Kinetic theory

- Overview:
  - Distribution functions
  - Basic equations:
    - Boltzmann
    - Vlasov
    - Fokker-Planck
  - Collisions:
    - Neutral meets neutral
    - Neutral meets charge
    - Charge meets charge

- Pre-requisites:
  - none
  - Thermal motion of the particles is considered

Phase space and distribution function

- Point in phase space:

$$\vec{Q} = (q_1, q_2, q_3; p_1, p_2, p_3) = (\vec{q}, \vec{p})$$

- Velocity in phase space:

$$\vec{C} = \frac{\mathrm{d}\vec{Q}}{\mathrm{d}t} = \left(\frac{\mathrm{d}q_1}{\mathrm{d}t}, \frac{\mathrm{d}q_2}{\mathrm{d}t}, \frac{\mathrm{d}q_3}{\mathrm{d}t}; \frac{\mathrm{d}p_1}{\mathrm{d}t}, \frac{\mathrm{d}p_2}{\mathrm{d}t}, \frac{\mathrm{d}p_3}{\mathrm{d}t} \right) = \left(\frac{\mathrm{d}\vec{q}}{\mathrm{d}t}, \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \right)$$

- Phase space density and particle density in 3D space: space:

$$n(\vec{q}, t) = \int_{-\infty}^{+\infty} f(\vec{q}, \vec{p}, t)\, \mathrm{d}^3\vec{p}$$

- Macroscopic quantities are averages of phase space density:

$$\langle a(\vec{q}, t) \rangle = \frac{1}{n(\vec{q}, t)} \int_{-\infty}^{\infty} a(\vec{q}, \vec{p}, t)\, f(\vec{q}, \vec{p}, t)\, \mathrm{d}^3\vec{p}$$

- Example: bulk velocity

$$\vec{u}(\vec{q}, t) = \langle \vec{v}(\vec{q}, t) \rangle = \frac{1}{n(\vec{q}, t)} \int_{-\infty}^{\infty} \vec{v}(\vec{p}, \vec{q}, t)\, f(\vec{q}, \vec{p}, t)\, \mathrm{d}^3\vec{p}$$

Maxwell distribution

- Kinetic energy in a volume element:

$$\left\langle \frac{m}{2}(\vec{v}-\vec{u})^2 \right\rangle = \frac{\int m(\vec{v}-\vec{u})^2 f(\vec{r},\vec{v},t)\mathrm{d}\vec{v}}{2\int f(\vec{r},\vec{v},t)\mathrm{d}\vec{v}}$$

- Fluctuating kinetic energy & pressure:

$$\frac{p}{n} = \frac{2}{N}\left\langle \frac{m}{2}(\vec{v}-\vec{u}_s)^2 \right\rangle$$

- Maxwell distribution:

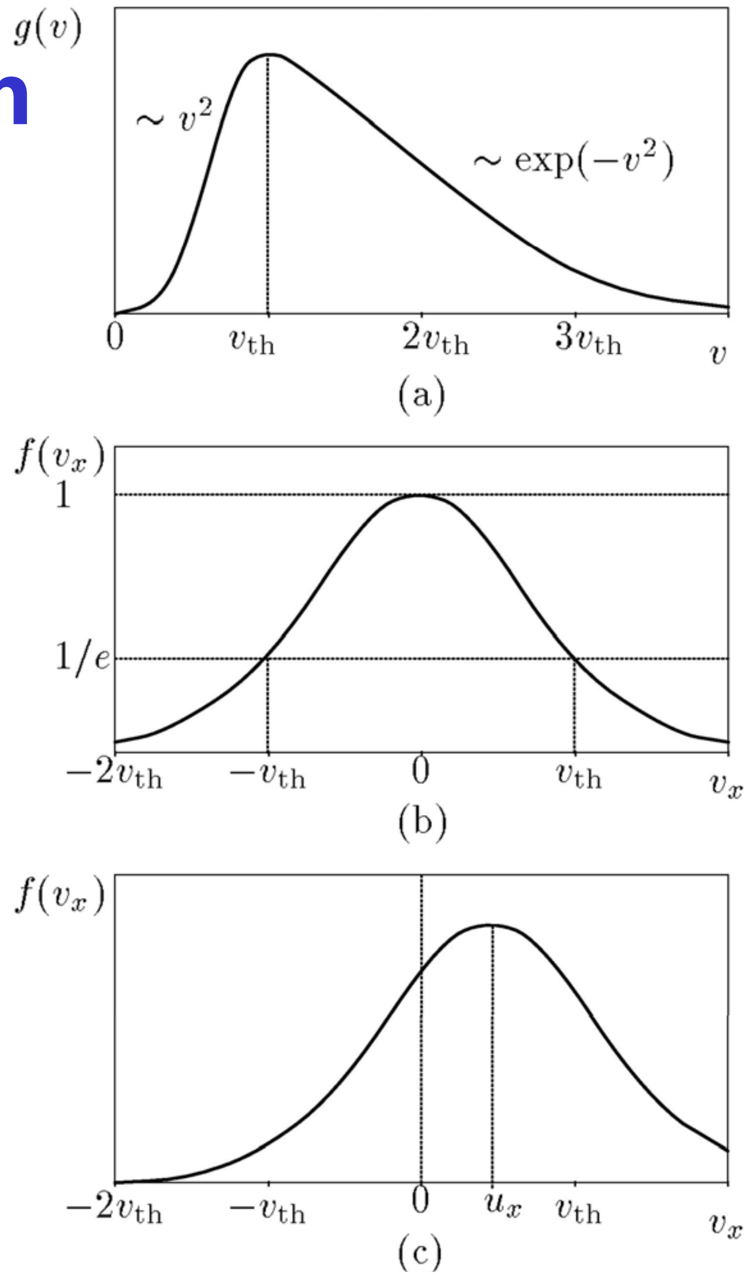
$$f(\vec{r},\vec{v},t) = n\sqrt{\left(\frac{m}{2\pi k_\mathrm{B}T}\right)^3}\exp\left\{-\frac{m(\vec{v}-\vec{u})^2}{2k_\mathrm{B}T}\right\}$$

- Thermal speed:

$$v_\mathrm{th} = \sqrt{\frac{2k_\mathrm{B}T}{m}}$$

- Speed instead of velocity:

$$\int\!\!\int f(\vec{r},\vec{v},t)\,\mathrm{d}\Omega_v\, v^2\,\mathrm{d}v = (4\pi f(\vec{r},|v|,t)v^2)\,\mathrm{d}v = g(\vec{r},v)\,\mathrm{d}v$$



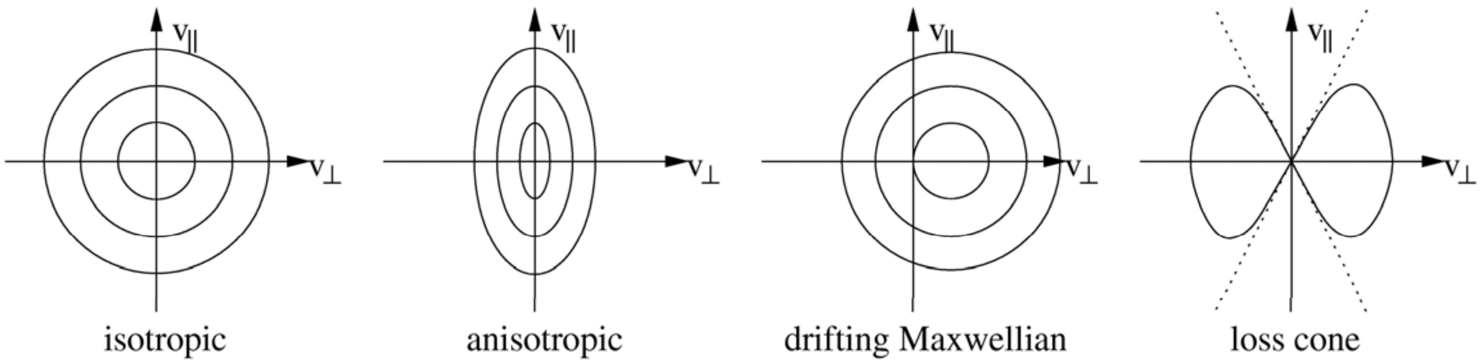
Other distribution functions

- Bi-Maxwellian

$$f(\vec{r},\vec{v},t) = \sqrt{\left(\frac{m}{2\pi k_B}\right)^3} \frac{n}{T_\perp \sqrt{T_\|}} \exp\left\{-\frac{m(v_\| - u_\|)^2}{2k_B T_\|}\right\} \exp\left\{-\frac{m(v_\perp - u_\perp)^2}{k_B T_\perp}\right\}$$

  - Allows for the consideration of different speeds parallel and perpendicular to the field:

$$\left\langle \tfrac{1}{2} m (v_\| - u_\|)^2 \right\rangle = \tfrac{1}{2} k_B T_{\|s} \quad \text{and} \quad \left\langle \tfrac{1}{2} m (v_\perp - u_\perp)^2 \right\rangle = k_B T_\perp$$



- Kappa-distribution: high energetic tail as power law in E:

$$f(\vec{r},\vec{v},t) = \frac{n_s}{\kappa} \sqrt{\left(\frac{m}{2\pi k_B T}\right)^3} \left[1 + \frac{m(\vec{v} - \vec{u})^2}{2\kappa E_T}\right]^{-\kappa - 1}$$

Distribution function and measured quantities

- Differential flux: number of particles in the energy band from E to E+ΔE, coming from the direction Ω within a solid angle ΔΩ, going through a surface dA perpendicular to Ω within a time interval dt:

$$J(E,\vec{\Omega},\vec{r},t)\,\mathrm{d}\vec{A}\,\mathrm{d}\vec{\Omega}\,\mathrm{d}t\,\mathrm{d}E$$

- Omnidirectional intensity:

$$J_{\mathrm{omni}}(E,\vec{r},t) = \frac{1}{4\pi}\int J(E,\vec{\Omega},\vec{r},t)\,\mathrm{d}\vec{\Omega}$$

- Relation between differential intensity and phase space density:

$$J(E,\vec{\Omega},\vec{r},t) = \frac{v^2}{m} f(\vec{r},\vec{p},t)$$

Basic equations

- The equation of motion in phase space described the evolution of a particle ensemble:
  - Boltzmann equation: most general form, yields the Maxwell-distribution
  - Vlasov equation: forces are electromagnetic only
  - Fokker-Planck equation: considers also collisions between the particles
- Basic concept: equation of continuity in phase space

Boltzmann equation

- Equation of continuity in phase space:

$$\frac{\partial f}{\partial t} + \nabla_6 \cdot (f\vec{C}) = 0 \quad \text{or} \quad \frac{\partial f}{\partial t} + \nabla_{\mathrm{r}} \cdot (\vec{v} f) + \nabla_{\mathrm{v}} \cdot (\vec{a} f) = 0$$

- Force independent of velocity $\Rightarrow$ collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_{\mathrm{r}} f + \vec{a} \cdot \nabla_{\mathrm{v}} f = 0 \quad \text{or} \quad \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{\vec{F}}{m} \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

- Or for short: $\frac{\mathrm{d}f}{\mathrm{d}t} = 0$, the medium in phase space can be regarded as an incompressible fluid.

- Collisions added: $\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{\vec{F}}{m} \cdot \frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}}$

- Reduced Boltzmann equation: changes in f due to collisions are small:

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}} = 0$$

Vlasov equation

- Insert Lorentz force into Boltzmann's equation:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{q}{m}\left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right) \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

- Problem: Boltzmann applies to forces which are independent of velocity;
  - Lorentz contains velocity explicitly.
  - but: we are looking at the gradient of the acceleration in velocity space, that is $\partial a_x / \partial v_x = 0$

- Jeans theorem: kinetic theory and orbit theory are equivalent:

$$\sum_i \left(\frac{\partial \gamma_i}{\partial t} + \vec{v} \cdot \nabla \gamma_i + \frac{\vec{F}}{m} \cdot \frac{\partial \gamma_i}{\partial \vec{v}}\right) = \sum_i \frac{\partial f}{\partial \gamma_i} \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} = 0$$

Fokker-Planck equation I

- Problem: collisions are not a deterministic but a stochastic process:
  - Probability distribution function for the change in velocity during a time interval Δt
  - Multiplying with phase speed density f(t-Δt) yields phase speed density f(t)

$$f(\vec{r},\vec{v},t) = \int f(\vec{r},\vec{v}-\Delta\vec{v},t-\Delta t)\ \psi(\vec{v}-\Delta\vec{v},\Delta\vec{v})\ \mathrm{d}(\Delta\vec{v})$$

- Limitation to small angel scattering $\Rightarrow$ Taylor expansion

$$f(\vec{r},\vec{v},t) = \int \left[f(\vec{r},\vec{v},t-\Delta t)\,\psi(\vec{v},\Delta\vec{v}) - \Delta\vec{v}\cdot\frac{\partial(f\psi)}{\partial\vec{v}}\right] \mathrm{d}(\Delta\vec{v}) + \int \left[\frac{\Delta\vec{v}\Delta\vec{v}}{2}\odot\frac{\partial^2(f\psi)}{\partial\vec{v}\partial\vec{v}}\right]\mathrm{d}(\Delta\vec{v})$$

- Normalization of probabilities (some scattering always takes place) allows simplification

$$f(\vec{r},\vec{v},t) = f(\vec{r},\vec{v},t-\Delta t) - \frac{\partial(f\langle\Delta\vec{v}\rangle)}{\partial\vec{v}} + \frac{1}{2}\frac{\partial}{\partial\vec{v}\partial\vec{v}}\odot(f\langle\Delta\vec{v}\Delta\vec{v}\rangle)$$

with $$\langle\Delta\vec{v}\rangle = \int \psi\Delta\vec{v}\,\mathrm{d}(\Delta\vec{v}) \quad \text{and} \quad \langle\Delta\vec{v}\Delta\vec{v}\rangle = \int \psi\Delta\vec{v}\Delta\vec{v}\,\mathrm{d}(\Delta\vec{v})$$

Fokker-Planck equation II

- Collisions (see above)

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \frac{f(\vec{r},\vec{v},t) - f(\vec{r},\vec{v},t-\Delta t)}{\Delta t}$$

Acceleration: friction allows for the acceleration of slow partices on the expense of the faster ones and vice versa

- Insert into Fokker-Planck:

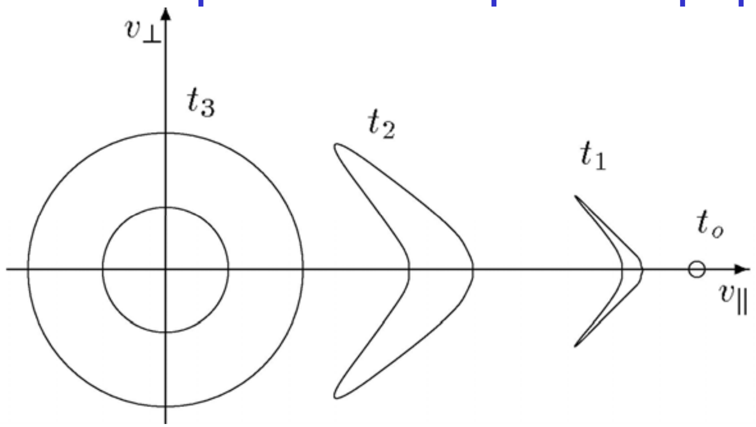
$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \Delta t = -\frac{\partial}{\partial \vec{v}} \cdot (f\langle\Delta\vec{v}\rangle) + \frac{1}{2}\frac{\partial^2}{\partial\vec{v}\partial\vec{v}} \odot (f\langle\Delta\vec{v}\Delta\vec{v}\rangle)$$

Diffusion in velocity space: widening of the distribution

or

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \Delta t = -\nabla_{\text{v}} \cdot (\mathtt{D} \cdot \nabla_{\text{v}} \cdot f)$$

- Evolution of a supra-thermal particle population

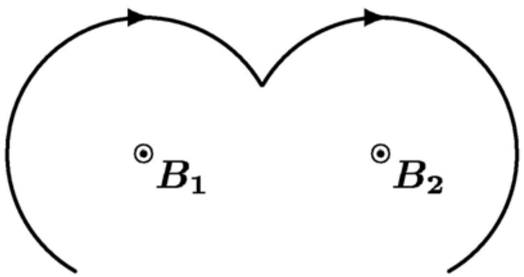


Monoenergetic beam evolves into an isotropic ring;

Evolution slow if the number of collisions with large changes in angle is small!!!

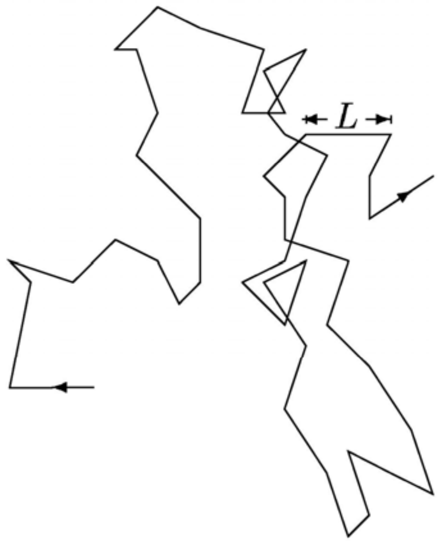
Collisions

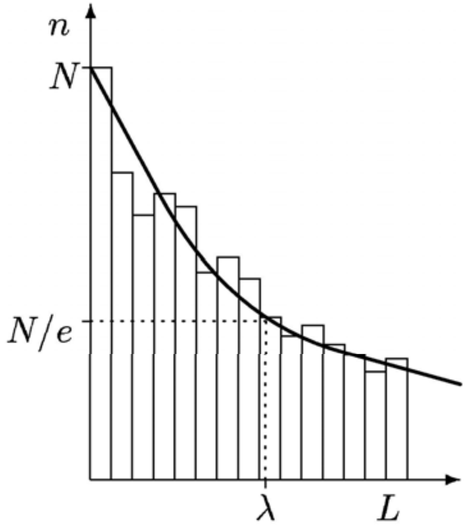
- Required to establish a maxwellian
- In Fokker-Planck's equation consequences of individual collisions are assumed to be small
- In a plasma collisions occur between
  - Two neutrals
  - Neutral and charged particle (identical with collision between two neutrals)
  - Two charges particles
- Consequence: Gyro-center shifted to a different line of force



Mean free path (neutrals)

- Definition mean free path λ from the distribution of path length between two subsequent collisions
- Reduction of a particle beam passing matter with particle number density n and scattering cross section σ



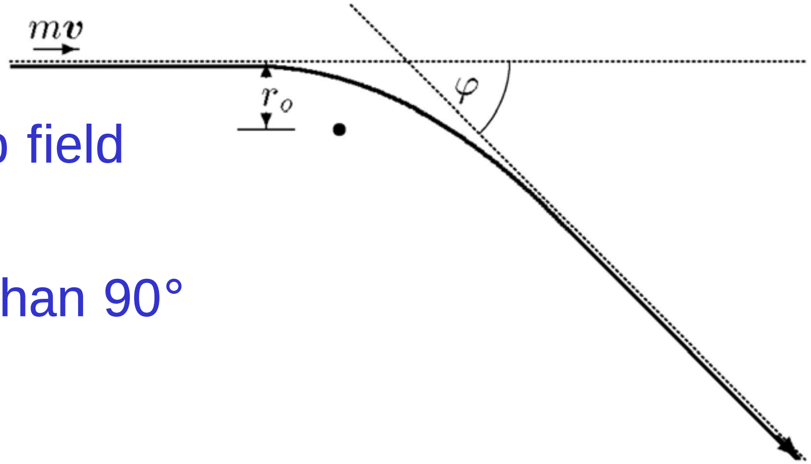


$$N(x) = N_o \exp(-\sigma n_s x) = N_o \exp(-x/\lambda) \qquad \lambda = \frac{1}{n_s \sigma}$$

- Time between two collisions and collision frequency:

$$\tau = \frac{\lambda}{\langle v \rangle} = \frac{1}{n_s \sigma \langle v \rangle} \qquad \nu_c = n_s \sigma \langle v \rangle$$

Coulomb collisions



- Scattering of a charge in the Coulomb field

- Cross section for scattering by more than 90°

$$\sigma_{>90^\circ} = \pi r_\mathrm{o}^2 = \frac{e^4}{16\pi m^2 v^4}$$

corresponding to a collision frequency:

$$\nu_{\mathrm{ei},>90^\circ} = n\sigma v = \frac{ne^4}{16\pi m^2 v^3}$$

- Plasma: ratio between scattering by small and large angles described by the Coulomb logarithm Λ:

$$\frac{\lambda_\mathrm{D}^2 - r_{90^\circ}^2}{r_{90^\circ}^2} \le \frac{\lambda_\mathrm{D}^2}{r_{90^\circ}^2} \approx \left(\frac{9}{Z_\mathrm{T} Z_\mathrm{F}}\right)^2 \left(\frac{4\pi}{3} n \lambda_\mathrm{D}^3\right)^2 =: \Lambda^2$$

Summary

- Interactions in a plasma are stochastic, therefore description by a distribution function
- formally: phase space density, Liouville´s theorem
- Evolution of phase space density can be described by
  - Boltzmann equation (most general equation)
    - Contains the Maxwell distribution
    - The medium in phase space can be treated like an incompressible fluid
  - Vlasov equation: electromagnetic forces
  - Fokker-Planck equation: collisions
- Collisions:
  - Important parameter: mean free path
  - In a plasma most collisions lead to deflections by small angles only. Reason: Debye screening.