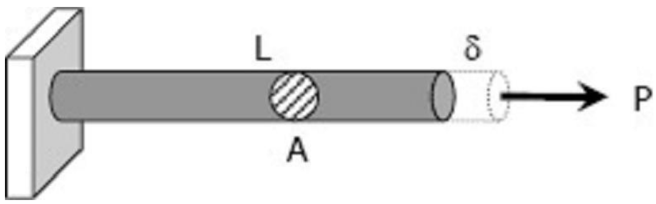
Chapter 2

Strain

Strain is the ratio of the change in length caused by the applied force, to the original length. It is known also as unit deformation.



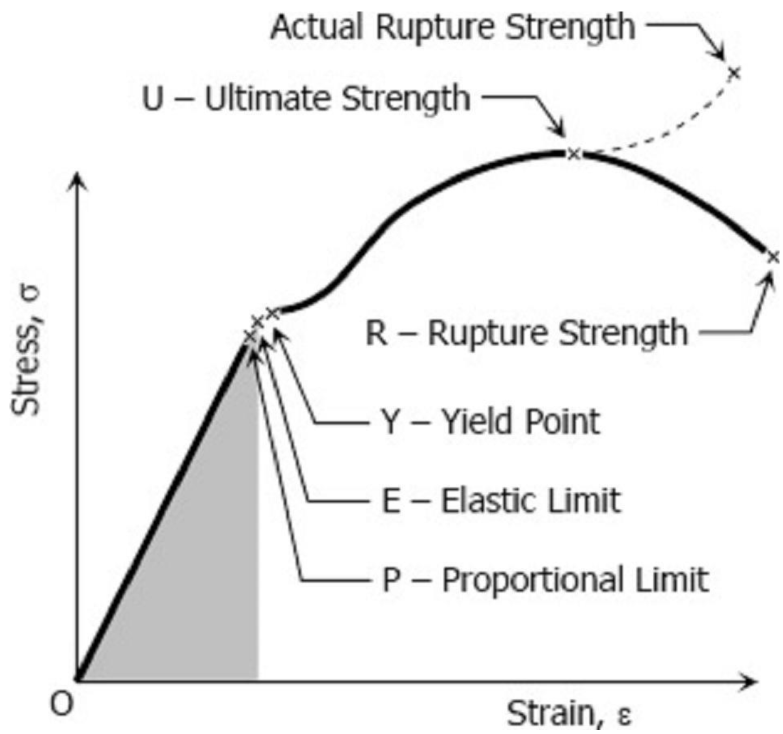
$$\varepsilon = \frac{\delta}{L}$$

Where δ the deformation and L is the original length, thus ε is dimensionless.

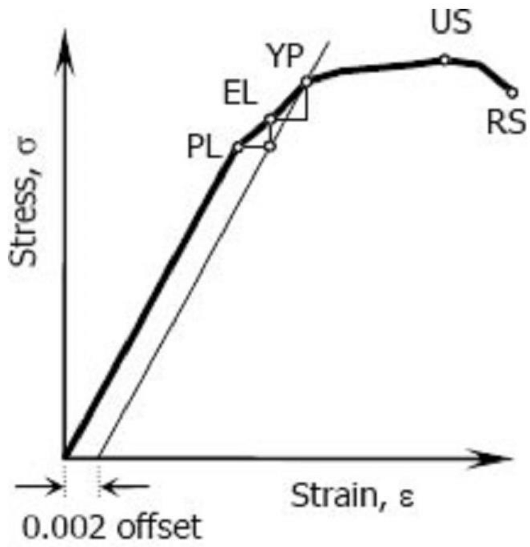
Stress-strain Diagram

Suppose that a metal specimen be placed in tension-compression-testing machine. As the axial load is gradually increased in increments, the total elongation over the gauge length is measured at each increment of the load and this is continued until failure of the specimen takes place. Knowing the original cross-sectional area and length of the specimen, the normal stress σ and the strain ε can be obtained. The graph of these quantities with the stress σ along the y-axis and the strain along the x-axis is called the **stress-strain diagram**. The stress-strain

diagram differs in form for various materials. The diagram shown below is that for a medium-carbon structural steel.



Metallic engineering materials are classified as either **ductile** or **brittle** materials. A ductile material is one having relatively large tensile strains up to the point of rupture like structural steel and aluminum, whereas brittle materials has a relatively small strain up to the point of rupture like cast iron and concrete. An arbitrary strain of 0.05 mm/mm is frequently taken as the dividing line between these two classes. To locate yield strength for brittle materials, a strain offset of 0.002 can be chosen as shown in the following figure:



Proportional Limit (Hooke's Law)

From the origin O to the point called proportional limit, the stress-strain curve is a straight line. This linear relation between elongation and the axial force causing is called **Hooke's Law** that within the proportional limit, the stress is directly proportional to strain or

$$\sigma \propto \varepsilon \text{ or } \sigma = k\varepsilon$$

The constant of proportionality *k* is called the **Modulus of Elasticity** *E* or **Young's Modulus** and is equal to the slope of the stress-strain diagram from O to P. Then

$$\sigma = E\varepsilon$$

Elastic Limit

The elastic limit is the limit beyond which the material will no longer go back to its original shape when the load is removed, or it is the maximum stress that may developed such that there is no permanent or residual deformation when the load is entirely removed. So, beyond elastic limit the material will no longer go back to its original shape when the load is removed.

Yield Point

Yield point is the point at which the material will have an appreciable elongation or yielding without any increase in load.

Ultimate Strength

The maximum ordinate in the stress-strain diagram is the ultimate strength or tensile strength.

Rapture Strength

Rapture strength is the strength of the material at rupture. This is also known as the breaking strength.

Modulus of Resilience

Modulus of resilience is the work done on a unit volume of material as the force is gradually increased from O to P, in N.m/m3. This may be calculated as the area under the stress-strain curve from the origin O to up to the elastic limit E (the shaded area in the figure). The resilience of the material is its ability to absorb energy without creating a permanent distortion.

Modulus of Toughness

Modulus of toughness is the work done on a unit volume of material as the force is gradually increased from O to R, in N.m/m3. This may be calculated as the area under the entire stress-strain curve (from O to R). The toughness of a material is its ability to absorb energy without causing it to break.

Working Stress, Allowable Stress, and Factor of Safety

Working stress is defined as the actual stress of a material under a given loading. The maximum safe stress that a material can carry is termed as the allowable stress. The allowable stress should be limited to values not exceeding the proportional limit. However, since proportional limit is difficult to determine accurately, the allowable tress is taken as either the yield point or ultimate strength divided by a factor of safety. The ratio of this strength (ultimate or yield strength) to allowable strength is called the factor of safety.

Axial Deformation

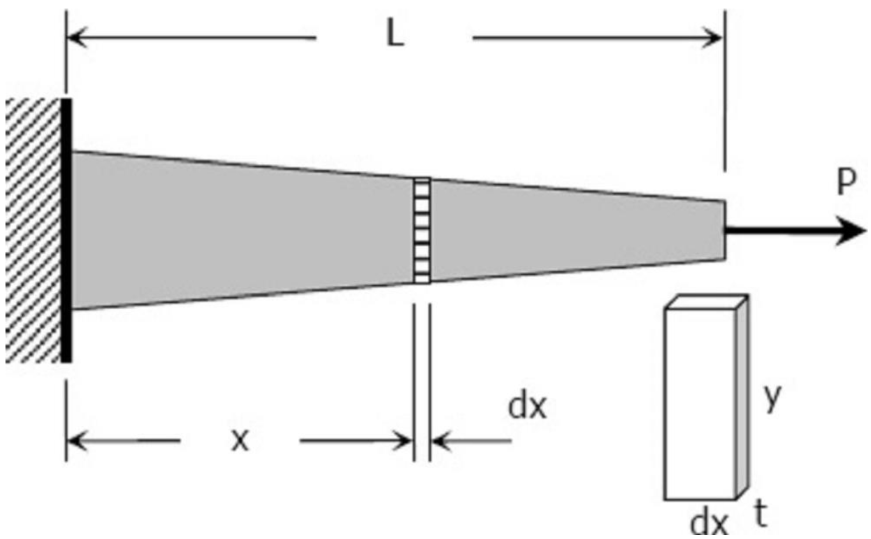
In the linear portion of the stress-strain diagram, the tress is proportional to strain and is given by:

$$\sigma = E\varepsilon$$

since $\sigma = P/A$ and $\varepsilon = \delta/L$, then $\dfrac{P}{A} = E \ \dfrac{\delta}{L}$

$$\delta = \frac{PL}{AE} = \frac{\sigma L}{E}$$

To use this formula, the load must be axial, the bar must have a uniform cross-sectional area, and the stress must not exceed the proportional limit. If however, the cross-sectional area is not uniform, the axial deformation can be determined by considering a differential length and applying integration.



$$\delta = \frac{P}{E}\int_0^L \frac{dx}{L}$$

Where $A=t\,y$ and y and t, if variable, must be expressed in terms of x.

For a rod of unit mass ρ suspended vertically from one end, the total elongation due to its own weight is:

$$\delta = \frac{\rho g L^2}{2E} = \frac{MgL}{2AE}$$

Where ρ is in kg/m3, *L* is the length of the rod in mm, *M* is the total mass of the rod in kg, *A* is the cross sectional area of the rod in mm^2, and g=9.81m/s^2.

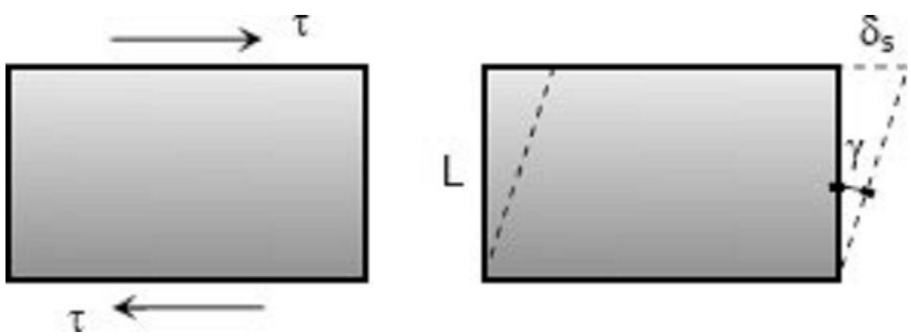
Stiffness, k

Stiffness is the ratio of the steady force acting on an elastic body to the resulting displacement. It has the unit of N/mm.

$$k = \frac{P}{\delta}$$

Shearing Deformation

Shearing forces cause shearing deformation. An element subject to shear does not change in length but undergoes a change in shape.



The change in angle at the corner of an original rectangular element is called the **shear strain** and is expressed as:

$$\gamma = \frac{\delta_s}{L}$$

The ratio of the shear stress τ and the shear strain γ is called the *modulus of elasticity in shear* or **modulus of rigidity** and is denoted as *G*, in MPa.

$$G = \frac{\tau}{\gamma}$$

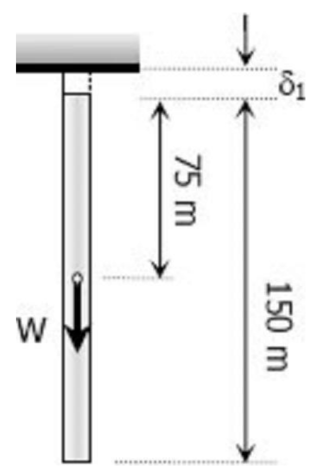
The relationship between the shearing deformation and the applied shearing force is:

$$\delta_s = \frac{VL}{A_s G} = \frac{\tau L}{G}$$

Where *V* is the shearing force acting over an area A_s.

<u>Problem 1</u>

The 150m length steel bar with a unit mass of 7850 kg/m3 and a cross-sectional area of 300 mm^2 is subjected to 20 KN tensile load at the lower end. Find the total elongation of the rod if the modulus of elasticity is 200 × 10^3 MN/m^2.



Sol.:

Elongation due to its own weight:

$$\delta_1 = \frac{PL}{AE}$$

Where:

P = W = 7850(1/1000)3(9.81)[300(150)(1000)]

P = 3465.3825 N

L = 75(1000) = 75 000 mm

A = 300 mm^2

E = 200 000 MPa

$$\delta_1 = \frac{3\,465.3825(75\,000)}{300(200\,000)}$$

δ_1 = 4.33 mm

Elongation due to applied load:

$$\delta_2 = \frac{PL}{AE}$$

Where:

P = 20 kN = 20 000 N

L = 150 m = 150 000 mm

A = 300 mm^2

E = 200 000 MPa

$$\delta_2 = \frac{20\,000(150\,000)}{300(200\,000)}$$

δ_2 = 50 mm

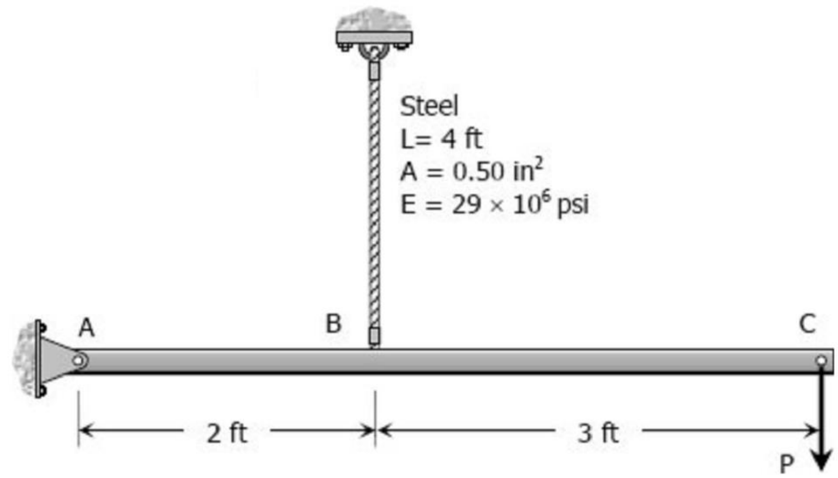
Total elongation:

$$\delta = \delta_1 + \delta_2$$

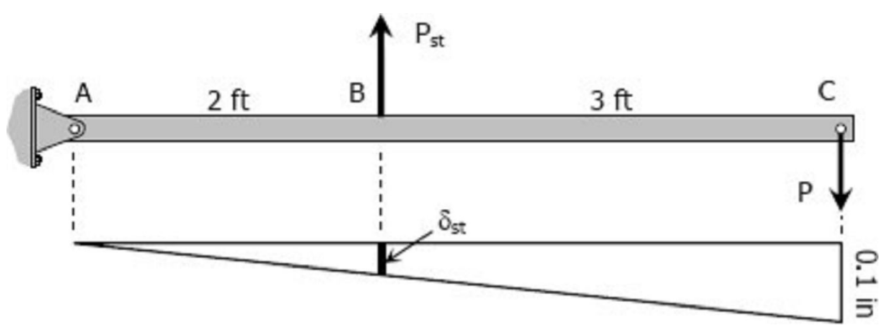
$$\delta = 4.33 + 50 = 54.33 \text{ mm} \rightarrow \textbf{\textit{answer}}$$

Problem 2

If the maximum stress in steel rod shown is 30 ksi and the maximum vertical movement at C is 0.10 inch, find the largest load P that can be applied at C.



Sol.:



Based on maximum stress of steel rod:

$$\Sigma M_A = 0$$
$$5P = 2P_{st}$$
$$P = 0.4P_{st}$$
$$P = 0.4\sigma_{at} A_{st}$$
$$P = 0.4[\,30(0.50)\,]$$
$$P = 6\,\text{kips}$$

Based on movement at C:

$$\frac{\delta_{st}}{2} = \frac{0.1}{5}$$

$$\delta_{st} = 0.04 \text{ in}$$

$$\frac{P_{st} L}{AE} = 0.04$$

$$\frac{P_{st}(4 \times 12)}{0.50(29 \times 10^6)} = 0.04$$

$$P_{st} = 12083.33 \text{ lb}$$

$$\Sigma M_A = 0$$

$$5P = 2P_{st}$$

$$P = 0.4P_{st}$$

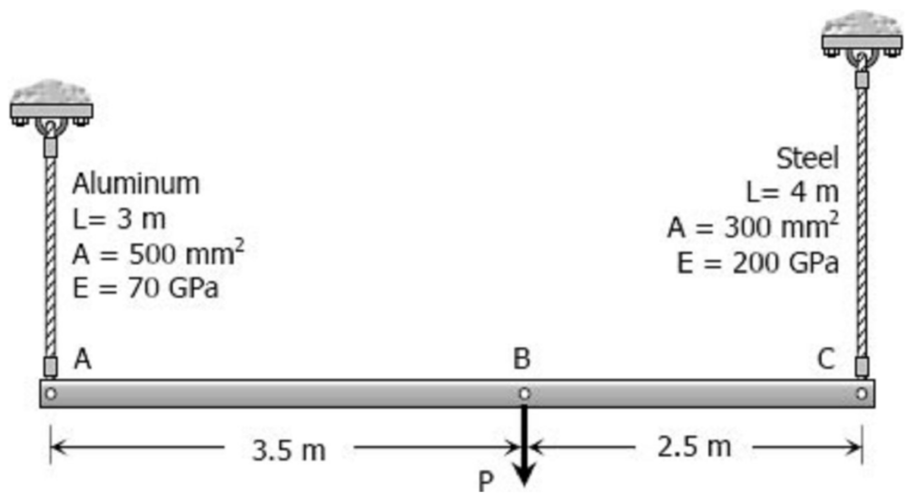
$$P = 0.4(12\,083.33)$$

$$P = 4833.33 \text{ lb} = 4.83 \text{ kips}$$

Use the smaller value, **P = 4.83 kips**

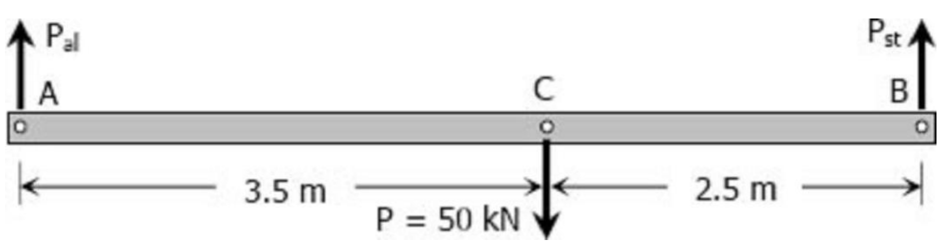
Problem 3

The rigid bar shown is horizontal before P (50 KN) is applied. Find the vertical movement of P.



Sol.:

Free body diagram:



For aluminum:

$$\Sigma M_B = 0$$
$$6P_{al} = 2.5(50)$$
$$P_{al} = 20.83 \text{ kN}$$

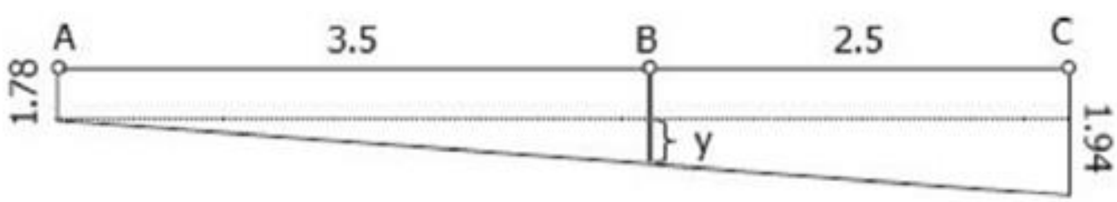
$$\delta = \frac{PL}{AE}$$
$$\delta_{al} = \frac{20.83(3)1000^2}{500(70\,000)}$$
$$\delta_{al} = 1.78 \text{ mm}$$

For steel:

$$\Sigma M_A = 0$$
$$6P_{st} = 3.5(50)$$
$$P_{st} = 29.17 \text{ kN}$$

$$\delta = \frac{PL}{AE}$$
$$\delta_{st} = \frac{29.17(4)1000^2}{300(200\,000)}$$
$$\delta_{st} = 1.94 \text{ mm}$$

Movement diagram:



$$\frac{y}{3.5} = \frac{1.94 - 1.78}{6}$$

$$y = 0.09 \text{ mm}$$

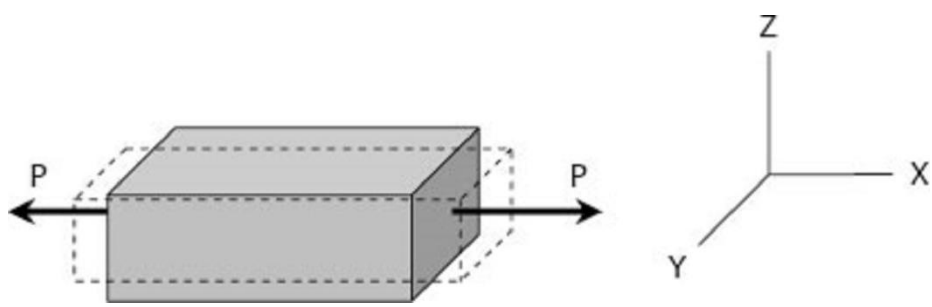
$$\delta_B = \text{ vertical movement of } P$$

$$\delta_B = 1.78 + y = 1.78 + 0.09$$

$$\delta_B = 1.87 \text{ mm} \rightarrow \textit{answer}$$

Poisson's Ratio

When a bar is subjected to a tensile loading there is an increase in length of the bar in the direction of the applied load, but there is also a decrease in a lateral dimension perpendicular to the load. The ratio of the sidewise deformation (or strain) to the longitudinal deformation (or strain) is called the Poisson’s ratio and is denoted by υ . For most steel, it lies in the range of 0.25 to 0.3, and 0.20 for concrete.



$$\nu = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

Where ε_x is strain in the x-direction and ε_y and ε_z are the strains in the perpendicular direction. The negative sign indicates a decrease in the transverse dimension when ε_x is positive.

Biaxial Deformation

If an element is subjected simultaneously by ensile stresses, σ_x and σ_y , in the *x* and *y* directions, the strain in the *x* direction is (σ_x /E) and the strain in the *y* direction is (σ_y /E). Simultaneously, the stress in the *y* direction will produce a lateral contraction on the *x* direction of the amount ($-\upsilon\varepsilon_y$) or ($-\upsilon\ \sigma_y$ /E). The resulting strain in the *x* direction will be:

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} \quad \text{or} \quad \sigma_x = \frac{(\varepsilon_x + \nu \varepsilon_y)E}{1 - \nu^2}$$

And

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} \quad \text{or} \quad \sigma_y = \frac{(\varepsilon_y + \nu \varepsilon_x)E}{1 - \nu^2}$$

Problem 4:

A welded steel cylindrical drum made of a 10-mm plate has an internal diameter of 1.20 m. Compute the change in diameter that would be caused by an internal pressure of 1.5 MPa. Assume that Poisson's ratio is 0.30 and E = 200 GPa.

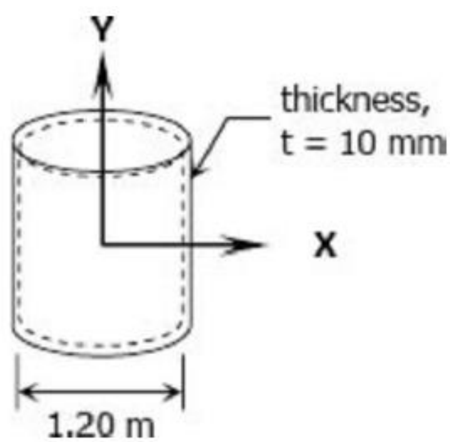
Sol.:

σ_y = longitudinal stress

$$\sigma_y = \frac{pD}{4t} = \frac{1.5(1200)}{4(10)}$$
$$\sigma_y = 45 \text{ MPa}$$

σ_x = tangential stress

$$\sigma_y = \frac{pD}{2t} = \frac{1.5(1200)}{2(10)}$$
$$\sigma_y = 90 \text{ MPa}$$



$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$
$$\varepsilon_x = \frac{90}{200\,000} - 0.3\left(\frac{45}{200\,000}\right)$$
$$\varepsilon_x = 3.825 \times 10^{-4}$$

$$\varepsilon_x = \frac{\Delta D}{D}$$
$$\Delta D = \varepsilon_x D == (3.825 \times 10^{-4})(120$$
$$\Delta D = 0.459 \text{ mm} \rightarrow \textit{answer}$$

Problem 5

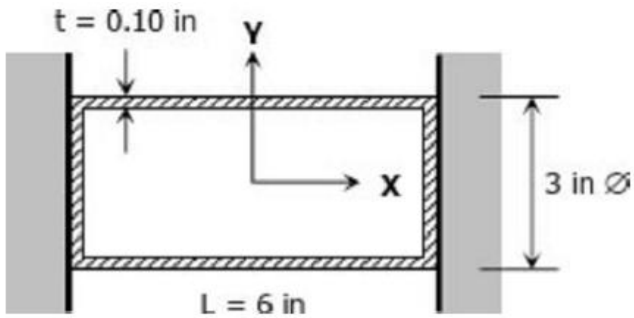
A 6-in.-long bronze tube, with closed ends, is 3 in. in diameter with a wall thickness of 0.10 in. with no internal pressure; the tube just fits between two rigid end walls. Calculate the longitudinal and tangential stresses for an internal pressure of 6000 psi. Assume ν = 1/3 and E = 12 × 10^6 psi.

Sol.:

$$\varepsilon = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} = 0$$

$$\sigma_x = \nu\,\sigma_y = \sigma_l \;\longrightarrow\; \text{longitudinal}$$

$$\sigma_t = \sigma_y \;\longrightarrow\; \text{tangential stress}$$



$$\sigma_t = \frac{pD}{2t} = \frac{6000(3)}{2(0.10)}$$

$$\sigma_t = 90\,000\,\text{psi} \;\longrightarrow\; \textit{answer}$$

$$\sigma_l = \nu\,\sigma_y = \tfrac{1}{3}(90\,000)$$

$$\sigma_l = 30\,000\,\text{psi} \;\longrightarrow\; \textit{answer}$$

Thermal Stress

Temperature changes cause the body to expand or contract. The amount δ_T, is given by:

$$\delta_T = \alpha L\,(T_f - T_i) = \alpha L\,\Delta T$$

Where α is the coefficient of thermal expansion in m/m°C, L is the length in meter, T_i and T_f are the initial and final temperatures, respectively in °C. If temperature deformation is permitted to occur freely, no load or stress will be induced in the structure.

In some cases where temperature deformation is not permitted, an internal stress is created. The internal stress created is termed as **thermal stress**. For a homogeneous rod mounted between unyielding supports as shown, the thermal stress is computed as:

Deformation due to temperature changes;

$$\delta_T = \alpha L\,\Delta T$$

Deformation due to equivalent axial stress;

$$\delta_P = \frac{PL}{AE} = \frac{\sigma L}{E}$$

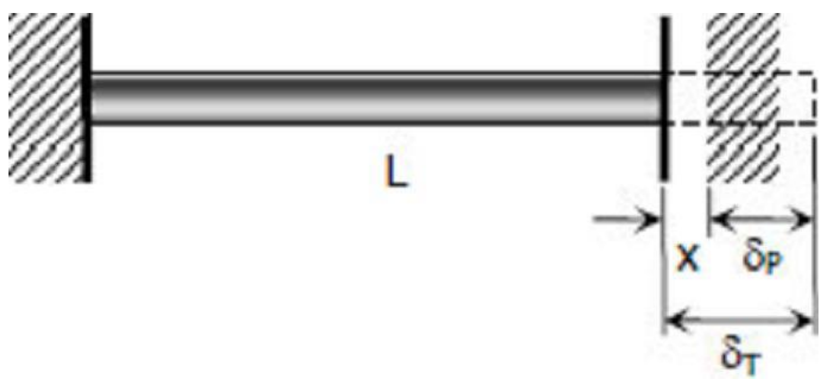
$$\delta_T = \delta_P$$

$$\alpha L \, \Delta T = \frac{\sigma L}{E}$$

$$\sigma = E \, \alpha \, \Delta T$$

Where σ is the thermal stress in MPa, *E* is the modulus of elasticity of the rod in MPa.

If the wall yields a distance of *x* as shown, the following calculations will be made:



$$\delta_T = x + \delta_P$$

$$\alpha \, L \, \Delta T = x + \frac{\sigma L}{E}$$

Where σ represents the thermal stress.

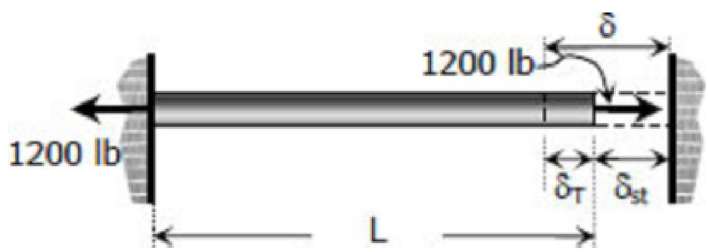
Take note that as the temperature rises above the normal, the rod will be in compression, and if the temperature drops below the normal, the rod is in tension.

Problem 6

A steel rod with a cross-sectional area of 0.25 in^2 is stretched between two fixed points. The tensile load at 70°F is 1200 lb. What will be the stress at 0°F? At what temperature will the stress be zero? Assume $\alpha = 6.5 \times 10^{-6}$ in/(in·°F) and $E = 29\times10^6$ psi.

Sol.:

For the stress at 0°C:



$$\delta = \delta_T + \delta_{st}$$

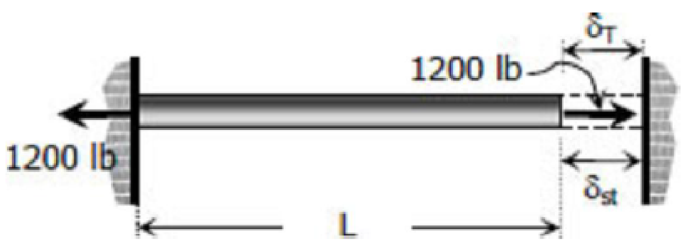
$$\frac{\sigma L}{E} = \alpha L (\Delta T) + \frac{PL}{AE}$$

$$\sigma = \alpha E (\Delta T) + \frac{P}{E}$$

$$\sigma = (6.5 \times 10^{-6})(29 \times 10^{6})(70) + \frac{1200}{0.25}$$

$$\sigma = 17\,995\,\text{psi} = 18 \text{ ksi} \quad \rightarrow \textit{\textbf{answer}}$$

For the temperature that causes zero stress:



$$\delta_T = \delta_{st}$$

$$\alpha L (\Delta T) = \frac{PL}{AE}$$

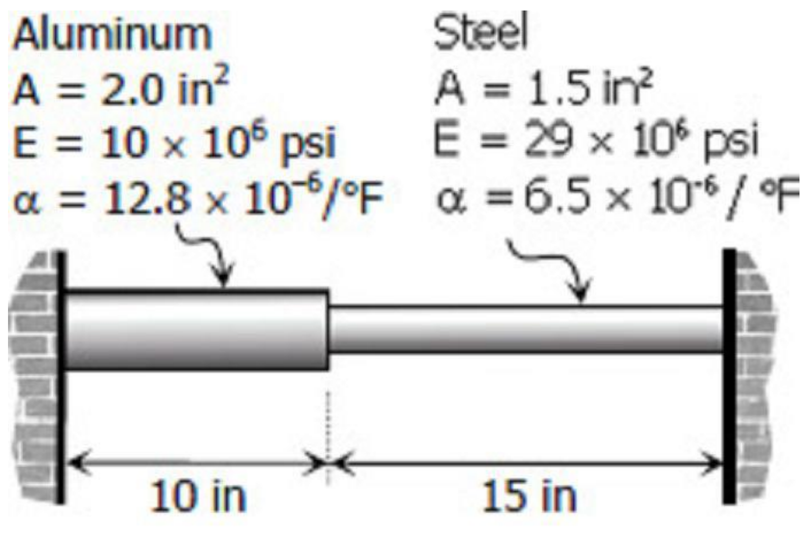
$$\alpha (\Delta T) = \frac{P}{AE}$$

$$(6.5 \times 10^{-6})(T - 70) = \frac{1200}{0.25(29 \times 10^{6})}$$

$$T = 95.46\,°C \quad \rightarrow \textit{\textbf{answer}}$$

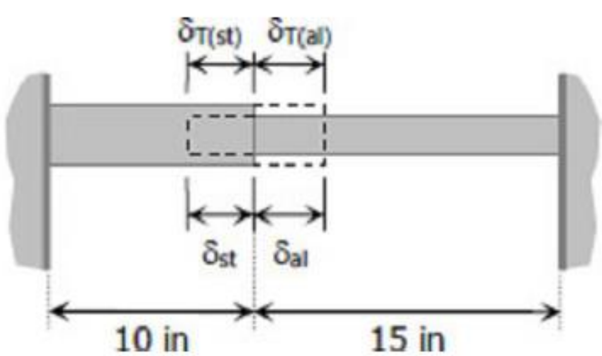
Problem 7

Calculate the increase in stress for each segment of the compound bar shown if the temperature increases by 100°F. Assume that the supports are unyielding and that the bar is suitably braced against buckling.



Sol.:

$$\delta_T = \alpha L \Delta T$$



$$\delta_{T(st)} = (6.5 \times 10^{-6})(15)(100)$$
$$\delta_{T(st)} = 0.00975$$

$$\delta_{T(al)} = (12.8 \times 10^{-6})(10)(100)$$
$$\delta_{T(al)} = 0.0128 \text{ in}$$

$$\delta_{st} + \delta_{al} = \delta_{T(st)} + \delta_{T(al)}$$

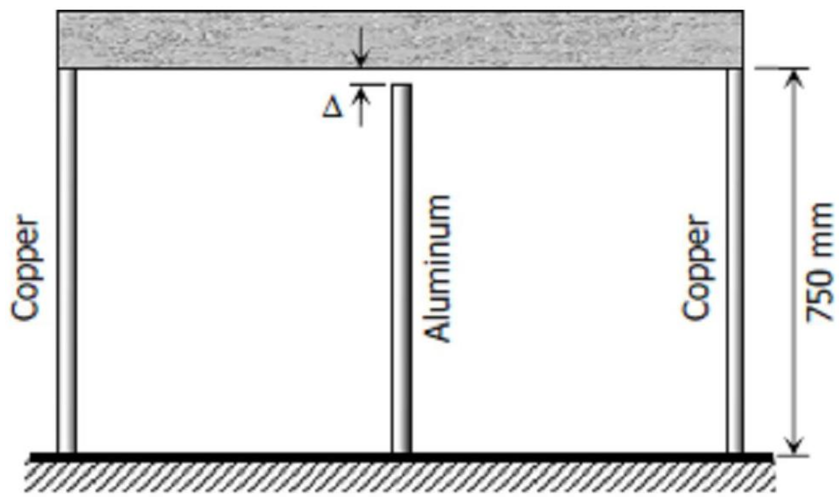
$$\left(\frac{PL}{AE}\right)_{st} + \left(\frac{PL}{AE}\right)_{al} = 0.00975 + 0.0128$$

where $P = P_{st} = P_{al}$
$$\frac{P(15)}{1.5(29 \times 10^6)} + \frac{P(10)}{2(10 \times 10^6)} = 0.02255$$
$$P = 26691.84 \text{ psi}$$

$$\sigma = \frac{P}{A}$$
$$\sigma_{st} = \frac{26\,691.84}{1.5} = 17\,794.56 \text{ psi} \rightarrow \textbf{\textit{answer}}$$
$$\sigma_{st} = \frac{26\,691.84}{2.0} = 13\,345.92 \text{ psi} \rightarrow \textbf{\textit{answer}}$$

Problem 8

For the figure shown, there is a gap between the aluminum bar and the rigid slab that is supported by two copper bars. At 10°C, Δ = 0.18 mm. Neglecting the mass of the slab, calculate the stress in each rod when the temperature in the assembly is increased to 95°C. For each copper bar, A = 500 mm^2, E = 120 GPa, and α=16.8μm/(m·°C). For the aluminum bar, A = 400 mm2, E = 70 GPa, and α=23.1μm/(m·°C).



Sol.:

Assuming complete freedom:

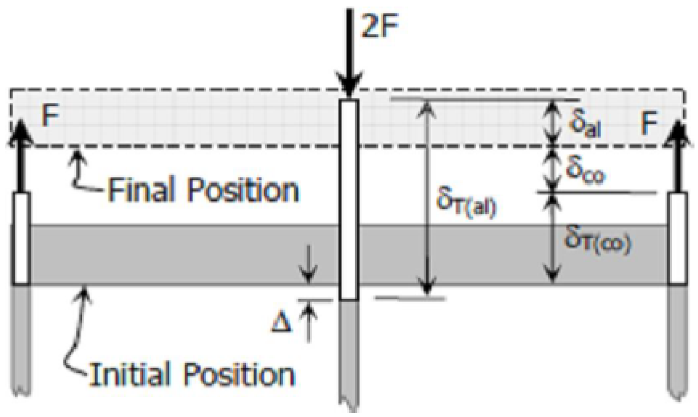
$$\delta_T = \alpha L \Delta T$$

$$\delta_{T(co)} = (16.8 \times 10^{-6})(750)(95-10)$$

$$\delta_{T(co)} = 1.071 \text{ mm}$$

$$\delta_{T(al)} = (23.1 \times 10^{-6})(750-0.18)(95-10)$$

$$\delta_{T(al)} = 1.472 \text{ mm}$$



From the figure:

$$\delta_{T(al)} - \delta_{al} = \delta_{T(co)} + \delta_{co}$$

$$1.472 - \left(\frac{PL}{AE}\right)_{al} = 1.071 + \left(\frac{PL}{AE}\right)_{co}$$

$$1.472 - \frac{2F(750-0.18)}{400(70\,000)} = 1.071 + \frac{F(750)}{500(120\,000)}$$

$$0.401 = (6.606 \times 10^{-5})\,F$$

$$F = 6070.37\text{ N}$$

$$P_{co} = F = 6070.37\text{ N}$$

$$P_{al} = 2F = 12\,140.74\text{ N}$$

$$\sigma = P/A$$

$$\sigma_{co} = \frac{6070.37}{500} = 12.14\text{ MPa} \rightarrow \textit{answer}$$

$$\sigma_{al} = \frac{12\,140.74}{400} = 30.35\text{ MPa} \rightarrow \textit{answer}$$