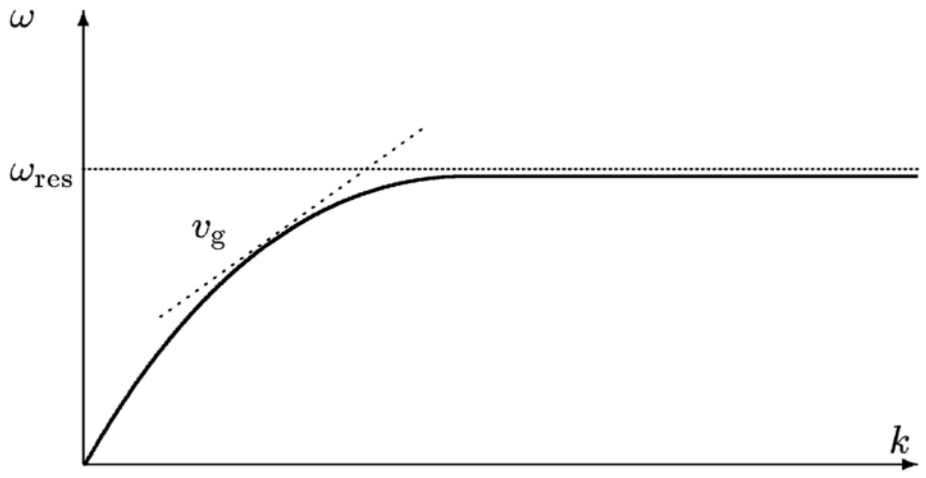
MHD waves

- Overview:
  - Linearization of the equations and Fourier transform
  - Magnetohydrodynamic waves
  - Electrostatic waves in non-magnetized plasmas
  - Electrostatic waves in magnetized plasmas
  - Electromagnetic waves in non-magnetized plasmas
  - Electromagnetic waves in magnetized plasmas

- Pre-requisites:
  - Generally as in magnetohydrodynamics
  - Additional pre-requisites depend on the special type of waves, e.g.
    - Which particle species is in motion,
    - Is the plasma cold or not?

Basics

- Wave equation: $\vec{B}(\vec{r},t) = \vec{B}_\mathrm{o} \exp\left\{ \mathrm{i}(\vec{k}\cdot\vec{r} - \omega t) \right\}$
- Phase speed: $\vec{v}_\mathrm{ph} = \frac{\omega}{k^2}\vec{k}$
- Group speed: $\vec{v}_\mathrm{g} = \frac{\partial \omega}{\partial \vec{k}}$
- Dispersion relation:



Linearization of MHD equations

- Assumptions as on side 30 of chap. 3
- Equations for averages and fluctuating quantities:

$$
\begin{aligned}
\nabla \times \vec{B}_\mathrm{o} &= \mu_\mathrm{o} \vec{j}_\mathrm{o} \\
\nabla \times \vec{E}_\mathrm{o} &= 0 \\
\nabla \cdot \vec{B}_\mathrm{o} &= 0 \\
\frac{\vec{j}_\mathrm{o}}{\sigma} &= \vec{E}_\mathrm{o} + \vec{u}_\mathrm{o} \times \vec{B}_\mathrm{o} \\
\varrho_\mathrm{o} \cdot (\vec{u}_\mathrm{o} \cdot \nabla)\vec{u}_\mathrm{o} &= -\nabla p_\mathrm{o} + \vec{j}_\mathrm{o} \times \vec{B}_\mathrm{o} \\
\nabla \cdot (\varrho_\mathrm{o} \vec{u}_\mathrm{o}) &= 0 \\
p_\mathrm{o} &= C \varrho_\mathrm{o}^{\gamma_\mathrm{a}}
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times \vec{B}_1 &= \mu_\mathrm{o} \vec{j}_1 \\
\nabla \times \vec{E}_1 &= -\frac{\partial \vec{B}_1}{\partial t} \\
\nabla \cdot \vec{B}_1 &= 0 \\
\vec{E}_1 &= -\vec{u}_1 \times \vec{B}_\mathrm{o} \\
\varrho_\mathrm{o} \frac{\partial \vec{u}_1}{\partial t} &= -\nabla p_1 + \vec{j}_1 \times \vec{B}_\mathrm{o} \\
\frac{\partial \varrho_1}{\partial t} &= -\nabla \cdot (\varrho_\mathrm{o} \vec{u}_1) \\
\frac{p_1}{p_\mathrm{o}} &= \gamma_\mathrm{a} \frac{\varrho_1}{\varrho_\mathrm{o}}
\end{aligned}
$$

Method: Fourier transform

- Example for in integral transform, here:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega(t-\tau)} f(\tau)\, d\omega\, d\tau$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) \cos(\omega(t-\tau))\, d\omega\, d\tau$$

$$f(t) = \int_{0}^{\infty} [a(\omega) \cos \omega t + b(\omega) \sin \omega t]\, d\omega$$

$$a(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t\, dt \qquad b(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t\, dt$$

- Goal: simple method to solve a differential equation
- Method:
  - Differential equation is transformed into an algebraic equation
  - The algebraic equation is solved
  - The solution of the algebraic equation is transformed back.

Rules

- Convolution: $$(f_1 * f_2)(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau$$
- Convolution theorem: the Fourier transform of the convolution of two functions f1 and f2 equals the product of the transforms of f1 and f2:

$$F[(f_1 * f_2)(t)] = F[f_1(t)] \cdot f[f_2(t)]$$

- Shifting theorem: the Fourier transform of a function shifted by t0 equals the transform of the original function multiplied by $\mathrm{e}^{-\mathrm{i}\omega t_0}$

$$F[f(t-t_0)] = F[f(t)]\,\mathrm{e}^{-\mathrm{i}\omega t_0}$$

- Linearity: the Fourier transform of a sum of functions equals the sum of the Fourier transforms of the functions:

$$F(f(t) + g(t)] = F[f(t)] + F[g(t)]$$

Application

- Examples for Fourier transforms

$f(t)$	$F(\omega)$
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$H(t)$	$\pi\delta(\omega) + \frac{1}{\mathrm{i}\omega}$
$\sin\omega_0 t$	$\mathrm{i}\pi\delta(\omega+\omega_0) - \mathrm{i}\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)$

- Application partial differential equations:

$$
\begin{array}{lcl}
\partial/\partial t & \rightarrow & -\mathrm{i}\omega \\
\nabla & \rightarrow & \mathrm{i}\vec{k} \\
\nabla\cdot & \rightarrow & \mathrm{i}\vec{k}\cdot \\
\nabla\times & \rightarrow & \mathrm{i}\vec{k}\times
\end{array}
$$

Magnetohydrodynamic waves

- Low frequency waves,
- Ion motion relevant, electrons are ignored,
- Cold plasma (no thermal motions),
- Alfven wave (transversal wave; direction of propagation parallel to the field),
- Magnetosonic wave (longitudinal wave; direction of propagation perpendicular to the field),
- Waves oblique to the magnetic field,
- Required concepts for graphical explanation: magnetic tension and magnetic pressure

Alfven waves 1

- Basic equations:

$$\begin{aligned}\varrho_\mathrm{o}\frac{\partial\vec{u}_1}{\partial t} &= \vec{j}_1\times\vec{B}_1 = \frac{1}{\mu_\mathrm{o}}(\nabla\times\vec{B}_1)\times\vec{B}_1\\ \frac{\partial\vec{B}_1}{\partial t} &= \nabla\times(\vec{u}_1\times\vec{B}_\mathrm{o})\\ \nabla\cdot\vec{B}_1 &= 0\end{aligned}$$

- Fourier transform and transformed system:

$$\begin{aligned}\partial/\partial t &\rightarrow -\mathrm{i}\omega\\ \nabla &\rightarrow \mathrm{i}\vec{k}\\ \nabla\cdot &\rightarrow \mathrm{i}\vec{k}\cdot\\ \nabla\times &\rightarrow \mathrm{i}\vec{k}\times\end{aligned}
\qquad
\begin{aligned}-\mathrm{i}\omega\varrho_\mathrm{o}\vec{u}_1 &= \frac{\mathrm{i}}{\mu_\mathrm{o}}(\vec{k}\times\vec{B}_1)\times\vec{B}_\mathrm{o}\\ \mathrm{i}\omega\vec{B}_1 &= \mathrm{i}\vec{k}\times(\vec{u}_1\times\vec{B}_\mathrm{o})\\ \vec{k}\cdot\vec{B}_1 &= 0\end{aligned}
\qquad
\begin{aligned}\omega\vec{u}_1 &= \frac{1}{\mu_\mathrm{o}\varrho_\mathrm{o}}(\vec{B}_\mathrm{o}\cdot\vec{B}_1)\vec{k}\\ \omega\vec{B}_1 &= (\vec{k}\cdot\vec{u}_1)\vec{B}_\mathrm{o}\\ \vec{k}\cdot\vec{B}_1 &= 0\end{aligned}$$

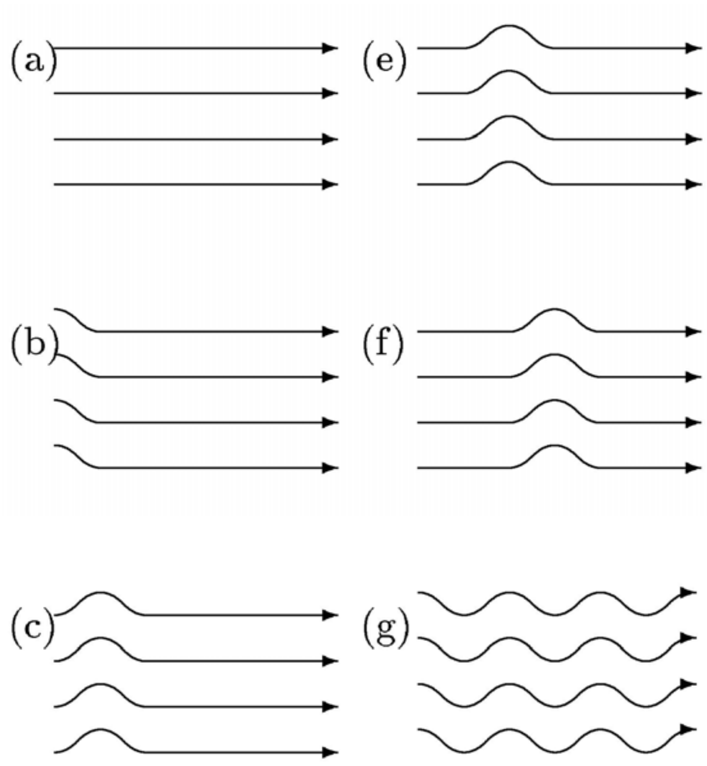
- Newly combined equation: $\omega^2 = \frac{B_\mathrm{o}^2}{\mu_\mathrm{o}\varrho_\mathrm{o}}k^2$

- Alfven speed: $v_\mathrm{A} = \frac{B_\mathrm{o}}{\sqrt{\mu_\mathrm{o}\varrho_\mathrm{o}}}$

Alfven wave 2

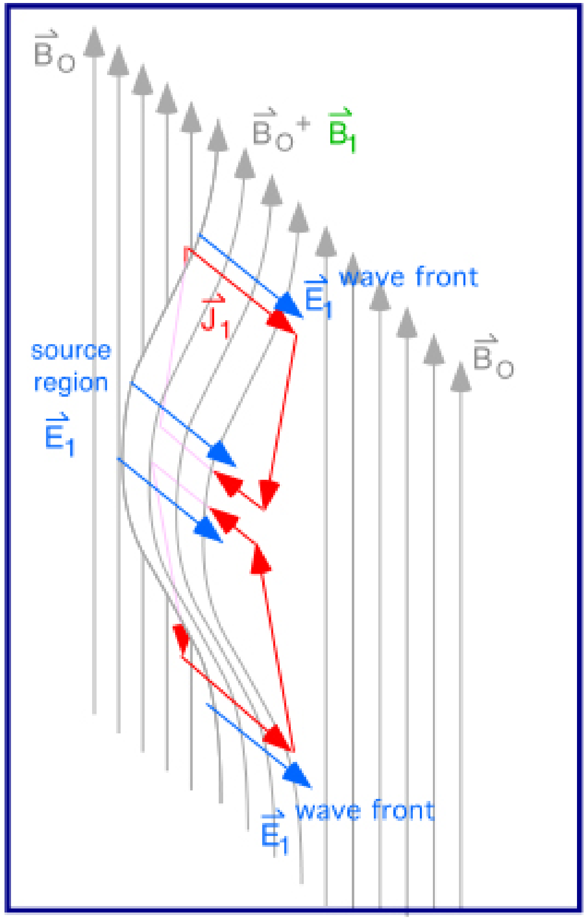
- Graphically: magnetic field line behaves like an oscillating string
- Restoring force: magnetic tension
- Propagation speed: Alfven speed

$$v_A = \frac{B_o}{\sqrt{\mu_o \varrho_o}}$$



Alfven wave 3

A 3-D View of Alfvén wave propagation



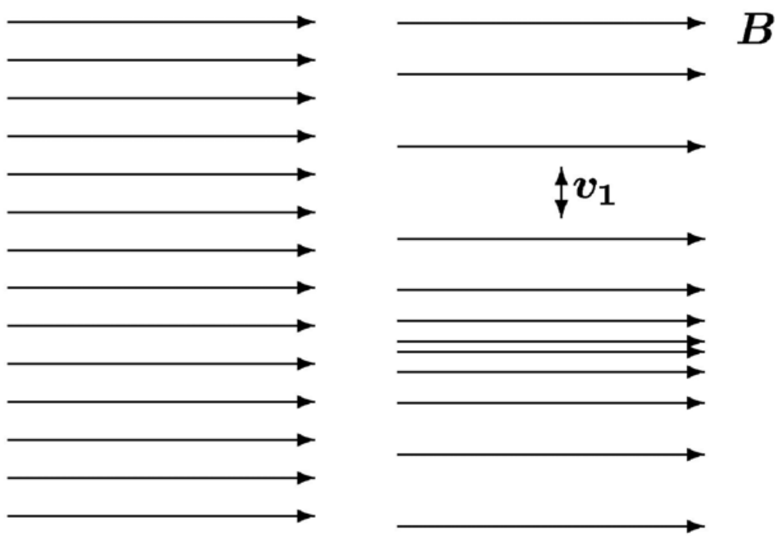
http://www.ss.ncu.edu.tw/~lyu/ResearchWorks/NewSubstormOnsetModel/AlfvenWave3D_Fig1_273k.jpg

- Field lines are not real and consequently cannot oscillate!
- But: a varying magnetic field creates an electromotoric force (Faraday)
- Owing to the high conductivity this leads to a current (Ohm)
- The current in turn gives rise to a magnetic field (Ampere)
- Geometry such that a wave front propagating parallel to the magnetic field results.

Magnetosonic wave

- Formally:
  - Equation of motion: u_1
  - Field equations: fields and currents expressed by u_1, insert into equation of motion
  - Equation of state (the waves allow for a compression of the medium):

$$p_1 = \gamma_a \frac{\varrho_1 p_o}{\varrho_o} = v_{\mathrm{s}}^2 \varrho_1 \qquad \text{mit} \qquad v_{\mathrm{s}} = \sqrt{\frac{\gamma_{\mathrm{a}} p_{\mathrm{o}}}{\varrho_{\mathrm{o}}}}$$



- Fourier-transform yields dispersion relation:

$$-\mathrm{i}\omega \varrho_{\mathrm{o}} \vec{u}_1 = -v_{\mathrm{s}}^2 \frac{\varrho_{\mathrm{o}} \mathrm{i} k^2}{\omega} \vec{u}_1 - \frac{B_{\mathrm{o}}^2 \mathrm{i} k^2}{\mu_{\mathrm{o}} \omega} \vec{u}_1$$

- Longitudinal wave with

$$v_{\mathrm{ms}}^2 = \frac{\omega^2}{k^2} = v_{\mathrm{s}}^2 + v_{\mathrm{A}}^2$$

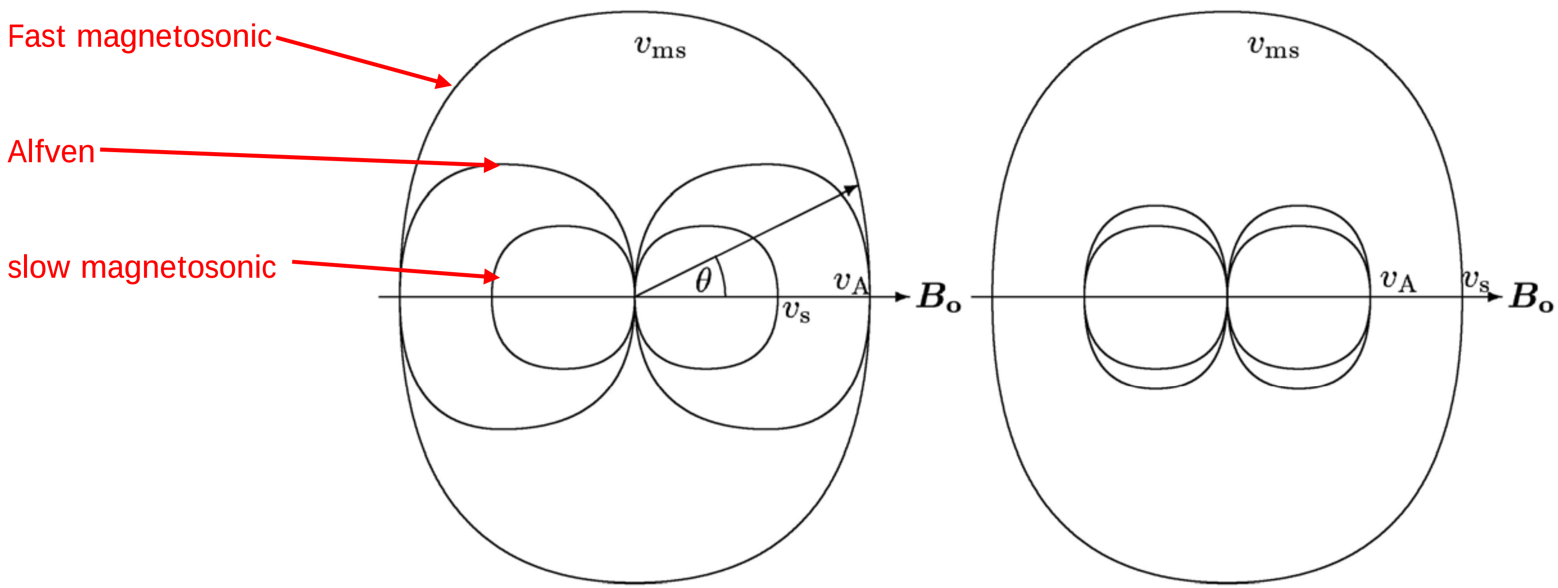
- Restoring force: magnetic pressure

Oblique MHD waves

- Formally: solve entire system of equations for average and fluctuating quantities:

$$u^4 - (v_A^2 + v_s^2)u^2 + v_A^2 v_s^2 \cos^2\theta = 0$$

- Hodograph: polar diagram in v with θ being the angle between direction of propagation and magnetic field



(a) $v_A > v_s$ (b) $v_A < v_s$

Electrostatic waves in non-magnetized plasmas

- Cause: local imbalance of charges in the quasi-neutral medium
- Consequence: oscillation of charges
  - Only the electric field is modified
  - No oscillating magnetic field components
  - $\Rightarrow$ electrostatic waves
- Except for ion waves, the ions are assumed to be infinitely heavy and thus do not participate in the motion; thus except ion waves the waves are high frequency waves.
- Fourier-transform of Faraday´s law for the fluctuating quantities yields

$$i\vec{k} \times \vec{E}_1 = i\omega \vec{B}_1 = 0 \qquad \Rightarrow \qquad \boxed{\vec{k} \| \vec{E}_1}$$

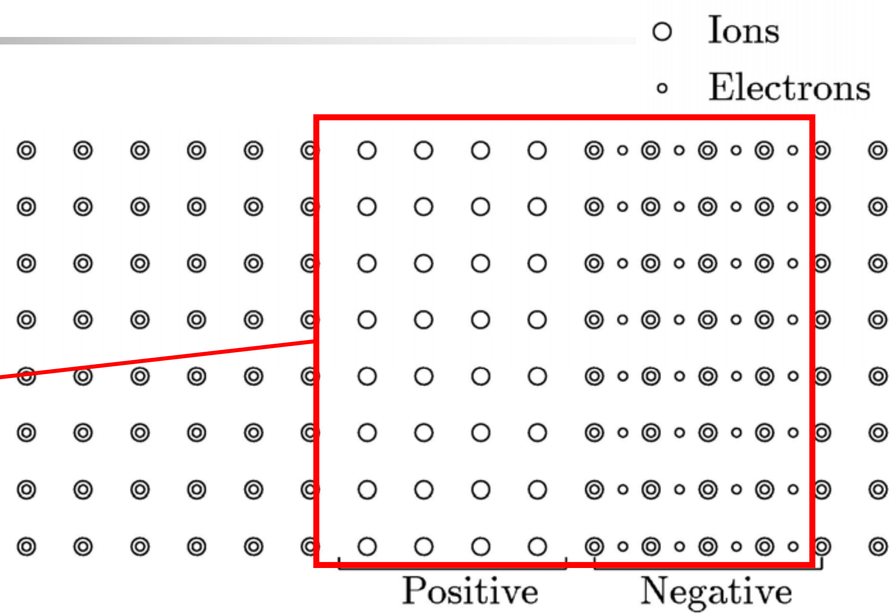
Wave vector parallel to the fluctuating electric field

Plasma oscillations

- Charge imbalance yields an electric field
- Equation of motion:

$$m_e \frac{\partial^2 x}{\partial t^2} = -eE$$

$$\frac{\partial^2 x}{\partial t^2} = -\frac{n_e e^2}{\varepsilon_o m_e} x = -\omega_{pe} x$$



- Electron plasma frequency:

$$\omega_{pe} = \sqrt{\frac{n_e e^2}{\varepsilon_o m_e}}$$

- Important tool in plasma diagnostics, for instance measurement of the electron density in the solar corona.

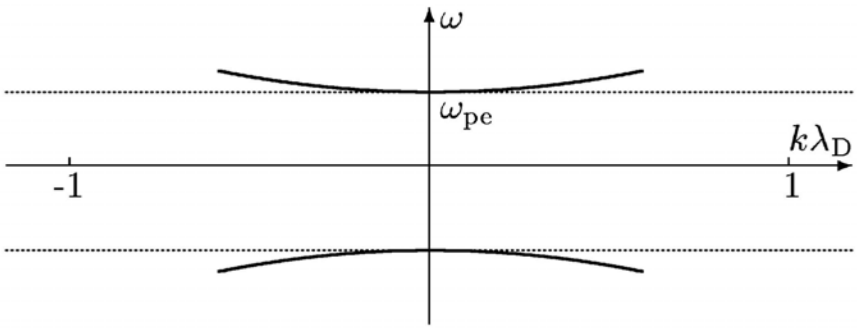
Electron plasma waves (Langmuir waves)

- Plasma oscillations:
  - Cold plasma,
  - Group speed 0: the disturbance does not propagate,
- Electron plasma waves:
  - Hot plasma (thermal motion),
  - Propagating disturbance.
- Plasma behaves adiabatically; maxwellian distribution:

$$\omega^2 = \omega_{\mathrm{pe}}^2 + \tfrac{5}{3}k^2 v_{\mathrm{th}}^2$$

- Dispersion relation in space plasmas (Bohm-Gross equation):

$$\omega^2 = \omega_{\mathrm{pe}}^2 + 3k^2 v_{\mathrm{th}}^2$$



- Group speed:

$$v_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{3k}{\omega} v_{th}^2 = \frac{3v_{\mathrm{th}}^2}{v_{\mathrm{ph}}}$$

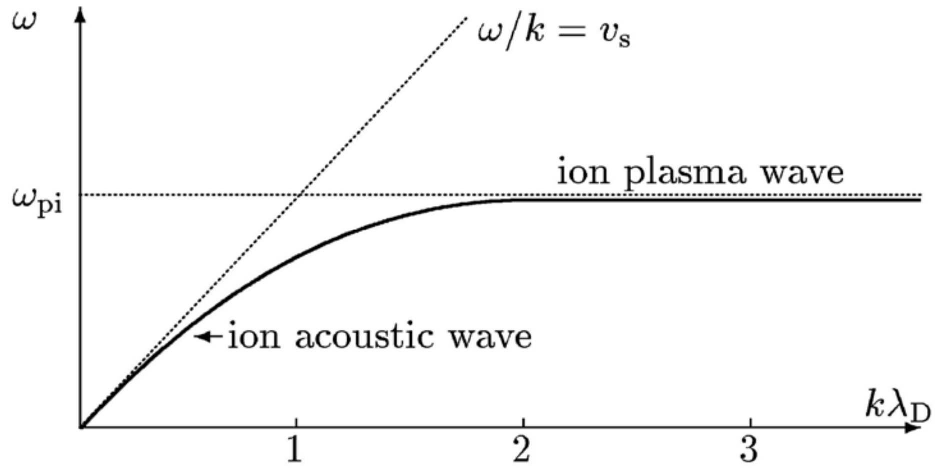
Ionacoustic Waves (Ion waves)

- Inertia of the ions $\Rightarrow$ low frequency waves
- Analogous: sound waves; in a rarefied plasma transport of momentum due to Coulomb collisions
- Equation of motion (2-fluid model) considers only inertial term, pressure gradient force and electromagnetic forces.
- Dispersion relation:

$$\frac{\omega^2}{k^2} = \frac{\gamma_e k_B T_e + \gamma_i k_B T_i}{m_i} = v_s$$

- Ion plasma frequency:

$$\omega^2 = \omega_{pi}^2 = \frac{n_i Z^2 e^2}{\varepsilon_o m_i}$$



Electrostatic waves in magnetized plasmas

- Different groups of waves:
  - Wave vector parallel to the average magnetic field
  - Wave vector perpendicular to the average magnetic field

- longitudinal and transversal describe the wave vector relative to the fluctuating electric field:
  - longitudinal waves are electrostatic waves.

- Waves oblique to the field can be described as a superposition of transversal and longitudinal waves.

Upper hybrid frequency

- Analogous electron oscillations:
  - Stationary ions
  - Cold plasma

- Dispersion relation: $\omega_{\mathrm{uh}}^2 = \omega_{\mathrm{pe}}^2 + \omega_{\mathrm{ce}}^2$

  - Limited to waves perpendicular to the field.
  - For waves parallel to the field the cyclotron frequency vanishes.

- Graphically: Superposition of two motions:
  - Compression/decompression of the electrons as in plasma oscillations,
  - Plus gyration: elliptical electron orbit instead of linear motion $\Rightarrow$ additional restoring force $\Rightarrow$ higher frequency

Ion cyclotron waves

- Low frequency waves (Ion motion);
- Assumption: wave vector roughly $\perp$ to B (small component $\parallel$ B required for free electron motion $\parallel$ B);
- Equation of motion: electromagnetic forces only;
- Dispersion relation:

$$\omega_{\mathrm{pi}}^2 = \omega_{\mathrm{ci}}^2 + k^2 v_{\mathrm{s}}^2$$

- Explanation: additional restoring force (Lorentz).

Lower hybrid frequency

- Wave vector perpendicular to B, thus no free electron motion parallel to B.
- Equation of motion for ions the same, that for electrons modified.
- Lower hybrid frequency: $\omega_{\mathrm{lh}} = \sqrt{\omega_{\mathrm{ce}} \cdot \omega_{\mathrm{ci}}}$
- Motion perpendicular to the field is sustained because the ion motion is determined by the ExB-drift.

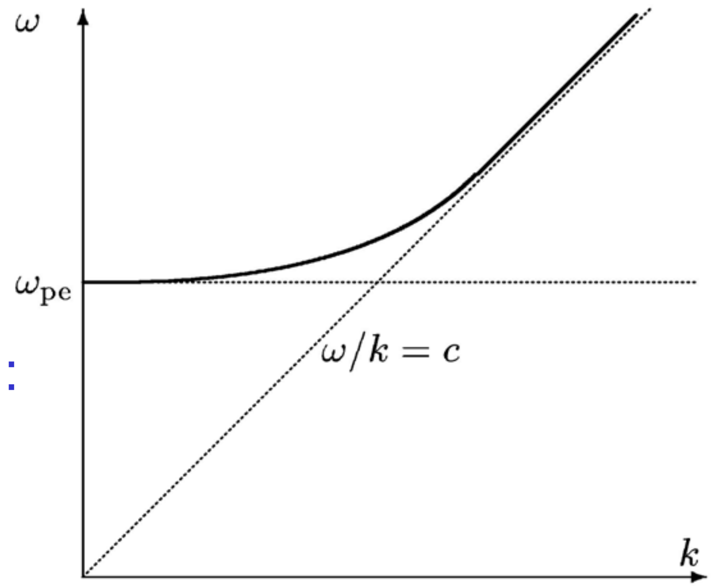
Electromagnetic waves in non-magnetized plasmas

- Electromagnetic wave: fluctuating electric and fluctuating magnetic field.
- Non-magnetized plasma: average B vanishes, fluctuating parts can exist.
- Dispersion relation:

$$\omega^2 = \omega_{\mathrm{pe}}^2 + k^2 c^2$$

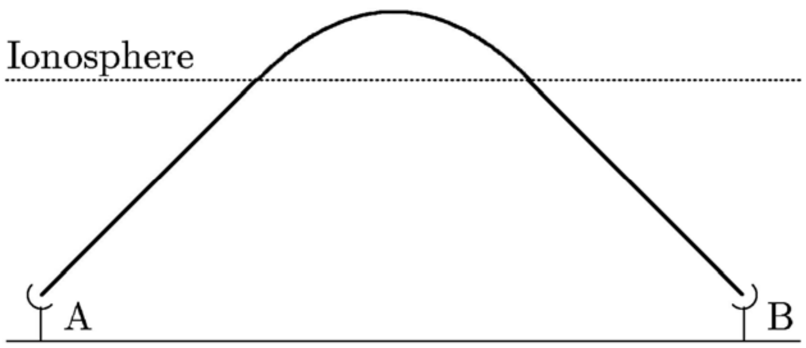
- Index of refraction (propagation for n > 0 only):

$$n = \sqrt{1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}}$$



Ionosounding

- Diagnostic tool for the ionosphere
- Absorption of radio waves at the local plasma frequency ⇒ allows to determine the electron density from the absorption between transmitter and receiver
- Alternatively: Variation of the index of refraction allows for reflection
  - Travel time yields height of reflection
  - Geometry yields index of refraction and thus electron density



Electromagnetic waves in non-magnetized plasmas

- Characterized by the direction of the wave vector relative to the undisturbed magnetic field B0 und the fluctuating electric field E1:

  - Electromagnetic waves perpendicular B0:
    - Ordinary waves (O-Waves): E1 ∥ B0,
    - Extraordinary waves: E1 ⊥ B0,
      - 2 Modes: left and right handed circularily polarized

  - Electromagnetic waves parallel to B0:
    - Whistler (R-Waves),
    - L-Waves

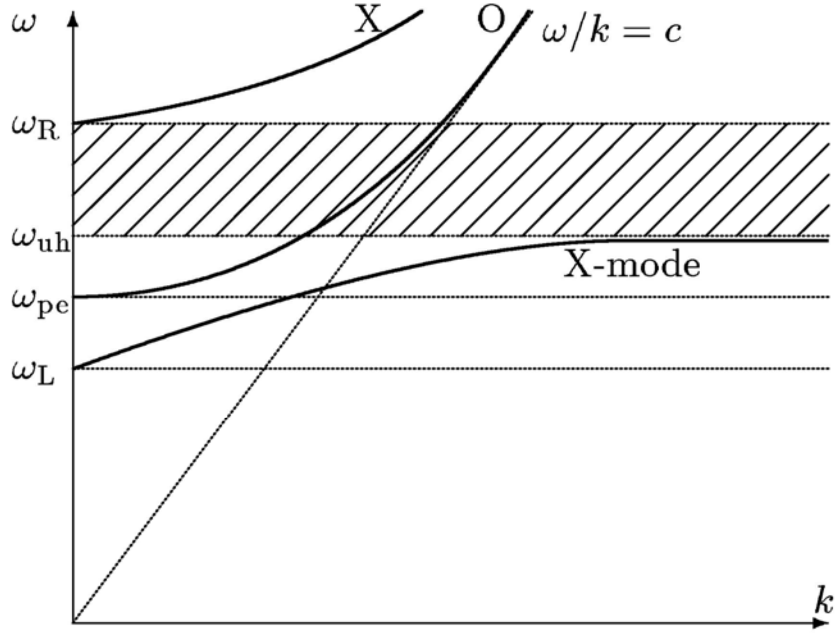
Electromagnetic waves perpendicular to the field

- Transversal wave with k ⊥ E1
- Ordinary wave: E1 ∥ B0
  - B does not influence the wave
  - Dispersion relation

$$\omega^2 = \omega_{\mathrm{pe}}^2 + c^2 k^2$$

- Extraordinary wave: E1 ⊥ B0
  - Dispersion relation:

$$\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2} \frac{\omega^2 - \omega_{\mathrm{pe}}^2}{\omega^2 - \omega_{\mathrm{uh}}^2} = n^2$$



  - mixed mode: transversal and longitudinal component, wave polarized elliptically.
  - Index of refraction n=0 for $\omega^{\mathrm{X}}_{(\mathrm{L,R})} = \pm \frac{\omega_{\mathrm{ce}}}{2} + \sqrt{\omega_{\mathrm{pe}}^2 + \frac{\omega_{\mathrm{ce}}^2}{4}}$
  - Existence of a stop band
  - Application: diagnostic of planetary waves (e.g. Jupiter)

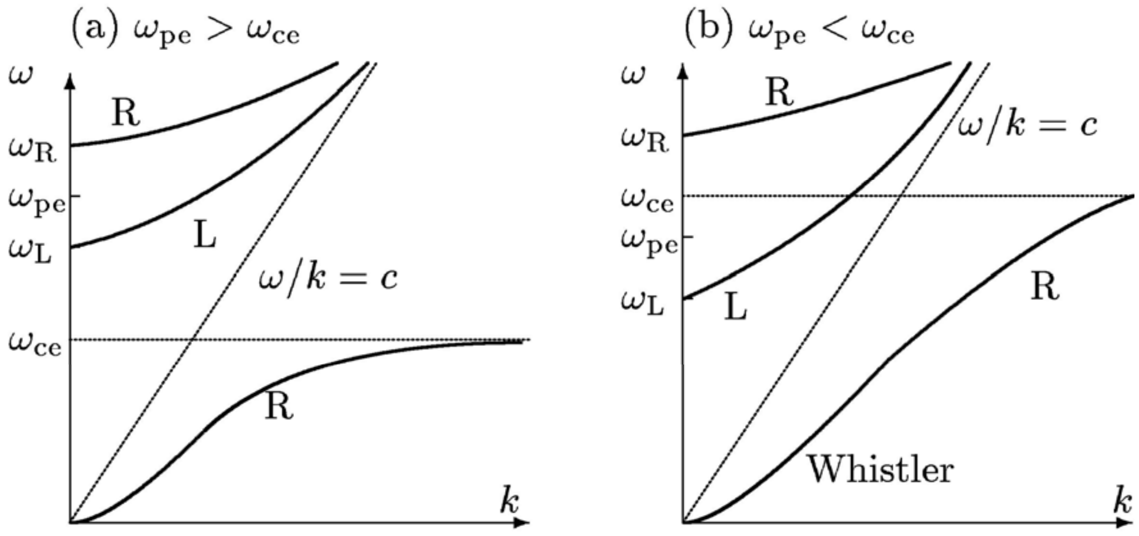
Electromagnetic waves parallel to the field: R- and L- waves

- Left and right-handed polarized waves
  - Dispersion relation:

$$\frac{\omega^2 - k^2c^2}{\omega_{pe}^2}\left(1 \pm \frac{\omega_{ce}}{\omega}\right) = 1$$

  - Index of refraction:

$$n^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_e^2/\omega^2}{1 \pm \omega_{ce}/\omega}$$



- Resonances:
  - R-wave (positive) at the cyclotron frequency (cyclotron resonance)
  - L-wave at the ion cyclotron frequency
- Cutoff-frequencies:

$$\omega_{(L,R)}^{\text{cutoff}} = \pm\frac{\omega_{ce}}{2} \pm \sqrt{\omega_{ce}^2 + \frac{\omega_{pe}^2}{4}}$$

- Low frequency part of the R-waves (Whistler): diagnostics of the magnetosphere

Summary

Wave	Geometry	Dispersion relation
Electron Waves (Electrostatic)		
Langmuir waves	$\vec{B}_o = 0$ or $\vec{k} \parallel \vec{B}_o$	$\omega^2 = \omega_{pe}^2 + 3k^2 v_{th}^2/2$
Upper hybrid waves	$\vec{k} \perp \vec{B}_o$	$\omega_{uh} = \omega_{pe}^2 + \omega_{ce}^2$
Ion Waves (Electrostatic)		
Ion acoustic waves	$\vec{B}_o = 0$ or $\vec{k} \parallel \vec{B}$	$\omega^2 = k^2 v_s^2$
Ion cyclotron waves	$\vec{k} \perp \vec{B}_o$	$\omega^2 = \omega_{ci}^2 + k^2 v_s^2$
Lower hybrid waves	$\vec{K} \perp \vec{B}_o$	$\omega_{lh}^2 = \omega_{ci} \cdot \omega_{ce}$
Electron Waves (Electromagnetic)		
Light waves	$\vec{B}_o = 0$	$\omega^2 = \omega_{pe}^2 + k^2 c^2$
O-waves	$\vec{k} \perp \vec{B}_o$, $\vec{E}_1 \parallel \vec{B}_o$	$\omega^2 = c^2 k^2 + \omega_{pe}^2$
X-waves	$\vec{k} \perp \vec{B}_o$, $\vec{E}_1 \perp \vec{B}_o$	$\omega^2 = c^2 k^2 + \omega_{pe}^2 \cdot (\omega^2 - \omega_{pe}^2)/(\omega^2 - \omega_h^2)$
Whistler (R-waves)	$\vec{k} \parallel \vec{B}_o$	$\omega^2 = c^2 k^2 - \omega_{pe}^2/[1 - (\omega_{ce}/\omega)]$
L-waves	$\vec{k} \parallel \vec{B}_o$	$\omega^2 = c^2 k^2 + \omega_{pe}^2/[1 + (\omega_{ce}/\omega)]$
Ion Waves (Electromagnetic)		
Alfvén waves	$\vec{k} \parallel \vec{B}_o$	$\omega^2 = k^2 v_A^2$
Magneto-sonic waves	$\vec{k} \perp \vec{B}_o$	$\omega^2 = c^2 k^2 \cdot (v_s^2 + v_A^2)/(c^2 + v_A^2)$