**Series Tests**

In this section the various tests mentioned in the previous section will be introduced, and a number of examples will be considered in class to illustrate the various tests.

**General (*n*th) Term Test** (also known as the **Divergence Test**)**:**

If , then the series  diverges.

**NOTE:** This test is a test for divergence only, and says nothing about convergence.

**Geometric Series Test:**

A geometric series has the form , where “*a*” is some fixed scalar (real number).

A series of this type will converge provided that ⏐*r*⏐< 1, and the sum is .

A proof of this result follows.

Consider the *k*th partial sum, and “*r*” times the *k*th  partial sum of the series



The difference between *rSk* and *Sk* is .

Provided that *r* ≠ 1, we can divide by (*r*  1), to obtain .

Since the only place that “*k*” appears on the right in this last equation is in the numerator, the limit of the sequence of partial sums {*Sk*} will exist iff the limit as *k* → ∞ exists as a finite number. This is possible iff ⏐*r*⏐< 1, and if this is true then the limit value of the sequence of partial sums, and hence the sum of the series, is .

**Telescoping Series:**

Generally, a telescoping series is a series in which the general term is a ratio of polynomials in powers of “*n*”. The method of partial fractions (learned when studying techniques of integration) is normally used to rewrite the general term, and then the sequence of partial sums is studied. This sequence will, most of the time, simplify to just a few terms, and the limit can then be determined. One example of a telescoping series will be presented here, and additional examples in class.

Sample Problem 3:

Evaluate .

The general term *an* can be rewritten as .

We now consider the partial sums *S*1, *S*2, *S*3, ..., *Sn*, ... until a pattern emerges and then the limit value *S* will be determined.



Since we have now determined the general pattern, the limit value *S* of the sequence of partial sums, and hence the sum of the series is seen to have a value of “1”.

**Integral Test:**

Given a series of the form , set *an = f*(*n*) where *f*(*x*) is a continuous function with positive values that are decreasing for *x* ≥ *k*. If the improper integral  exists as a finite real number, then the given series converges. If the improper integral above does not have a finite value, then the series above diverges.

If the improper integral exists, then the following inequality is always true



By adding the terms from *n = k* to *n = p* to each expression in the inequalities above it is possible to put both upper and lower bounds on the sum of the series. Also it is possible to estimate the error generated in estimating the sum of the series by using only the first “*p*” terms. If the error is represented by *Rp*, then it follows that .

**Comparison Tests:**

There are four comparison tests that are used to test series. There are two convergence tests, and two divergence tests. In order to use these tests it is necessary to know a number of convergent series and a number of divergent series. For the tests that follow we shall assume that  is some known convergent series, that  is some known divergent series, and that  is the series to be tested. Also it is to be assumed that for *n* ∈ {1, 2, 3, ..., (*k*1)} the values of *an* are finite, and that each of the series contains only positive terms.

**Standard Comparison Tests:**

**Convergence Test:** If  is a convergent series and *an* ≤ *cn* for all *n* ≥ *k*

then  is a convergent series.

**Divergence Test:** If  is a divergent series and *an* ≥ *dn* for all *n* ≥ *k*,

then  is a divergent series.

**Limit Comparison Tests:**

**Convergence Test:** If  is a convergent series and 

where 0 ≤ *L* < ∞, then  is a convergent series.

**Divergence Test:** If  is a divergent series and 

where 0 < *L* ≤ ∞, then  is a divergent series.

The choice for the reference series  or  is often the geometric series 

or the **hyperharmonic** series (or *p*-series) .

The *p*-series converges absolutely when *p* > 1 and diverges otherwise.

A special case is the **harmonic** series , which diverges (*p* = 1).

[The alternating *p*-series converges absolutely when *p* > 1 ,

converges conditionally when 0 < *p* ≤ 1 and diverges otherwise.]

**Alternating Series Test:**

Given a series  = *a*1 + *a*2 + *a*3 + ... + *a*(*k*1) + where *a*1 , *a*2 , *a*3 , ... , *a*(*k*1) can be any finite real numbers, and  for all *n* ≥ *k* ,

if , then the series converges. If , then the series diverges.

**Ratio Test:**

Given a series with no restriction on the values of the *an*’s except that they are finite, and that , the series converges absolutely whenever 0 ≤ *L* < 1, diverges whenever

1 < *L* ≤ ∞, and the test fails if *L* = 1.

**Root Test:**

Given a series with no restriction on the values of the *an*’s except that they are finite, and that , the series converges absolutely whenever 0 ≤ *L* < 1, diverges whenever

1 < *L* ≤ ∞, and the test fails if *L* = 1.

**Absolute and Conditional Convergence:**

A convergent series that contains an infinite number of both negative and positive terms must be tested for absolute convergence.

A series of the form  is **absolutely convergent** iff  the series of absolute values is convergent.

If  is convergent, but  the series of absolute values is divergent, then the series  is **conditionally convergent**.

A shortcut:

In some cases it is easier to show that  is convergent.

It then follows immediately that the original series is absolutely converge