Magnetohydrodynamics (MHD)

- Overview:
  - Basic equations
  - Magnetic pressure and tension (magnetohydrostatics)
  - Frozen-in magnetic fields (magnetohydrokinematics)
  - Dissipation of fields (magnetohydrokinematics)
  - Merging of magnetic fields (reconnection)
  - Magnetohydrodynamic dynamo

- Prerequisites:
  - Particles are treated as fluid (no distribution function, mono-energetic particle ensembles $\rightarrow$ kinetic theory)
  - The electromagnetic field is not prescribed; instead, it is determined by the positions and motion of the charges $\Rightarrow$ self-consistent solutions required (time dependent!)

General comments

- Equation of motion: transition from a single particle to a particle density and thus also to a force density

$$nm\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = nq\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

with u=<v> being the bulk speed.

- Additional internal forces (interaction between the charges)

- The total temporal derivative is the sum of local temporal derivative and advection

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \underbrace{\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial\varepsilon}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial\varepsilon}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial\varepsilon}{\partial z} + \frac{\partial\varepsilon}{\partial t}}_{\text{Chain rule } \varepsilon=\varepsilon(x,y,z,t)} = (\vec{u}\cdot\nabla)\varepsilon + \frac{\partial\varepsilon}{\partial t}$$

Local temporal derivative

Advection

Basic assumptions MHD

- The medium can neither be polarized nor magnetized:

  $$\varepsilon=\mu=1.$$

- Bulk speeds and speeds of changes of properties are small compared with the speed of light:

  $$u/c<<1$$

  $$v\_phase/c <<1$$

  As a consequence, electromagnetic waves cannot be described within the framework of MHD.
- The conductivity is infinitely high:

  $$\sigma\rightarrow\infty$$

  As a consequence, strong electric fields cancel immediately due to rearrangement of the charges.
- MHD is based on the conservation laws for mass, momentum and energy.

Basic equations of MHD I

- Maxwell's laws:

$$\nabla \cdot \vec{E} = \varrho / \varepsilon_o \qquad \text{und} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{und} \qquad \nabla \times \vec{B} = \mu_o \vec{j} + \varepsilon_o \mu_o \frac{\partial \vec{E}}{\partial t}$$

- Ohm's law: $\vec{j} = \sigma \left(\vec{E} + \vec{u} \times \vec{B} \right)$

Lorentz
Gravitation
Coriolis
Centrifugal

- Equation of motion (Navier-Stokes):

$$\begin{aligned} \varrho \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} &= \varrho \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} \right) \\ &= -\nabla \mathrm{P} + \varrho \vec{E} + \vec{j} \times \vec{B} + \varrho \vec{g} - 2\varrho \vec{\Omega} \times \vec{u} - \varrho \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \end{aligned}$$

Stress tensor

$$\varrho \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = \varrho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} \right] = -\nabla p + \varrho \nu \nabla^2 \vec{u} + \vec{j} \times \vec{B} + \varrho \vec{g}$$

Pressure gradient force
Gravitation
Friction
Lorentz-force

Basic equations of MHD II

- Equation of motion: simple, no Coulomb collisions, no friction between different components of the plasma, no sources or sinks (ionization or recombination),
- Equation of continuity: $\frac{\partial \varrho}{\partial t} = -\nabla(\varrho \vec{u}) = -\nabla \vec{j}$
- Equation of state: $p = C(T)\,\varrho$
- Characterization: plasma-β

$$S = \frac{B^2/2\mu_o}{\varrho u^2/2} = \frac{\text{magnetic field energy density}}{\text{kinetic energy density}}$$

$$\beta_{\|} = \frac{2\mu_o p_{\|}}{B^2} \quad \text{and} \quad \beta_{\perp} = \frac{2\mu_o p_{\perp}}{B^2}$$

Basic equations MHD 2-Fluid

- Maxwell's laws:

$$\nabla \cdot \vec{E} = (\varrho_\mathrm{i} + \varrho_\mathrm{e})/\varepsilon_\mathrm{o} \qquad \text{und} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{und} \qquad \nabla \times \vec{B} = \mu_\mathrm{o}(\vec{j}_\mathrm{i} + \vec{j}_\mathrm{e}) + \varepsilon_\mathrm{o}\mu_\mathrm{o}\frac{\partial \vec{E}}{\partial t}$$

Hall effect (Lorentz-force)

- Ohm´s law:

Diffusive streaming caused by the pressure gradient

Current acceleration

$$\frac{m_\mathrm{e}}{e^2 n}\frac{\partial \vec{j}}{\partial t} = \vec{E} + \vec{u} \times \vec{B} - \frac{\vec{j} \times \vec{B}}{en} + \frac{\nabla p_\mathrm{e}}{en} - \frac{\vec{j}}{\sigma}$$

- Equation of motion:

Friction (Coulomb-collisions)

$$m_j n_j \frac{\mathrm{d}\vec{u}_j}{\mathrm{d}t} = q_j n_j \left(\vec{E} + \vec{u}_j \times \vec{B}\right) - \nabla p_j \pm \beta(\vec{u}_\mathrm{i} - \vec{u}_\mathrm{e}), \quad j = \mathrm{i}, \mathrm{e}$$

- Equation of continuity: $\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \vec{u}_j) = 0 \;, \quad j = \mathrm{i}, \mathrm{e}$

- Equation of state: $p_j = p_j(\varrho_j, T_j) \;, \quad j = \mathrm{i}, \mathrm{e}$

Magnetohydrostatics

- Pre-requisite: the medium is at rest and the inertia term in the equation of motion vanishes
- Formally: energy balance between particles and fields
- Concepts:
  - Magnetic pressure: magnetic lines of force repulse each other
  - Magnetic tension: magnetic lines of force tend to shorten

Magnetic pressure

- Fundamentals: equation of motion (without inertia term) + Ampere´s law:

$$\vec{f} = \vec{j} \times \vec{B} \;=\; -\frac{1}{\mu_o}\vec{B} \times \left(\nabla \times \vec{B}\right) = -\frac{1}{\mu_o}\nabla(\vec{B}\vec{B}) + \frac{1}{\mu_o}\vec{B}(\nabla \cdot \vec{B})$$

$$= -\frac{1}{\mu_o}\nabla(\vec{B}\vec{B}) + \frac{1}{2\mu_o}\nabla \vec{B}^2$$

dyadic produkt (magnetic stresstensor),

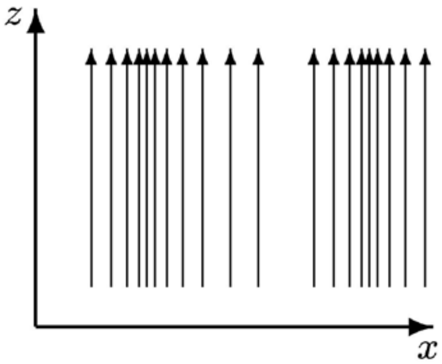
Tension will be considered later

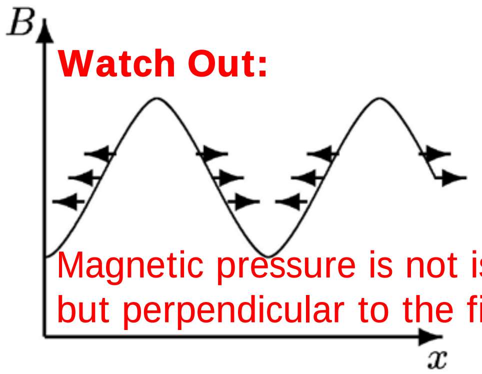
- Formally: magnetic pressure (it is an energy density)

$$\boxed{p_{\mathrm{M}} = \frac{B^2}{2\mu_o}}$$

- Graphically:

$$f_x = -\frac{1}{\mu_o} B \frac{\partial B}{\partial x} = -\frac{\partial}{\partial x}\frac{B^2}{2\mu_o}$$





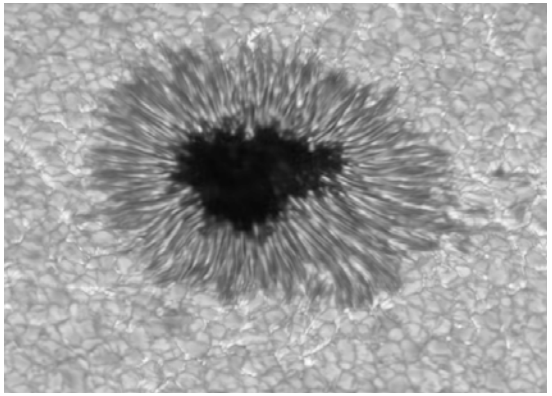
Watch Out:

Magnetic pressure is not isotropic but perpendicular to the field!!

- Example: homogenous field, B=5T, magnetic pressure 100 times the atmospheric pressure at the surface

Sunspot and magnetic pressure

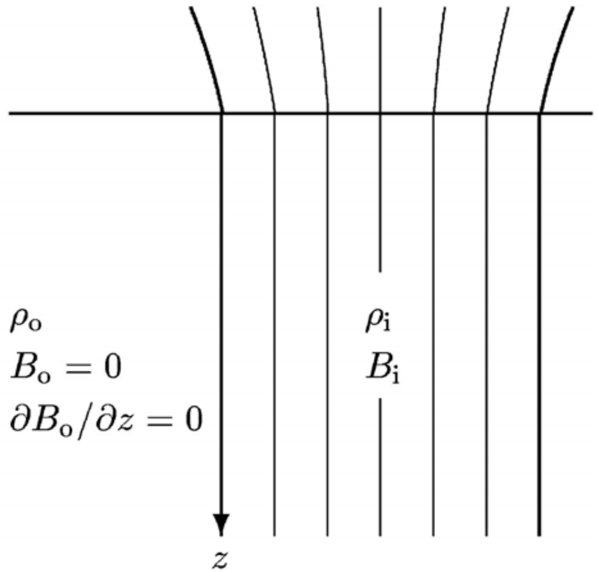
- Observation:
  - cold, therefore dark (4000 K instead of 6000 k)
  - Strong magnetic field (3000 instead of a few G)
  - Dark is relative (comparable to the full moon)



- Model: pressure balance + hydrostatic equation + $\partial B/\partial z=0$ + equation of state:

$$p_\mathrm{i} + B_\mathrm{i}^2/(2\mu_\mathrm{o}) = p_\mathrm{o}$$

$$\boxed{\text{can only be fulfilled if temperature inside the spot is lower than outside (as observed)}}$$



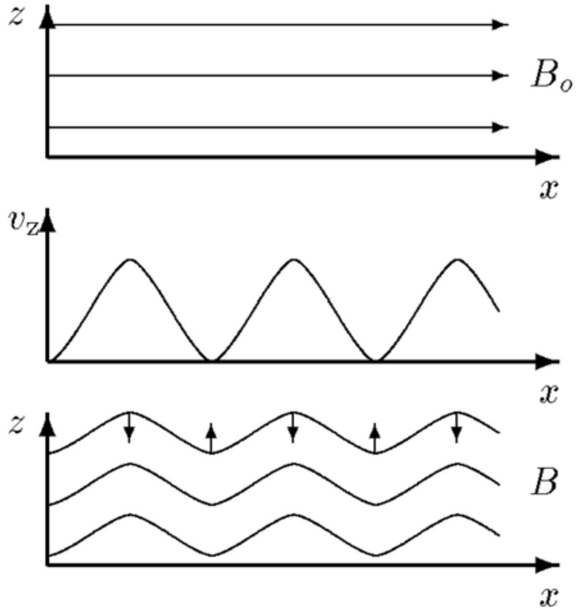
Magnetic tension

- Basics: a magnetic field is deformed by a plasma flow perpendicular to it

$$\vec{f} = \vec{j} \times \vec{B} \quad = \quad -\frac{1}{\mu_o}\vec{B} \times \left(\nabla \times \vec{B}\right)$$

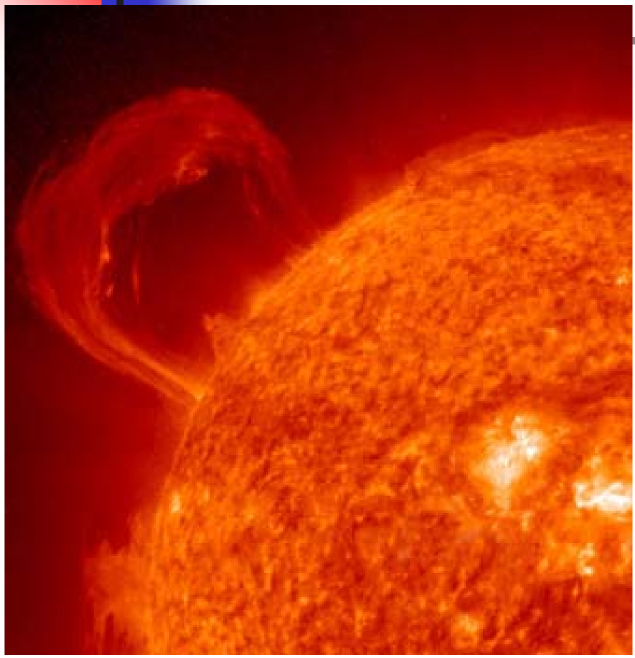
$$\vec{B} = \vec{B}_o + \frac{\partial \vec{B}}{\partial t}\mathrm{d}t = \vec{B}_o + \nabla \times (\vec{u} \times \vec{B})\mathrm{d}t$$

$$f_z = \frac{1}{\mu_o}\frac{\partial^2 u_z}{\partial x^2}B_o^2\mathrm{d}t$$

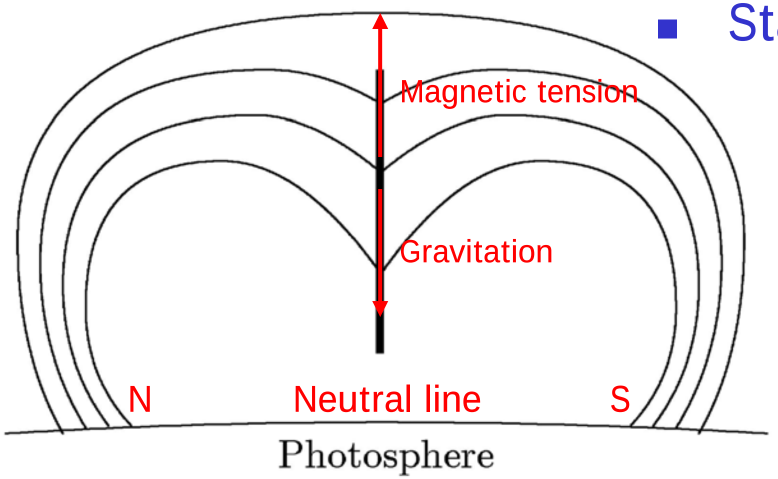


- Magnetic field lines have a tendency to shorten (thus as magnetic pressure magnetic tension is anisotropic!)
- Application: Alfven-wave

Filament and magnetic tension



- Observation Filament/Protuberance:
  - cold because dark before the visible disk (7000 K vs. 1 Mio K)
  - Dense because bright at the limb (ca. 100fach)
  - height 30 Mm (about 100 times the scale height)
  - Stable over many solar rotations until released explosively as CME
  - Roughly oriented along the neutral line.



- Starting point:

$$\nabla p = \frac{1}{\mu_o}(\nabla \times \vec{B}) \times \vec{B} + \varrho \vec{g} \quad \text{und} \quad p = n k_B T$$

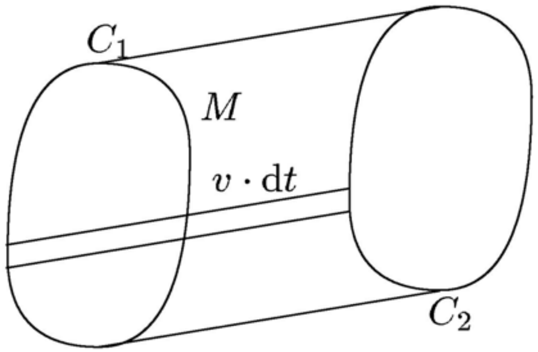
allow calculation of the filament's details, in particular the bending of the field lines at the filament

Magnetohydrokinematics

- Formal requirements:
  - Prescribed velocity field (no solution of the equation of motion required)
  - Sought-after: resulting changes in the electromagnetic field
  - assumption: electromagnetic fields do not retroactive affect the velocity field (plasma-β is large)
  - Relevant equations: Maxwell and Ohm

- Fundamental concepts:
  - Frozen-in magnetic fields: a moving plasma convects a magnetic field with itself (prerequisite: infinite consuctivity)
  - Dissipation of magnetic fields: the time constant of magnetic field dissipation decreases with decreasing scale length (small scale fields vanish faster than large scale ones; again conductivity high)

Frozen-in magnetic fields I



- Magnetic flux through area S bounded by line C:

$$\Phi = \int \vec{B}\,\mathrm{d}\vec{S}$$

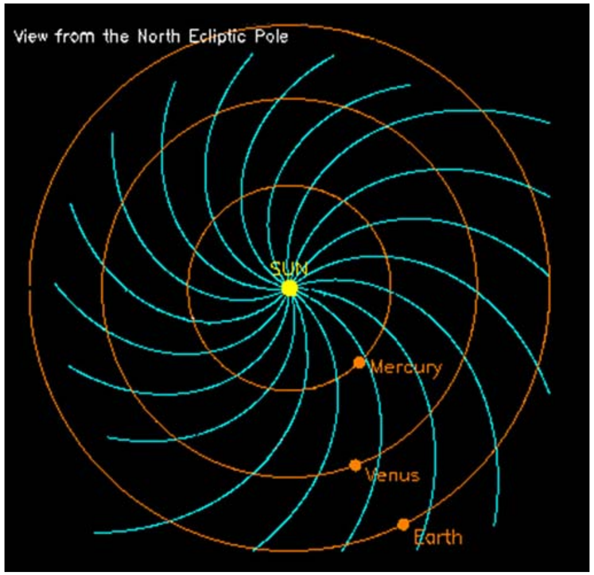
- C moves: change in magnetic flux

$$\mathrm{d}\Phi = \Phi_2 - \Phi_1 = \mathrm{d}t \int_{S_1} \frac{\partial \vec{B}}{\partial t}\,\mathrm{d}\vec{S}_1 + \int_{M} \vec{B}\,\mathrm{d}\vec{S}_M$$

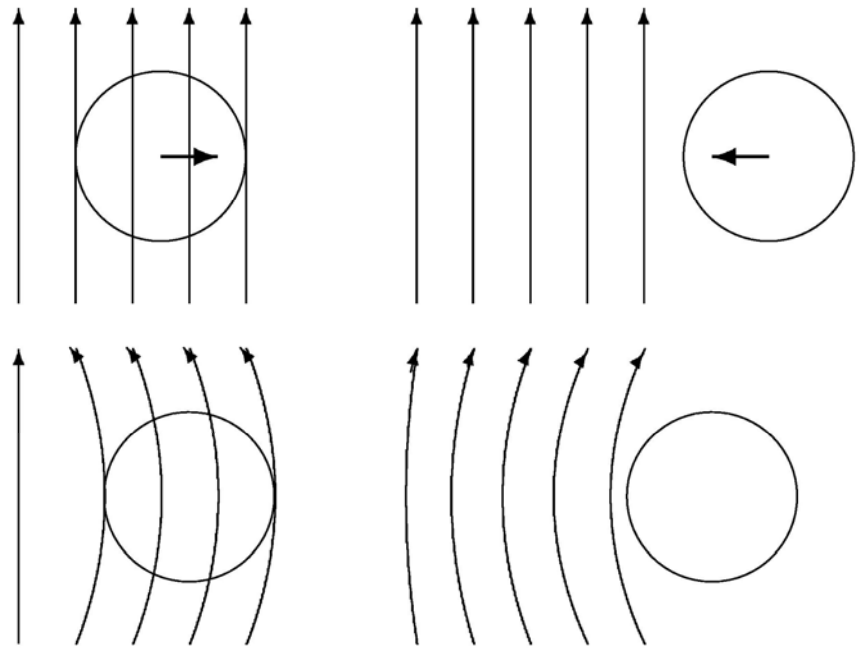
- rewrite:

$$\begin{aligned} \frac{\mathrm{d}\Phi}{\mathrm{d}t} &= \int_{S_1} \frac{\partial B}{\partial t}\,\mathrm{d}\vec{S}_1 + \int_{C_1} \vec{B} \cdot \vec{u} \times \mathrm{d}\vec{l}_1 \\ &= \int_{S_1} \left[\frac{\partial B}{\partial t} - \nabla \times (\vec{u} \times \vec{B}) \right] \mathrm{d}\vec{S}_1 \\ &= -\int_{S_1} \nabla \times \vec{j}\frac{1}{\sigma}\mathrm{d}\vec{S}_1 = -\int_{C_1} \frac{1}{\sigma}\,\vec{j} \cdot \mathrm{d}\vec{l}_1 \end{aligned}$$

- Frozen-in magnetic field: $\boxed{\sigma \to \infty}$: $\Rightarrow$ $\Phi = \mathrm{const}$
- Application: interplanetary magnetic field (archimedian spirals)



Frozen-in magnetic fields II



Frozen-in field:

$\Phi \neq 0$

Frozen-out field:

$\Phi = 0$

- Frozen-in magnetic field: the magnetic field moves with the plasma (e.g. interplanetary magnetic field)
- Frozen-out field: a non-magnetized plasma does not penetrate a magnetic field, e.g. the solar wind does not penetrate the magnetosphere.

Deformation of magnetic field lines

- assumption: σ constant in time and space
- Starting point: Faraday's and Ohm's law

$$\begin{aligned}
\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{u} \times \vec{B}) &= -\frac{1}{\sigma} \nabla \times \vec{j} \\
&= -\frac{1}{\sigma} \nabla \times \frac{\nabla \times \vec{B}}{\mu_o} = -\frac{1}{\mu_o \sigma} \nabla \times \left(\nabla \times \vec{B} \right) \\
&= \frac{1}{\mu_o \sigma} \nabla^2 \vec{B}
\end{aligned}$$

- Stationary equation: $-\nabla \times (\vec{u} \times \vec{B}) = \frac{1}{\mu_o \sigma} \nabla^2 \vec{B}$

Solution: a plasma flow $\perp$ B deforms the field until

$$u_\perp \approx \frac{L}{\tau} \approx \frac{1}{\mu_o \sigma L}$$

is fulfilled. Now the plasma flows perpendicular to B.

Dissipation of magnetic fields

- Assumption: no external velocity field

$$\frac{\partial \vec{B}}{\partial t} = \frac{1}{\mu_o \sigma} \nabla^2 \vec{B}$$

- formally analogc $\frac{\partial T}{\partial t} = \chi \nabla^2 T \,, \quad \frac{\partial \vec{\omega}}{\partial t} = \nu \nabla^2 \vec{\omega}$

Heat conduction vorticity equation

- 1D case: B=B(Bx(y,t),0,0)

$$\frac{\partial B_x}{\partial t} = \frac{1}{\mu_o \sigma} \frac{\partial^2 B_x}{\partial y^2} \,, \qquad \left(\text{formal analog} : \frac{\partial T}{\partial t} = \chi \frac{\partial^2 T}{\partial y^2} \right)$$

- Scale analysis → magnetic diffusion coefficient Dm

$$\frac{B}{\tau} \approx \frac{1}{\mu_o \sigma} \frac{B}{L^2}$$

Dissipation time: $\tau \approx \mu_o \sigma L^2 = L^2 / D_M \qquad \text{mit} \qquad D_M = 1/\mu_o \sigma$

- Example: Sun 1.2E10a, sunspot 1000a

Magnetohydrodynamics

- Pre-requisites:
  - Same as in MHD (see above)
  - Self consistent solutions for the field equations and the equation of motion

- Concepts:
  - Reconnection (merging of fields): conversion of energy stored in the magnetic field into kinetic energy of the plasma
  - MHD Dynamo: conversion of kinetic energy of a plasma into magnetic field energy

Reconnection I

- Idea: explosive release of energy stored in the magnetic field; allows for the slow accumulation of field energy over long time periods and a sudden release of this energy

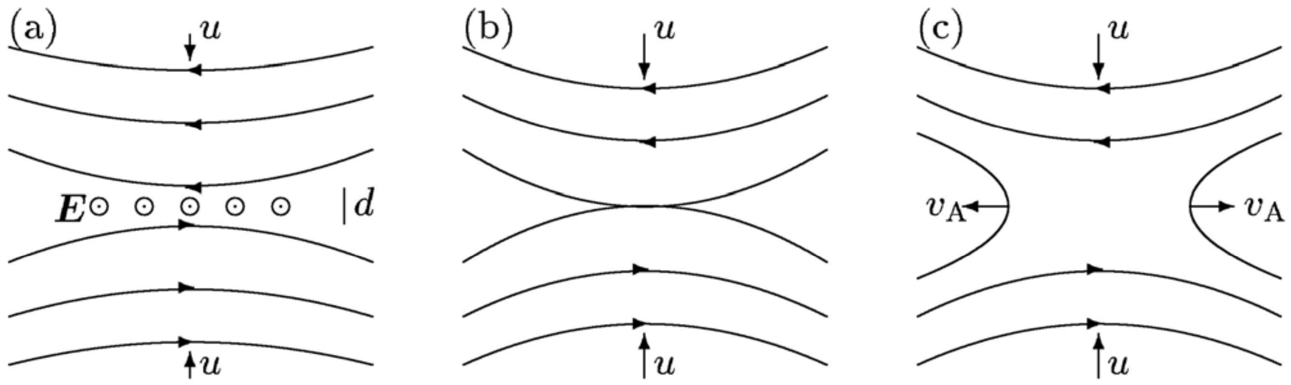
- Observational evidence:
  - Restructuring of the coronal magnetic field following flares and coronal mass ejections
  - Flux transfer events at the magnetopause
  - Particle acceleration in the tail of the magnetosphere

- Pre-requisite: infinitely high conductivity (frozen-in magnetic fields)

- Problem: special topologies required

Reconnection II

- Topology:

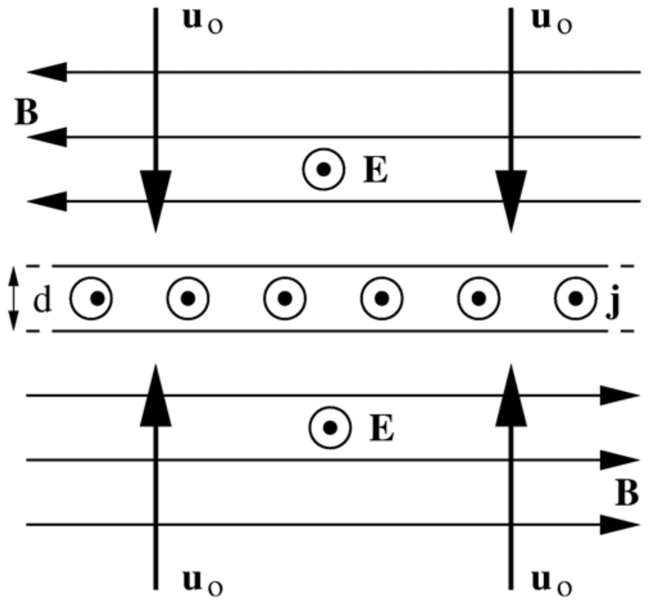


- Current in the neutral sheet: $j = \frac{1}{\mu_o} \frac{\Delta B}{d}$

- Development:
  - ⇒ Frozen-in field
  - ⇒ Plasma flow convects field towards the neutral sheet
  - ⇒ Increase of the neutral sheet current
  - ⇒ Dissipation of the field; magnetic field energy is converted into plasma energy

Details in the neutral sheet

- Assumption: neutral sheet infinitely thin, uniform conductivity
- Magnetic field parallel to the flow can be determined as



$$B_\perp = B_\mathrm{o}\,\mathrm{erf}(\xi) \quad \text{with} \quad \xi = \sqrt{\frac{\mu_\mathrm{o}\sigma}{t}}\,l_\perp$$

$$\mathrm{erf} = \frac{2}{\sqrt{\pi}} \int_0^\xi \mathrm{e}^{\xi^2}\,\mathrm{d}\xi$$

- Time dependence of neutral sheet thickness: $d = \sqrt{\frac{4t}{\mu_\mathrm{o}\sigma}}$

- Energy available for conversion

$$\frac{\partial W_\mathrm{B}}{\partial t} = -\int \vec{E}\cdot\vec{j}\,\mathrm{d}l_\perp$$

Steady reconnection

- Assumption: onset of reconnection does not change the general configuration of fields and plasma
- Ohm's and Ampere's laws:

$$\vec{E} = \frac{\vec{j}}{\sigma} - \vec{u} \times \vec{B} = \frac{\nabla \times \vec{B}}{\mu_o \sigma} - \vec{u} \times \vec{B} = \vec{u}_o \times \vec{B}_o$$

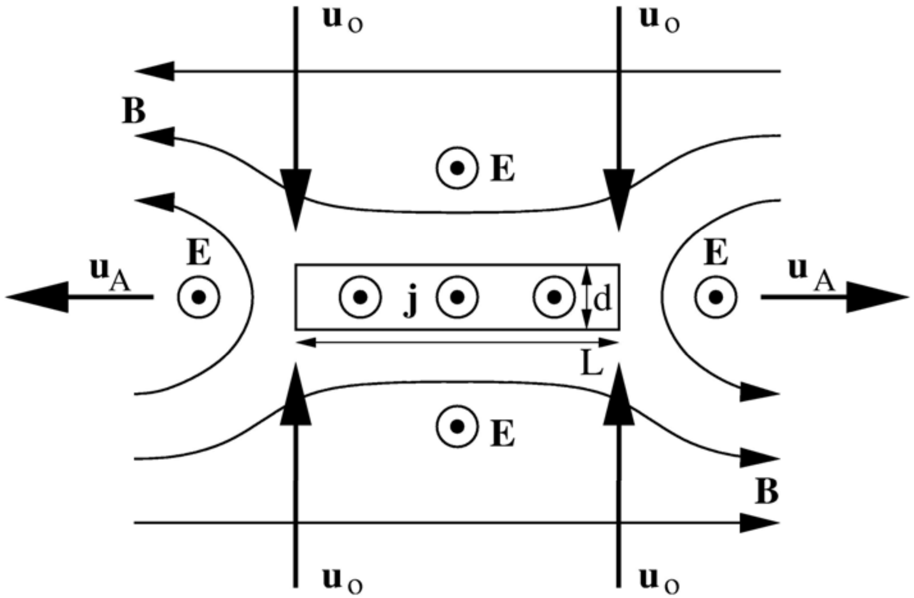
- stationary: Faraday's law yields E = const, thus

$$\vec{E} = \frac{\nabla \times \vec{B}}{\mu_o \sigma}$$

- Thickness of the neutral sheet:

$$d \approx \frac{1}{\mu_o \sigma u_o}$$

Energy balance



- Continuity magnetic field:

$$\vec{u} \times \vec{B} = \vec{u}_\mathrm{A} \times \vec{B}_\mathrm{A} = \vec{u}_\mathrm{o} \times \vec{B}_\mathrm{o}$$

- skalar (u⊥B):

$$uB = u_\mathrm{A} B_\mathrm{A} = u_\mathrm{o} B_\mathrm{o}$$

- Continuity mass: $u_\mathrm{A} d = u_\mathrm{o} L$

- combined: $u_\mathrm{o} = \frac{u_\mathrm{A}\, d}{L} = u_\mathrm{A}^2 \frac{L}{\sigma \mu_\mathrm{o} u_\mathrm{o}} = \frac{u_\mathrm{A}}{\sqrt{R_\mathrm{M}^\mathrm{out}}}$ and $B_\mathrm{A} = B_\mathrm{o} \frac{u_\mathrm{o}}{u_\mathrm{A}} = B \frac{d}{L}$

- Energy balance: $2 L u_\mathrm{o} \left(\frac{1}{2} \varrho u_\mathrm{o}^2 + \frac{B_\mathrm{o}}{2\mu_\mathrm{o}} \right) = 2 d u_\mathrm{A} \left(\frac{1}{2} \varrho u_\mathrm{A}^2 + \frac{B_\mathrm{A}}{2\mu_\mathrm{o}} \right)$

$$\varrho u_\mathrm{o}^2 + \frac{B_\mathrm{o}}{\mu_\mathrm{o}} = \varrho u_\mathrm{A}^2 + \frac{B_\mathrm{A}}{\mu_\mathrm{o}} \qquad u_\mathrm{A}^2 = u_\mathrm{o}^2 + v_\mathrm{A}^2 \left(1 - \frac{u_\mathrm{o}^2}{u_\mathrm{A}^2} \right)$$

- Rate of reconnection: $R_\mathrm{P} = \frac{\pi}{8} \ln \left(\sqrt{\frac{1}{L \sigma v_{\mathrm{A,in}} \mu_\mathrm{o}}} \right)$

Rate of reconnection

- Sweet-Parker Reconnection:

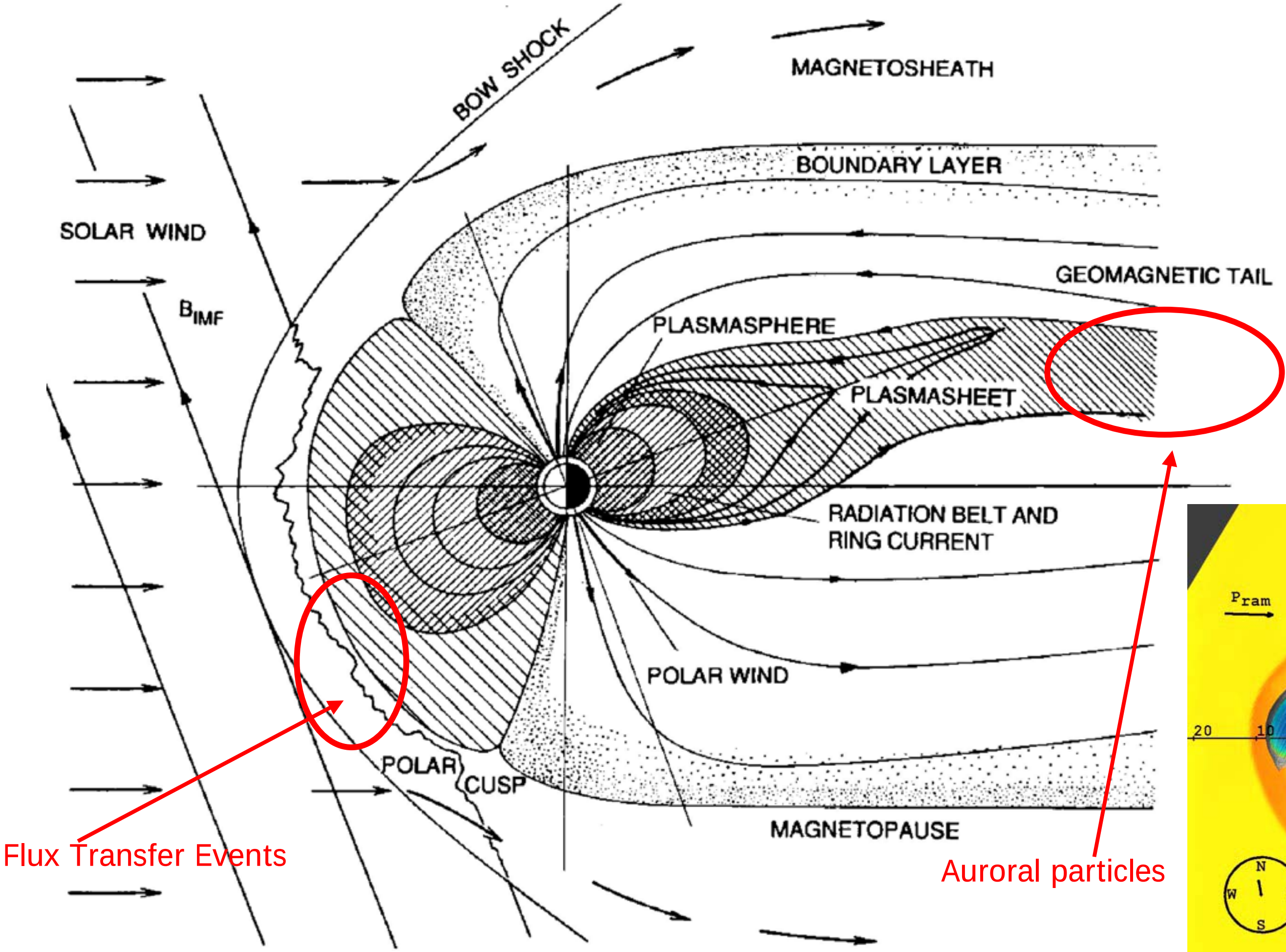
$$R_{\mathrm{SP}} = \sqrt{\frac{1}{L\sigma v_{\mathrm{A,in}}\mu_{\mathrm{o}}}}$$

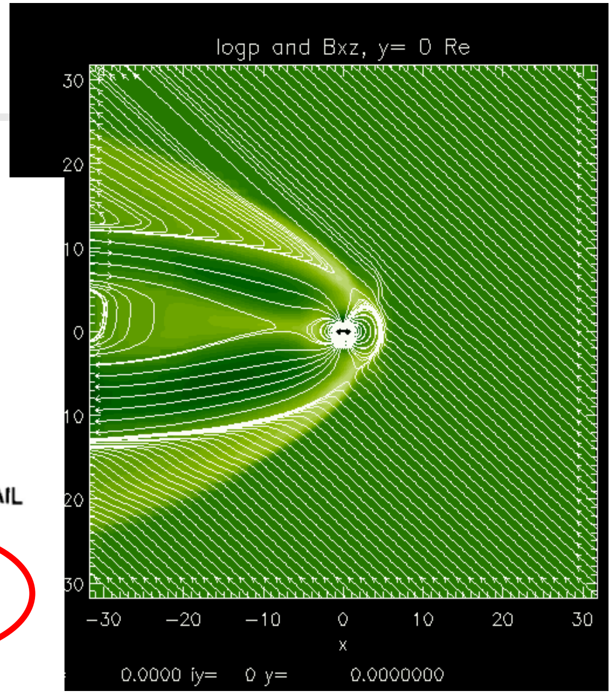
  - Slow process
  - Roughly half of the magnetic field energy is converted into kinetic energy
  - Flux-Transfer Events
- Petchek Reconnection:

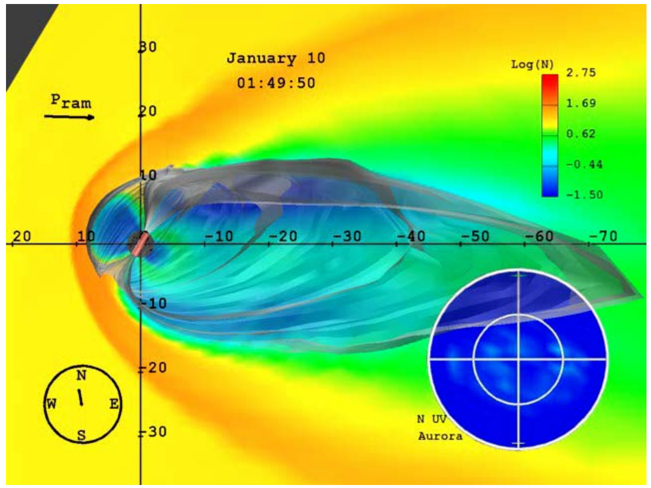
$$R_{\mathrm{P}} = \frac{\pi}{8}\ln\left(\sqrt{\frac{1}{L\sigma v_{\mathrm{A,in}}\mu_{\mathrm{o}}}}\right)$$

  - Smaller spatial scales
  - Faster because of smaller scales
  - About 3/5 of the field energy is converted into plasma kinetic energy
  - Shocks
  - Flares and coronal mass ejections

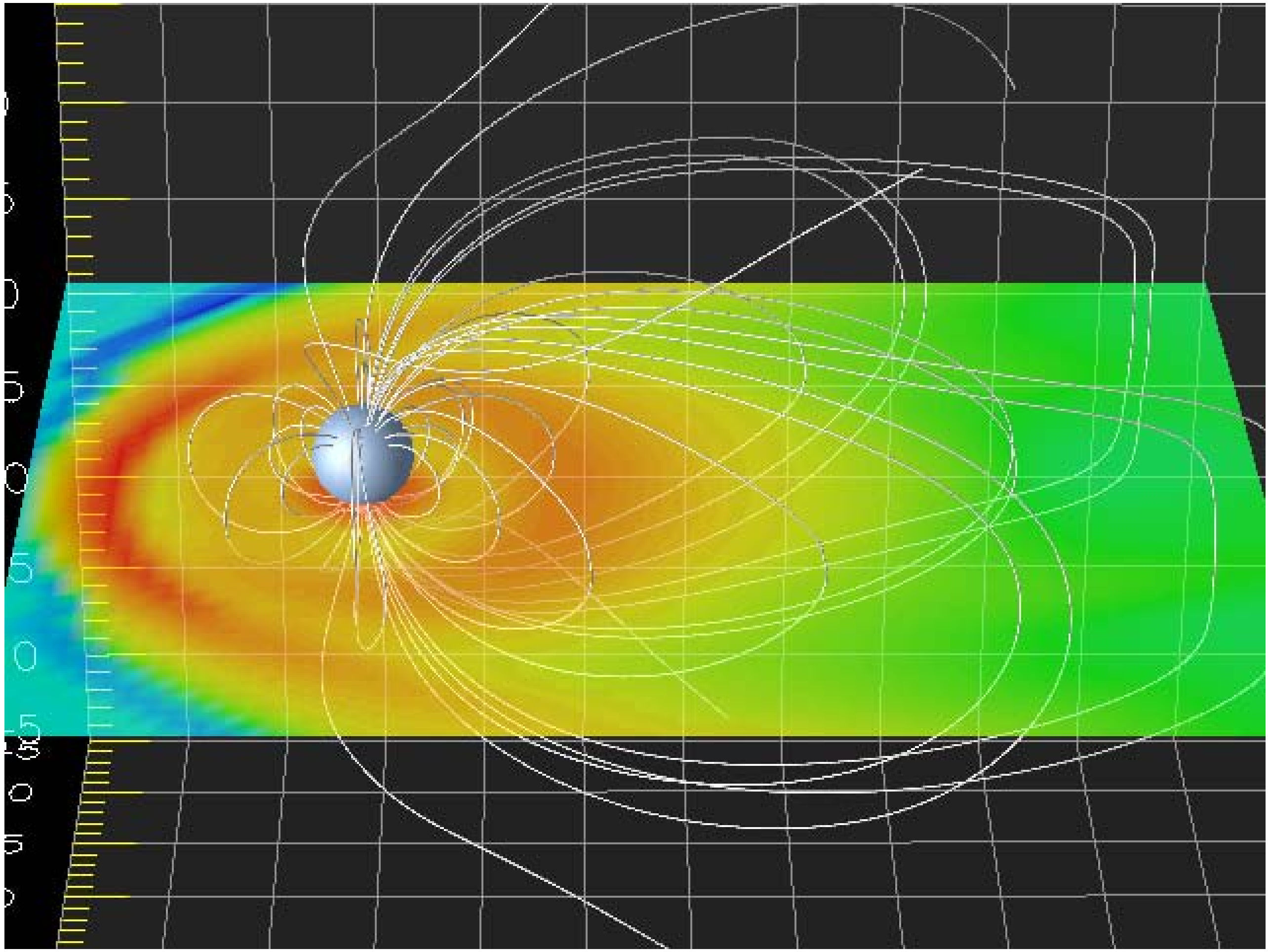
Reconnection in the magnetosphere







Simulation reconnection

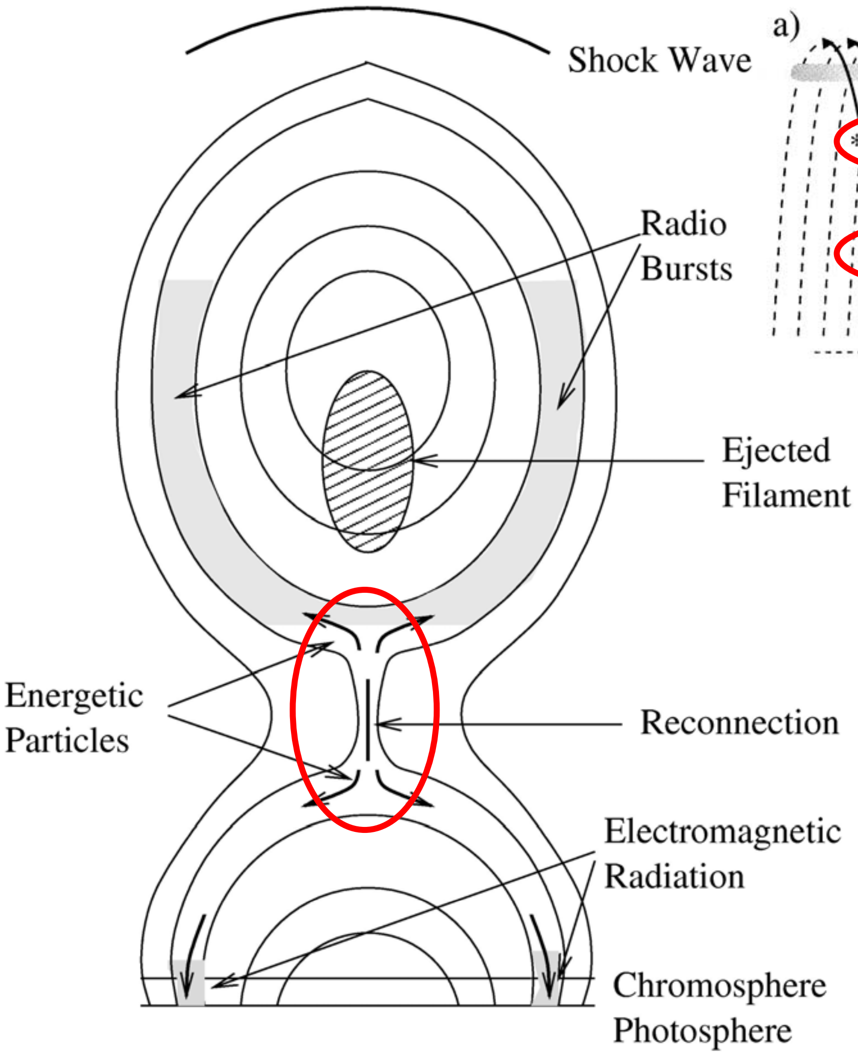


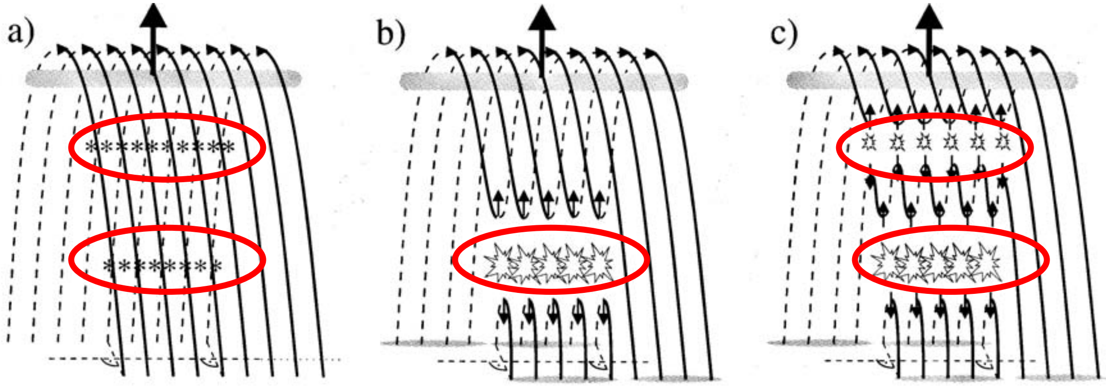
http://www.ess.washington.edu/Space/SpaceModel/tailreconnection.html

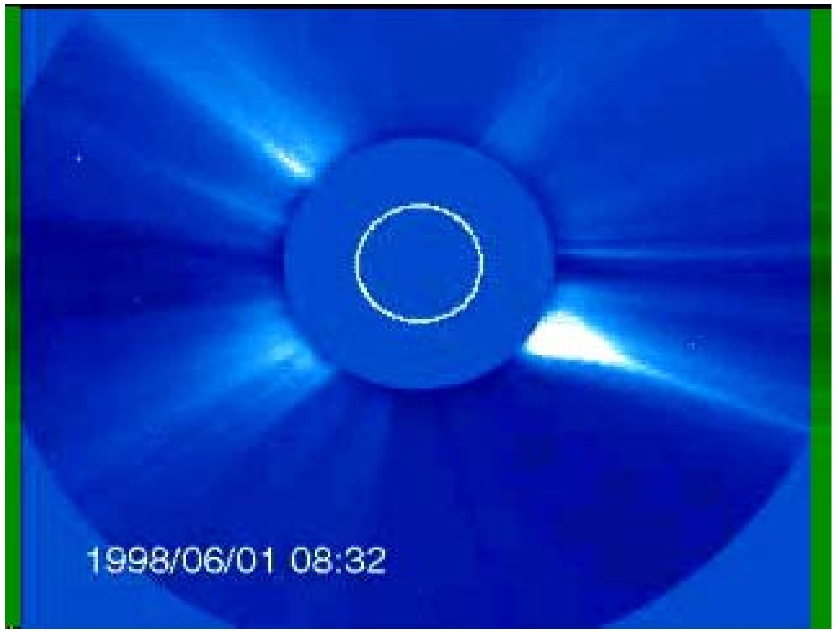
Magnetic field convection:

- Flux Transfer Events at the dayside
- Convection of the field lines with the solar wind to the nightside
- Reconnection inside the tail
- Rotation of these newly closed field lines to the dayside

Reconnection on the Sun







MHD dynamo I

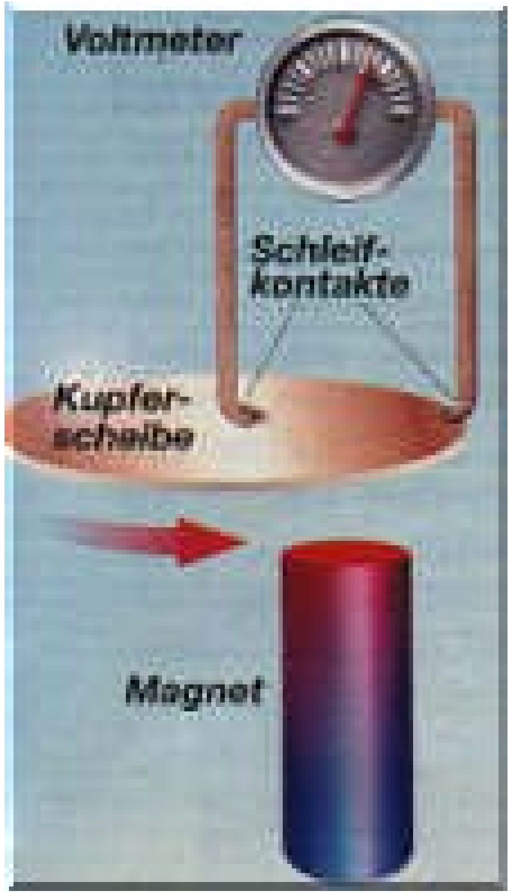
- Idea: conversion of rotational energy into magnetic field energy

- Observational evidence:
  - Planetary magnetic fields increase with the rate of rotation
  - Magnetic moment parallel to the axis of rotation
  - Polarity reversals in planetary and stellar magnetic fields

- Pre-requisite:
  - High conductivity (frozen-in fields)

- Problem: it is difficult to model the reversal with realistic plasma parameters

MHD dynamo II



- Currents induce magnetic fields $\Rightarrow$ moving plasma creates B.
- homogenous dynamo: no coils, wires etc., instead rotating homogenous medium:
  - Seed field parallel to axis of rotation;
  - Lorentz force induces radial electromotoric force (current);
  - Clever geometry of the contacts: amplification of the seed field;
  - Conversion of mechanical energy into field energy.
- Fields are axis symmetric: unipolarer inductor?
- Cowlings Theorem: a stationary axis symmetric field cannot be maintained by a finite velocity field (can be derived from the induction equation)
- Sun: v = 1E-9 m/s sufficient to maintain a dynamo

Dynamo formally

- Induction equation

$$\frac{\partial \vec{B}}{\partial t} = \eta \nabla^2 \vec{B} + \nabla \times (\vec{u} \times \vec{B})$$

In case of a vanishing velocity field dissipation (time scales Earth: 24 000 years)

Conversion of mechanical energy into field energy

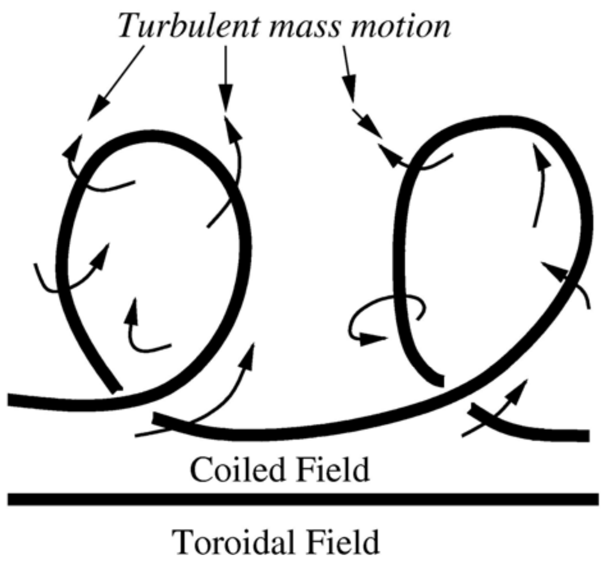
- Frozen-in fields (conservation of magnetic flux) combined with conservation of mass:

$$\frac{B}{\varrho l} = \mathrm{const}$$

dynamo in incompressible turbulent magnetic fields!

$\rightarrow$ Kinematic dynamo: seed field and velocity field are prescribed, the generated field does not act back onto the prescribed field.

Statistical dynamo I



- Turbulent motion in the Sun's convection zone:
  - Average magnetic field: axis symmetric
  - Superposed turbulent field: not axis symmetric

$$\langle \vec{u} \times \vec{B} \rangle = \vec{u}_o \times \vec{B}_o + \langle \vec{u}_1 \times \vec{B}_1 \rangle$$

- Correlation function (product of the fluctuating quantities):
  - Does not vanish because the fluctuations are not statistically independent (frozen-in field)
  - Can be approximated as

$$\langle \vec{u}_1 \times \vec{B}_1 \rangle = \alpha \vec{B}_o - \beta \nabla \times \vec{B}_o$$

  - α,β can be determined from the properties of the fluctuating quantities

Fluctuating quantities

- The physical quantities can be described as a sum of the average quantities and their fluctuations::

$$\vec{B} = \vec{B}_o + \vec{B}_1 \,, \qquad \vec{E} = \vec{E}_o + \vec{E}_1 \,, \qquad \vec{u} = \vec{u}_o + \vec{u}_1$$

$$\vec{j} = \vec{j}_o + \vec{j}_1 \,, \qquad \varrho = \varrho_o + \varrho_1 \,, \qquad p = p_o + p_1$$

- The averages of the fluctuations vanish:

$$\langle \vec{u}_1 \rangle = \langle \vec{j}_1 \rangle = \langle \vec{E}_1 \rangle = \langle \vec{B}_1 \rangle = \langle \varrho_1 \rangle = \langle p_1 \rangle = 0$$

- The fluctuations are small compared with the averages:

$$u_1 < u_o \,, \quad B_1 < B_o \,, \quad E_1 < E_o \,, \quad j_1 < j_o \,, \quad \varrho_1 < \varrho_o \,, \quad p_1 < p_o$$

Reynolds-axioms I

- Average of the sum of fluctuating quantities:

$$\langle \vec{A} + \vec{B} \rangle = \langle \vec{A} \rangle + \langle \vec{B} \rangle = \vec{A}_\mathrm{o} + \vec{B}_\mathrm{o}$$

- Average of the product of fluctuating and average quantities:

$$\begin{aligned}
\langle A_\mathrm{o}\, B_1 \rangle &= \langle A_\mathrm{o} \rangle \langle B_1 \rangle = A_\mathrm{o}\, 0 = 0 \\
\langle \vec{A}_\mathrm{o} \cdot \vec{B}_1 \rangle &= \langle \vec{A}_\mathrm{o} \rangle \cdot \langle \vec{B}_1 \rangle = \vec{A}_\mathrm{o} \cdot 0 = 0 \\
\langle \vec{A} \times \vec{B}_1 \rangle &= \langle \vec{A}_\mathrm{o} \times \langle \vec{B}_1 \rangle = \vec{A}_\mathrm{o} \times 0 = 0
\end{aligned}$$

- Average of the product of average quantities:

$$\begin{aligned}
\langle \langle A \rangle \langle B \rangle \rangle &= \langle A \rangle \langle B \rangle = A_\mathrm{o} B_\mathrm{o} \\
\langle \langle \vec{A} \rangle \langle \vec{B} \rangle \rangle &= \langle \vec{A} \rangle \cdot \langle \vec{B} \rangle = \vec{A}_\mathrm{o} \cdot \vec{B}_\mathrm{o} \\
\langle \langle \vec{A} \rangle \times \langle \vec{B} \rangle \rangle &= \langle \vec{A} \rangle \times \langle \vec{B} \rangle = \vec{A}_\mathrm{o} \times \vec{B}_\mathrm{o}\ .
\end{aligned}$$

Reynolds-axioms II

- Products of instantaneous quantities:

$$\begin{aligned}
\langle AB \rangle &= \langle (A_o + A_1)\,(B_o + B_1) \rangle \langle A_o B_o + A_o B_1 + A_1 B_o + A_1 B_1 \rangle \\
&= \langle A_o B_o \rangle + \langle A_o B_1 \rangle + \langle A_1 B_o \rangle + \langle A_1 B_1 \rangle = A_o B_o + \langle A_1 B_1 \rangle \\
\langle \vec{A} \cdot \vec{B} \rangle &= \vec{A}_o \cdot \vec{B}_o + \langle \vec{A}_1 \cdot \vec{B}_1 \rangle \\
\langle \vec{A} \times \vec{B} \rangle &= \vec{A}_o \times \vec{B}_o + \langle \vec{A}_1 \times \vec{B}_1 \rangle \,.
\end{aligned}$$

- Covariance or correlation product:

$$\begin{aligned}
\langle x_1 y_1 \rangle &= \langle (x - x_1)(y - y_1) \rangle - x_o y_o \\
\langle \vec{x}_1 \cdot \vec{y}_1 \rangle &= \langle (\vec{x} - \vec{x}_1) \cdot (\vec{y} - \vec{y}_1) \rangle - \vec{x}_o \cdot \vec{y}_o \\
\langle \vec{x}_1 \times \vec{y}_1 \rangle &= \langle (\vec{x} - \vec{x}_1) \times (\vec{y} - \vec{y}_1) \rangle - \vec{x}_o \times \vec{y}_o \,.
\end{aligned}$$

- Differentiation and integration:

$$\left\langle \frac{\partial \vec{A}}{\partial \zeta} \right\rangle = \frac{\partial \langle \vec{A} \rangle}{\partial \zeta} = \frac{\partial \vec{A}_o}{\partial \zeta}$$

$$\left\langle \int \vec{A}\,\mathrm{d}\zeta \right\rangle = \int \langle \vec{A} \rangle\,\mathrm{d}\zeta = \int \vec{A}_o\,\mathrm{d}\zeta$$

Statistical dynamo II

- Derivation of the correlation function from the field equations using average and fluctuating quantities :
  - Ohm's law:

$$\begin{aligned}\vec{j} &= \vec{j}_o + \vec{j}_1 \\ &= \sigma\left(\vec{E}_o + \vec{E}_1 + \vec{u}_o \times \vec{B}_o + \vec{u}_o \times \vec{B}_1 + \vec{u}_1 \times \vec{B}_o + \vec{u}_1 \times \vec{B}_1\right)\end{aligned}$$

$$\vec{j}_o = \sigma(\vec{E}_o + \vec{u}_o \times \vec{B}_o + \vec{u}_1 \times \vec{B}_1)$$

  - Induction equation:

$$\begin{aligned}\frac{1}{\mu_o\sigma}\nabla^2\vec{B}_o &+ \nabla \times (\vec{u}_1 \times \vec{B}_o) - \frac{\partial \vec{B}_o}{\partial t} \\ &= -\frac{1}{\mu_o\sigma}\nabla^2\vec{B}_1 - \nabla \times (\vec{u}_1 \times \vec{B}_1) + \frac{\partial \vec{B}_1}{\partial t}\end{aligned}$$

  - Proportionalities: $\langle \vec{u}_1 \times \vec{B}_1 \rangle \sim \alpha \vec{B}_o$

$$\langle \vec{u}_1 \times \vec{B}_1 \rangle \sim \beta \nabla \times \vec{B}_o$$

Correlation function

- Correlation function:

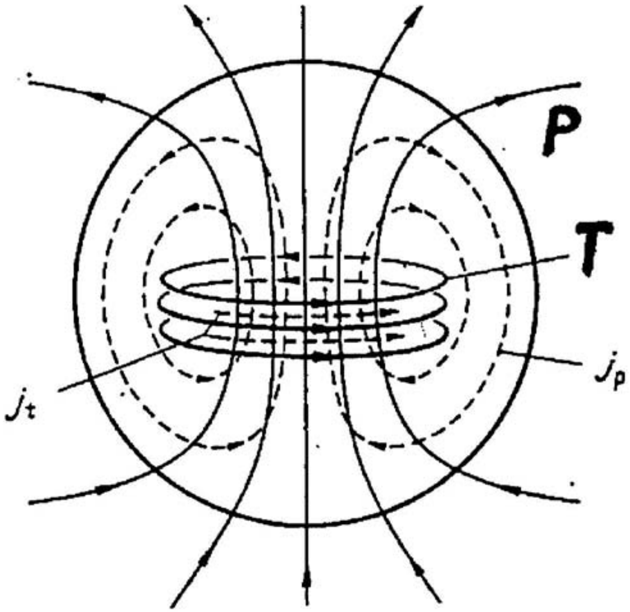
$$\frac{\partial \vec{B}_o}{\partial t} - \nabla \times (\vec{u}_o \times \vec{B}_o + \alpha \vec{B}_o) = -(\eta + \beta) \nabla \times (\nabla \times \vec{B}_o\rangle$$

  - β-term: increase in magnetic diffusivity due to turbulent motion
  - α-term: deviation from axis symmetry
- Difference compared to the diffusion equation:

$$\begin{aligned} \frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{u} \times \vec{B}) &= -\frac{1}{\sigma} \nabla \times \vec{j} \\ &= -\frac{1}{\sigma} \nabla \times \frac{\nabla \times \vec{B}}{\mu_o} = -\frac{1}{\mu_o \sigma} \nabla \times \left(\nabla \times \vec{B} \right) \\ &= \frac{1}{\mu_o \sigma} \nabla^2 \vec{B} \end{aligned}$$

Cowlings-theorem does not apply here!

Toroidal and poloidal fields



- Magnetic field divergence free → sum of a toroidal and a poloidal field

$$\vec{B} = \vec{T} + \vec{P} = \vec{T}(\Psi_{\mathrm{T}}) + \vec{P}(\Psi_{\mathrm{P}})$$

- Generating functions:

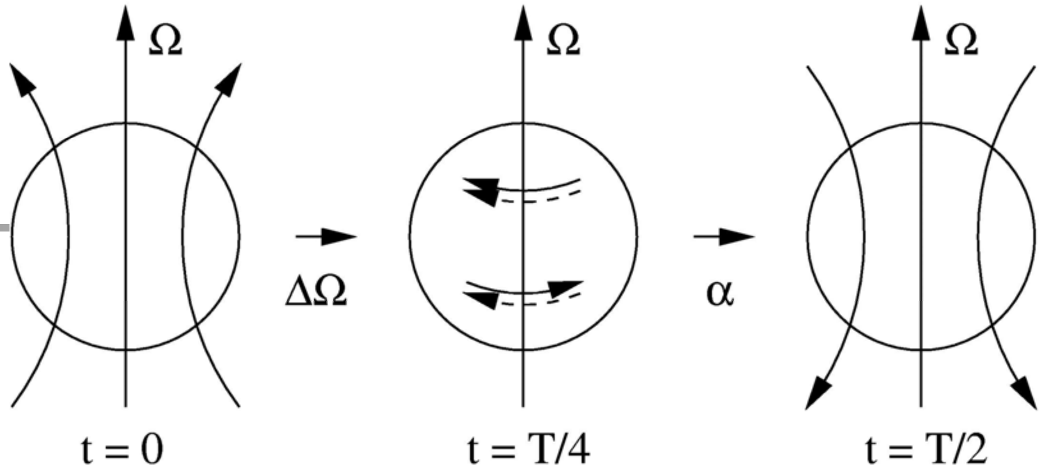
$$\begin{aligned} \vec{T}(\Psi_{\mathrm{T}}) &= \nabla \times (\Psi_{\mathrm{T}}\vec{r}) = \nabla \Psi_{\mathrm{T}} \\ \vec{P}(\Psi_{\mathrm{P}}) &= \nabla \times \nabla \times (\Psi_{\mathrm{P}}\vec{r}) \end{aligned}$$

- A toroidal field can be generated by a poloidal current system and vice versa:

$$\nabla \times \vec{T}(\Psi) = \vec{P}(\Psi) \quad \text{and} \quad \nabla \times \vec{P}(\Psi) = \vec{T}(-\nabla^2\Psi)$$

- Formally e.g. spherical harmonics in the Bullard-Gellman-ansatz (does not converge)

αΩ-Dynamo



- αΩ-Dynamo
  - differential rotation Ω
  - α-effect

$$\langle\vec{B}\rangle = \langle\vec{B}_{\rm tor}\rangle + \langle\vec{B}_{\rm pol}\rangle = B\cdot\vec{e}_\Phi + \nabla\times A\cdot\vec{e}_\Phi$$

- Ohm's law:

$$\vec{j}_{\rm o} = \sigma\left\{\vec{E}_{\rm o} + (\vec{u}_{\rm o}\times\vec{B}_{\rm o}) + \alpha\vec{B}_{\rm o} - \beta(\nabla\times\vec{B}_{\rm o})\right\}$$

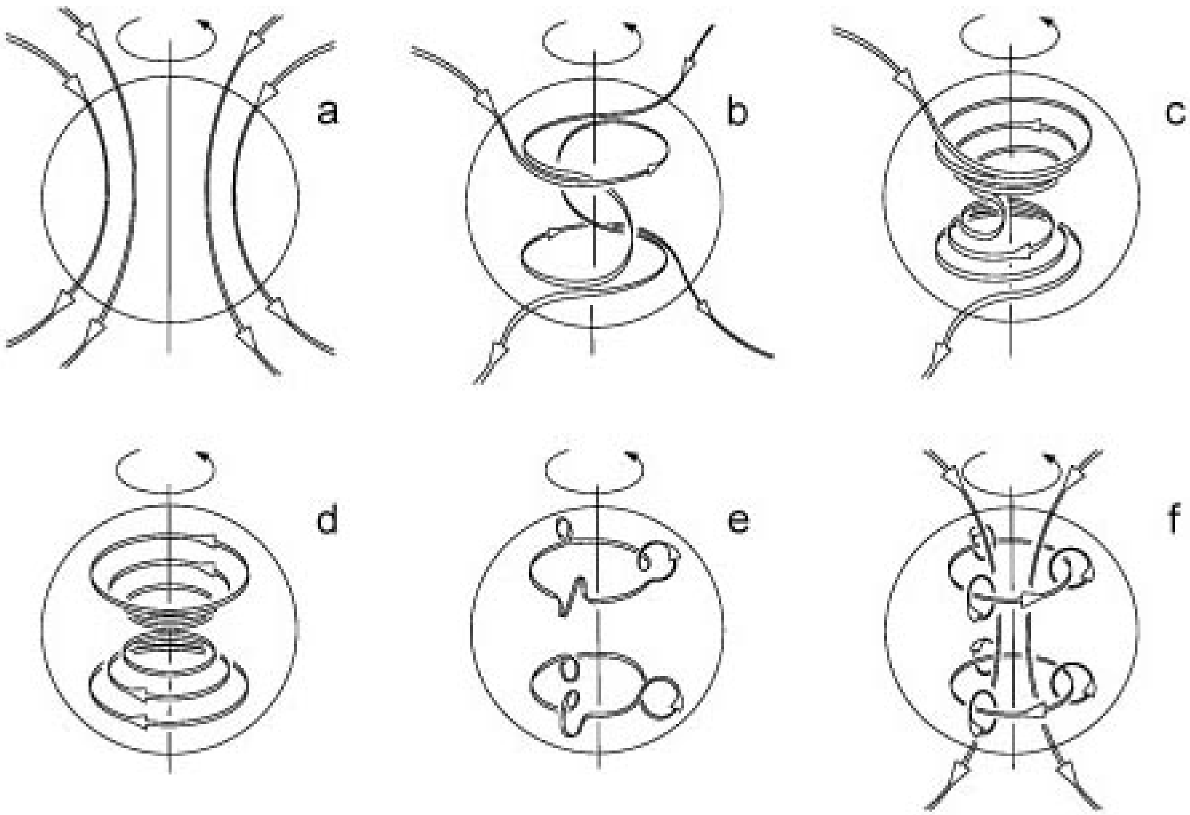
- add Faraday and turbulent conductivity:

$$\vec{j} = \sigma_{\rm T}\,(\vec{E}_{\rm o} + \vec{v}_{\rm o}\times\vec{B}_{\rm o} + \alpha\vec{B}_{\rm o}) \quad \text{mit} \quad \frac{1}{\sigma_{\rm T}} = \frac{1}{\sigma} + \beta$$

- turbulent dissipation time:

$$\tau_{\rm T} \approx \mu_{\rm o}\sigma_{\rm T}L^2$$

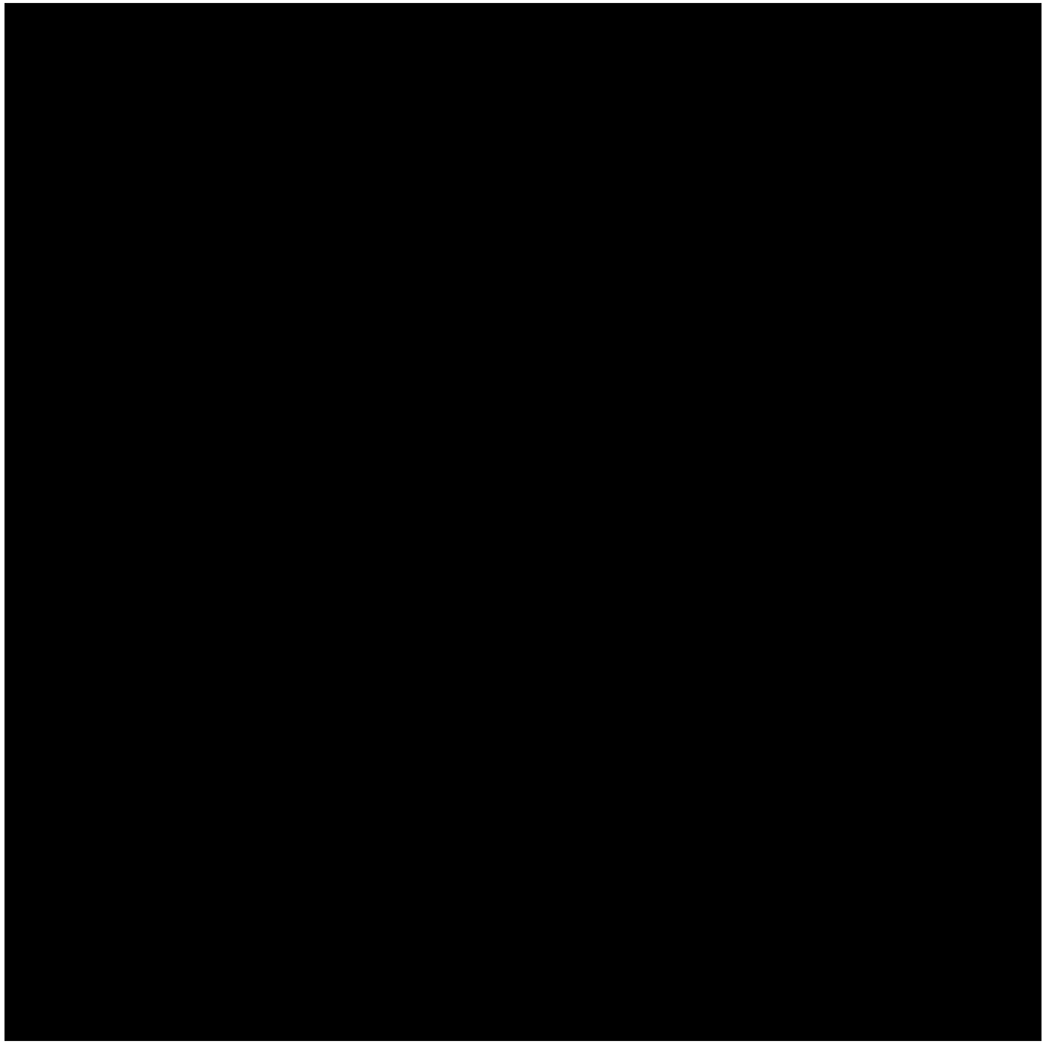
αΩ-Dynamo (details)



Love, J. J., 1999. Astronomy & Geophysics, 40, 6.14-6.19.

- Generation of a toroidal from a poloidal fied (ω-effect) trivial
- Problem: generation of the poloidal field from the toroidal one (ω-effect)
- Consequence: the simulation of the field reversal with realistic parameters is difficult!

Dynamo-simulation



http://www.psc.edu/research/graphics/gallery/CORRECTno_earth.mpg

Summary MHD

- Basic equations: conservation of mass, charge momentum and energy; field equations: equation of state
- 1-fluid or 2-fluid model
  - In many cases the ions are a stationary background and only the electrons are mobile (1 fluid model sufficient)
- General assumption:
  - Very high conductivity
- Basic concepts (derived from 1-fluid model):
  - Magnetic tension and pressure (MHStatic)
  - Frozen-in fields and dissipation of magnetic fields (MHKinematic)
  - Reconnection and magnetohydrodynamic dynamo (MHDynamic)