq. (5.5.1).

To compute the power radiated by our simplified antenna we first need to calculate $\ddot{\boldsymbol{p}}(t)$, the second time derivative of the dipole moment vector. For this it is efficient to turn to Eq. (5.4.12),

$$\dot{\boldsymbol{p}}(t) = \int \boldsymbol{j}(t, \boldsymbol{x})\, d^3x,$$

in which we substitute Eq. (5.5.3). We have

$$\dot{\boldsymbol{p}}(t) = I_0\hat{\boldsymbol{z}}\cos\omega t \int_{-\ell}^{\ell}\big(1 - |z|/\ell\big)\, dz,$$

and evaluating the integral gives

$$\dot{\boldsymbol{p}}(t) = I_0\ell\hat{\boldsymbol{z}}\cos\omega t.$$

Taking an additional derivative and making the substitution $t \to u = t - r/c$ yields

$$\ddot{\boldsymbol{p}}(u) = -I_0\omega\ell\hat{\boldsymbol{z}}\sin(\omega u),$$

or

$$\ddot{\boldsymbol{p}}(u) = -(I_0 c)(k\ell)\hat{\boldsymbol{z}}\sin(\omega u) \tag{5.5.5}$$

after using Eq. (5.5.2).

Since the vector $\ddot{\boldsymbol{p}}(u)$ is always aligned with the z axis, we have that $\Theta(u) = \theta$, where θ is the angle between the vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{z}}$. According to Eq. (5.4.22), therefore, the angular distribution of the power is given by

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{\mu_0}{(4\pi)^2 c}\big|\ddot{\boldsymbol{p}}(u)\big|^2 \sin^2\theta \\ &= \frac{\mu_0}{(4\pi)^2 c}(I_0 c)^2(k\ell)^2 \sin^2(\omega u)\sin^2\theta.\end{aligned}$$

After averaging over a complete wave cycle, this reduces to

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 c}{32\pi^2}\big(I_0 k\ell\big)^2 \sin^2\theta. \tag{5.5.6}$$

To obtain the total power radiated we must integrate over the angles. Using $\oint \sin^2\theta\, d\Omega = 8\pi/3$, we arrive at our final result

$$\langle P \rangle = \frac{\mu_0 c}{12\pi}\big(I_0 k\ell\big)^2. \tag{5.5.7}$$

For a fixed frequency ω, the power increases like the square of the feed current I_0. For a fixed current, the power increases like the square of the frequency, so long as the condition $k\ell \ll 1$ is satisfied. From Eq. (5.5.6) we learn that most of the energy is radiated in the directions perpendicular to the antenna; none of the energy propagates along the axis.

5.6 Classical atom

The classical picture of a hydrogen atom contains a proton at the centre and an electron on a circular orbit around it. Supposing for simplicity that the proton is at rest, the system's dipole moment is $\boldsymbol{p}(t) = -e\boldsymbol{a}(t)$ where $\boldsymbol{a}(t)$ is the electron's position vector. Since the electron is accelerated by the Lorentz force $\boldsymbol{F}$ exerted by the proton, we have $\ddot{\boldsymbol{p}}(t) = -e\ddot{\boldsymbol{a}} = -(e/m)\boldsymbol{F}$. Because this is nonzero, the atom emits electromagnetic radiation. The atom therefore loses energy to the radiation, and as a result, the orbital radius a gradually decreases; it eventually goes to zero. As is well known, the classical model leads to an unstable atom. In this section we calculate the lifetime of the classical atom.

The Coulomb force exerted by the proton is

$$\boldsymbol{F} = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{a^2}\hat{\boldsymbol{a}},$$

where $\hat{\boldsymbol{a}} = \boldsymbol{a}(t)/a$. This gives

$$\ddot{\boldsymbol{p}}(t) = \frac{1}{4\pi\epsilon_0}\frac{e}{m}\frac{e^2}{a^2}\hat{\boldsymbol{a}}(t) \tag{5.6.1}$$

for the second derivative of the dipole moment vector. According to the electric-dipole approximation, the total energy radiated by the atom per unit time is given by Eq. (5.4.21),

$$P = \frac{\mu_0}{6\pi c}\bigl|\ddot{\boldsymbol{p}}(u)\bigr|^2.$$

Substituting Eq. (5.6.1) then gives

$$P = \frac{\mu_0}{6\pi c}\frac{e^6}{(4\pi\epsilon_0)^2 m^2 c a^4}. \tag{5.6.2}$$

Energy conservation dictates that the energy carried away by the radiation must come at the expense of the atom's orbital energy; we therefore have

$$\frac{dE_{\rm orb}}{dt} = -P. \tag{5.6.3}$$

The orbital energy of a classical atom is

$$E_{\rm orb} = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{2a}. \tag{5.6.4}$$

Combining the last three equations, it is easy to show that the orbital radius must decrease, at a rate given by

$$\frac{da}{dt} = -\frac{4}{3}\frac{1}{(4\pi\epsilon_0)^2}\frac{e^2}{m^2c^3a^2}. \tag{5.6.5}$$

To obtain the lifetime we integrate Eq. (5.6.5), starting with the initial condition that at $t = 0$, the orbital radius is equal to Bohr's radius

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}.$$

(In this step we make a leap and momentarily leave the classical realm, since the Bohr radius is very much a consequence of quantum mechanics. But we may still think of a_0 as giving an appropriate initial condition, as it sets the atomic

length scale.) Integration of Eq. (5.6.5) is elementary, and we obtain $\tau \equiv \int_0^\tau dt = \int_{a_0}^0 (dt/da)\, da$ as

$$\tau = \frac{(4\pi\epsilon_0)^5 c^3 \hbar^6}{4me^{10}}.$$

To put this in friendlier terms we introduce

$$\alpha \equiv \frac{e^2}{(4\pi\epsilon_0)\hbar c} \simeq \frac{1}{137} \equiv \text{fine structure constant} \tag{5.6.6}$$

and

$$\lambda_{\rm C} \equiv \frac{2\pi\hbar}{mc} \simeq 2.43 \times 10^{-12}\ \text{m} \equiv \text{electron's Compton wavelength.} \tag{5.6.7}$$

The lifetime can then be expressed as

$$\tau = \frac{1}{8\pi\alpha^5}\frac{\lambda_{\rm C}}{c} \simeq 1.6 \times 10^{-11}\ \text{s.} \tag{5.6.8}$$

This result shows that under classical laws, the hydrogen atom would be extremely short lived. Equation (5.6.8), by the way, gives an order-of-magnitude estimate of the time required by an atom to make a transition from an excited state to the ground state.

The results of this section, especially Eq. (5.6.5), leave us with a curious paradox. Ignoring the fact that the classical laws of electrodynamics do not apply to a real hydrogen atom, we seem to find an inconsistency in our description of the classical atom. On the one hand, the fact that the electron feels the Coulomb field of the proton guarantees that the electron's motion will be circular and that its orbital radius will not change with time. (The most general orbit allowed by the Coulomb force is actually an ellipse, but the point remains that the size of this ellipse cannot change with time.) On the other hand, we know that the electron is accelerated, that it emits electromagnetic waves, that the radiation carries energy away from the system, and that the orbital radius cannot possibly stay constant. The paradox is therefore this: The electron's equations of motion do not account for the decrease in orbital radius; they unambiguously predict that the radius should be constant. Is classical electrodynamics internally inconsistent?

Later on, in Chapter 6, we will show that the electron's *radiation reaction* is actually contained in Maxwell's theory. The effect is there, but it is well hidden, and it is a subtle matter to reveal it.

5.7 Magnetic-dipole and electric-quadrupole radiation

[The material presented in this section is also covered in Sec. 9.3 of Jackson's text.]

Electric-dipole radiation corresponds to the leading-order approximation of the electromagnetic field in an expansion in powers of v_c/c, where v_c is a typical internal velocity of the source. In some cases, however, the dipole moment $\boldsymbol{p}$ either vanishes or does not depend on time, and the leading term is actually zero. In such cases, or when higher accuracy is required, we need to compute the next term in the expansion. This is our task in this section. We will see that at the next-to-leading-order, the wave-zone fields depend on the dipole moment vector $\boldsymbol{p}$, the magnetic moment vector $\boldsymbol{m}$, and the electric dipole moment tensor Q_{ab}. We recall the definitions of these objects:

$$\boldsymbol{p}(t) \;=\; \int_V \rho(t, \boldsymbol{x'})\boldsymbol{x'}\, d^3x', \tag{5.7.1}$$

$$\boldsymbol{m}(t) = \frac{1}{2}\int_V \boldsymbol{x'} \times \boldsymbol{j}(t, \boldsymbol{x'})\, d^3x', \tag{5.7.2}$$

$$Q_{ab}(t) = \int_V \rho(t, \boldsymbol{x'})\big(3x'_a x'_b - r'^2\delta_{ab}\big)\, d^3x', \tag{5.7.3}$$

where V is the volume that contains the charge and current distribution; this volume is bounded by the surface S.

In the foregoing manipulations we will introduce the vector $\boldsymbol{Q}(t, \boldsymbol{x})$ defined by

$$Q_a(t, \hat{\boldsymbol{r}}) = Q_{ab}(t) r_b, \tag{5.7.4}$$

where $r_b \equiv x_b/r$ are the components of the unit vector $\hat{\boldsymbol{r}} = \boldsymbol{x}/r$, and in which summation over the repeated index b is understood.

5.7.1 Wave-zone fields

To make our calculations more efficient we first derive a few useful results that follow from the fact that in this section, we place ourselves exclusively in the wave zone. So throughout this discussion we shall assume that

$$r \gg \frac{\lambda_c}{2\pi} = \frac{c}{\omega_c}. \tag{5.7.5}$$

Notice that this condition is *independent* of any assumption that concerns v_c: there is a wave zone whether or not the source is moving slowly. But if $v_c/c \ll 1$, then we also have that $r_c \ll \lambda_c/(2\pi) \ll r$, as was shown in Sec. 5.4.1. In this subsection we rely on the wave-zone condition of Eq. (5.7.5), but we do not rely on the slow-motion assumption; this is will be incorporated at a later stage.

We can anticipate that the wave-zone potentials will have the form of a spherical wave. For example, we shall verify that the vector potential is given by

$$\boldsymbol{A}(t, \boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{w}(u)}{r} \tag{5.7.6}$$

in the wave zone, where $u = t - r/c$ and $\boldsymbol{w}$ is a vector that will be determined. We will not need an expression for the scalar potential; as we shall see, in the wave zone $\boldsymbol{E}$ can be obtained directly from $\boldsymbol{B}$, which is determined by the vector potential.

Given the form of the vector potential, its curl is

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{\mu_0}{4\pi}\boldsymbol{\nabla} \times \frac{\boldsymbol{w}(t - r/c)}{r} = \frac{\mu_0}{4\pi}(\boldsymbol{\nabla} r) \times \frac{\partial}{\partial r}\frac{\boldsymbol{w}(t - r/c)}{r}.$$

The gradient of r is the vector $\hat{\boldsymbol{r}}$, and the derivative of $\boldsymbol{w}(u)/r$ with respect to r is $-c^{-1}\dot{\boldsymbol{w}}(u)/r$, up to terms of order $1/r^2$ that can be neglected in the wave zone. We therefore have

$$\boldsymbol{B}(t, \boldsymbol{x}) = -\frac{\mu_0}{4\pi}\frac{\hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}}(u)}{cr} \tag{5.7.7}$$

for the wave-zone magnetic field.

Later we will find that $\boldsymbol{w}$ is not just a function of retarded time u, but that it depends also on $\hat{\boldsymbol{r}}$: $\boldsymbol{w} = \boldsymbol{w}(u, \hat{\boldsymbol{r}})$. This does not affect our result for the magnetic field, nor any other result derived in this subsection. To see this, let us calculate more carefully the x derivative of the vector $\boldsymbol{w}$:

$$\nabla_x \boldsymbol{w} = \frac{\partial \boldsymbol{w}}{\partial u}\nabla_x u + \frac{\partial \boldsymbol{w}}{\partial \hat{\boldsymbol{r}}} \cdot \nabla_x \hat{\boldsymbol{r}}.$$

In the first term we substitute $\nabla_x u = -c^{-1}\nabla_x r = -c^{-1} r_x$ and we recover our previous expression. We then recognize that $\nabla_x \hat{\boldsymbol{r}}$ is of order $1/r$ and that the

second term can be neglected in the wave zone. We conclude that the dependence of $\boldsymbol{w}$ on $\hat{\boldsymbol{r}}$ is invisible to $\boldsymbol{\nabla}$ so long as all computations are carried out in the wave zone.

To get the electric field we go back to one of Maxwell's equations,

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t}.$$

Because the current density vanishes outside the volume V, we have that $\boldsymbol{j} = 0$ in the wave zone, and

$$\frac{\partial \boldsymbol{E}}{\partial t} = c^2 \boldsymbol{\nabla} \times \boldsymbol{B} = -\frac{\mu_0 c}{4\pi} \boldsymbol{\nabla} \times \left[\frac{\hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}}(u)}{r} \right].$$

By the same argument as the one just given, we can treat $\hat{\boldsymbol{r}}$ as a constant vector when evaluating the curl, and we can also pull the $1/r$ factor in front of the derivative operator. This yields

$$\boldsymbol{\nabla} \times \left[\frac{\hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}}(u)}{r} \right] = (\boldsymbol{\nabla} r) \times \left[\frac{\hat{\boldsymbol{r}}}{r} \times \frac{\partial \dot{\boldsymbol{w}}}{\partial r} \right] = -\frac{1}{c} \hat{\boldsymbol{r}} \times \left[\frac{\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{w}}(u)}{r} \right],$$

and we obtain

$$\frac{\partial \boldsymbol{E}}{\partial t} = \frac{\mu_0}{4\pi} \frac{\hat{\boldsymbol{r}} \times \left[\hat{\boldsymbol{r}} \times \ddot{\boldsymbol{w}}(u) \right]}{r}.$$

Integrating this with respect to t gives

$$\boldsymbol{E}(t, \boldsymbol{x}) = -\frac{\mu_0}{4\pi} \frac{\left[\hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}}(u) \right] \times \hat{\boldsymbol{r}}}{r} = c \boldsymbol{B}(t, \boldsymbol{x}) \times \hat{\boldsymbol{r}} \tag{5.7.8}$$

for the wave-zone electric field.

Using Eqs. (5.7.7) and (5.7.8) it is easy to show that the wave-zone Poynting vector is given by

$$\boldsymbol{S}(t, \boldsymbol{x}) = \frac{c}{\mu_0} |\boldsymbol{B}|^2 \hat{\boldsymbol{r}} = \frac{\mu_0}{16\pi^2 c} \frac{\left| \hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}}(u) \right|^2}{r^2} \hat{\boldsymbol{r}}. \tag{5.7.9}$$

Evaluating this on a sphere of constant r gives

$$dP = \boldsymbol{S} \cdot d\boldsymbol{a} = \frac{\mu_0}{16\pi^2 c} \left| \hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}}(u) \right|^2 d\Omega,$$

or

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c} \left| \hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}}(u) \right|^2, \tag{5.7.10}$$

where $d\Omega = \sin\theta \, d\theta d\phi$ is an element of solid angle. Equation (5.7.10) gives the radiation's angular distribution, and here it becomes important to specify the dependence of $\boldsymbol{w}$ on the radial vector $\hat{\boldsymbol{r}}$. Notice that the fields and the radiation's angular profile depend only on $\hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}}$, and that they are insensitive to an eventual component of $\boldsymbol{w}$ along $\hat{\boldsymbol{r}}$. We will use this observation later, and discard from $\boldsymbol{w}$ any longitudinal component.

From these results we infer, once more, that irrespective of the exact expression for the vector $\boldsymbol{w}(u, \hat{\boldsymbol{r}})$, the fields are both transverse (orthogonal to $\hat{\boldsymbol{r}}$) and mutually orthogonal, that their amplitudes differ by a factor of c, that they are in phase, and that the field energy travels in the radial direction. To be more concrete we shall have to calculate the vector $\boldsymbol{w}$.

5.7.2 Charge-conservation identities

Before we get to this we need to establish the useful identities

$$\int_V \boldsymbol{j}(t,\boldsymbol{x'})\,d^3x' = \dot{\boldsymbol{p}}(t) \tag{5.7.11}$$

and

$$\int_V \boldsymbol{j}(t,\boldsymbol{x'})(\hat{\boldsymbol{r}}\cdot\boldsymbol{x'})\,d^3x' = \boldsymbol{m}(t)\times\hat{\boldsymbol{r}} + \frac{1}{6}\dot{\boldsymbol{Q}}(t,\hat{\boldsymbol{r}}) + \frac{1}{6}\hat{\boldsymbol{r}}\frac{d}{dt}\int_V \rho(t,\boldsymbol{x'})r'^2\,d^3x', \tag{5.7.12}$$

where $\boldsymbol{p}(t)$ is the electric dipole moment, $\boldsymbol{m}(t)$ the magnetic dipole moment, and $\boldsymbol{Q}(t,\hat{\boldsymbol{r}})$ the vector related to the electric quadrupole moment which we introduced in Eq. (5.7.3).

Equation (5.7.11) was already derived near the end of Sec. 5.4.3; refer back to Eq. (5.4.12). Equation (5.7.12) is a generalization of Eq. (3.4.4), which covers static situations (and which was derived in Sec. 3.4), and we derive it using similar methods.

Consider the z component of Eq. (5.7.12). We write the left-hand side as

$$I_z = \boldsymbol{e}\cdot\int_V j_z(t,\boldsymbol{x})\boldsymbol{x}\,d^3x,$$

in which we have changed the integration variables from $\boldsymbol{x'}$ to $\boldsymbol{x}$ and replaced the constant vector $\hat{\boldsymbol{r}}$ by the arbitrary constant vector $\boldsymbol{e}$. More explicitly, I_z is given by

$$I_z = e_x\int_V xj_z\,d^3x + e_y\int_V yj_z\,d^3x + e_z\int_V zj_z\,d^3x.$$

Consider now the relation

$$\begin{aligned}\boldsymbol{\nabla}\cdot(xz\boldsymbol{j}) &= xz\boldsymbol{\nabla}\cdot\boldsymbol{j} + x\boldsymbol{j}\cdot\boldsymbol{\nabla}z + z\boldsymbol{j}\cdot\boldsymbol{\nabla}x\\ &= -xz\frac{\partial\rho}{\partial t} + xj_z + zj_x.\end{aligned}$$

Integrating both sides over V and using Gauss' theorem (together with the statement that $\boldsymbol{j}\cdot d\boldsymbol{a} = 0$ on the bounding surface S) gives

$$\int_V xj_z\,d^3x = -\int_V zj_x\,d^3x + \frac{d}{dt}\int_V \rho xz\,d^3x.$$

We obtain

$$\int_V yj_z\,d^3x = -\int_V zj_y\,d^3x + \frac{d}{dt}\int_V \rho yz\,d^3x$$

in a similar way. Consider now

$$\boldsymbol{\nabla}\cdot(z^2\boldsymbol{j}) = z^2\boldsymbol{\nabla}\cdot\boldsymbol{j} + 2z\boldsymbol{j}\cdot\boldsymbol{\nabla}z = -z^2\frac{\partial\rho}{\partial t} + 2zj_z;$$

integrating this over V leads to

$$\int_V zj_z\,d^3x = \frac{1}{2}\frac{d}{dt}\int_V \rho z^2\,d^3x.$$

Let us now substitute these results into our previous expression for I_z. We have

$$\begin{aligned}I_z &= \frac{1}{2}e_x\left[\int_V xj_z\,d^3x - \int_V zj_x\,d^3x + \frac{d}{dt}\int_V \rho xz\,d^3x\right]\\ &\quad + \frac{1}{2}e_y\left[\int_V yj_z\,d^3x - \int_V zj_y\,d^3x + \frac{d}{dt}\int_V \rho yz\,d^3x\right] + \frac{1}{2}e_z\frac{d}{dt}\int_V \rho z^2\,d^3x\\ &= -\frac{1}{2}e_x\int_V(\boldsymbol{x}\times\boldsymbol{j})_y\,d^3x + \frac{1}{2}e_y\int_V(\boldsymbol{x}\times\boldsymbol{j})_x\,d^3x\\ &\quad + \frac{1}{2}\frac{d}{dt}\left[e_x\int_V \rho xz\,d^3x + e_y\int_V \rho yz\,d^3x + e_z\int_V \rho z^2\,d^3x\right].\end{aligned}$$

In this last expression, the first two terms are equal to $-e_x m_y + e_y m_x = (\boldsymbol{m} \times \boldsymbol{e})_z$. The bracketed terms are related to Q_z: Keeping in mind that we use $\boldsymbol{e}$ as a substitute for $\hat{\boldsymbol{r}}$, we have

$$\begin{aligned} Q_z(t, \boldsymbol{e}) &= Q_{zx} e_x + Q_{zy} e_y + Q_{zz} e_z \\ &= e_x \int_V \rho(3xz)\, d^3x + e_y \int_V \rho(3yz)\, d^3x + e_z \int_V \rho(3z^2 - r^2)\, d^3x \\ &= 3[\ ] - e_z \int_V \rho r^2\, d^3x, \end{aligned}$$

where [ ] stands for the bracketed terms in our previous expression for I_z. Using these new results we finally obtain

$$I_z = (\boldsymbol{m} \times \boldsymbol{e})_z + \frac{1}{6} \frac{d}{dt} \left(Q_z + e_z \int_V \rho r^2\, d^3x \right),$$

which is the same statement (after the substitutions $\boldsymbol{x} \to \boldsymbol{x'}$ and $\boldsymbol{e} \to \hat{\boldsymbol{r}}$) as Eq. (5.7.12).

5.7.3 Vector potential in the wave zone

We are now ready to calculate the vector $\boldsymbol{w}$. We begin with the exact expression for the vector potential,

$$\boldsymbol{A}(t, \boldsymbol{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{j}(t - |\boldsymbol{x} - \boldsymbol{x'}|/c, \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|}\, d^3x'.$$

In this we substitute the approximations

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{x'}|} \simeq \frac{1}{r}$$

and

$$t - |\boldsymbol{x} - \boldsymbol{x'}|/c \simeq u + \hat{\boldsymbol{r}} \cdot \boldsymbol{x'}/c,$$

where $u = t - r/c$ is retarded time, which are appropriate in the wave zone. This gives

$$\boldsymbol{A}(t, \boldsymbol{x}) = \frac{\mu_0}{4\pi} \frac{1}{r} \int_V \boldsymbol{j}\big(u + \hat{\boldsymbol{r}} \cdot \boldsymbol{x'}/c, \boldsymbol{x'}\big)\, d^3x' \tag{5.7.13}$$

for vector potential. Notice that this expression, though restricted to the wave zone, is otherwise general: it is not yet limited by a slow-motion assumption.

We now incorporate the slow-motion approximation by Taylor-expanding the current density:

$$\boldsymbol{j}(u + \hat{\boldsymbol{r}} \cdot \boldsymbol{x'}/c) = \boldsymbol{j}(u) + \frac{1}{c}(\hat{\boldsymbol{r}} \cdot \boldsymbol{x'}) \frac{\partial \boldsymbol{j}}{\partial u}(u) + \cdots.$$

Relative to the leading term, the second term is smaller by a factor of order $c^{-1} r_c/t_c \equiv v_c/c$. This term was neglected in our discussion of electric-dipole radiation, but we shall now keep it. We will, however, neglect all remaining terms, which contribute at order $(v_c/c)^2$.

Substituting the expansion for $\boldsymbol{j}$ into Eq. (5.7.13) gives

$$\boldsymbol{A}(t, \boldsymbol{x}) = \frac{\mu_0}{4\pi} \frac{1}{r} \left[\int_V \boldsymbol{j}(u, \boldsymbol{x'})\, d^3x' + \frac{1}{c} \frac{\partial}{\partial u} \int_V \boldsymbol{j}(u, \boldsymbol{x'})(\hat{\boldsymbol{r}} \cdot \boldsymbol{x'})\, d^3x' + \cdots \right].$$

According to Eq. (5.7.11), the first integral is equal to $\dot{\boldsymbol{p}}(u)$. And according to Eq. (5.7.12), the second integral is equal to

$$\boldsymbol{m}(u) \times \hat{\boldsymbol{r}} + \frac{1}{6}\dot{\boldsymbol{Q}}(u, \hat{\boldsymbol{r}}) + \frac{1}{6}\hat{\boldsymbol{r}}\frac{d}{du}\int_V \rho(u, \boldsymbol{x'})r'^2\, d^3x'.$$

The quantity within the square brackets defines the vector $\boldsymbol{w}$, and we have

$$\boldsymbol{w} = \dot{\boldsymbol{p}}(u) + \frac{1}{c}\left[\dot{\boldsymbol{m}}(u) \times \hat{\boldsymbol{r}} + \frac{1}{6}\ddot{\boldsymbol{Q}}(u, \hat{\boldsymbol{r}})\right] + \frac{1}{6c}\hat{\boldsymbol{r}}\frac{d^2}{du^2}\int_V \rho(u, \boldsymbol{x'})r'^2\, d^3x'.$$

The last term is proportional to $\hat{\boldsymbol{r}}$. But we have seen in Sec. 5.7.1 that the electric and magnetic fields, as well as the radiated power, depend only on $\hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}}$. This means that any component of $\boldsymbol{w}$ along $\hat{\boldsymbol{r}}$ will not contribute to the wave-zone fields. We can therefore discard this last term in our expression for $\boldsymbol{w}$.

We have obtained

$$\boldsymbol{w}(u, \hat{\boldsymbol{r}}) = \dot{\boldsymbol{p}}(u) + \frac{1}{c}\left[\dot{\boldsymbol{m}}(u) \times \hat{\boldsymbol{r}} + \frac{1}{6}\ddot{\boldsymbol{Q}}(u, \hat{\boldsymbol{r}})\right], \tag{5.7.14}$$

and in terms of this the wave-zone vector potential is given by

$$\boldsymbol{A}(t, \boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{w}(u, \hat{\boldsymbol{r}})}{r}, \tag{5.7.15}$$

an expression that was already anticipated in Eq. (5.7.3). The first term on the right-hand side of Eq. (5.7.14) gives rise to electric-dipole radiation; this is the leading term in the expansion of the fields in powers of v_c/c. The second and third terms give rise to magnetic-dipole radiation and electric-quadrupole radiation, respectively; these contribute at order v_c/c beyond the leading term.

To calculate the radiated power we shall need

$$\hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}} = \hat{\boldsymbol{r}} \times \ddot{\boldsymbol{p}}(u) + \frac{1}{c}\Big\{\ddot{\boldsymbol{m}}(u) - \big[\hat{\boldsymbol{r}} \cdot \ddot{\boldsymbol{m}}(u)\big]\hat{\boldsymbol{r}}\Big\} + \frac{1}{6c}\hat{\boldsymbol{r}} \times \boldsymbol{Q}^{(3)}(u, \hat{\boldsymbol{r}}), \tag{5.7.16}$$

where $\boldsymbol{Q}^{(3)} \equiv \partial^3\boldsymbol{Q}/\partial u^3$ and we have used the vectorial identity $\hat{\boldsymbol{r}} \times (\ddot{\boldsymbol{m}} \times \hat{\boldsymbol{r}}) = \ddot{\boldsymbol{m}} - (\hat{\boldsymbol{r}} \cdot \ddot{\boldsymbol{m}})\hat{\boldsymbol{r}}$.

5.7.4 Radiated power (magnetic-dipole radiation)

Let us calculate the power radiated by a distribution of charges and currents for which

$$\ddot{\boldsymbol{p}}(u) = \boldsymbol{Q}^{(3)}(u, \hat{\boldsymbol{r}}) = 0. \tag{5.7.17}$$

Such a source will emit magnetic-dipole radiation.

Under the conditions of Eq. (5.7.17), Eq. (5.7.16) becomes

$$\hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}} = \frac{1}{c}\Big[\ddot{\boldsymbol{m}} - \big(\hat{\boldsymbol{r}} \cdot \ddot{\boldsymbol{m}}\big)\hat{\boldsymbol{r}}\Big]$$

so that

$$\big|\hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}}\big|^2 = \frac{1}{c^2}\Big[|\ddot{\boldsymbol{m}}|^2 - (\hat{\boldsymbol{r}} \cdot \ddot{\boldsymbol{m}})^2\Big],$$

which we rewrite in component form as

$$\big|\hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}}\big|^2 = \frac{1}{c^2}\ddot{m}_a\ddot{m}_b\big(\delta_{ab} - r_a r_b\big),$$

where $r_a = x_a/r$ are the components of the vector $\hat{\boldsymbol{r}}$, and where summation over both repeated indices is understood. Substituting this into Eq. (5.7.10) gives

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c^3}\ddot{m}_a\ddot{m}_b\big(\delta_{ab} - r_a r_b\big),$$

and integrating over the angles yields

$$P = \frac{\mu_0}{4\pi c^3}\ddot{m}_a\ddot{m}_b\big(\delta_{ab} - \langle r_a r_b\rangle\big),$$

where

$$\langle r_a r_b\rangle \equiv \frac{1}{4\pi}\int r_a r_b\, d\Omega = \frac{1}{3}\delta_{ab}.$$

These angular integrations are worked out in Sec. 5.7.7.

Our final result for the radiated power is therefore

$$P_{\text{magnetic-dipole}} = \frac{\mu_0}{6\pi c^3}\big|\ddot{\boldsymbol{m}}(u)\big|^2. \tag{5.7.18}$$

We might compare this result with Eq. (5.4.21),

$$P_{\text{electric-dipole}} = \frac{\mu_0}{6\pi c}\big|\ddot{\boldsymbol{p}}(u)\big|^2.$$

In orders of magnitude we have that the electric dipole moment is $p \sim (\rho r_c)(r_c^3) \sim \rho r_c^4$ and that $\ddot{p} \sim \rho r_c^4/t_c^2$. On the other hand, $m \sim (jr_c)(r_c^3) \sim jr_c^4 \sim \rho v_c r_c^4$, and $\ddot{m} \sim \rho v_c r_c^4/t_c^2$. The ratio of powers is then

$$\frac{P_{\text{magnetic-dipole}}}{P_{\text{electric-dipole}}} \sim \left(\frac{\ddot{m}}{c\ddot{p}}\right)^2 \sim \left(\frac{v_c}{c}\right)^2.$$

So for slowing-moving sources, the power emitted in magnetic-dipole radiation is typically smaller than the power emitted in electric-dipole radiation by a factor of order $(v_c/c)^2 \ll 1$.

5.7.5 Radiated power (electric-quadrupole radiation)

Let us now calculate the power radiated by a distribution of charges and currents for which

$$\ddot{\boldsymbol{p}}(u) = \ddot{\boldsymbol{m}}(u) = 0. \tag{5.7.19}$$

Such a source will emit electric-quadrupole radiation.

Under the conditions of Eq. (5.7.19), Eq. (5.7.16) becomes

$$\hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}} = \frac{1}{6c}\hat{\boldsymbol{r}} \times \boldsymbol{Q}^{(3)},$$

so that

$$\big|\hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}}\big|^2 = \frac{1}{36c^2}\big(\hat{\boldsymbol{r}} \times \boldsymbol{Q}^{(3)}\big)\cdot\big(\hat{\boldsymbol{r}} \times \boldsymbol{Q}^{(3)}\big).$$

The inner product can be expressed as

$$(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{r}})\big(\boldsymbol{Q}^{(3)}\cdot\boldsymbol{Q}^{(3)}\big) - \big(\hat{\boldsymbol{r}}\cdot\boldsymbol{Q}^{(3)}\big)\big(\hat{\boldsymbol{r}}\cdot\boldsymbol{Q}^{(3)}\big) = \big|\boldsymbol{Q}^{(3)}\big|^2 - \big(\hat{\boldsymbol{r}}\cdot\boldsymbol{Q}^{(3)}\big)^2,$$

and we have

$$\big|\hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}}\big|^2 = \frac{1}{36c^2}Q_a^{(3)}Q_b^{(3)}\big(\delta_{ab} - r_a r_b\big).$$

But the vector $\boldsymbol{Q}^{(3)}$ depends on $\hat{\boldsymbol{r}}$, because by the definition of Eq. (5.7.4), $Q_a = Q_{ac}r_c$. Writing also $Q_b = Q_{bd}r_d$, we arrive at

$$\big|\hat{\boldsymbol{r}} \times \dot{\boldsymbol{w}}\big|^2 = \frac{1}{36c^2}Q_{ac}^{(3)}Q_{bd}^{(3)}\big(\delta_{ab}r_c r_d - r_a r_b r_c r_d\big).$$

Substituting this into Eq. (5.7.10) gives

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c}\frac{1}{36c^2}Q_{ac}^{(3)}Q_{bd}^{(3)}\big(\delta_{ab}r_c r_d - r_a r_b r_c r_d\big),$$

and integrating over the angles yields

$$P = \frac{\mu_0}{4\pi c}\frac{1}{36c^2}Q^{(3)}_{ac}Q^{(3)}_{bd}\big(\delta_{ab}\langle r_c r_d\rangle - \langle r_a r_b r_c r_d\rangle\big).$$

The angular integrals

$$\langle r_c r_d\rangle \equiv \frac{1}{4\pi}\int r_c r_d\, d\Omega = \frac{1}{3}\delta_{cd}$$

and

$$\langle r_a r_b r_c r_d\rangle \equiv \frac{1}{4\pi}\int r_a r_b r_c r_d\, d\Omega = \frac{1}{15}\Big(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\Big)$$

are evaluated in Sec. 5.7.7. Taking these results into account, we obtain

$$\begin{aligned}
\Gamma &\equiv Q^{(3)}_{ac}Q^{(3)}_{bd}\big(\delta_{ab}\langle r_c r_d\rangle - \langle r_a r_b r_c r_d\rangle\big)\\
&= \frac{1}{3}Q^{(3)}_{ac}Q^{(3)}_{bd}\delta_{ab}\delta_{cd} - \frac{1}{15}Q^{(3)}_{ac}Q^{(3)}_{bd}\big(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\big)\\
&= \frac{1}{3}Q^{(3)}_{ac}Q^{(3)}_{ac} - \frac{1}{15}Q^{(3)}_{ac}Q^{(3)}_{ac} - \frac{1}{15}Q^{(3)}_{aa}Q^{(3)}_{bb} - \frac{1}{15}Q^{(3)}_{ac}Q^{(3)}_{ca}\\
&= \left(\frac{1}{3} - \frac{2}{15}\right)Q^{(3)}_{ac}Q^{(3)}_{ac}\\
&= \frac{1}{5}Q^{(3)}_{ac}Q^{(3)}_{ac},
\end{aligned}$$

where we have used the property that Q_{ab} is a symmetric, tracefree tensor (so that $Q_{ca} = Q_{ac}$ and $Q_{aa} = 0$).

Our final result for the radiated power is therefore

$$P_{\text{electric-quadrupole}} = \frac{\mu_0}{720\pi c^3}Q^{(3)}_{ab}(u)Q^{(3)}_{ab}(u), \tag{5.7.20}$$

where summation over the two repeated indices is understood. In orders of magnitude, $Q_{ab} \sim (\rho r_c^2)(r_c^3) \sim \rho r_c^5$, and $Q^{(3)}_{ab} \sim \rho r_c^5/t_c^3 \sim \ddot{p} r_c/t_c \sim \ddot{p} v_c$. We therefore have

$$\frac{P_{\text{electric-quadrupole}}}{P_{\text{electric-dipole}}} \sim \left(\frac{v_c}{c}\right)^2;$$

for slowing-moving distributions, the power emitted in electric-quadrupole radiation is smaller than the power emitted in electric-dipole radiation by a factor of order $(v_c/c)^2 \ll 1$. We see also that electric-quadrupole radiation is of the same order of magnitude as magnetic-dipole radiation.

5.7.6 Total radiated power

In situations in which all three types of radiation contribute, the *total power* is the sum of individual contributions,

$$P = \frac{\mu_0}{6\pi c}\left[\big|\ddot{\boldsymbol{p}}(u)\big|^2 + \frac{1}{c^2}\big|\ddot{\boldsymbol{m}}(u)\big|^2 + \frac{1}{120c^2}Q^{(3)}_{ab}(u)Q^{(3)}_{ab}(u) + O\big(v_c^4/c^4\big)\right]. \tag{5.7.21}$$

There is no interference between the different terms in Eq. (5.7.16), because the angular profiles associated with each term are orthogonal.

5.7.7 Angular integrations

In this subsection we establish the formulae

$$\langle r_a \rangle = 0, \tag{5.7.22}$$

$$\langle r_a r_b \rangle = \frac{1}{3}\delta_{ab}, \tag{5.7.23}$$

$$\langle r_a r_b r_c \rangle = 0, \tag{5.7.24}$$

$$\langle r_a r_b r_c r_d \rangle = \frac{1}{15}\Big(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\Big), \tag{5.7.25}$$

where $r_a = x_a/r$ are the components of the vector $\hat{\boldsymbol{r}}$, and $\langle \cdots \rangle \equiv (4\pi)^{-1}\int(\cdots)\,d\Omega$ is the average of the quantity $(\cdots)$ over a sphere of constant r.

These relations can be derived by brute-force computation, by simply substituting $\hat{\boldsymbol{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and integrating over $d\Omega = \sin\theta\, d\theta d\phi$. But there is an easier, prettier way that relies on arguments of symmetry.

Take the zero result for $\langle r_a \rangle$. If these integrals were not zero, they would have to be equal to some vector $\boldsymbol{a}$ that does not depend on the angles θ and ϕ (because they have been integrated over) nor on r (because r appears nowhere on the left-hand side of the equation). But there is no such vector available, because there is no preferred direction in a homogeneous three-dimensional space. The right-hand side must therefore be zero.

To arrive at the result for $\langle r_a r_b \rangle$ we observe that the right-hand side must be a constant tensor that is symmetric in the pair of indices a and b. The only such tensor available is δ_{ab}, and we must therefore have $\langle r_a r_b \rangle = \lambda\delta_{ab}$, for some constant λ. To determine λ we simply take the trace of the left-hand side,

$$\delta_{ab}\langle r_a r_b \rangle = \langle \delta_{ab} r_a r_b \rangle = \langle 1 \rangle = 1,$$

and the trace of the right-hand side,

$$\lambda\delta_{ab}\delta_{ab} = 3\lambda.$$

Equating the results produces $\lambda = 1/3$, and this agrees with the result of Eq. (5.7.23).

The zero result for $\langle r_a r_b r_c \rangle$ follows from the fact that we do not have a vector that could be combined with δ_{ab} to form a symmetric, three-index tensor.

To derive the result for $\langle r_a r_b r_c r_d \rangle$ we note that the right-hand side must be a four-index tensor that is symmetric in all pairs of indices. We must form this tensor with two copies of δ_{ab}, and it is easy to see that it must be equal to $\lambda'(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})$. To determine λ' we evaluate the double trace of the left-hand side, which gives 1, and the double trace of the right-hand side, which gives

$$\lambda'\delta_{ab}\delta_{cd}\big(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\big) = \lambda'(3\times 3 + 3 + 3) = 15\lambda'.$$

Equating the results produces $\lambda' = 1/15$, in agreement with Eq. (5.7.24).

5.8 Pulsar spin-down

As an application of magnetic-dipole radiation we consider the *oblique-rotator model* of a *pulsar*, a rotating neutron star that emits pulses of electromagnetic radiation at regular intervals. The most famous pulsar is located in the Crab nebula, and the pulsar is believed to be the source of energy for this supernova remnant.

The Crab pulsar is observed to spin down: the frequency of the pulses decreases with time. If we associate the pulse frequency with the neutron star's spin frequency,

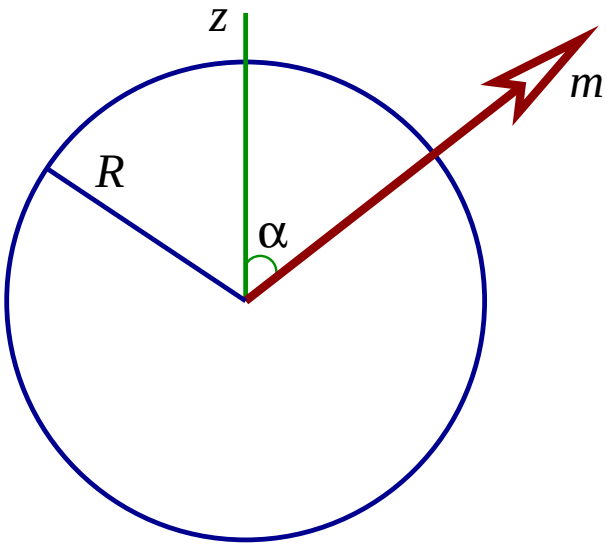

Figure 5.4: The neutron star rotates with angular velocity Ω around the z axis. The magnetic moment $\boldsymbol{m}$ makes an angle of α with respect to this axis, and it rotates along with the star.

then we can interpret this phenomenon in terms of the pulsar losing *rotational energy*, which is given by

$$E_{\rm rot} = \frac{1}{2} I \Omega^2, \tag{5.8.1}$$

where I is the star's moment of inertia and Ω its angular velocity. If P denotes the pulses period (which is not to be confused with the power radiated by electromagnetic waves), then $\Omega = 2\pi/P$. If the neutron star is modeled as a solid sphere of uniform mass density, then $I = \frac{2}{5} M R^2$, where M is the pulsar's mass and R its radius. The loss of rotational energy translates into a decrease of Ω, or an increase of P:

$$\dot{E}_{\rm rot} = I \Omega \dot{\Omega} = -(2\pi)^2 I \frac{\dot{P}}{P^3}. \tag{5.8.2}$$

For the Crab we have the observational values $\dot{P} \simeq 4 \times 10^{-13}$ s/s and $P \simeq 0.03$ s. If we take the neutron star to have a mass of 1.4 solar masses and a radius of 12 km, then the rate of loss of rotational energy is

$$\dot{E}_{\rm rot} \simeq -7 \times 10^{31} \text{ J/s}.$$

This is comparable to what is required to power the Crab nebula. The energetics of the Crab can therefore be explained by the pulsar losing its rotational energy.

What is responsible for this loss? We can imagine that the neutron star is carrying a magnetic field, and that the spinning motion of this field produces magnetic-dipole radiation. The energy carried off by the radiation will then come at the expense of the star's rotational energy. For this model to work we need the field's orientation to differ from the star's rotational axis — otherwise we would have a stationary situation analogous to the one analyzed in Sec. 3.3. This is the oblique-rotator model depicted in Fig. 5.4.

We imagine that the neutron star maintains a magnetic dipole moment $\boldsymbol{m}$ that is oriented at an angle α with respect to the rotation axis. This vector rotates along with the star, also with an angular velocity Ω. If the rotation axis is aligned with the z direction, we can write

$$\boldsymbol{m}(t) = m_0 \big(\sin\alpha \cos\Omega t \, \hat{\boldsymbol{x}} + \sin\alpha \sin\Omega t \, \hat{\boldsymbol{y}} + \cos\alpha \, \hat{\boldsymbol{z}} \big), \tag{5.8.3}$$

where m_0 is the magnitude of the pulsar's magnetic moment. This can be related to the magnitude of the magnetic field on the surface of the neutron star. Because the star is located well within the near zone, we can use Eq. (3.4.2) to describe the surface field:

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}}{R^3},$$

where R is the star's radius. The field is maximum at the magnetic pole, where $\hat{\boldsymbol{r}}$ is aligned with $\boldsymbol{m}$. We therefore have

$$B_{\max} = \frac{\mu_0}{4\pi}\frac{2m_0}{R^3}, \tag{5.8.4}$$

a relationship between m_0, the star's radius, and the maximum surface magnetic field.

The time-changing magnetic moment of Eq. (5.8.3) produces magnetic-dipole radiation. This radiation takes energy away from the star, at a rate given by Eq. (5.7.18),

$$\dot{E}_{\text{rad}} = \frac{\mu_0}{6\pi c^3}\big|\ddot{\boldsymbol{m}}(u)\big|^2,$$

where $u = t - r/c$ is retarded time. Substituting Eq. (5.8.3) gives

$$\dot{E}_{\text{rad}} = \frac{\mu_0}{6\pi c^3}\big(m_0\Omega^2\sin\alpha\big)^2, \tag{5.8.5}$$

or

$$\dot{E}_{\text{rad}} = \frac{2\pi}{3\mu_0 c^3}\big(B_{\max}\Omega^2 R^3\sin\alpha\big)^2 \tag{5.8.6}$$

after involving Eq. (5.8.4).

To produce an equality between $\dot{E}_{\text{rad}}$ and $|\dot{E}_{\text{rot}}| \simeq 7\times 10^{31}$ J/s, we need a magnetic field such that

$$B_{\max}\sin\alpha \simeq 5\times 10^8 \text{ T}.$$

This is very large, but not unreasonable. A main-sequence star typically supports a magnetic field of 0.1 T. When the star collapses to form a neutron star, the magnetic field is frozen in and the magnetic flux $4\pi R^2 B$ is conserved. Because R decreases by a factor of 10^5 during the collapse, B increases by a factor of 10^{10}, and fields of the order of 10^8 T can easily be achieved for neutron stars.

5.9 Problems

1. A point electric dipole is rotating around the z axis at an angular velocity ω, so that

   $$\boldsymbol{p}(t) = p_0\big[\cos(\omega t)\,\hat{\boldsymbol{x}} + \sin(\omega t)\,\hat{\boldsymbol{y}}\big],$$

   where p_0 is a constant.

   a) Calculate the wave-zone electric and magnetic fields of this rotating dipole. Express the fields in terms of the unit vectors $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$.

   b) Calculate $\langle dP/d\Omega\rangle$, the angular profile of the radiated power, averaged over a complete wave cycle.

   c) Calculate the total radiated power. How does it compare with the power radiated by an *oscillating* dipole?

2. Two equal charges q are moving with constant angular velocity Ω on a circle of radius b. The charges are situated at diametrically opposite points on the circle. Calculate, to leading order in a slow-motion approximation, the total power radiated by this distribution of charges.

3. The current density inside a centre-fed, linear antenna is given by

$$\boldsymbol{j}(t,\boldsymbol{x}) = I\cos(\omega t)\sin(k\ell - k|z|)\delta(x)\delta(y)\,\hat{\boldsymbol{z}},$$

where I is the current's peak amplitude, ω the oscillation frequency, ℓ the antenna's half-length, and $k = \omega/c$. In this problem we do *not* impose the slow-motion approximation, so that $k\ell$ is *not* assumed to be small.

a) Show that in the wave zone ($kr \gg 1$), the vector potential can be expressed as

$$\boldsymbol{A}(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi}\,\frac{\boldsymbol{w}(u,\theta)}{r},$$

and find an expression for the vector $\boldsymbol{w}(u,\theta)$. Here, $u = t - r/c$ is retarded time and θ is the angle between $\boldsymbol{x}$ and the z axis.

b) Calculate $\langle dP/d\Omega\rangle$, the angular profile of the radiated power, averaged over a complete wave cycle. For the two special cases $k\ell = \pi/2$ (half-wave antenna) and $k\ell = \pi$ (full-wave antenna), provide parametric plots of this angular profile (like those displayed in Jackson's Figure 9.7). Compare these profiles with the $\sin^2\theta$ dependence obtained for electric-dipole radiation.

c) From the general expression for $\langle dP/d\Omega\rangle$ derived in part **b)**, obtain the short-antenna limit $k\ell \ll 1$. Make sure that your result matches the one derived in the text.

4. An oscillating current $I(t) = I_0 e^{-\gamma t}\sin(\omega t)$ flows in a circular loop of radius a placed in the equatorial plane of the coordinate system (r,θ,ϕ). The current is started at $t = 0$, and it decays exponentially due to dissipation within the loop; we assume that $\gamma \ll \omega$, so that the time scale for dissipation is much longer than the oscillation period. The current density is given by

$$\boldsymbol{j}(t,\boldsymbol{x}) = \frac{I(t)}{a}\,\delta(r-a)\delta(\theta - \pi/2)\,\hat{\boldsymbol{\phi}}.$$

Calculate $\langle P\rangle$, the total power radiated by the loop, averaged over an oscillation period. Calculate also the total amount of energy radiated by the loop.

5. A nearly spherical surface described by

$$r = R(t,\theta) \equiv a(1 + \epsilon\cos\theta\cos\omega t),$$

where a, ω, and ϵ are constants, supports a nearly uniform distribution of charge. It is assumed that $\epsilon \ll 1$, and the charge density is given by

$$\rho(t,\boldsymbol{x}) = \frac{q}{4\pi a^2}\delta\big(r - R(t,\theta)\big),$$

where q is the total charge.

Calculate $\langle P\rangle$, the electromagnetic power emitted by this distribution of charge, averaged over a complete wave cycle. You may work to leading order in the slow-motion approximation. You may also work to leading order in ϵ.

6. A ring of radius b is placed in the x-y plane, and the origin of the coordinate system is placed at the centre of the ring. Initially, the ring is at rest and it supports a linear charge density proportional to $\sin\phi$, where ϕ is the angle from the x-axis. The ring is then made to rotate with a uniform angular velocity Ω, so that the linear charge density becomes proportional to $\sin(\phi - \Omega t)$. The ring's volume charge density is described by

$$\rho(t,r,\theta,\phi) = \frac{\lambda}{b}\delta(r-b)\delta(\theta - \tfrac{\pi}{2})\sin(\phi - \Omega t).$$

(a) Calculate, to leading order in a slow-motion approximation, the total power radiated by the ring. At what angle θ is most of the radiation emitted?

(b) Find an exact expression for $\Phi(t, \mathbf{0})$, the scalar potential at the centre of the ring.

(c) Find an exact expression for $\boldsymbol{A}(t, \mathbf{0})$, the vector potential at the centre of the ring.

7. Three charges are located along the z axis: a charge $+2q$ stays at the origin $z = 0$, a charge $-q$ is at $z = +a\cos\omega t$, and the final charge $-q$ is at $z = -a\cos\omega t$; a and ω are constants. It is assumed that $a\omega \ll 2\pi c$ and that the charges simply go through one another when they meet at $z = 0$.

   For this situation calculate $\langle dP/d\Omega\rangle$, the angular profile of the radiated power, averaged over a complete wave cycle. (Express this in terms of the angle θ from the z axis.) Calculate also $\langle P\rangle$, the total radiated power. What is the frequency of the electromagnetic waves emitted by this system of charges?

8. A current I is suddenly established in an infinite wire that extends along the z axis. The current density is given by

$$\boldsymbol{j}(t, \boldsymbol{x}) = I\theta(t)\delta(x)\delta(y)\hat{\boldsymbol{z}},$$

   where $\theta(t)$ is the step function. Because $\boldsymbol{\nabla}\cdot\boldsymbol{j} = 0$, it is consistent to set $\rho(t, \boldsymbol{x}) = 0$.

   Calculate, without approximations, the vector potential $\boldsymbol{A}(t, \boldsymbol{x})$ associated with this distribution of current. From your answer you should be able to verify that at late times, the electric and magnetic fields that are produced by the distribution of current have the components

$$E_z \simeq -\frac{\mu_0}{4\pi}\frac{2I}{t}, \qquad B_x \simeq -\frac{\mu_0}{4\pi}\frac{2Iy}{x^2+y^2}, \qquad B_y \simeq \frac{\mu_0}{4\pi}\frac{2Ix}{x^2+y^2}.$$

   The electric field eventually disappears, and the magnetic field settles down to the value it would have for an unchanging current.

Chapter 6
Electrodynamics of point charges

6.1 Lorentz transformations

[The material presented in this section is also covered in Secs. 11.3, 11.9, and 11.10 of Jackson's text.]

In this final chapter we will look at the electrodynamics of charge and current distributions for which the characteristic speed v_c is close to the speed of light. To keep the discussion concrete we will restrict it to the electromagnetic field produced by a single point particle in arbitrary (relativistic) motion. To set the stage we begin with a review of the Lorentz transformations of special relativity.

We consider a reference frame S' that moves with constant speed v relative to another frame S; we take the motion to take place in the x direction, so that the frame's velocity vector is $\boldsymbol{v} = v\hat{\boldsymbol{x}}$. The coordinates (t', x', y', z') of an event seen in S' are related to the coordinates (t, x, y, z) of the same event seen in S by the well-known relations

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right), \tag{6.1.1}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{6.1.2}$$

These are the *Lorentz transformations* between the two reference frames. The reversed transformation is obtained by reversing the sign of v:

$$x = \gamma(x' + vt'), \qquad y = y', \qquad z = z', \qquad t = \gamma\left(t' + \frac{v}{c^2}x'\right). \tag{6.1.3}$$

We now would like to derive how the electric and magnetic fields transform under such a change of reference frame.

We note first that Lorentz transformations alter the differential operators. For example, applying the chain rule on an arbitrary function ψ gives

$$\frac{\partial\psi}{\partial t'} = \frac{\partial\psi}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial t'} + \frac{\partial\psi}{\partial z}\frac{\partial z}{\partial t'},$$

or

$$\frac{\partial\psi}{\partial t'} = \gamma\left(\frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial x}\right)$$

after using Eq. (6.1.3). Doing a similar calculation for all other partial derivatives, we obtain

$$\frac{\partial}{\partial x'} = \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right), \qquad \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial t'} = \gamma\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right). \tag{6.1.4}$$

It follows from this that the wave operator is a Lorentz invariant:

$$\begin{aligned}
\Box'\psi &= -\frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2} + \frac{\partial^2\psi}{\partial x'^2} + \frac{\partial^2\psi}{\partial y'^2} + \frac{\partial^2\psi}{\partial z'^2} \\
&= -\frac{1}{c^2}\gamma^2\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\psi \\
&\quad + \gamma^2\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\psi + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \\
&= -\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \\
&= \Box\psi.
\end{aligned}$$

In other words,

$$\Box' = \Box. \tag{6.1.5}$$

To figure out how the electric and magnetic fields transform under a change of reference frame, we first work out how ρ and $\boldsymbol{j}$ should transform. This will tell us how the potentials Φ and $\boldsymbol{A}$ transform, and from this we will be able to get to the fields.

It is sufficient to consider a simple special case: a point charge q moving in the x direction with a speed v (the same speed that enters in the Lorentz transformation). As seen from S we have $\rho = q\delta(x - vt)\delta(y)\delta(z)$, $j_x = qv\delta(x - vt)\delta(y)\delta(z)$, and $j_y = j_z = 0$. As seen from S', however, the particle is not moving, and $\rho' = q\delta(x')\delta(y')\delta(z')$, while $j'_x = j'_y = j'_z = 0$. We can expect that the Lorentz transformation will involve a linear mixture of ρ and j_x, and that the other components of the current density will be left alone. Rewriting ρ' in terms of the unprimed coordinates gives

$$\begin{aligned}
\rho' &= q\delta[\gamma(x + vt)]\delta(y)\delta(z) \\
&= \frac{q}{\gamma}\delta(x - vt)\delta(y)\delta(z).
\end{aligned}$$

This expression for ρ' involves a factor of $1/\gamma$ instead of the expected γ, and j_x has not yet made an appearance. But we can write it in the equivalent form

$$\begin{aligned}
\rho' &= q\gamma(1 - v^2/c^2)\delta(x - vt)\delta(y)\delta(z) \\
&= \gamma\left[q\delta(x - vt)\delta(y)\delta(z) - \frac{v}{c^2}qv\delta(x - vt)\delta(y)\delta(z)\right] \\
&= \gamma\left(\rho - \frac{v}{c^2}j_x\right),
\end{aligned}$$

and this is the expected expression of a Lorentz transformation. This shows that ρ transforms as t, and we further expect that j_x will transform as x, that is, $j'_x = \gamma(j_x - v\rho)$. This is confirmed by the fact that the right-hand side is zero for the situation considered here. We conclude that under a change of reference frame, the charge and current densities transform as

$$j'_x = \gamma(j_x - v\rho), \qquad j'_y = j_y, \qquad j'_z = j_z, \qquad \rho' = \gamma\left(\rho - \frac{v}{c^2}j_x\right). \tag{6.1.6}$$

While the derivation of these transformation rules was restricted to a very simple situation, it is possible to show that they are in fact completely general.

To see how the potentials transform we recall the wave equations $\Box(\Phi/c^2) = -\rho/(\epsilon_0 c^2) = -\mu_0\rho$ and $\Box\boldsymbol{A} = -\mu_0\boldsymbol{j}$ from Sec. 5.1. Because $\Box$ is an invariant, these equations tell us that Φ/c^2 must transform as ρ, and that $\boldsymbol{A}$ must transform as $\boldsymbol{j}$. Thus,

$$A'_x = \gamma\left(A_x - \frac{v}{c^2}\Phi\right), \qquad A'_y = A_y, \qquad A'_z = A_x, \qquad \Phi' = \gamma(\Phi - vA_x) \quad (6.1.7)$$

are the Lorentz transformations for the scalar and vector potentials. These are designed to leave the form of the wave equations unchanged under a change of reference frame. You should check that the Lorenz gauge condition,

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0,$$

also takes the same form in S'. All the equations satisfied by the potentials are therefore preserved under a Lorentz transformation. This implies that Maxwell's equations themselves are invariant under a change of reference frame.

We can now figure out how the fields transform. We simply write down how the fields in S' are related to the potentials,

$$\boldsymbol{E'} = -\frac{\partial\boldsymbol{A'}}{\partial t'} - \boldsymbol{\nabla'}\Phi', \qquad \boldsymbol{B'} = \boldsymbol{\nabla'}\times\boldsymbol{A'},$$

and we substitute Eqs. (6.1.4) and (6.1.7). For example,

$$\begin{aligned} E'_x &= -\gamma^2\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\left(A_x - \frac{v}{c^2}\Phi\right) - \gamma^2\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\left(\Phi - vA_x\right) \\ &= -\frac{\partial A_x}{\partial t} - \frac{\partial\Phi}{\partial x} \\ &= E_x. \end{aligned}$$

Or

$$\begin{aligned} E'_y &= -\gamma\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)A_y - \gamma\frac{\partial}{\partial y}\left(\Phi - vA_x\right) \\ &= -\gamma\left(\frac{\partial A_y}{\partial t} + \frac{\partial\Phi}{\partial y}\right) - \gamma v\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\ &= \gamma\big(E_y - vB_z\big). \end{aligned}$$

Another example is

$$B'_x = \frac{\partial A'_z}{\partial y'} - \frac{\partial A'_y}{\partial z'} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = B_x.$$

Yet another is

$$\begin{aligned} B'_y &= \frac{\partial A'_x}{\partial z'} - \frac{\partial A'_z}{\partial x'} \\ &= \gamma\frac{\partial}{\partial z}\left(A_x - \frac{v}{c^2}\Phi\right) - \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)A_z \\ &= \gamma\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \gamma\frac{v}{c^2}\left(-\frac{\partial A_z}{\partial t} - \frac{\partial\Phi}{\partial z}\right) \\ &= \gamma\left(B_y + \frac{v}{c^2}E_z\right). \end{aligned}$$

The complete summary of these results is that the electric field transforms as

$$E'_x = E_x, \qquad E'_y = \gamma\big(E_y - vB_z\big), \qquad E'_z = \gamma\big(E_z + vB_y\big) \tag{6.1.8}$$

under a change of reference frame, while the magnetic field transforms as

$$B'_x = B_x, \qquad B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right). \tag{6.1.9}$$

Two observers in relative motion will therefore disagree as to the value of the electric and magnetic fields.

There are interesting special cases to the foregoing results. Suppose first that $\boldsymbol{B} = \boldsymbol{0}$ in some reference frame S. In another reference frame S' we have $E'_x = E_x$, $E'_y = \gamma E_y$, $E'_z = \gamma E_z$, $B'_x = 0$, $B'_y = \gamma(v/c^2)E_z = (v/c^2)E'_z = -(\boldsymbol{v}\times\boldsymbol{E'})_y/c^2$, and $B'_z = -\gamma(v/c^2)E_y = -(v/c^2)E'_y = -(\boldsymbol{v}\times\boldsymbol{E'})_z/c^2$. In other words,

$$\boldsymbol{B} = \boldsymbol{0} \quad \rightarrow \quad \boldsymbol{B'} = -\frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E'}. \tag{6.1.10}$$

A pure electric field in S is therefore perceived as a mixture of electric and magnetic fields in S'. Suppose next that $\boldsymbol{E} = \boldsymbol{0}$ in S. Then $B'_x = B_x$, $B'_y = \gamma B_y$, $B'_z = \gamma B_z$, $E'_x = 0$, $E'_y = -\gamma v B_z = -vB'_z = (\boldsymbol{v}\times\boldsymbol{B'})_y$, and $E'_z = \gamma v B_y = vB'_y = (\boldsymbol{v}\times\boldsymbol{B'})_z$. In other words,

$$\boldsymbol{E} = \boldsymbol{0} \quad \rightarrow \quad \boldsymbol{E'} = \boldsymbol{v}\times\boldsymbol{B'}. \tag{6.1.11}$$

A pure magnetic field in S is therefore perceived as a mixture of electric and magnetic fields in S'. We conclude that $\boldsymbol{E}$ and $\boldsymbol{B}$ are observer-dependent fields; only the electromagnetic field as a whole has an observer-independent existence.

6.2 Fields of a uniformly moving charge

[The material presented in this section is also covered in Sec. 11.10 of Jackson's text.]

We can use the Lorentz transformations of Eqs. (6.1.8) and (6.1.9) to calculate the electric and magnetic fields of a point charge q that moves with constant speed v in the x direction. The charge's position vector is

$$\boldsymbol{r}(t) = (vt, b, 0), \tag{6.2.1}$$

where b is the charge's displacement above the x axis. We want $\boldsymbol{E}$ and $\boldsymbol{B}$ as measured at O, the origin of the reference frame S. We shall first do the calculation in S', which moves along with the particle.

As seen in the reference frame S', the charge's position vector is $\boldsymbol{r'} = (0, b, 0)$, but the observer at O has a position $\boldsymbol{r'}_O = (-vt', 0, 0)$, because relative to S', O moves in the negative x direction with speed v. In S' there is only an electric field $\boldsymbol{E'}$, and its value at O is given by the standard expression

$$\boldsymbol{E'} = -\frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{r'} - \boldsymbol{r'}_O}{|\boldsymbol{r'} - \boldsymbol{r'}_O|^3}.$$

We have $\boldsymbol{r'} - \boldsymbol{r'}_O = (vt', b, 0)$, $|\boldsymbol{r'} - \boldsymbol{r'}_O| = \sqrt{b^2 + v^2t'^2}$, and the primed components of the electric field are

$$E'_x = -\frac{q}{4\pi\epsilon_0}\frac{vt'}{(b^2 + v^2t'^2)^{3/2}}, \qquad E'_y = -\frac{q}{4\pi\epsilon_0}\frac{b}{(b^2 + v^2t'^2)^{3/2}}, \qquad E'_z = 0. \tag{6.2.2}$$

As measured in S', the electric field lines are distributed isotropically around $\boldsymbol{r'}$, the charge's position.

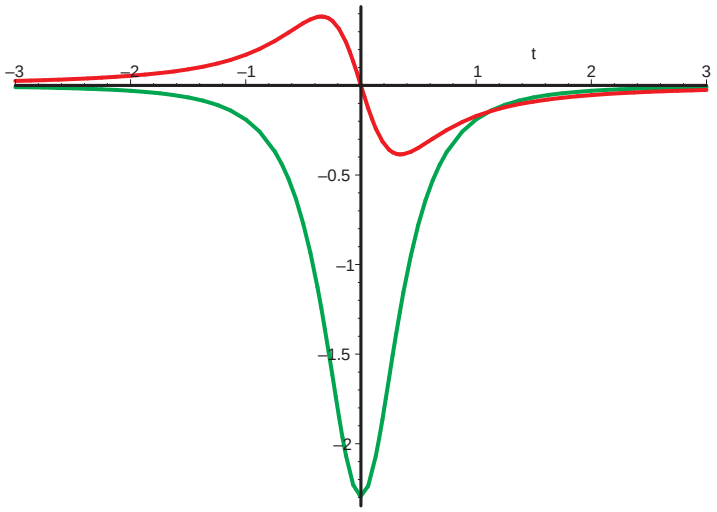

Figure 6.1: Electric field of a uniformly moving charge, as measured by an observer at the origin of a fixed reference frame. The x component of the field goes through zero at $t = 0$, while the y component is always negative and achieves its maximum (in magnitude) at that time.

We now transform the field to the frame of reference S. The coordinates of O are $x = y = z = 0$, and the time coordinates are therefore related by $t' = \gamma t$. The unprimed fields are obtained by inverting Eqs. (6.1.8) and (6.1.9), which is done by reversing the sign of v. Because $\boldsymbol{B'} = \mathbf{0}$ we have $E_x = E'_x$, $E_y = \gamma E'_y$, $E_z = E'_z = 0$, $B_x = B'_x = 0$, $B_y = -(\gamma v/c^2)E'_z = 0$, and $B_z = (\gamma v/c^2)E'_y$. We therefore obtain

$$E_x = -\frac{q}{4\pi\epsilon_0}\frac{\gamma v t}{(b^2+\gamma^2 v^2 t^2)^{3/2}}, \qquad E_y = -\frac{q}{4\pi\epsilon_0}\frac{\gamma b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}, \qquad E_z = 0 \tag{6.2.3}$$

and

$$B_x = 0, \qquad B_y = 0, \qquad B_z = \frac{v}{c^2}E_y \tag{6.2.4}$$

for the fields measured at O, in the S frame. Plots of E_x and E_y as functions of time are shown in Fig. 6.1.

The fields can also be expressed in terms of the charge's position vector, $\boldsymbol{r}(t) = (vt, b, 0)$. For this purpose we let ψ be the angle between the vectors $\boldsymbol{r}$ and $\boldsymbol{v}$. This is defined by

$$\cos\psi = \frac{\boldsymbol{v}\cdot\boldsymbol{r}(t)}{vr(t)}, \tag{6.2.5}$$

where $r \equiv |\boldsymbol{r}| = \sqrt{b^2+v^2t^2}$. We have $\boldsymbol{v}\cdot\boldsymbol{r} = v^2 t$, so $\cos\psi = vt/\sqrt{b^2+v^2t^2}$, which implies $\sin\psi = b/\sqrt{b^2+v^2t^2}$. We also have

$$\begin{aligned} b^2+\gamma^2v^2t^2 &= \gamma^2(b^2/\gamma^2+v^2t^2) \\ &= \gamma^2\left(b^2+v^2t^2-\frac{v^2}{c^2}b^2\right) \\ &= \gamma^2(b^2+v^2t^2)\left(1-\frac{v^2}{c^2}\frac{b^2}{b^2+v^2t^2}\right) \\ &= \gamma^2 r^2\left(1-\frac{v^2}{c^2}\sin^2\psi\right), \end{aligned}$$

and this can be substituted into Eqs. (6.2.3). The end result is

$$\boldsymbol{E} = -\frac{q}{4\pi\epsilon_0}\frac{1}{\gamma^2\big[1-(v/c)^2\sin^2\psi\big]^{3/2}}\frac{\boldsymbol{r}}{r^3} \tag{6.2.6}$$

for the electric field at O, and

$$\boldsymbol{B} = \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E} \tag{6.2.7}$$

for the magnetic field.

We see from Eq. (6.2.6) that $\boldsymbol{E}$ is always directed along $-\boldsymbol{r}(t)$, a vector that points from the charge to O. The electric field therefore points away from the charge, and this means that the electric field lines originate at the position of the moving charge. The field lines, however, are not distributed isotropically around the charge; this is revealed by the presence of $\sin^2\psi$ in the expression for the electric field. When $\psi = 0$, that is, when $\boldsymbol{r}$ is directed along $\boldsymbol{v}$, we have that $[1-(v/c)^2\sin^2\psi]^{-3/2} = 1$, and the electric field is reduced by a relativistic factor of $\gamma^{-2} < 1$. But when instead $\psi = \pi/2$, so that $\boldsymbol{r}$ is orthogonal to $\boldsymbol{v}$, then $[1-(v/c)^2\sin^2\psi]^{-3/2} = \gamma^3 > 1$, and the field is amplified by a factor $\gamma > 1$. The electric field, as seen in S, is strongest in the directions transverse to the motion, and weakest in the longitudinal direction.

The results derived in this section are at once remarkable and unremarkable. It is *unremarkable* that the field lines should emanate from the position of the charge — we have the same picture in the charge's rest frame. And it is *unremarkable* that the field lines should be distorted by a Lorentz contraction in the direction of the motion. But it is *remarkable* that the field registered at O does not involve a time delay associated with the propagation of a signal from charge to observer.

Allowing for such a delay would suggest that the field at time t should point not toward the charge at its *current position* $\boldsymbol{r}(t)$, but instead toward an *earlier position* $\boldsymbol{r}(t')$. In this picture, a signal originates at the charge when it was at the earlier position, and it propagates to O in a time $\Delta t \equiv t - t'$ determined by the condition that the signal travels at the speed of light; since the distance traveled is $|\boldsymbol{r}(t')|$, we must have $t - t' = |\boldsymbol{r}(t')|/c$, which in principle can be solved for t'. During the time in which the signal propagates, the particle moves to its current position, but the field registered at O should not yet be aware of this; it should know only of the earlier position, and $\boldsymbol{E}$ should therefore point in the direction of $\boldsymbol{r}(t')$. But this is not what we have found: the field does point in the direction of $\boldsymbol{r}(t)$, the current position of the charge. Our picture, according to which information should be delayed, is curiously wrong. We will return to this issue later on, and investigate it more fully.

6.3 Fields of an arbitrarily moving charge

[The material presented in this section is also covered in Secs. 6.5 and 14.1 of Jackson's text.]

We now allow the point charge q to move on an arbitrary trajectory $\boldsymbol{r}(t)$. We let $\boldsymbol{v}(t) = d\boldsymbol{r}/dt$ be its velocity vector, and $\boldsymbol{a}(t) = d\boldsymbol{v}/dt$ its acceleration vector. The charge density is

$$\rho(t,\boldsymbol{x}) = q\delta\big(\boldsymbol{x}-\boldsymbol{r}(t)\big), \tag{6.3.1}$$

and the current density is

$$\boldsymbol{j}(t,\boldsymbol{x}) = q\boldsymbol{v}(t)\delta\big(\boldsymbol{x}-\boldsymbol{r}(t)\big). \tag{6.3.2}$$

To obtain the fields produced by this particle we shall first calculate the potentials.

6.3.1 Potentials

To calculate the potentials associated with the densities of Eqs. (6.3.1) and (6.3.2) we employ a technique that was first put to use in Secs. 5.2 and 5.3. For example, we express the vector potential as

$$\boldsymbol{A}(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \boldsymbol{j}(t',\boldsymbol{x'})\frac{\delta(t-t'-|\boldsymbol{x}-\boldsymbol{x'}|/c)}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'dt',$$

in terms of the retarded Green's function obtained in Sec. 1.9. Substituting Eq. (6.3.2) yields

$$\boldsymbol{A}(t,\boldsymbol{x}) = \frac{q\mu_0}{4\pi}\int \boldsymbol{v}(t')\delta\big(\boldsymbol{x'}-\boldsymbol{r}(t')\big)\frac{\delta(t-t'-|\boldsymbol{x}-\boldsymbol{x'}|/c)}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'dt',$$

and integrating over d^3x' returns

$$\boldsymbol{A}(t,\boldsymbol{x}) = \frac{q\mu_0}{4\pi}\int \boldsymbol{v}(t')\frac{\delta(t-t'-|\boldsymbol{x}-\boldsymbol{r}(t')|/c)}{|\boldsymbol{x}-\boldsymbol{r}(t')|}\,dt'. \tag{6.3.3}$$

The same manipulations produce

$$\Phi(t,\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\int \frac{\delta(t-t'-|\boldsymbol{x}-\boldsymbol{r}(t')|/c)}{|\boldsymbol{x}-\boldsymbol{r}(t')|}\,dt' \tag{6.3.4}$$

for the scalar potential. At this stage it is convenient to introduce the quantities

$$\boldsymbol{R}(t') = \boldsymbol{x}-\boldsymbol{r}(t'), \qquad R(t') = |\boldsymbol{x}-\boldsymbol{r}(t')|, \qquad \hat{\boldsymbol{R}}(t') = \frac{\boldsymbol{R}(t')}{R(t')}. \tag{6.3.5}$$

In terms of these the argument of the δ-function is $t-t'-R(t')/c$.

To evaluate the integrals of Eqs. (6.3.3) and (6.3.4) we let $s \equiv t'+R(t')/c-t$ be the new variable of integration. We have $ds/dt' = 1+c^{-1}dR/dt'$, and it is easy to verify that $dR/dt' = -\boldsymbol{R}\cdot\boldsymbol{v}/R = -\hat{\boldsymbol{R}}\cdot\boldsymbol{v}$. With this we obtain $ds/dt' = \kappa$, where

$$\kappa \equiv 1-\boldsymbol{v}(t')\cdot\hat{\boldsymbol{R}}(t')/c. \tag{6.3.6}$$

The integral for the vector potential is then

$$\begin{aligned}\boldsymbol{A} &= \frac{q\mu_0}{4\pi}\int \frac{\boldsymbol{v}\delta(s)}{R}\frac{dt'}{ds}\,ds\\ &= \frac{q\mu_0}{4\pi}\int \frac{\boldsymbol{v}\delta(s)}{\kappa R}\,ds\\ &= \frac{q\mu_0}{4\pi}\left[\frac{\boldsymbol{v}}{\kappa R}\right]_{s=0}.\end{aligned}$$

Similarly,

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\kappa R}\right]_{s=0}.$$

The condition $s=0$ gives $R(t')/c = t-t'$, or

$$|\boldsymbol{x}-\boldsymbol{r}(t')| = c(t-t'), \tag{6.3.7}$$

and this implicit equation can in principle be solved for t'. The meaning of Eq. (6.3.7) is clear: The signal received at position $\boldsymbol{x}$ and time t depends on the state of motion of the particle at an earlier time t'; the delay $(t-t')$ is set by the time required by light to travel the distance $R(t')$. We shall call t' the *retarded time* corresponding

to the spacetime point $(t, \boldsymbol{x})$, and Eq. (6.3.7) will be referred to as the *retarded condition*.

We shall write the potentials as

$$\Phi(t, \boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\kappa R}\right]_{\rm ret} \tag{6.3.8}$$

and

$$\boldsymbol{A}(t, \boldsymbol{x}) = \frac{q\mu_0}{4\pi}\left[\frac{\boldsymbol{v}}{\kappa R}\right]_{\rm ret}, \tag{6.3.9}$$

indicating that the quantities within the square brackets must be evaluated at the time t' determined by the retarded condition of Eq. (6.3.7). These expressions are known as the *Liénard-Wiechert potentials*. Apart from the presence of κ in the denominator, the potentials take their familiar form from time-independent situations, and they implement the notion that the potentials here and now should depend on the *retarded* position and velocity of the particle. Given that the retardation effect is present in the Liénard-Wiechert potentials, it is all the more curious that the effect was not seen in the preceding section. (Stay tuned.)

The factor of $1/\kappa$ accounts for the fact that

$$\int \rho(t - |\boldsymbol{x} - \boldsymbol{x'}|/c, \boldsymbol{x'})\, d^3x'$$

is *not equal* to the charge q. While we do have $\int \rho(t, \boldsymbol{x'})\, d^3x' = q$, in the previous expression we sample the charge density at different times while performing the integration, and the result is not q. Instead,

$$\begin{aligned}\int \rho(t - |\boldsymbol{x} - \boldsymbol{x'}|/c, \boldsymbol{x'})\, d^3x' &= \int \rho(t', \boldsymbol{x'})\delta(t - t' - |\boldsymbol{x} - \boldsymbol{x'}|/c)\, d^3x'dt' \\ &= q\int \delta(t - t' - |\boldsymbol{x} - \boldsymbol{r}(t')|/c)\, dt' \\ &= \left[\frac{q}{\kappa}\right]_{\rm ret},\end{aligned}$$

and this result clarifies the origin of $\kappa = 1 - \boldsymbol{v}\cdot\hat{\boldsymbol{R}}/c$.

6.3.2 Fields

Having obtained the potentials, we can calculate the fields by straightforward differentiation,

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\Phi, \qquad \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}.$$

The dependence of $\boldsymbol{A}$ on t comes from its *explicit* dependence on t' and the retarded condition $|\boldsymbol{x} - \boldsymbol{r}(t')| = c(t - t')$. Its dependence on $\boldsymbol{x}$ comes from the *explicit* dependence of R and κ on position, but there is also an *implicit* dependence contributed by the retarded condition. This mixture of explicit and implicit dependence complicates the task of taking derivatives. But we can simplify things by going back to earlier forms of Eqs. (6.3.3) and (6.3.4),

$$\Phi(t, \boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\int \frac{1}{R(t')}\delta(s)\, dt' \tag{6.3.10}$$

and

$$\boldsymbol{A}(t, \boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\frac{1}{c^2}\int \frac{\boldsymbol{v}(t')}{R(t')}\delta(s)\, dt', \tag{6.3.11}$$

where $s = t' + R(t')/c - t$. Their advantage is that t' inside the integrand is no longer restricted, so that the dependence of the potentials on t and $\boldsymbol{x}$ is fully explicit. It will then be straightforward (if tedious) to differentiate first, and integrate next (thereby setting t' by the retarded condition). Before we proceed we recall the definitions of Eqs. (6.3.5) and (6.3.6): $\boldsymbol{R}(t') = \boldsymbol{x} - \boldsymbol{r}(t')$ and $\kappa = 1 - \boldsymbol{v}(t') \cdot \hat{\boldsymbol{R}}(t')/c = ds/dt'$.

We begin with the scalar potential. We have

$$\boldsymbol{\nabla}\Phi = \frac{q}{4\pi\epsilon_0} \int \left[\boldsymbol{\nabla}\Big(\frac{1}{R}\Big)\delta(s) + \frac{1}{R}\delta'(s)\boldsymbol{\nabla}s \right] dt',$$

and since s depends on $\boldsymbol{x}$ through R, we can replace $\boldsymbol{\nabla}s$ by $\boldsymbol{\nabla}R/c$. This gives

$$\begin{aligned}
\boldsymbol{\nabla}\Phi &= \frac{q}{4\pi\epsilon_0} \int \left[\boldsymbol{\nabla}\Big(\frac{1}{R}\Big)\delta(s) + \frac{\boldsymbol{\nabla}R}{cR}\delta'(s) \right] dt' \\
&= \frac{q}{4\pi\epsilon_0} \int \left[\frac{1}{\kappa}\boldsymbol{\nabla}\Big(\frac{1}{R}\Big)\delta(s) + \frac{\boldsymbol{\nabla}R}{c\kappa R}\delta'(s) \right] ds \\
&= \frac{q}{4\pi\epsilon_0} \int \left[\frac{1}{\kappa}\boldsymbol{\nabla}\Big(\frac{1}{R}\Big) - \frac{d}{ds}\left(\frac{\boldsymbol{\nabla}R}{c\kappa R}\right) \right] \delta(s)\, ds,
\end{aligned}$$

where we have integrated the second term by parts. The derivative with respect to s can now be expressed in terms of d/dt', and we finally obtain

$$\boldsymbol{\nabla}\Phi = \frac{q}{4\pi\epsilon_0} \int \left[\frac{1}{\kappa}\boldsymbol{\nabla}\Big(\frac{1}{R}\Big) - \frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\boldsymbol{\nabla}R}{\kappa R}\right) \right] \delta(s)\, ds.$$

Leaving this result alone for the time being, we move on and calculate $\partial\boldsymbol{A}/\partial t$. We have

$$\begin{aligned}
\frac{\partial \boldsymbol{A}}{\partial t} &= \frac{q}{4\pi\epsilon_0}\frac{1}{c^2} \int \frac{\boldsymbol{v}}{R}\delta'(s)\frac{\partial s}{\partial t}\, dt' \\
&= -\frac{q}{4\pi\epsilon_0}\frac{1}{c^2} \int \frac{\boldsymbol{v}}{\kappa R}\delta'(s)\, ds \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{c^2} \int \left[\frac{d}{ds}\left(\frac{\boldsymbol{v}}{\kappa R}\right) \right] \delta(s)\, ds \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{c^2} \int \left[\frac{1}{\kappa}\frac{d}{dt'}\left(\frac{\boldsymbol{v}}{\kappa R}\right) \right] \delta(s)\, ds.
\end{aligned}$$

In the first step we used the fact that the only t dependence is contained in s. In the second step we substituted $\partial s/\partial t = -1$. In the third step we integrated by parts. And in the fourth step we replaced the s-derivative by a t'-derivative.

Collecting these results gives us an expression for the electric field:

$$\begin{aligned}
\boldsymbol{E} &= \frac{q}{4\pi\epsilon_0} \int \left[-\frac{1}{\kappa}\boldsymbol{\nabla}\Big(\frac{1}{R}\Big) + \frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\boldsymbol{\nabla}R}{\kappa R}\right) - \frac{1}{c^2\kappa}\frac{d}{dt'}\left(\frac{\boldsymbol{v}}{\kappa R}\right) \right] \delta(s)\, ds \\
&= \frac{q}{4\pi\epsilon_0} \int \left[-\frac{1}{\kappa}\boldsymbol{\nabla}\Big(\frac{1}{R}\Big) + \frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\boldsymbol{\nabla}R - \boldsymbol{v}/c}{\kappa R}\right) \right] \delta(s)\, ds.
\end{aligned}$$

This can be integrated at once, and we obtain

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0} \left[-\frac{1}{\kappa}\boldsymbol{\nabla}\Big(\frac{1}{R}\Big) + \frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\boldsymbol{\nabla}R - \boldsymbol{v}/c}{\kappa R}\right) \right]_{\text{ret}}.$$

This expression badly needs to be simplified. It is easy to establish that $\boldsymbol{\nabla}R = \hat{\boldsymbol{R}}$, so that $\boldsymbol{\nabla}R^{-1} = -\hat{\boldsymbol{R}}/R^2$, and we already have obtained $dR/dt' = -\boldsymbol{v} \cdot \hat{\boldsymbol{R}}$ in the preceding subsection. We now set to work on

$$\frac{d}{dt'}\left(\frac{\boldsymbol{\nabla}R - \boldsymbol{v}/c}{\kappa R}\right) = \frac{1}{\kappa R}\left(\frac{d\hat{\boldsymbol{R}}}{dt'} - \frac{\boldsymbol{a}}{c}\right) - \frac{\hat{\boldsymbol{R}} - \boldsymbol{v}/c}{\kappa^2 R^2}\frac{d}{dt'}\Big(\kappa R\Big),$$

where $\boldsymbol{a} = d\boldsymbol{v}/dt'$ is the particle's acceleration. We have $d\hat{\boldsymbol{R}}/dt' = d(\boldsymbol{R}/R)/dt' = -\boldsymbol{v}/R + (\boldsymbol{R}/R^2)(\boldsymbol{v}\cdot\hat{\boldsymbol{R}}) = -[\boldsymbol{v} - (\boldsymbol{v}\cdot\hat{\boldsymbol{R}})\hat{\boldsymbol{R}}]/R$, and also $d(\kappa R)/dt' = d(R - \boldsymbol{v}\cdot\boldsymbol{R}/c)/dt' = -\boldsymbol{v}\cdot\hat{\boldsymbol{R}} - (\boldsymbol{a}\cdot\boldsymbol{R} - \boldsymbol{v}\cdot\boldsymbol{v})/c = -\boldsymbol{v}\cdot\hat{\boldsymbol{R}} - \boldsymbol{a}\cdot\boldsymbol{R}/c + v^2/c$. Plugging these results back into our previous expression gives

$$\begin{aligned}
\frac{1}{c}\frac{d}{dt'}\left(\frac{\boldsymbol{\nabla}R - \boldsymbol{v}/c}{\kappa R}\right) &= -\frac{1}{\kappa R^2}\Big[\boldsymbol{v}/c - (\boldsymbol{v}\cdot\hat{\boldsymbol{R}}/c)\hat{\boldsymbol{R}}\Big] - \frac{\boldsymbol{a}}{\kappa R c^2} \\
&\quad - \frac{\hat{\boldsymbol{R}} - \boldsymbol{v}/c}{\kappa^2 R^2}\Big(-\boldsymbol{v}\cdot\hat{\boldsymbol{R}}/c + v^2/c^2\Big) + \frac{\hat{\boldsymbol{R}} - \boldsymbol{v}/c}{\kappa^2 R^2}\frac{\boldsymbol{a}\cdot\boldsymbol{R}}{c^2} \\
&= -\frac{1}{\kappa R^2}\Big[\boldsymbol{v}/c - (1-\kappa)\hat{\boldsymbol{R}}\Big] + \frac{\hat{\boldsymbol{R}} - \boldsymbol{v}/c}{\kappa^2 R^2}\Big(1 - \kappa - v^2/c^2\Big) \\
&\quad + \frac{1}{\kappa^2 R c^2}\Big[(\hat{\boldsymbol{R}} - \boldsymbol{v}/c)(\boldsymbol{a}\cdot\hat{\boldsymbol{R}}) - \kappa\boldsymbol{a}\Big] \\
&= \frac{\hat{\boldsymbol{R}} - \boldsymbol{v}/c}{\kappa R^2} - \frac{\hat{\boldsymbol{R}}}{R^2} + \frac{\hat{\boldsymbol{R}} - \boldsymbol{v}/c}{\kappa^2 R^2}\Big(1 - v^2/c^2\Big) - \frac{\hat{\boldsymbol{R}} - \boldsymbol{v}/c}{\kappa R^2} \\
&\quad + \frac{1}{\kappa^2 R c^2}\Big[(\hat{\boldsymbol{R}} - \boldsymbol{v}/c)(\boldsymbol{a}\cdot\hat{\boldsymbol{R}}) - \kappa\boldsymbol{a}\Big].
\end{aligned}$$

The first and fourth terms cancel out, and the expression within the square brackets is $\hat{\boldsymbol{R}}\times[(\hat{\boldsymbol{R}} - \boldsymbol{v}/c)\times\boldsymbol{a}] = (\hat{\boldsymbol{R}}\cdot\boldsymbol{a})(\hat{\boldsymbol{R}} - \boldsymbol{v}/c) - \hat{\boldsymbol{R}}\cdot(\hat{\boldsymbol{R}} - \boldsymbol{v}/c)\boldsymbol{a} = (\hat{\boldsymbol{R}}\cdot\boldsymbol{a})(\hat{\boldsymbol{R}} - \boldsymbol{v}/c) - \kappa\boldsymbol{a}$. This gives

$$\frac{1}{c}\frac{d}{dt'}\left(\frac{\boldsymbol{\nabla}R - \boldsymbol{v}/c}{\kappa R}\right) = -\frac{\hat{\boldsymbol{R}}}{R^2} + \frac{\hat{\boldsymbol{R}} - \boldsymbol{v}/c}{\gamma^2\kappa^2 R^2} + \frac{\hat{\boldsymbol{R}}\times\big[(\hat{\boldsymbol{R}} - \boldsymbol{v}/c)\times\boldsymbol{a}\big]}{\kappa^2 c^2 R}.$$

Substituting all this into our previous result for the electric field returns

$$\boldsymbol{E}(t,\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\boldsymbol{R}} - \boldsymbol{v}/c}{\gamma^2\kappa^3 R^2} + \frac{\hat{\boldsymbol{R}}\times\big[(\hat{\boldsymbol{R}} - \boldsymbol{v}/c)\times\boldsymbol{a}\big]}{\kappa^3 c^2 R}\right]_{\text{ret}}, \tag{6.3.12}$$

our final expression for the electric field of a moving point charge. The electric field is naturally decomposed into a "velocity field" that decays as $1/R^2$ and an "acceleration field" that decays as $1/R$. The velocity field represents that part of the electric field that stays bound to the particle as it moves; it is the particle's generalized Coulomb field. The acceleration field, on the other hand, represents electromagnetic radiation that propagates independently of the charge.

To obtain the magnetic field we take the curl of Eq. (6.3.11), which gives

$$\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{A} &= \frac{q}{4\pi\epsilon_0}\frac{1}{c^2}\int\left[\boldsymbol{\nabla}\Big(\frac{1}{R}\Big)\times\boldsymbol{v}\delta(s) + \frac{\boldsymbol{\nabla}s\times\boldsymbol{v}}{R}\delta'(s)\right]dt' \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{c^2}\int\left[\frac{1}{\kappa}\boldsymbol{\nabla}\Big(\frac{1}{R}\Big)\times\boldsymbol{v}\delta(s) + \frac{\boldsymbol{\nabla}R\times\boldsymbol{v}}{c\kappa R}\delta'(s)\right]ds \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{c^2}\int\left[\frac{1}{\kappa}\boldsymbol{\nabla}\Big(\frac{1}{R}\Big)\times\boldsymbol{v} - \frac{d}{ds}\left(\frac{\boldsymbol{\nabla}R\times\boldsymbol{v}}{c\kappa R}\right)\right]\delta(s)\,ds \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{c^2}\int\left[\frac{1}{\kappa}\boldsymbol{\nabla}\Big(\frac{1}{R}\Big)\times\boldsymbol{v} - \frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\boldsymbol{\nabla}R\times\boldsymbol{v}}{\kappa R}\right)\right]\delta(s)\,ds,
\end{aligned}$$

so that

$$\boldsymbol{B} = \frac{q}{4\pi\epsilon_0}\frac{1}{c^2}\left[\frac{1}{\kappa}\boldsymbol{\nabla}\Big(\frac{1}{R}\Big)\times\boldsymbol{v} - \frac{1}{c\kappa}\frac{d}{dt'}\left(\frac{\boldsymbol{\nabla}R\times\boldsymbol{v}}{\kappa R}\right)\right]_{\text{ret}}.$$

We now work on simplifying this expression. We have $\boldsymbol{\nabla}(1/R)\times\boldsymbol{v} = -(\hat{\boldsymbol{R}}\times\boldsymbol{v})/R^2$, and $d(\hat{\boldsymbol{R}}\times\boldsymbol{v}/\kappa R)/dt'$ can be expressed as

$$\frac{d}{dt'}\frac{(\hat{\boldsymbol{R}} - \boldsymbol{v}/c)\times\boldsymbol{v}}{\kappa R} = \frac{d}{dt'}\left(\frac{\hat{\boldsymbol{R}} - \boldsymbol{v}/c}{\kappa R}\right)\times\boldsymbol{v} + \frac{\hat{\boldsymbol{R}} - \boldsymbol{v}/c}{\kappa R}\times\boldsymbol{a},$$

which allows us to use results that were generated during the calculation of the electric field. We have

$$\begin{aligned}\frac{1}{c}\frac{d}{dt'}\frac{(\hat{\boldsymbol{R}}-\boldsymbol{v}/c)\times\boldsymbol{v}}{\kappa R} &= \left[-\frac{\hat{\boldsymbol{R}}}{R^2}+\frac{\hat{\boldsymbol{R}}-\boldsymbol{v}/c}{\gamma^2\kappa^2R^2}+\frac{(\hat{\boldsymbol{R}}\cdot\boldsymbol{a})(\hat{\boldsymbol{R}}-\boldsymbol{v}/c)-\kappa\boldsymbol{a}}{\kappa^2Rc^2}\right]\times\boldsymbol{v}\\ &\quad+\frac{\hat{\boldsymbol{R}}-\boldsymbol{v}/c}{c\kappa R}\times\boldsymbol{a}\\ &= -\frac{\hat{\boldsymbol{R}}\times\boldsymbol{v}}{R^2}+\frac{\hat{\boldsymbol{R}}\times\boldsymbol{v}}{\gamma^2\kappa^2R^2}+\frac{(\hat{\boldsymbol{R}}\cdot\boldsymbol{a})\hat{\boldsymbol{R}}\times\boldsymbol{v}}{\kappa^2Rc^2}-\frac{\boldsymbol{a}\times\boldsymbol{v}}{\kappa Rc^2}\\ &\quad+\frac{\hat{\boldsymbol{R}}\times\boldsymbol{a}}{c\kappa R}+\frac{\boldsymbol{a}\times\boldsymbol{v}}{\kappa Rc^2}\\ &= -\frac{\hat{\boldsymbol{R}}\times\boldsymbol{v}}{R^2}+\frac{\hat{\boldsymbol{R}}\times\boldsymbol{v}}{\gamma^2\kappa^2R^2}+\frac{(\hat{\boldsymbol{R}}\cdot\boldsymbol{a})\hat{\boldsymbol{R}}\times\boldsymbol{v}}{\kappa^2Rc^2}+\frac{\hat{\boldsymbol{R}}\times\boldsymbol{a}}{c\kappa R}.\end{aligned}$$

Collecting these results, we arrive at

$$\boldsymbol{B}=\frac{q}{4\pi\epsilon_0}\frac{1}{c^2}\left[-\frac{\hat{\boldsymbol{R}}\times\boldsymbol{v}}{\gamma^2\kappa^3R^2}-\frac{(\hat{\boldsymbol{R}}\cdot\boldsymbol{a})\hat{\boldsymbol{R}}\times\boldsymbol{v}}{\kappa^3Rc^2}-\frac{\hat{\boldsymbol{R}}\times\boldsymbol{a}}{c\kappa^2R}\right]_{\text{ret}},$$

or

$$\boldsymbol{B}(t,\boldsymbol{x})=-\frac{q}{4\pi\epsilon_0}\frac{1}{c^2}\left[\frac{\hat{\boldsymbol{R}}\times\boldsymbol{v}}{\gamma^2\kappa^3R^2}+\frac{(\hat{\boldsymbol{R}}\cdot\boldsymbol{a})\hat{\boldsymbol{R}}\times\boldsymbol{v}/c+\kappa\hat{\boldsymbol{R}}\times\boldsymbol{a}}{c\kappa^3R}\right]_{\text{ret}}. \tag{6.3.13}$$

A little more algebra reveals that this can also be expressed as

$$\boldsymbol{B}=\frac{1}{c}\big[\hat{\boldsymbol{R}}\big]_{\text{ret}}\times\boldsymbol{E}. \tag{6.3.14}$$

The magnetic field also is decomposed into a velocity field that stays bound to the particle, and an acceleration field that represents electromagnetic radiation.

Our general results for the electric and magnetic fields allow us to resolve the issue that was first brought up near the end of Sec. 6.2: where does the electric field point? Does it point toward $\boldsymbol{r}(t')$, the *retarded position* of the particle, as it was argued it might do? Or does it point toward $\boldsymbol{r}(t)$, its *current position*, as we actually found in Sec. 6.2? The answer is that in general, the electric field follows neither option.

Consider our expression (6.3.12) for the *bound* electric field. (We exclude the radiative field from this discussion, as we expect it to behave largely independently of the charge.) It shows that the field points in the direction opposite to the vector $-\boldsymbol{R}+R\boldsymbol{v}/c$ evaluated at the retarded time t'. If the observation point is at the origin of the coordinates ($\boldsymbol{x}=\boldsymbol{0}$), then $\boldsymbol{R}=-\boldsymbol{r}$ and the bound field points in the direction opposite to

$$\boldsymbol{r}(t')+(\Delta t)\boldsymbol{v}(t'),$$

where Δt is determined by the retarded condition

$$\Delta t\equiv t-t'=r(t')/c.$$

This result is interesting: the field points not in the direction opposite to the charge's retarded position $\boldsymbol{r}(t')$, nor in the direction opposite to the charge's current position $\boldsymbol{r}(t)$, but in the direction opposite to the charge's *anticipated position* $\boldsymbol{r}_a\equiv\boldsymbol{r}(t')+(\Delta t)\boldsymbol{v}(t')$. This is where the charge *would be* after a time Δt *if* it were moving with a *constant velocity* $\boldsymbol{v}(t')$; the time interval corresponds to the time required for a light signal to travel from the charge's retarded position to the observation point. (The situation is described in Fig. 6.2.) So the field brings information to the

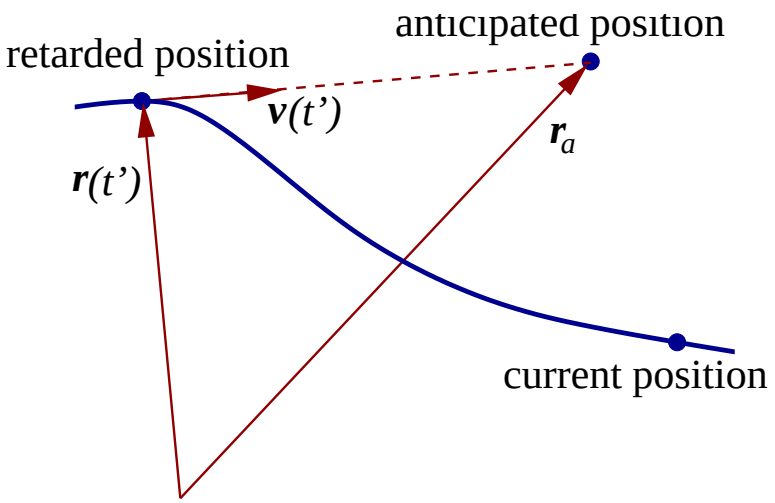

Figure 6.2: Retarded position, current position, and anticipated position of an accelerated charge.

observer that does indeed originate from the retarded position, but the field orients itself *as though it were anticipating the motion of the particle during the time the information propagates from the charge to the observer*. Because the extrapolation is based only on partial information (the values of $\boldsymbol{r}$ and $\boldsymbol{v}$ at the retarded time, and nothing more), it is not fully accurate, and the field misses the actual position of the particle.

Notice that uniform motion is a special case: when the velocity is constant, the charge's anticipated position $\boldsymbol{r}_a$ *coincides* with its current position $\boldsymbol{r}(t)$, and the field "miraculously" orients itself in this direction. This explains the result found in Sec. 6.2: the information needed time to reach the observer, but it was sufficient to correctly anticipate the actual motion of the charge.

6.3.3 Uniform motion

Let us verify that the general expression of Eq. (6.3.12) reduces to Eq. (6.2.3) in the case of uniform motion in the x direction.

We set $\boldsymbol{r}(t) = (vt, b, 0)$, $\boldsymbol{r}(t') = (vt', b, 0) \equiv \boldsymbol{r'}$, and $\boldsymbol{v}(t) = \boldsymbol{v}(t') = v\hat{\boldsymbol{x}}$. We once more take the observation point to be at $\boldsymbol{x} = 0$. Because $\boldsymbol{a} = \boldsymbol{0}$, Eq. (6.3.12) reduces to

$$\boldsymbol{E} = -\frac{q}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{r}}' + \boldsymbol{v}/c}{\gamma^2\kappa^3 r'^2} = -\frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{r}' + r'\boldsymbol{v}/c}{\gamma^2\kappa^3 r'^3},$$

or

$$\boldsymbol{E} = -\frac{q}{4\pi\epsilon_0}\frac{\gamma(\boldsymbol{r}' + r'\boldsymbol{v}/c)}{(\gamma\kappa r')^3}. \tag{6.3.15}$$

We recall that $\kappa = 1 + \boldsymbol{v}\cdot\hat{\boldsymbol{r}}'/c$.

The retarded condition is

$$r' = \sqrt{b^2 + (vt')^2} = c(t - t'),$$

and it gives, after squaring,

$$(1 - v^2/c^2)t'^2 - 2tt' + t^2 - b^2/c^2 = 0. \tag{6.3.16}$$

This is a quadratic equation for t', which we do not need to solve.

The numerator of Eq. (6.3.15) involves the vector $\boldsymbol{r'} + r'\boldsymbol{v}/c$. Its x component is $vt' + r'v/c = vt' + c(t-t')v/c = vt$, and this vector therefore coincides with $\boldsymbol{r}$. In the denominator we have $(\gamma\kappa r')^2$ raised to the power 3/2. This is

$$\begin{aligned}(\gamma\kappa r')^2 &= \gamma^2(r' + \boldsymbol{v}\cdot\boldsymbol{r'}/c)^2 \\ &= \gamma^2(r' + v^2t'/c)^2 \\ &= \gamma^2c^2(t - t' + v^2t'/c^2)^2 \\ &= \gamma^2c^2(t - t'/\gamma^2)^2 \\ &= c^2(\gamma^2t^2 - 2tt' + t'^2/\gamma^2).\end{aligned}$$

We now use Eq. (6.3.16) to eliminate t' and obtain

$$\begin{aligned}(\gamma\kappa r')^2 &= c^2\left[b^2/c^2 + \gamma^2t^2(1 - 1/\gamma^2)\right] \\ &= b^2 + \gamma^2v^2t^2.\end{aligned}$$

Substituting these results into Eq. (6.3.15) gives

$$\boldsymbol{E} = -\frac{q}{4\pi\epsilon_0}\frac{\gamma\boldsymbol{r}(t)}{(b^2 + \gamma^2v^2t^2)^{3/2}}, \tag{6.3.17}$$

the same statement as Eq. (6.2.3).

6.3.4 Summary

The scalar and vector potentials of a point charge in arbitrary motion described by the position vector $\boldsymbol{r}(t)$ are given by the Liénard-Wiechert expressions

$$\Phi(t,\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\kappa R}\right]_{\text{ret}} \tag{6.3.18}$$

and

$$\boldsymbol{A}(t,\boldsymbol{x}) = \frac{q\mu_0}{4\pi}\left[\frac{\boldsymbol{v}}{\kappa R}\right]_{\text{ret}}, \tag{6.3.19}$$

where $\boldsymbol{R}(t') = \boldsymbol{x} - \boldsymbol{r}(t')$, $R = |\boldsymbol{R}|$, $\hat{\boldsymbol{R}} = \boldsymbol{R}/R$,

$$\kappa = 1 - \boldsymbol{v}(t')\cdot\hat{\boldsymbol{R}}(t')/c, \tag{6.3.20}$$

and where t' is determined by the retarded condition

$$|\boldsymbol{x} - \boldsymbol{r}(t')| = c(t - t'). \tag{6.3.21}$$

The electric field is

$$\boldsymbol{E}(t,\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\boldsymbol{R}} - \boldsymbol{v}/c}{\gamma^2\kappa^3R^2} + \frac{\hat{\boldsymbol{R}}\times\left[(\hat{\boldsymbol{R}} - \boldsymbol{v}/c)\times\boldsymbol{a}\right]}{\kappa^3c^2R}\right]_{\text{ret}}, \tag{6.3.22}$$

and the magnetic field is

$$\boldsymbol{B} = \frac{1}{c}\left[\hat{\boldsymbol{R}}\right]_{\text{ret}}\times\boldsymbol{E}. \tag{6.3.23}$$

Both fields are naturally decomposed into velocity (bound) fields that decay as $1/R^2$, and acceleration (radiative) fields that decay as $1/R$.

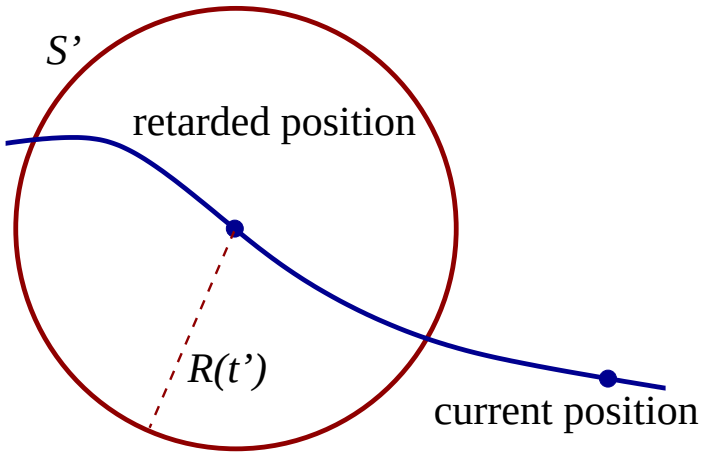

Figure 6.3: Sphere S' centered at the retarded position $\boldsymbol{r}(t')$ of the accelerated charge. The sphere has a constant radius $R(t') \equiv |\boldsymbol{x} - \boldsymbol{r}(t')|$.

6.4 Radiation from an accelerated charge

[The material presented in this section is also covered in Secs. 14.2, 14.3, and 14.4 of Jackson's text.]

At large distances from the particle — in the wave zone — we can neglect the velocity fields and approximate the electric field by

$$\boldsymbol{E}(t,\boldsymbol{x}) = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\boldsymbol{R}}\times\left[(\hat{\boldsymbol{R}}-\boldsymbol{v}/c)\times\boldsymbol{a}\right]}{\kappa^3 c^2 R}\right]_{\rm ret}, \tag{6.4.1}$$

and the magnetic field by

$$\boldsymbol{B} = \frac{1}{c}\left[\hat{\boldsymbol{R}}\right]_{\rm ret}\times\boldsymbol{E}. \tag{6.4.2}$$

We recall that $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{r}(t')$, $R = |\boldsymbol{R}|$, $\hat{\boldsymbol{R}} = \boldsymbol{R}/R$, $\kappa = 1 - \boldsymbol{v}(t')\cdot\hat{\boldsymbol{R}}(t')/c$, and that t' is determined by the retarded condition $t - t' = R(t')/c$. In this section we use these expressions to calculate how much power is radiated by the accelerated charge. We also determine the angular distribution of the radiated power.

6.4.1 Angular profile of radiated power

Because our expressions for the fields refer directly to the charge's retarded position, it is a relatively simple task to calculate the power that crosses a sphere S' centered at this position (see Fig. 6.3); it would be much more complicated to adopt any other reference point. The sphere S' has a constant radius $R(t')$ and its element of surface area is $d\boldsymbol{a'} = R^2(t')\hat{\boldsymbol{R}}(t')\,d\Omega'$, where $d\Omega'$ is an element of solid angle on the sphere.

The energy crossing an element of S' per unit time t is, by definition, $\boldsymbol{S}(t')\cdot d\boldsymbol{a'}$, where $\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{B}/\mu_0$ is the Poynting vector. But because all quantities are expressed in terms of retarded time t' instead of t, it is convenient to convert this "per-unit-t-time" flux to a "per-unit-t'-time" measure. We have that

$$\boldsymbol{S}(t')\cdot d\boldsymbol{a'}\frac{dt}{dt'}$$

is the energy crossing an element of S' per unit t'-time, and the conversation factor is $dt/dt' = 1 + c^{-1}dR(t')/dt' = 1 - \boldsymbol{v}(t')\cdot\hat{\boldsymbol{R}}(t')/c \equiv \kappa$. The displayed quantity can

therefore be expressed as $\boldsymbol{S}(t')\cdot\hat{\boldsymbol{R}}(t')R^2(t')\kappa\,d\Omega'$, which is a definition for $dP' = (dP'/d\Omega')\,d\Omega'$. We have arrived at

$$\begin{aligned}\frac{dP'}{d\Omega'} &= \kappa R^2 \boldsymbol{S}\cdot\hat{\boldsymbol{R}} \\ &= \text{energy crossing an element of } S' \\ &\quad \text{per unit } t'\text{-time and unit solid angle.}\end{aligned} \tag{6.4.3}$$

Because it is understood that all quantities refer to the retarded time t', we no longer have a need for the redundant notation $[\,]_{\rm ret}$. Integrating Eq. (6.4.3) over $d\Omega'$ gives the total power crossing S'. Integrating this over dt' (which involves displacing S' in response to the particle's motion during integration) returns the total energy radiated by the charge.

The Poynting vector is given by

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B} = \frac{1}{\mu_0 c}\boldsymbol{E}\times(\hat{\boldsymbol{R}}\times\boldsymbol{E}) = \frac{1}{\mu_0 c}|\boldsymbol{E}|^2\hat{\boldsymbol{R}},$$

because $\boldsymbol{E}$ is orthogonal to $\hat{\boldsymbol{R}}$. This gives

$$\frac{dP'}{d\Omega'} = \frac{1}{\mu_0 c}\kappa R^2|\boldsymbol{E}|^2 = \frac{1}{\mu_0 c}\frac{q^2}{(4\pi\epsilon_0)^2}\frac{\big|\hat{\boldsymbol{R}}\times[(\hat{\boldsymbol{R}}-\boldsymbol{v}/c)\times\boldsymbol{a}]\big|^2}{\kappa^5 c^4},$$

or

$$\frac{dP'}{d\Omega'} = \frac{\mu_0 q^2}{(4\pi)^2 c}\frac{\big|\hat{\boldsymbol{R}}\times[(\hat{\boldsymbol{R}}-\boldsymbol{v}/c)\times\boldsymbol{a}]\big|^2}{\kappa^5}. \tag{6.4.4}$$

The general expression for the angular profile is complicated, and it depends on the individual directions of $\boldsymbol{v}$, $\boldsymbol{a}$, and $\boldsymbol{R}$. To gain insight into this it will help to look at specific simple situations; we shall do so in the following subsections. To ease the notation we shall now agree to drop the primes on $dP'/d\Omega'$.

6.4.2 Slow motion: Larmor formula

Our first special case will be the limit $v \ll c$. In this slow-motion approximation we can neglect $\boldsymbol{v}/c$ in front of $\hat{\boldsymbol{R}}$ in Eq. (6.4.4), and also set $\kappa \simeq 1$. We then have

$$\begin{aligned}\frac{dP}{d\Omega} &\simeq \frac{\mu_0 q^2}{(4\pi)^2 c}\big|\hat{\boldsymbol{R}}\times(\hat{\boldsymbol{R}}\times\boldsymbol{a})\big|^2 \\ &\simeq \frac{\mu_0 q^2}{(4\pi)^2 c}\big|\boldsymbol{a}-(\boldsymbol{a}\cdot\hat{\boldsymbol{R}})\hat{\boldsymbol{R}}\big|^2 \\ &\simeq \frac{\mu_0 q^2}{(4\pi)^2 c}\Big[|\boldsymbol{a}|^2-(\boldsymbol{a}\cdot\hat{\boldsymbol{R}})^2\Big].\end{aligned}$$

If θ denotes the angle between the vectors $\boldsymbol{a}$ and $\hat{\boldsymbol{R}}$, then $|\boldsymbol{a}|^2-(\boldsymbol{a}\cdot\hat{\boldsymbol{R}})^2 = |\boldsymbol{a}|^2(1-\cos^2\theta) = |\boldsymbol{a}|^2\sin^2\theta$, and we finally obtain

$$\frac{dP}{d\Omega} \simeq \frac{\mu_0}{(4\pi)^2 c}\big(q|\boldsymbol{a}|\big)^2\sin^2\theta. \tag{6.4.5}$$

This expression could have been derived on the basis of the electric-dipole approximation of Sec. 5.4. According to the results of that section, in particular Eq. (5.4.22), the angular profile of radiation produced by a slowly-moving source is given by

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c}|\ddot{\boldsymbol{p}}|^2\sin^2\Theta,$$

where $\boldsymbol{p}$ is the dipole moment of the charge distribution, and Θ the angle between the vectors $\hat{\boldsymbol{r}} \equiv \boldsymbol{x}/|\boldsymbol{x}|$ and $\ddot{\boldsymbol{p}}$. Here $\boldsymbol{p} = q\boldsymbol{r}(t)$, so that $\ddot{\boldsymbol{p}} = q\boldsymbol{a}$. And because the sphere S' is centered on the particle, the vector $\hat{\boldsymbol{R}}$ plays the same role as $\hat{\boldsymbol{r}}$, so that $\Theta \equiv \theta$. The electric-dipole approximation therefore produces the same result as in Eq. (6.4.5).

Integration of $dP/d\Omega$ over all angles yields

$$P \simeq \frac{\mu_0}{6\pi c}\big(q|\boldsymbol{a}|\big)^2. \tag{6.4.6}$$

This result is known as *Larmor's formula*.

6.4.3 Linear motion

In this subsection we no longer restrict the size of v/c, but we assume that $\boldsymbol{v}$ and $\boldsymbol{a}$ are pointing in the same direction. This condition might hold at all times, but it is sufficient to assume that it holds only momentarily. At this moment the particle is moving on a straight line. We now let θ be the angle between $\hat{\boldsymbol{R}}$ and the common direction of $\boldsymbol{v}$ and $\boldsymbol{a}$.

We have that $\boldsymbol{v} \times \boldsymbol{a} = \boldsymbol{0}$, so Eq. (6.4.4) becomes

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{(4\pi)^2 c} \frac{\big|\hat{\boldsymbol{R}} \times (\hat{\boldsymbol{R}} \times \boldsymbol{a})\big|^2}{\kappa^5}.$$

We already know that $|\hat{\boldsymbol{R}}\times(\hat{\boldsymbol{R}}\times\boldsymbol{a})|^2 = |\boldsymbol{a}|^2 \sin^2\theta$, and we recall that $\kappa = 1-\boldsymbol{v}\cdot\hat{\boldsymbol{R}}/c = 1 - v\cos\theta/c$. Substituting these results gives

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c}\big(q|\boldsymbol{a}|\big)^2 \frac{\sin^2\theta}{(1 - v\cos\theta/c)^5}. \tag{6.4.7}$$

For $v/c \ll 1$ this reproduces the $\sin^2\theta$ behaviour encountered in the preceding subsection. For v/c approaching unity, however, the angular profile is strongly modified — it tips forward in the direction of $\boldsymbol{v}$ ($\theta = 0$), and the power radiated at small angles is very large (see Fig. 6.4). For such values of v/c, the radiation is *beamed* in the forward direction.

The angular function is

$$f = \frac{\sin^2\theta}{(1 - v\cos\theta/c)^5}. \tag{6.4.8}$$

Differentiating with respect to $\mu \equiv \cos\theta$ and putting the result to zero indicates that the maximum of the distribution is at the value of μ that solves $(3v/c)\mu^2 + 2\mu - 5v/c = 0$. So most of the radiation is emitted at angles close to $\theta_{\rm max}$, which is determined by

$$\cos\theta_{\rm max} = \frac{\sqrt{1 + 15(v/c)^2} - 1}{3(v/c)}. \tag{6.4.9}$$

For $v/c \ll 1$, this expression is well approximated by $\cos\theta_{\rm max} \simeq \frac{5}{2}(v/c) - \frac{75}{8}(v/c)^3 + \cdots$, and $\theta_{\rm max} \simeq \pi/2$: most of the radiation is emitted in the directions transverse to $\boldsymbol{v}$. This is the usual nonrelativistic situation. For v/c close to unity, however, so that $\gamma \gg 1$, Eq. (6.4.9) is well approximated by $\cos\theta_{\rm max} \simeq 1 - \frac{1}{8}\gamma^{-2} - \frac{43}{512}\gamma^{-4} + \cdots$, and $\theta_{\rm max}$ is very small: most of the radiation is emitted in the forward direction. For such small angles we have $\cos\theta_{\rm max} = 1 - \frac{1}{2}\theta_{\rm max}^2 + \cdots$, and this gives us the approximate relation

$$\theta_{\rm max} \simeq \frac{1}{2\gamma}, \qquad \gamma \gg 1. \tag{6.4.10}$$

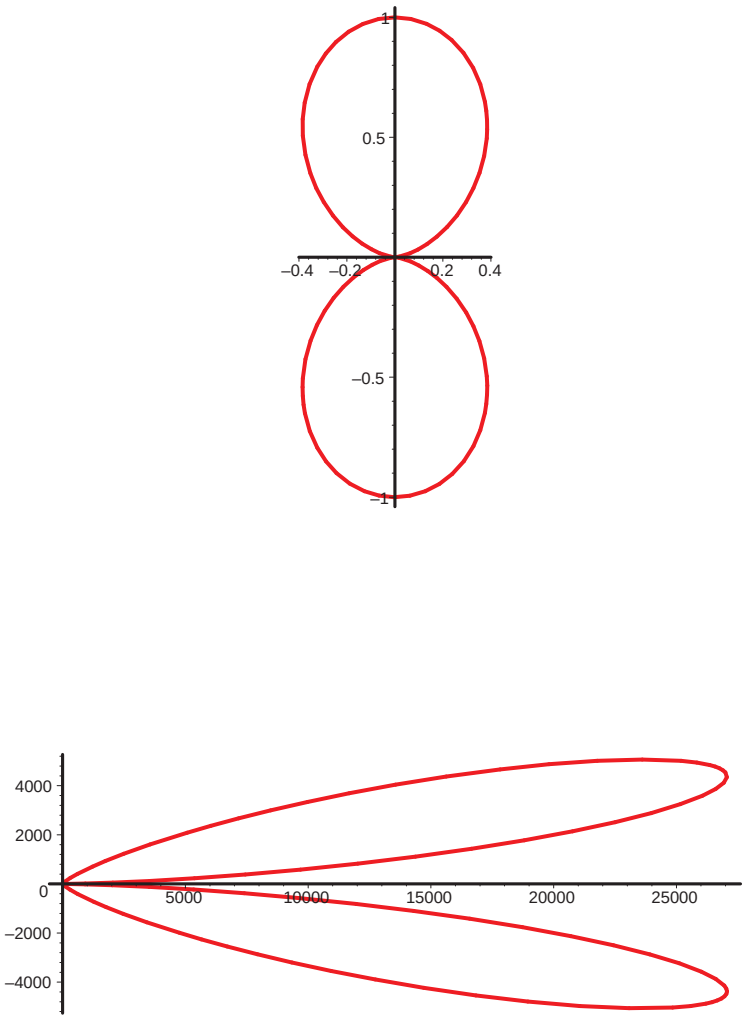

Figure 6.4: The first plot shows the angular profile of radiation produced by a nonrelativistic particle in linear motion; the radiation propagates mostly in the directions transverse to the charge's velocity (the horizontal direction in the plot). The second plot shows the angular profile of radiation produced by a relativistic particle in linear motion; the radiation is beamed in the forward direction with an angular width of order $1/(2\gamma)$. Notice the different scales in the two diagrams.

This is the statement that when the charge moves with a highly relativistic speed, the radiation is strongly beamed in the direction of the velocity vector.

The total power radiated is obtained by integrating Eq. (6.4.7) with respect to $d\Omega = \sin\theta\, d\theta d\phi$. We have

$$P = \frac{\mu_0}{(4\pi)^2 c}\big(q|\boldsymbol{a}|\big)^2 2\pi \int_0^\pi \frac{\sin^3\theta\, d\theta}{(1 - v\cos\theta/c)^5},$$

and the integral evaluates to $\frac{4}{3}\gamma^6$. The final answer is therefore

$$P = \frac{\mu_0}{6\pi c}\big(q|\boldsymbol{a}|\big)^2 \gamma^6. \tag{6.4.11}$$

This generalizes the Larmor formula of Eq. (6.4.6). It is instructive to express this in terms of the force acting on the particle. This force is $\boldsymbol{F} = d\boldsymbol{p}/dt$, where $\boldsymbol{p} = m\gamma\boldsymbol{v}$ is the particle's relativistic momentum. For linear motion in the x direction we have $p_x = mv/(1 - v^2/c^2)^{1/2}$ and

$$\frac{dp_x}{dt} = \frac{ma}{(1 - v^2/c^2)^{1/2}} + \frac{(mv)(v/c^2)a}{(1 - v^2/c^2)^{3/2}} = \frac{ma}{(1 - v^2/c^2)^{3/2}}.$$

This result can be expressed as $\boldsymbol{F} = m\gamma^3\boldsymbol{a}$, and Eq. (6.4.11) becomes

$$P = \frac{\mu_0}{6\pi c}\left(\frac{q|\boldsymbol{F}|}{m}\right)^2. \tag{6.4.12}$$

This is the total power radiated by a charge in instantaneous linear motion.

6.4.4 Circular motion

We now take $\boldsymbol{a}$ and $\boldsymbol{v}$ to be orthogonal vectors. This condition might hold at all times, but it is sufficient to assume that it holds only momentarily. At this moment the particle's motion is circular. For concreteness we let $\boldsymbol{a}$ be directed along the x axis, and $\boldsymbol{v}$ points in the z direction. We let the angles (θ, ϕ) give the direction of the vector $\hat{\boldsymbol{R}}$ relative to this set of axes. We therefore have $\boldsymbol{a} = a\hat{\boldsymbol{x}}$, $\boldsymbol{v} = v\hat{\boldsymbol{z}}$, and

$$\hat{\boldsymbol{R}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}.$$

According to this, $\boldsymbol{v}\cdot\hat{\boldsymbol{R}} = v\cos\theta$, $\boldsymbol{a}\cdot\hat{\boldsymbol{R}} = a\sin\theta\cos\phi$, $\kappa = 1 - v\cos\theta/c$, and

$$\begin{aligned}\hat{\boldsymbol{R}}\times\big[(\hat{\boldsymbol{R}} - \boldsymbol{v}/c)\times\boldsymbol{a}\big] &= (\hat{\boldsymbol{R}}\cdot\boldsymbol{a})(\hat{\boldsymbol{R}} - \boldsymbol{v}/c) - \hat{\boldsymbol{R}}\cdot(\hat{\boldsymbol{R}} - \boldsymbol{v}/c)\boldsymbol{a} \\ &= (\hat{\boldsymbol{R}}\cdot\boldsymbol{a})(\hat{\boldsymbol{R}} - \boldsymbol{v}/c) - \kappa\boldsymbol{a}.\end{aligned}$$

The squared norm of this vector is

$$\begin{aligned}\big|\hat{\boldsymbol{R}}\times[(\hat{\boldsymbol{R}} - \boldsymbol{v}/c)\times\boldsymbol{a}]\big|^2 &= (\hat{\boldsymbol{R}}\cdot\boldsymbol{a})^2(\hat{\boldsymbol{R}} - \boldsymbol{v}/c)\cdot(\hat{\boldsymbol{R}} - \boldsymbol{v}/c) \\ &\quad - 2\kappa(\hat{\boldsymbol{R}}\cdot\boldsymbol{a})(\hat{\boldsymbol{R}} - \boldsymbol{v}/c)\cdot\boldsymbol{a} + \kappa^2 a^2 \\ &= (\hat{\boldsymbol{R}}\cdot\boldsymbol{a})^2(1 - 2\hat{\boldsymbol{R}}\cdot\boldsymbol{v}/c + v^2/c^2 - 2\kappa) + \kappa^2 a^2 \\ &= -(\hat{\boldsymbol{R}}\cdot\boldsymbol{a})^2(1 - v^2/c^2) + \kappa^2 a^2 \\ &= \kappa^2 a^2\left[1 - \frac{(\hat{\boldsymbol{R}}\cdot\hat{\boldsymbol{a}})^2}{\kappa^2\gamma^2}\right].\end{aligned}$$

Substituting this result into Eq. (6.4.4) gives

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c}\frac{q^2 a^2}{\kappa^3}\left[1 - \frac{(\hat{\boldsymbol{R}}\cdot\hat{\boldsymbol{a}})^2}{\kappa^2\gamma^2}\right],$$

Figure 6.5: Angular profile of radiation produced by a relativistic charge in circular motion. The radiation is beamed in the (horizontal) direction of the velocity vector.

or

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c}\frac{q^2|\boldsymbol{a}|^2}{(1-v\cos\theta/c)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-v\cos\theta/c)^2}\right]. \tag{6.4.13}$$

This angular profile is rather complex, and quantitatively different from the one encountered in the preceding subsection; but it still features a strong beaming in the forward direction (direction of $\boldsymbol{v}$) when v is close to the speed of light. This is illustrated in Fig. 6.5.

The total power radiated is obtained by integrating $dP/d\Omega$ over the angles θ and ϕ. The result is

$$P = \frac{\mu_0}{6\pi c}\big(q|\boldsymbol{a}|\big)^2\gamma^4. \tag{6.4.14}$$

This is a different generalization of the Larmor formula of Eq. (6.4.6). We may also express this in terms of the force $\boldsymbol{F}$ acting on the particle. In this case the force changes only the direction of the vector $\boldsymbol{v}$, but not its magnitude. So $\boldsymbol{F} = d\boldsymbol{p}/dt = d(m\gamma\boldsymbol{v})/dt = m\gamma\boldsymbol{a}$, and Eq. (6.4.14) becomes

$$P = \frac{\mu_0}{6\pi c}\left(\frac{q|\boldsymbol{F}|}{m}\right)^2\gamma^2. \tag{6.4.15}$$

This is the total power radiated by a charge in instantaneous circular motion. You should compare this with Eq. (6.4.12) and notice the extra factor of γ^2.

6.4.5 Synchrotron radiation

We continue our discussion of circular motion and provide additional details. We now assume that the charge moves at all times on a circle of radius ρ at a constant angular velocity Ω. Its position vector is

$$\boldsymbol{r}(t') = \rho\big[\cos(\Omega t')\hat{\boldsymbol{x}} + \sin(\Omega t')\hat{\boldsymbol{y}}\big], \tag{6.4.16}$$

its velocity vector is

$$\boldsymbol{v}(t') = v\big[-\sin(\Omega t')\hat{\boldsymbol{x}} + \cos(\Omega t')\hat{\boldsymbol{y}}\big], \tag{6.4.17}$$

where $v = \Omega\rho$, and its acceleration vector is

$$\boldsymbol{a}(t') = -v\Omega\big[\cos(\Omega t')\hat{\boldsymbol{x}} + \sin(\Omega t')\hat{\boldsymbol{y}}\big]. \tag{6.4.18}$$

We wish to calculate the electric field that would be measured by a detector situated far away, in the wave zone. We place this detector on the y axis, at position $\boldsymbol{x} = r\hat{\boldsymbol{y}}$, and we assume that $r \gg \rho$.

The wave-zone electric field is given by Eq. (6.4.1), which we first evaluate as a function of retarded time t'. The relation between t' and true time t is given by Eq. (6.4.20). It is useful to recall that t' is time measured by a clock that moves with the charged particle, while t is time measured by a clock attached to the detector. As we shall see, these measurements of time differ substantially when v is comparable to c, and this leads to an interesting modulation of the electromagnetic wave.

We use the condition $r \gg \rho$ to calculate $\boldsymbol{E}$ approximately. We shall need some level of precision in the calculation of $t(t')$, because it is this function which determines the phasing of the electric field. A much cruder approximation will suffice when calculating the field's amplitude.

Working to first order in ρ/r, it is easy to see that the length of the vector $\boldsymbol{R} \equiv \boldsymbol{x} - \boldsymbol{r}(t')$ is given by $R(t') = r - \rho\sin\Omega t'$. Equation (6.3.21) then gives $t = t' + r/c - (\rho/c)\sin\Omega t'$, or

$$\Omega(t - r/c) = \Omega t' - \frac{v}{c}\sin\Omega t'. \tag{6.4.19}$$

This gives the relationship between $t - r/c$, the retarded time measured at the detector, and t', the time measured at the particle. From this it follows that

$$\kappa = \frac{dt}{dt'} = 1 - \frac{v}{c}\cos\Omega t'. \tag{6.4.20}$$

We see that κ will deviate strongly from 1 when v is close to the speed of light and when $|\cos\Omega t'|$ is close to unity.

To perform the remaining calculations we allow ourselves to neglect all terms linear in ρ/r. We take $\hat{\boldsymbol{R}} = \hat{\boldsymbol{y}}$,

$$\hat{\boldsymbol{R}} - \boldsymbol{v}/c = (v/c)\sin(\Omega t')\hat{\boldsymbol{x}} - \big[1 - (v/c)\cos(\Omega t')\big]\hat{\boldsymbol{y}},$$

and $\hat{\boldsymbol{R}}\times(\hat{\boldsymbol{R}} - \boldsymbol{v}/c)\times\boldsymbol{a} = -v\Omega(v/c - \cos\Omega t')\hat{\boldsymbol{x}}$. Substituting all this into Eq. (6.4.1) gives $\boldsymbol{E} = E\hat{\boldsymbol{x}}$, with

$$E = -\frac{q}{4\pi\epsilon_0}\frac{v\Omega}{c^2 r}\frac{v/c - \cos\Omega t'}{[1 - (v/c)\cos\Omega t']^3}. \tag{6.4.21}$$

The electric field can be expressed in terms of detector time t by inverting Eq. (6.4.19). Its behaviour is displayed in Fig. 6.6 for selected values of $\gamma = [1 - (v/c)^2]^{-1/2}$. The plots reveal that for γ moderately larger than unity, the wave deviates strongly from a simple sinusoid. The phase modulations which originate from the discrepancy between t and t' produce a wave that is characterized by two distinct timescales: a long one associated with the rotational frequency Ω, and a short one associated with the pulse duration. The frequency spectrum of the radiation is presented in Fig. 6.7. These plots show that in the relativistic regime, the radiation is carried by a large number of harmonics of the fundamental frequency Ω; as γ increases the dominant harmonic is shifted to higher frequencies.

To understand these features it is helpful to consider the *extreme relativistic regime*, in which $\gamma \gg 1$. In this regime the basic mechanism behind the narrowing of the pulses is very easy to identify: When $v/c \to 1$ the denominator of Eq. (6.4.21)

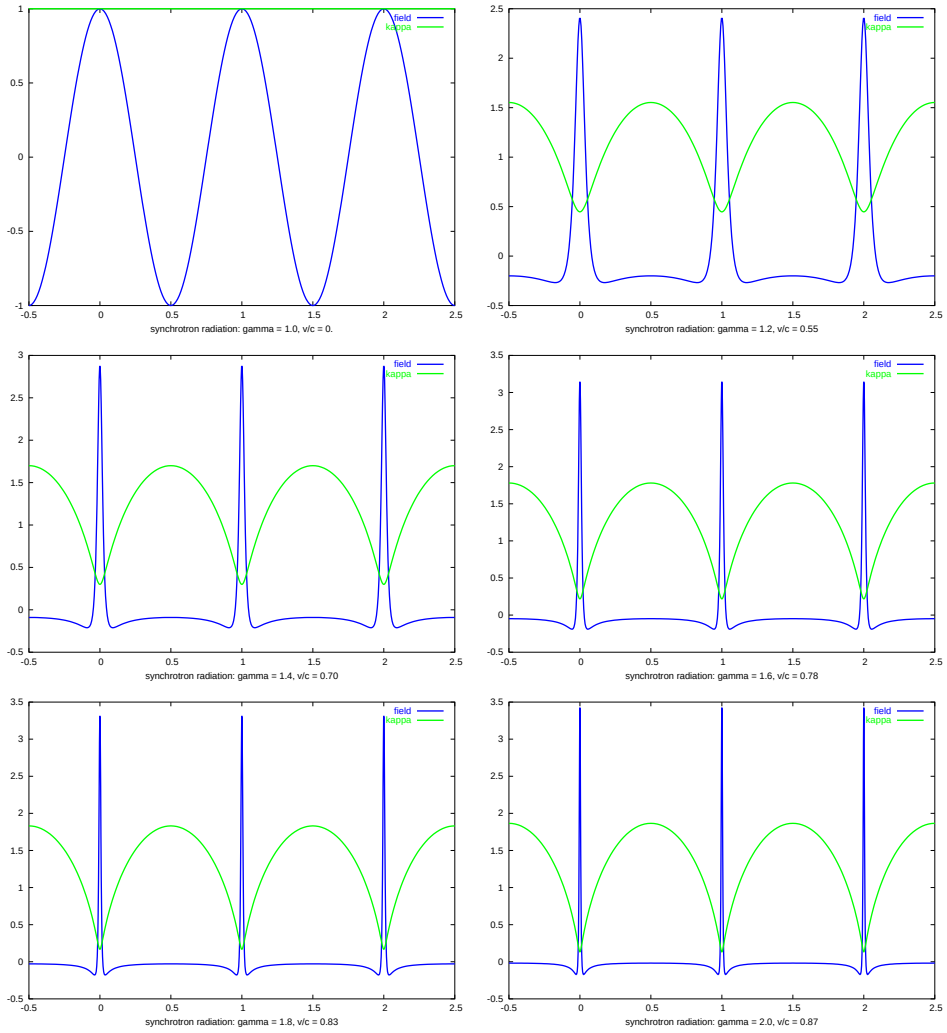

Figure 6.6: Electric field produced by a charged particle in circular motion, as a function of detector time t, for selected values of the relativistic γ factor. The plots also show $\kappa = dt/dt'$, whose deviation from 1 measures the magnitude of the relativistic time-dilation effect. The relativistic modulation of the wave pulse for large values of v/c is clearly apparent. In particular, the plots reveal that the radiation is characterized by two distinct time scales: a long one associated with the rotational frequency Ω, and a short one associated with the pulse duration.

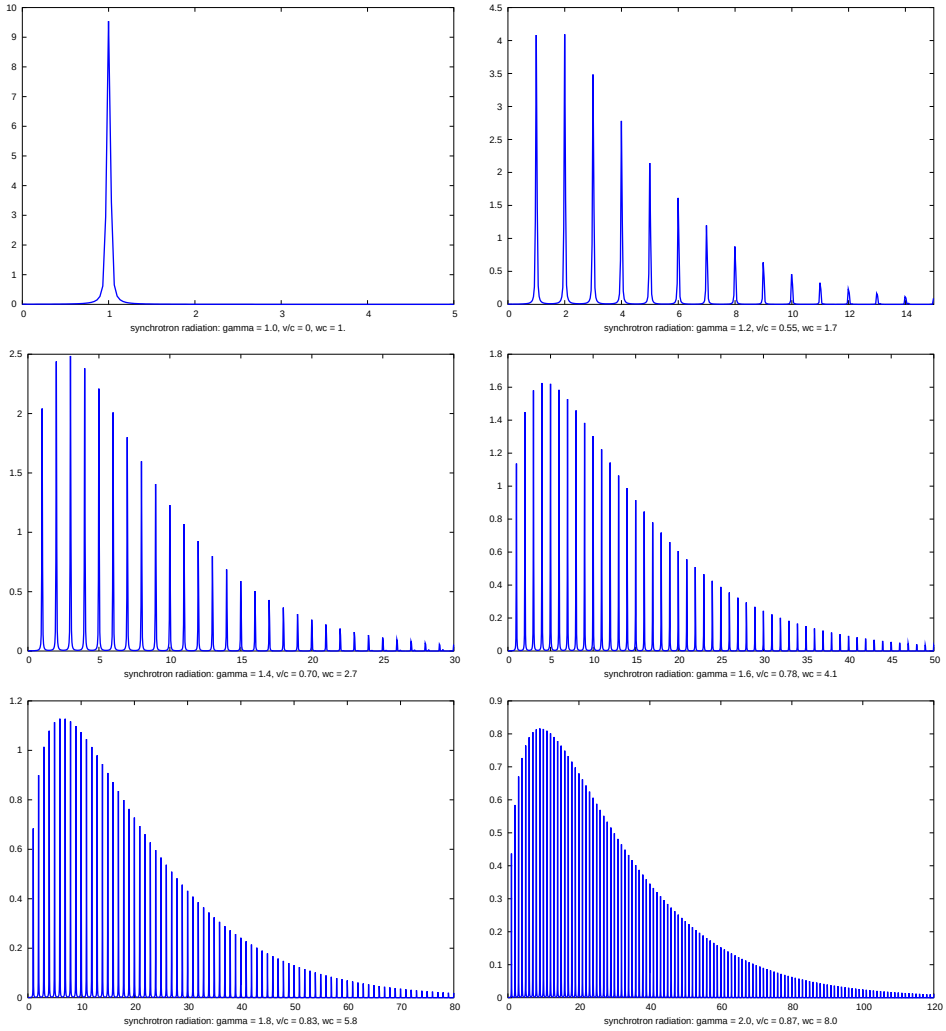

Figure 6.7: Fourier transform of the electric field produced by a charged particle in circular motion, for selected values of the relativistic γ factor. The plots reveal that the number of relevant harmonics of the fundamental frequency Ω increases with γ. In addition, the dominant harmonic is shifted to higher frequencies.

approaches zero whenever $\cos\Omega t' = 1$, which happens whenever $\Omega t'$ is a multiple of 2π. Most of the radiation is therefore emitted during a very short portion of the particle's orbit, just as the particle crosses the x axis in the positive sense. This radiation is strongly beamed in the forward direction, and it travels along the y axis toward the detector. During the remaining portion of the orbit little radiation is emitted, and the detector must wait for the completion of a full orbit before receiving the next pulse.

To express these ideas mathematically we write $v/c = 1 - \frac{1}{2}\gamma^{-2} + O(\gamma^{-4})$ and we approximate $\sin\Omega t' \simeq \Omega t'$ and $\cos\Omega t' \simeq 1 - \frac{1}{2}(\Omega t')^2$. From Eq. (6.4.20) we obtain

$$\kappa \simeq \frac{1}{2\gamma^2}\Big[1 + (\gamma\Omega t')^2\Big],$$

which shows that κ is small (of order $\gamma^{-2} \ll 1$) for a duration of order $(\gamma\Omega)^{-1}$ in terms of particle time t'. We also obtain

$$v/c - \cos\Omega t' \simeq -\frac{1}{2\gamma^2}\Big[1 - (\gamma\Omega t')^2\Big],$$

and Eq. (6.4.21) reduces to

$$E \simeq \frac{q}{4\pi\epsilon_0}\frac{v\Omega}{c^2 r}\,4\gamma^4\frac{1 - (\gamma\Omega t')^2}{[1 + (\gamma\Omega t')^2]^3}. \tag{6.4.22}$$

Again, this shows that in terms of particle time t', the radiation is "on" for a duration of order $(\gamma\Omega)^{-1}$.

To see how long the pulse lasts in terms of detector time t, we approximate Eq. (6.4.19) as $\Omega(t - r/c) \simeq \frac{1}{2}\gamma^{-2}\Omega t'$, which gives

$$\gamma\Omega t' \simeq 2\gamma^3\Omega(t - r/c). \tag{6.4.23}$$

This relation reveals that the pulse duration, as measured by a clock attached to the detector (and as displayed in Fig. 6.6) is given by

$$\Delta t_{\rm pulse} \sim \frac{1}{\gamma^3\Omega}. \tag{6.4.24}$$

It is reduced by a factor $\gamma^{-3} \ll 1$ relative to the orbital timescale associated with the angular velocity Ω. As a consequence, the characteristic frequency ω_c of the radiation is shifted relative to the fundamental frequency Ω. It is given by $\omega_c \sim \gamma^3\Omega$, and the displacement of the spectrum toward higher frequencies when γ increases is seen very clearly in Fig. 6.7.

6.4.6 Extremely relativistic motion

To conclude this section we note that at any point on the arbitrary trajectory of a point charge, the applied force can be decomposed into a component that is parallel to the velocity vector $\boldsymbol{v}$, and a component $\boldsymbol{F}_\perp$ that is perpendicular to it ($\boldsymbol{F}_\perp \cdot \boldsymbol{v} = 0$). When the motion is extremely relativistic, $\gamma \gg 1$ and the total power emitted by the particle is well approximated by Eq. (6.4.15),

$$P = \frac{\mu_0}{6\pi c}\left(\frac{q|\boldsymbol{F}_\perp|}{m}\right)^2\gamma^2.$$

The contribution from the parallel force is smaller by a factor of $1/\gamma^2 \ll 1$, and it can be neglected. In this limit, therefore, the parallel force plays no role in the energetics of the radiation.

6.5 Radiation reaction

[The material presented in this section is also covered in Secs. 16.1, 16.2, and 16.3 of Jackson's text.]

In Sec. 5.6 we saw that a classical electron orbiting a classical proton emits electromagnetic radiation. As a result, the classical hydrogen atom loses energy at a rate determined by Larmor's formula,

$$\frac{dE}{dt} = -P = -\frac{\mu_0}{6\pi c}\big(q|\boldsymbol{a}|\big)^2.$$

This loss of energy translates into a decrease in the electron's orbital radius. The orbit gradually shrinks, and the classical atom is unstable to the emission of electromagnetic radiation.

It is natural to assume that the electron is subjected only to the electric field produced by the proton. This picture, however, is incomplete: So long as the electron feels only the proton's Coulomb field, its motion must stay circular (or more generally, elliptical). The electron's inspiraling motion cannot take place in this picture, in contradiction with our previous description. We are forced to conclude that the proton's electric field cannot be the only field acting on the electron. The electron must also be subjected to its *own electric field*. It is the electron's self-field which provides the force that drives the electron's inspiraling motion.

This conclusion is difficult to understand, because the electron's own field diverges at the electron's position. How can it then produce a (presumably finite) *radiation-reaction force* that drives the inspiral? The answer is that the divergent part of the electron's self-field exerts no force — it just travels along with the electron. The part of the self-field that is responsible for the radiation reaction is finite.

To figure out what this part is, let us ask what would happen if instead of imposing outgoing-wave boundary conditions on our spherical-wave solutions to Maxwell's equations, we were to impose incoming-wave boundary conditions. In this case we would have incoming waves bringing energy to the system instead of outgoing waves taking energy away. The atom's energy would then increase, and the electron would move *out* instead of in. We would, in fact, have a mirror image (more precisely, a time reversal) of the correct picture, and the radiation-reaction force would be acting in the opposite direction (pushing the orbit out instead of in).

We can figure out which part of the electron's field is producing the radiation-reaction force by examining what happens when we switch from outgoing-wave boundary conditions to incoming-wave boundary conditions. We will see that while part of the field is not affected by this switch, the remaining part changes sign. And we will see that while the unaffected part is infinite at the electron's position, the remaining part is finite. It is this part of the field that is responsible for the radiation reaction.

Let us examine the vector potential first. The physical solution to Maxwell's equations is the *retarded solution*,

$$\boldsymbol{A}_-(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{j}(t-|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x',$$

where $\boldsymbol{j}$ is the electron's current density. This describes waves that are properly outgoing in the wave zone. These waves remove energy from the system, and the radiation reaction drives the electron inward. An unphysical solution to Maxwell's equations is the *advanced solution*,

$$\boldsymbol{A}_+(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{j}(t+|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x',$$

and this describes waves that are *incoming* in the wave zone. These waves bring energy to the system, and the radiation reaction drives the electron outward. To isolate the part of the (retarded) vector potential that is responsible for the radiation reaction, we must examine both solutions simultaneously, and determine what changes when we choose one sign over the other.

We want to evaluate $\boldsymbol{A}_\pm$ near the electron, in the near zone. We can therefore take $|\boldsymbol{x}-\boldsymbol{x'}|$ to be small, and Taylor-expand the current density:

$$\boldsymbol{j}(t\pm|\boldsymbol{x}-\boldsymbol{x'}|/c)=\boldsymbol{j}(t)\pm\frac{|\boldsymbol{x}-\boldsymbol{x'}|}{c}\frac{\partial\boldsymbol{j}}{\partial t}(t)+\cdots.$$

Relative to the leading term, the second term is of order $(r_c/t_c)/c=v_c/c$, and we shall take this to be small — we are working under a slow-motion approximation. We have

$$\boldsymbol{A}_\pm=\frac{\mu_0}{4\pi}\int\frac{\boldsymbol{j}(t,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'\pm\frac{\mu_0}{4\pi c}\frac{d}{dt}\int\boldsymbol{j}(t,\boldsymbol{x'})\,d^3x'+\cdots.$$

We see that the first term is the same irrespective of the choice of boundary conditions (outgoing waves versus incoming waves); it cannot, therefore, contribute to a radiation-reaction force. The second term, however, changes sign under a change of boundary conditions, and the force it contributes will also change sign; this must be the vector potential's contribution to the reaction-reaction force. Because the retarded solution provides the correct sign, we shall write

$$\boldsymbol{A}_{\rm rr}(t,\boldsymbol{x})=-\frac{\mu_0}{4\pi c}\frac{d}{dt}\int\boldsymbol{j}(t,\boldsymbol{x'})\,d^3x'+\cdots, \tag{6.5.1}$$

and we will see that the fields this potential generates produce the radiation-reaction force that acts on the electron.

For the scalar potential we have the two solutions

$$\Phi_\pm(t,\boldsymbol{x})=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(t\pm|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x',$$

and we Taylor-expand the charge density:

$$\begin{aligned}\rho(t\pm|\boldsymbol{x}-\boldsymbol{x'}|/c) &= \rho(t)\pm\frac{|\boldsymbol{x}-\boldsymbol{x'}|}{c}\frac{\partial\rho}{\partial t}(t)+\frac{|\boldsymbol{x}-\boldsymbol{x'}|^2}{2c^2}\frac{\partial^2\rho}{\partial t^2}(t)\\ &\quad\pm\frac{|\boldsymbol{x}-\boldsymbol{x'}|^3}{6c^3}\frac{\partial^3\rho}{\partial t^3}(t)+\cdots.\end{aligned}$$

This gives

$$\begin{aligned}\Phi_\pm &= \frac{1}{4\pi\epsilon_0}\biggl[\int\frac{\rho(t,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\,d^3x'\pm\frac{1}{c}\frac{d}{dt}\int\rho(t,\boldsymbol{x'})\,d^3x'\\ &\quad+\frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int\rho(t,\boldsymbol{x'})|\boldsymbol{x}-\boldsymbol{x'}|\,d^3x'\\ &\quad\pm\frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int\rho(t,\boldsymbol{x'})|\boldsymbol{x}-\boldsymbol{x'}|^2\,d^3x'+\cdots\biggr].\end{aligned}$$

The first and third terms are insensitive to boundary conditions; they have nothing to do with radiation reaction. The second and fourth terms, on the other hand, change sign under a switch of boundary conditions, and they are associated with radiation reaction. The second term, however, vanishes by virtue of charge conservation, and we conclude that the radiation-reaction part of the scalar potential is given by (recall that $c^{-2}=\epsilon_0\mu_0$)

$$\Phi_{\rm rr}(t,\boldsymbol{x})=-\frac{\mu_0}{4\pi c}\frac{1}{6}\frac{\partial^3}{\partial t^3}\int\rho(t,\boldsymbol{x'})|\boldsymbol{x}-\boldsymbol{x'}|^2\,d^3x'+\cdots. \tag{6.5.2}$$

Up to now the charge and current densities have been left unspecified, and our considerations have been general: Eqs. (6.5.1) and (6.5.2) give the radiation-reaction potentials for an arbitrary distribution of change and current. These expressions will apply, in particular, to a point charge q if we set $\rho(t,\boldsymbol{x}') = q\delta(\boldsymbol{x}' - \boldsymbol{r}(t))$ and $\boldsymbol{j}(t,\boldsymbol{x}') = q\boldsymbol{v}(t)\delta(\boldsymbol{x}' - \boldsymbol{r}(t))$, where $\boldsymbol{v} = d\boldsymbol{r}/dt$. Substituting this into our expressions for the potentials returns

$$\Phi_{\rm rr}(t,\boldsymbol{x}) = -\frac{\mu_0}{4\pi c}\frac{q}{6}\frac{\partial^3}{\partial t^3}\big|\boldsymbol{x} - \boldsymbol{r}(t)\big|^2$$

and

$$\boldsymbol{A}_{\rm rr}(t,\boldsymbol{x}) = -\frac{\mu_0}{4\pi c}q\frac{d}{dt}\boldsymbol{v}(t).$$

We have $|\boldsymbol{x}-\boldsymbol{r}|^2 = (\boldsymbol{x}-\boldsymbol{r})\cdot(\boldsymbol{x}-\boldsymbol{r})$, and differentiating once with respect to t gives $-2(\boldsymbol{x}-\boldsymbol{r})\cdot\boldsymbol{v}$. Differentiating once more gives $-2(\boldsymbol{x}-\boldsymbol{r})\cdot\boldsymbol{a}+2\boldsymbol{v}\cdot\boldsymbol{v}$, and differentiating a third time yields $-2(\boldsymbol{x}-\boldsymbol{r})\cdot\dot{\boldsymbol{a}}+2\boldsymbol{v}\cdot\boldsymbol{a}+4\boldsymbol{v}\cdot\boldsymbol{a} = -2(\boldsymbol{x}-\boldsymbol{r})\cdot\dot{\boldsymbol{a}}+6\boldsymbol{v}\cdot\boldsymbol{a}$. We finally arrive at

$$\Phi_{\rm rr}(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi c}q\left[\frac{1}{3}(\boldsymbol{x}-\boldsymbol{r})\cdot\dot{\boldsymbol{a}} - \boldsymbol{v}\cdot\boldsymbol{a}\right] \tag{6.5.3}$$

and

$$\boldsymbol{A}_{\rm rr}(t,\boldsymbol{x}) = -\frac{\mu_0}{4\pi c}q\boldsymbol{a}. \tag{6.5.4}$$

Here we have defined $\boldsymbol{a} = d\boldsymbol{v}/dt$ and $\dot{\boldsymbol{a}} = d\boldsymbol{a}/dt$. These are the radiation-reaction potentials acting on a point charge q moving on a trajectory $\boldsymbol{r}(t)$.

The radiation-reaction part of the electric field is given by

$$\boldsymbol{E}_{\rm rr} = -\frac{\partial\boldsymbol{A}_{\rm rr}}{\partial t} - \boldsymbol{\nabla}\Phi_{\rm rr} = \frac{\mu_0}{4\pi c}q\left(\dot{\boldsymbol{a}} - \frac{1}{3}\dot{\boldsymbol{a}}\right),$$

or

$$\boldsymbol{E}_{\rm rr} = \frac{\mu_0}{6\pi c}q\dot{\boldsymbol{a}}. \tag{6.5.5}$$

Because $\boldsymbol{A}_{\rm rr}$ does not actually depend on $\boldsymbol{x}$, we also have

$$\boldsymbol{B}_{\rm rr} = \boldsymbol{0}. \tag{6.5.6}$$

The radiation-reaction force is therefore $\boldsymbol{F}_{\rm rr} = q\boldsymbol{E}_{\rm rr}$, or

$$\boldsymbol{F}_{\rm rr} = \frac{\mu_0}{6\pi c}q^2\dot{\boldsymbol{a}}. \tag{6.5.7}$$

This force acts on the point charge, and it originates from the charge's own electric field!

It is easy to verify that the radiation-reaction force does negative work on the particle and takes energy away from it. As the charge moves with velocity $\boldsymbol{v}$ the force does work at a rate

$$\dot{W} = \boldsymbol{F}_{\rm rr}\cdot\boldsymbol{v} = \frac{\mu_0}{6\pi c}q^2\dot{\boldsymbol{a}}\cdot\boldsymbol{v}.$$

We write

$$\dot{\boldsymbol{a}}\cdot\boldsymbol{v} = \frac{d}{dt}(\boldsymbol{a}\cdot\boldsymbol{v}) - |\boldsymbol{a}|^2,$$

and we average $\dot{W}$ over a time interval T. This gives

$$\begin{aligned}
\langle\dot{W}\rangle &\equiv \frac{1}{T}\int_0^T \dot{W}\,dt \\
&= \frac{\mu_0}{6\pi c}q^2\left[\frac{1}{T}\int_0^T \frac{d}{dt}(\boldsymbol{a}\cdot\boldsymbol{v})\,dt - \frac{1}{T}\int_0^T |\boldsymbol{a}|^2\,dt\right] \\
&= \frac{\mu_0}{6\pi c}q^2\left[\frac{1}{T}(\boldsymbol{a}\cdot\boldsymbol{v})\Big|_0^T - \big\langle|\boldsymbol{a}|^2\big\rangle\right].
\end{aligned}$$

Supposing now that the motion is periodic with period T (as it would be in the case of atomic motion), or else that it begins and ends with a vanishing acceleration, we may discard the boundary terms and obtain

$$\langle \dot{W} \rangle = -\frac{\mu_0}{6\pi c} q^2 \langle |\boldsymbol{a}|^2 \rangle = -\langle P \rangle, \tag{6.5.8}$$

where P is the power radiated by the point charge, as given by the Larmor formula of Eq. (6.4.6). The work done by the radiation-reaction force therefore matches the energy taken away by electromagnetic radiation, and we have energy conservation. (It is noteworthy that this statement holds only on the average.)

We have a complete resolution of the paradox stated at the beginning of this section. The electron's own field, or at least the finite part that is sensitive to the choice of boundary conditions, acts on the electron and prevents it from keeping its circular motion. (Indeed, it is easy to check that for a circular orbit, the radiation-reaction force of Eq. (6.5.7) points in the direction of $-\boldsymbol{v}$ and drives the orbit to a smaller radius.) Furthermore, the work done by this radiation-reaction force matches the amount of energy taken away by the radiation, as it well should if energy is to be conserved in the process.

6.6 Problems

1. A point charge q moves on a circle of fixed radius a with an angular frequency $\omega(t)$ that changes slowly with time. (The time scale over which ω changes is much longer than the rotational period.) We take $a\omega$ to be smaller than c, but we do not assume that it is much smaller than c; the charge has a Lorentz γ-factor that can be large compared with unity.

   The charge emits electromagnetic radiation. It therefore loses energy and this causes the angular frequency ω to change with time. You are asked to derive an equation for $d\omega/dt$ and to express your result in terms of ω, a, and other relevant constants. (You may need to know that the charge's relativistic energy is $E = \gamma mc^2$, where m is the charge's rest mass.)

2. (This problem is adapted from Jackson's problem 14.12.) A charge q moves in simple harmonic motion along the z axis. Its position is given as a function of t' by $z(t') = z_0 \cos(\omega t')$, where z_0 and ω are constants. It is assumed that $\beta \equiv z_0\omega/c$ is fairly close to unity, so that the motion is relativistic.

   a) Show that the time-averaged angular profile of the radiation is described by

   $$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0}{32\pi^2 c} (q z_0 \omega^2)^2 \frac{1 + (\beta/2)^2 \cos^2\theta}{(1 - \beta^2 \cos^2\theta)^{7/2}} \sin^2\theta,$$

   where θ is the angle from the z axis. The time average is defined by $\langle (\cdots) \rangle \equiv T^{-1} \int_0^T (\cdots)\, dt'$, where $T \equiv 2\pi/\omega$ is the period.

   b) Produce a parameteric plot of the angular function for $\beta = 0.95$. Comment on its appearance in view of the beaming effect discussed in Sec. 6.3.3.

3. A charge q moves on a trajectory described by $z(t') = \sqrt{b^2 + (ct')^2}$, where b is a constant. The particle comes in from infinity, it turns around at $z = b$ when $t' = 0$, and it eventually returns to infinity.

   Calculate P, the total power emitted by the particle. Show in particular that this does not depend on time t'.

4. (This problem is adapted from Jackson's problem 14.7.) A nonrelativistic particle of charge q, mass m, and initial speed v is incident on a fixed charge Q with an impact parameter b that is large enough to ensure that the particle's deflection in the course of the collision is very small.

   Calculate the total energy radiated by the particle. Because the deflection angle is so small, it is sufficient to assume that the particle's trajectory is a straight line and that its velocity vector is constant. But this approximation must be introduced only *after* you have calculated the particle's acceleration.