

# PLASMA PHYSICS

## V. WALL PHENOMENA: DIFFUSION & SHEATHS

### Debye Length: characteristic length of a plasma

The screening of electrostatic fields in by the charges in a plasma leads to the *Debye length*  $\lambda_D$ .

First, consider a positive charge  $q$  all by itself. The potential at a distance  $r$  from the charge is

$$\phi = \frac{q}{4\pi\epsilon_0 r}.$$

Now, consider a positive charge  $q$  in the middle of a plasma. It attracts electrons into its vicinity and repels positive ions. We will calculate  $\phi$  for this case.

If we allow the particle to have both kinetic and potential energy, the probability factor

becomes  $\exp\left(-\frac{\frac{1}{2}mv^2 + q\phi}{kT}\right) dv_x dv_y dv_z$ .  $\phi$  depends on position so the probability

depends on position.

The particle density is given by  $n = \int f(v) dv_x dv_y dv_z$  so  $n \propto \exp\left(-\frac{q\phi}{kT}\right)$

for electrons  $n_e = n_0 \exp\left(-\frac{e\phi}{kT}\right)$

for ions (we will suppose they are singly-ionized)  $n_i = n_0 \exp\left(-\frac{e\phi}{kT}\right)$

Gauss' Law can be written as

$$\nabla \cdot \mathbf{E} = \frac{\sigma}{\epsilon_0}$$

$\mathbf{E} = -\nabla\phi$  so

$$-\nabla^2\phi = \frac{\sigma}{\epsilon_0}.$$

This is *Poisson's equation*.

The charge density is  $\sigma = -en_e + en_i = en_0\left(-\exp\frac{e\phi}{kT} + \exp\frac{-e\phi}{kT}\right)$ .

Assume that this potential term is very small,  $e\phi \ll kT$

$$\sigma \cong -en_0\left(1 + \frac{e\phi}{kT}\right) + en_0\left(1 - \frac{e\phi}{kT}\right) = -\frac{2n_0e^2\phi}{kT}.$$

I am going to use spherical coordinates (and assume spherical symmetry)

$$\nabla^2 \phi \equiv \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right).$$

Poisson's equation becomes

$$-\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = -\frac{2n_0 e^2 \phi}{\epsilon_0 kT}$$

with solution

$$\phi = \frac{q}{4\pi\epsilon_0 r} \exp \left( -\frac{r}{\sqrt{\frac{\epsilon_0 kT}{2n_0 e^2}}} \right).$$

The potential falls away exponentially.

If we call  $\lambda_D = \sqrt{\frac{\epsilon_0 kT}{n_0 e^2}}$  the *Debye length* then  $\phi = \frac{q}{4\pi\epsilon_0 r} \exp \left( -\frac{\sqrt{2}r}{\lambda_D} \right).$

Beyond a few Debye lengths, shielding by the plasma is quite effective and the potential due to our charge is negligible.

This provides condition to determine if we have a plasma or not.

- (i) the system must be large enough  $L \gg \lambda_D$ , and
- (ii) there must be enough electrons to produce shielding  $N_D \gg 1$ , where  $N_D$  is the number of electrons in a Debye sphere.

Suppose there is a local concentration of charge. If plasma dimensions are much greater than  $\lambda_D$ , then on the whole plasma is still neutral (we can describe the plasma as *quasineutral*) and we can take  $n_e \cong n_i \cong n_0$ .

If we put an electrode into a plasma, it becomes shielded by a *sheath* of thickness  $\approx \lambda_D$ .

$$\lambda_D = 69.0 \sqrt{\frac{T}{n_e}} \text{ m } (T \text{ in K, } n_e \text{ in m}^{-3})$$

*You do.* Show  $\omega_{pe} \lambda_D = \sqrt{\frac{kT}{m_e}}$ . We can now use  $\lambda_D$  and  $\omega_{pe}$  as the length and time scales to classify plasmas.

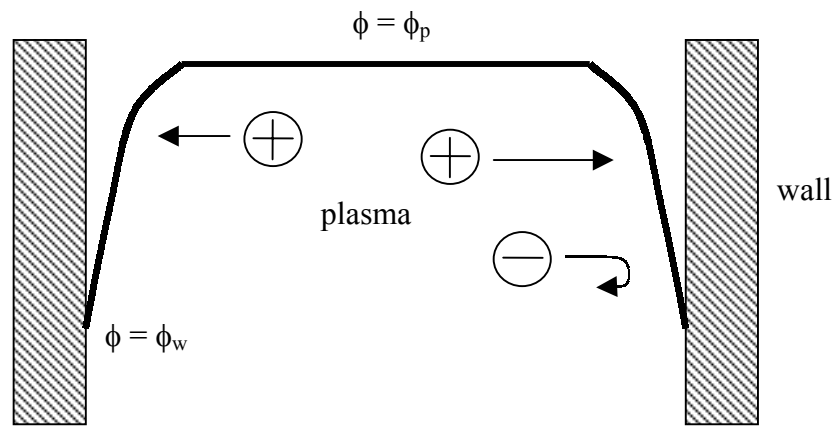
## The Necessity for Sheaths

In practical plasma devices all have walls. Our analysis of plasmas up to this point has assumed that we are in the bulk of the plasma far from any boundary. In this chapter we characterise the behaviour at walls and boundaries. This analysis is particularly important for materials processing plasmas because the surface of the material being processed interacts with the plasma like a wall or solid boundary.

Because of the shielding that occurs over distances greater than the Debye length the bulk of the plasma will be at roughly constant potential (this potential is known as the “plasma potential”). When ions and electrons hit a wall they will be lost from the plasma. Because electrons have much higher thermal velocities than ions they will make more collisions with the wall and be lost faster than ions while the wall is at plasma potential.

When the plasma is first ignited there is a net electron current to the chamber walls. Over time this leaves the plasma bulk with a net positive charge and thus raises its potential above that of the walls. That is the potential of the walls  $\phi_w$  becomes negative with respect to the plasma. This potential drop provides a barrier for electrons and attracts ions. The equilibrium potential drop is that required to equalise electron and ion losses.

The potential drop is confined to a layer of the order of some Debye lengths next to the wall due to the effects of Debye shielding. This layer of charge imbalance, which must exist on all cold walls with which the plasma makes contact, is called a *sheath*.



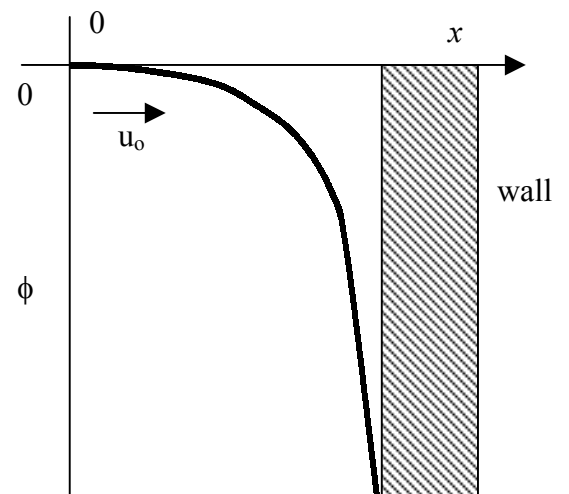
## The Collisionless Planar Sheath

Consider the situation shown in the diagram to the left. Assuming ions enter the sheath at plane  $x = 0$  with velocity  $u_0$ , we wish to calculate the potential as a function of  $x$ .

Conservation of energy requires that

$$\frac{1}{2} m_i u^2 = \frac{1}{2} m_i u_0^2 - e\phi(x)$$

where  $m$  is the ion mass,  $\phi(x)$  is the potential at position  $x$  and  $u(x)$  the velocity. Note that this



expression assumes the ion makes no collisions in moving between 0 and  $x$ . In general if the mean free path of the ions is greater than the distance  $x$  this assumption will be valid.

The ion continuity equation gives  $n_0 u_0 = n_i(x) u(x)$

Combining these equations gives  $n_i(x) = n_0 \left( 1 - \frac{2e\phi}{m_i u_0^2} \right)^{1/2}$  (1)

Now to find an expression for the potential,  $\phi(x)$ , from the uncompensated charge in the sheath we must solve Poisson's equation

$$\epsilon_0 \frac{\partial^2 \phi}{\partial x^2} = e(n_e - n_i) \quad (2)$$

so we need an expression for  $n_e$ .

Assuming that if there is a magnetic field present, it is along the  $x$  direction, the fluid momentum equation for the electrons simplifies to:

$$m_e n_e \left[ \frac{\partial v_x}{\partial t} + (\mathbf{v} \cdot \nabla) v_x \right] = e n_e E_x - \frac{\partial p}{\partial x} \Rightarrow \left[ \frac{\partial v_x}{\partial t} + (\mathbf{v} \cdot \nabla) v_x \right] = \frac{e}{m_e} E_x - \frac{kT_e}{m_e n_e} \frac{\partial n_e}{\partial x}$$

On all but very short timescales the electrons can be viewed as massless (ie. they have no inertia), so take the limit as  $m_e \rightarrow 0$ . Then the terms on the RHS dominate and we can neglect the left-hand side.

Hence 
$$\frac{e}{m_e} E = \frac{kT_e}{m_e n_e} \frac{\partial n_e}{\partial x}$$

so 
$$e \frac{\partial \phi}{\partial x} = \frac{kT_e}{n_e} \frac{\partial n_e}{\partial x}$$

Integrating gives 
$$\frac{e\phi}{kT_e} = \ln n_e + C$$

Therefore 
$$n_e = n_0 \exp\left(\frac{e\phi}{kT_e}\right) \quad (3)$$

This is called **Boltzman's relation** for electrons !!!

Physical interpretation:- because electrons are so light they would accelerate indefinitely if the forces on them didn't balance. Thus a gradient in density will cause movement, instantaneously setting up a charge separation with ions, that gives a balancing  $E$  field.

Combining equations (1), (2) and (3) above gives:

$$\varepsilon_0 \frac{\partial^2 \phi}{\partial x^2} = e(n_e(x) - n_i(x)) = en_0 \left[ \exp\left(\frac{e\phi}{kT_e}\right) - \left(1 - \frac{2e\phi}{m_i u_0^2}\right)^{-1/2} \right]$$

In order for the sheath to perform its function and repel electrons the potential must be monotonically decreasing with increasing  $x$ . This will only be the case if  $n_i(x) > n_e(x)$  for all  $x$  in the sheath. This condition corresponds to:

$$\exp\left(\frac{e\phi}{kT_e}\right) < \left(1 - \frac{2e\phi}{m_i u_0^2}\right)^{-1/2}$$

For small  $|\phi|$

$$1 + \frac{e\phi}{kT_e} < 1 + \frac{e\phi}{m_i u_0^2}$$

And since  $\phi < 0$

$$\frac{1}{kT_e} > \frac{1}{m_i u_0^2} \quad \Rightarrow \quad u_0^2 > \frac{kT_e}{m_i}$$

The inequality

$$u_0 > \sqrt{\frac{kT_e}{m_i}}$$

is known as the ***Bohm sheath criterion***.

and

$$u_B = \sqrt{\frac{kT_e}{m_i}}$$

is known as the ***Bohm velocity***.

To give the ions this minimum directed velocity  $u_B$  there must be a small finite electric field in the plasma over a region prior to the sheath. This region, which is typically wider than the sheath, is often referred to as the *presheath*. If the ions have finite temperature, the critical drift velocity  $u_B$  will be somewhat lower.

The potential drop in the presheath responsible for accelerating the ions to the Bohm velocity must be of order

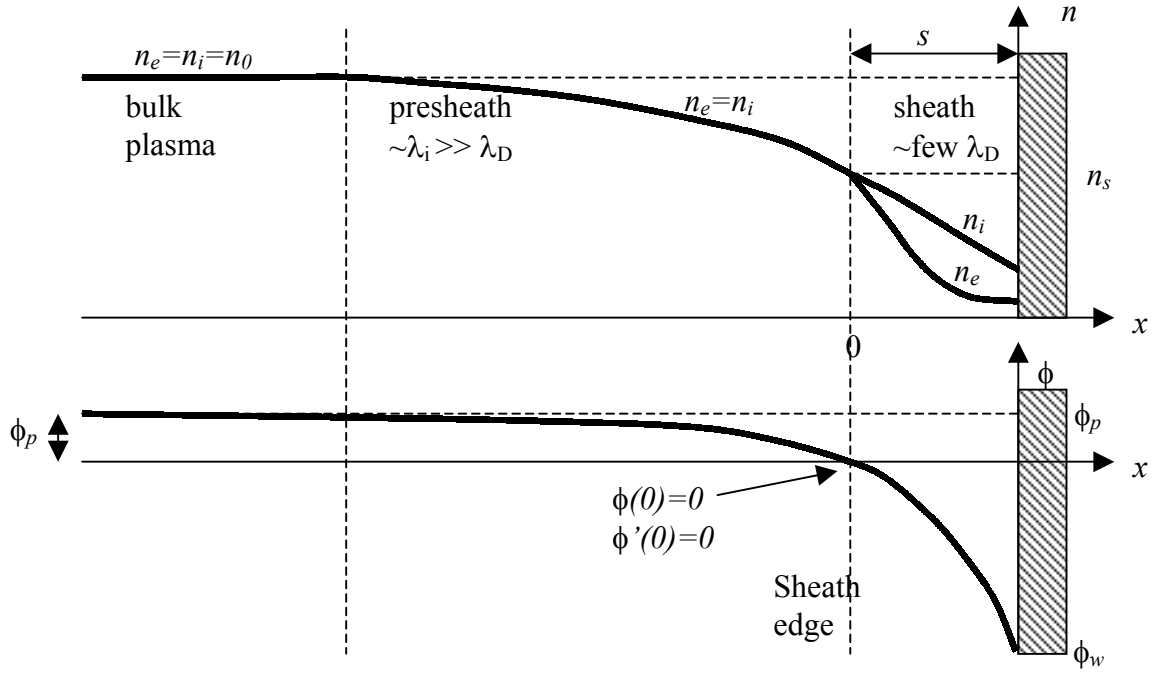
$$\frac{1}{2} m_i u_B^2 = e\phi_p \quad \Rightarrow \quad \phi_p = \frac{kT_e}{2e} \text{ which is half the}$$

electron temperature when quoted in electron volts.

The ratio of the density at the sheath edge to that in the plasma can then be calculated according to the Boltzman relation.

$$n_s = n_p \exp\left(\frac{e\phi_p}{kT_e}\right) = n_p \exp\left(\frac{1}{2}\right) \approx 0.61 n_p$$

where  $n_p$  is the density in the bulk plasma and  $n_s$  is the density at the boundary of the sheath and presheath.



## Sheath Potential at a Floating Wall

In equilibrium the net current to a floating wall must be zero. This means that ion and electron fluxes to the wall must balance one another. In the absence of collisions the ion flux must be constant throughout the sheath and is thus given by  $\Gamma_i = n_s u_B$  and the

electron flux at the wall is given by  $\Gamma_e = \frac{1}{4} n_s \bar{v}_e \exp\left(\frac{e\phi_w}{kT_e}\right)$ , where  $\bar{v}_e = \left(\frac{8kT_e}{\pi m_e}\right)^{1/2}$  is the mean electron speed and  $\phi_w$  is the potential of the wall with respect to the sheath-presheath edge.

Substituting for the Bohm velocity and equating the fluxes gives

$$n_s \left(\frac{kT_e}{m_i}\right)^{1/2} = \frac{1}{4} n_s \left(\frac{8kT_e}{\pi m_e}\right)^{1/2} \exp\left(\frac{e\phi_w}{kT_e}\right)$$

Solving for  $\phi_w$  gives

$$\phi_w = -\frac{kT_e}{e} \ln \sqrt{\frac{m_i}{2\pi m_e}}$$

Thus the floating wall potential is negative and is linearly related to the electron temperature with a factor proportional to the logarithm of the square root of the ion-electron mass ratio.

## The High-Voltage Sheath

When the voltage on an electrode in contact with the plasma is highly negative compared to the plasma-sheath edge,  $n_e = n_0 \exp\left(\frac{eV_0}{kT_e}\right) \rightarrow 0$  since  $eV_0 \gg kT_e$ . In this case all of the electrons will be driven out of the region near the electrode leaving only ions in that part of the sheath.

### Matrix Sheath

On time scales which are small compared to the time it takes ions to respond to electric fields, ( $\sim 1/\omega_{pi}$ ), the ions remain in position as the electrons are expelled. If the ion density at the sheath edge is  $n_s$  and we choose  $x = 0$  at the plasma sheath boundary, we have

$$\frac{d^2\phi}{dx^2} = -\frac{en_s}{\epsilon_0}$$

Integration gives

$$\phi = -\frac{en_s}{\epsilon_0} \frac{x^2}{2}.$$

Setting  $\phi = -V_0$  at  $x = s$ , we obtain the thickness of the matrix sheath

$$s = \left( \frac{2\epsilon_0 V_0}{en_s} \right)^{\frac{1}{2}} = \lambda_D \left( \frac{2eV_0}{kT_e} \right)^{\frac{1}{2}}$$

The matrix sheath can have a thickness of tens or even hundreds of Debye lengths with the application of voltages between a few hundred volts up to say 30 kV. Such voltages are often applied to substrates in the Plasma Immersion Ion Implantation (PIII) surface processing technique.

### Child Law Sheath

If the high voltage is applied over longer time scales ( $> 1/\omega_{pi}$ ) the ions will be accelerated by the electric field. Since the applied potential  $V_0$  is large we can neglect the initial ion energy,  $\frac{1}{2}m_i u_B^2$ , in comparison and the ion energy and flux conservation equations can be written as

$$\frac{1}{2}m_i u^2(x) = -e\phi(x) \quad \text{and} \quad J_0 = en_i(x)u(x),$$

where  $J_0$  is the ion current which must remain constant throughout the sheath.

Solving for  $n(x)$  gives

$$n_i(x) = \frac{J_0}{e} \left( -\frac{2e\phi}{m_i} \right)^{-1/2}$$

Since there are no electrons in the high voltage sheath, we can write Poisson's equation

as:

$$\frac{d^2\phi}{dx^2} = -\frac{en_i(x)}{\epsilon_0} = \frac{J_0}{e} \left( -\frac{2e\phi}{m_i} \right)^{-1/2}$$

Multiplying by  $d\phi/dx$  and integrating from 0 to  $x$  gives  $\frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 = 2 \frac{J_0}{\epsilon_0} \left( \frac{2e}{m_i} \right)^{-1/2} (-\phi)^{1/2}$

since  $d\phi/dx = -E \sim 0$  and  $\phi \sim 0$  at  $x = 0$ , (i.e.  $E$  and  $\phi$  small compared to the values in the bulk of the high voltage sheath).

Taking the negative square root (since  $d\phi/dx$  is negative) and integrating again gives

$$-\phi^{3/4} = \frac{3}{2} \left( \frac{J_0}{\epsilon_0} \right)^{1/2} \left( \frac{2e}{m_i} \right)^{-1/4} x \quad (1)$$

Letting  $\phi = -V_0$  at  $x = s$  and solving for  $J_0$  gives

$$J_0 = \frac{4}{9} \epsilon_0 \left( \frac{2e}{m_i} \right)^{1/2} \frac{V_0^{3/2}}{s^2} \quad (2)$$

This is known as the Child law of space-charge-limited current. It gives the current between two electrodes as a function of the potential difference between them if the electrode spacing  $s$  is fixed. However, in the case of a plasma sheath the ion current available from the plasma is fixed and given by  $J_0 = en_s u_B$ , where  $n_s$  is the ion density at the edge of the sheath and  $u_B$  is the Bohm speed. In this case the sheath width  $s$  adjusts so that the Child law is satisfied. Substituting for  $J_0$ ,  $u_B$ , and the electron Debye length yields:

$$s = \frac{\sqrt{2}}{3} \lambda_D \left( \frac{2eV_0}{kT_e} \right)^{3/4} \quad (3)$$

Thus the Child law sheath is larger than the matrix sheath by a factor of order  $\left( \frac{eV_0}{kT_e} \right)^{1/4}$ .

Substituting equation 2 into equation 1 gives the following expression for the potential in the high voltage sheath as a function of position:

$$\phi = -V_0 \left( \frac{x}{s} \right)^{4/3} \quad (4)$$

The electric field  $E = d\phi/dx$  is

$$E = \frac{4}{3} \frac{V_0}{s} \left( \frac{x}{s} \right)^{1/3} \quad \text{and the ion density is} \quad n_i = \frac{\epsilon_0}{e} \frac{dE}{dx} = \frac{4}{9} \frac{\epsilon_0}{e} \frac{V_0}{s^2} \left( \frac{x}{s} \right)^{-2/3}$$

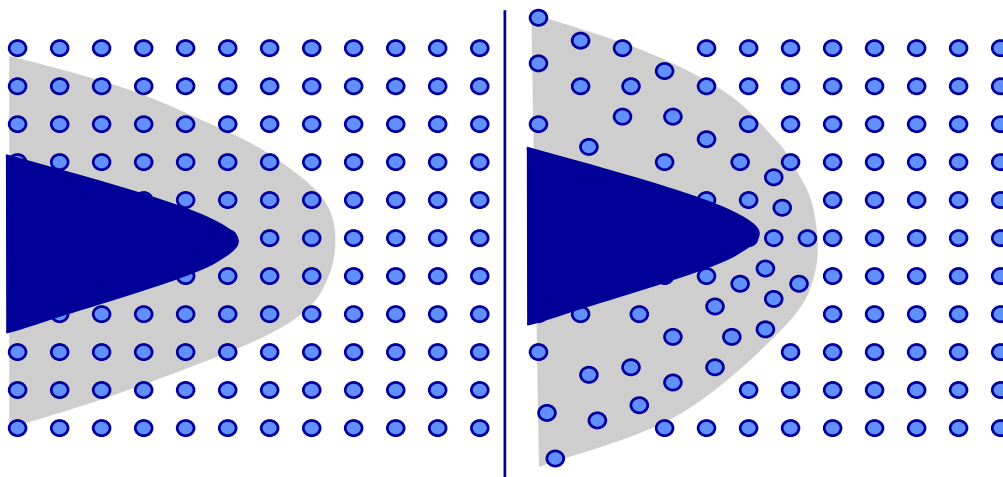
The Child law derived here is valid only if the sheath potentials are large compared to the electron temperature given in eV. Since ion motion was assumed collisionless in the



derivation it is not appropriate for high pressure discharges (i.e. when the mean free path is less than the dimensions of the sheath).

## Sheaths at Curved Surfaces

The derivations above have assumed a planar sheath and so could be done in one-dimensional space. Calculations for sheaths around curved or pointed objects are much more complicated so we shall use physical principles and what we know of planar sheaths to get some insight into this situation. Plasma deposition and ion implantation processes using high bias must often treat complex shapes such as medical devices.



The matrix sheath around a curved object is shown schematically in the figure on the left. The ions (shown as dots) have not moved from their positions in the bulk plasma since this is a very short time after the application of the high voltage bias. The electrons however have been repelled from the region as shown by the shading. As time goes on and the ions respond to the electric field in the matrix sheath they are accelerated into the surface of the electrode. At the surfaces that are flat as the ions accelerate their density is considerably reduced so that the sheath must expand in order to contain enough ions to shield the applied potential out of the plasma. Around the curve the ions also accelerate, however, due to the geometry they converge on the curved tip. If the curvature is high enough this effect may compensate for the density reduction due to ion acceleration and the density in the sheath may not decrease. If the curvature is stronger still the convergence of ions may over shadow the density reduction due to acceleration and the sheath may actually contract as shown on the right.

## Diffusion and mobility – phenomena in the bulk plasma

Suppose the plasma density in the bulk plasma is higher near the centre vessel than it is near the sheath edges (i.e. the bulk plasma has a non-constant density). Ions and electrons will collide with each other and with neutral particles and diffuse outwards. Suppose there is an electric field. The ions and electrons will move but the neutral particles will not. Diffusion and mobility in the plasma bulk leads to the loss of ions and electrons from the plasma.

## Diffusion

First, consider a gas of two kinds of neutral particles,  $A$  and  $B$ . The  $B$  particles are in the minority.

There are two ways we might look at diffusion.

- (i) Suppose the background of  $A$  particles is uniform, but the density of the  $B$  particles is not. The  $B$  particles collide with the  $A$  particles until the non-uniformity is smoothed out.

The continuity and momentum equations for the  $B$  particles are

$$\frac{\partial \rho_B}{\partial t} + \nabla \cdot (\rho_B \mathbf{v}_B) = 0$$

$$\rho_B \frac{\partial \mathbf{v}_B}{\partial t} + \rho_B \mathbf{v}_B \cdot \nabla \mathbf{v}_B = -\nabla p_B - \rho_B \mathbf{v}_{BA} (\mathbf{v}_B - \mathbf{v}_A)$$

Let us make some simplifications.

The non-uniformity is small so  $n_B = n_B^0 + n_B^1$ , where  $^0$  indicates the uniform, constant part and  $^1$  indicates a small first-order part that varies in space and time.

There are no zero order drifts so  $\mathbf{v}_A^0 = \mathbf{0}$  and  $\mathbf{v}_B^0 = \mathbf{0}$ .

We will use  $p_B = n_B k T_B$ .

Note that collisions between particles of the same type do not contribute to diffusion.

To first order, the equations become

$$\frac{\partial n_B^1}{\partial t} + n_B^0 \nabla \cdot \mathbf{v}_B^1 = 0 \quad (1)$$

$$\frac{\partial \mathbf{v}_B^1}{\partial t} = -\frac{k T_B}{m_B n_B^0} \nabla n_B^1 - \mathbf{v}_{BA} \mathbf{v}_B^1 \quad (2)$$

Take  $\nabla \cdot$  of (2) and substitute using (1).

$$\frac{\partial n_B^1}{\partial t} = D_B \nabla^2 n_B^1$$

$\boxed{D = \frac{kT}{m\nu}}$  is the *diffusion coefficient*.

(Note a  $\frac{1}{\mathbf{v}_{BA}} \frac{\partial^2 n_B^1}{\partial t^2}$  term has been dropped.)

The units of  $D$  are  $\text{m}^2 \text{s}^{-1}$ .

### The meaning of $D$

- (a) In terms of a scale length  $L$ , any initial non-uniformity is smoothed out in a time  $T \approx \frac{L^2}{D}$ .

(b)  $D \approx \frac{v_{rms}^2}{\nu}$ . Then  $D \approx \frac{\lambda_m^2}{\tau}$ . i.e., the diffusion coefficient is based on a random walk with a step equal to the mean free path between collisions.

*You do.* Show this.

(ii) A steady state where there is a density gradient of the  $B$  particles. There will clearly be a steady flow or flux of these  $B$  particles.

The momentum equation is (we will drop the subscripts and the superscripts)

$$\mathbf{0} = -\frac{kT}{mn} \nabla n - \mathbf{v} \mathbf{v} \quad \text{or} \quad \mathbf{v} = -D \frac{\nabla n}{n}.$$

The flux of the  $B$  particles is

$$\Gamma = n\mathbf{v} = -D\nabla n.$$

This equation  $\Gamma = -D\nabla n$  is known as *Fick's Law*.

## Mobility

Next, suppose the  $B$  particles are electrons and there is an  $E$  field. The  $A$  particles are still neutrals.

The momentum equation for electrons is

$$\mathbf{0} = -en\mathbf{E} - \nabla p_e - m_e n \mathbf{v} \mathbf{v}_e$$

Rearrange

$$\mathbf{v}_e = -\frac{e}{m_e \nu} \mathbf{E} - \frac{kT_e}{m_e \nu} \frac{\nabla n}{n} \quad \text{or} \quad \mathbf{v}_e = -\mu_e \mathbf{E} - D_e \frac{\nabla n}{n}$$

$\mu = \frac{e}{m\nu}$  is the *mobility*.  $\mu$  and  $D$  are known as *transport coefficients*.

The units of  $\mu$  are  $\text{m}^2 \text{V}^{-1} \text{s}^{-1}$ .

These drift velocities are  $\ll$  the random velocities of the particles.

Free diffusion is driven by the density gradient and drift is driven by the electric field.

## Weakly-ionized plasma, no magnetic field

### Ambipolar diffusion

In a plasma there are ions and electrons. The electrons tend to diffuse more rapidly than the heavier ions. If this results in  $n_e$  being different from  $n_i$ , an  $E$  field is established. This  $E$  field accelerates the ions and slows down the electrons, so, to a good approximation, they diffuse together. (This is the *first* key idea concerning diffusion of a weakly-ionized plasma.)

Write down the equations for ions and electrons

$$\mathbf{v}_e = -\mu_e \mathbf{E} - D_e \frac{\nabla n}{n}$$

$$\mathbf{v}_i = \mu_i \mathbf{E} - D_i \frac{\nabla n}{n}.$$

Remember that in a weakly-ionized plasma  $\nu_e$  and  $\nu_i$  are electron-neutral and ion-neutral collision frequencies. Electron-ion collisions can be ignored.

Set the electron flux equal to the ion flux

$$\Gamma = -n\mu_e \mathbf{E} - D_e \nabla n = n\mu_i \mathbf{E} - D_i \nabla n.$$

Solve for  $\mathbf{E}$  and substitute to obtain

$$\Gamma = -D_a \nabla n.$$

where  $D_a = \frac{\mu_e D_i + \mu_i D_e}{\mu_e + \mu_i}$  is the *ambipolar diffusion coefficient*.

From above  $\mu_e = \frac{e}{m_e \nu_e}$ ,  $\mu_i = \frac{e}{m_i \nu_i}$  and  $\nu \propto \nu \propto \sqrt{\frac{kT}{m}}$ , so  $\mu_i \ll \mu_e$ .

$$D_a \cong \left(1 + \frac{T_e}{T_i}\right) D_i.$$

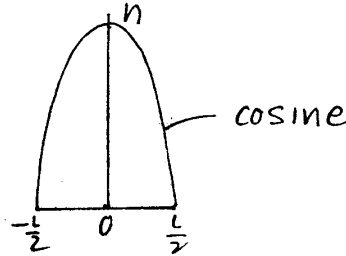
You do. Show this.

If  $T_i = T_e$ ,

$$D_a \cong 2D_i$$

*Example*

Let us apply  $\frac{\partial n}{\partial t} = D_a \nabla^2 n$  to a plasma slab where the initial electron density profile is shown in the sketch.



We will treat this as a 1-dimensional problem so

$$\frac{\partial n}{\partial t} = D_a \frac{\partial^2 n}{\partial x^2}.$$

This can be solved by separation of variables. Write  $n = f(x)g(t)$  and substitute. This leads to

$$n = A e^{-\frac{t}{T}} \cos\left(\frac{x}{\sqrt{D_a T}}\right) + B e^{-\frac{t}{T}} \sin\left(\frac{x}{\sqrt{D_a T}}\right).$$

Now substitute the initial conditions. This gives

$$n = n_0 e^{-\frac{t}{T}} \cos\left(\frac{\pi x}{L}\right) \text{ where } T = \frac{L^2}{\pi D_a}.$$

The electron density profile remains the same but the peak decreases exponentially with time.

You do. Show this.

## Weakly-ionized plasma in a magnetic field.

First consider electrons.

Start with the momentum equation,

$$\mathbf{0} = -en(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \nabla p_e - m_e n \mathbf{v} \nabla \mathbf{v}_e$$

The  $z$ -component equation yields the same  $\mu_e$  and  $D_e$  as before.

The  $x$ - and  $y$ -component equations are

$$v_x = -\mu_e E_x - \frac{D_e}{n} \frac{\partial n}{\partial x} - \frac{\omega_{ce}}{v} v_y$$

$$v_y = -\mu_e E_y - \frac{D_e}{n} \frac{\partial n}{\partial y} + \frac{\omega_{ce}}{v} v_x$$

This yields

$$\mathbf{v}_\perp = -\mu_{\perp e} \mathbf{E} - D_{\perp e} \frac{\nabla n}{n} + \frac{\mathbf{v}_{E \times B} + \mathbf{v}_{dia}}{1 + 1/\omega_{ce}^2 \tau^2}$$

where  $\mu_{\perp e} = \frac{\mu_e}{1 + \omega_{ce}^2 \tau^2}$  and  $D_{\perp e} = \frac{D_e}{1 + \omega_{ce}^2 \tau^2}$ .

If  $\omega_{ce}^2 \tau^2 \ll 1$ ,  $B$  is small and has little effect on diffusion.

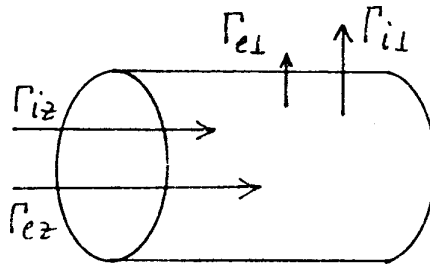
If  $\omega_{ce}^2 \tau^2 \gg 1$ ,  $B$  is large and diffusion across  $B$  is retarded.

This is the *second* key idea. Mobility and diffusion across the magnetic field are smaller. In this case

$$D_{\perp e} \approx \frac{\frac{\lambda_m^2}{\tau}}{\omega_{ce}^2 \tau^2} \approx \frac{\frac{\lambda_m^2}{\tau}}{\left(\frac{v}{r_{Le}}\right)^2 \left(\frac{\lambda_m}{v}\right)^2} \approx \frac{r_{Le}^2}{\tau},$$

i.e., the diffusion coefficient is based on a random walk with a step equal to the Larmor radius.

Diffusion of ions and electrons is ambipolar but is complicated. Whether the diffusion of a particle is primarily along  $z$  or perpendicular depends sensitively on the plasma boundaries.



## Fully ionized plasma in a magnetic field

Collisions between electrons and ions. Can derive an expression for  $\nu_{ei}$  and obtain an expression for the resistivity

$$\eta_{\parallel} = 5.25 \times 10^{-5} \frac{Z \ln \Lambda}{T_e^{3/2}} \text{ and } \eta_{\perp} = 2\eta_{\parallel} \text{ (} T_e \text{ in eV).}$$

It is usually adequate to take  $\ln \Lambda \approx 10$ .

Note that  $\eta$  is independent of  $n$ , decreases rapidly as  $T_e$  increases.

Start with the single fluid equations (so diffusion in this case is automatically ambipolar)

$$\mathbf{0} = \mathbf{j} \times \mathbf{B} - \nabla p$$

$$\mathbf{0} = \mathbf{E} + \mathbf{v} \times \mathbf{B} - \eta \mathbf{j}$$

Multiply the second by  $\times \mathbf{B}$  and substitute using the first.

$$\mathbf{v}_{\perp} = -\frac{\eta_{\perp}}{B^2} \nabla p + \mathbf{v}_{E \times B} = -\frac{\eta_{\perp} n k (T_e + T_i)}{B^2} \frac{\nabla n}{n} + \mathbf{v}_{E \times B}.$$

So we can define a  $D_{\perp}$  for a fully-ionized plasma.

Compare this with  $D_{\perp}$  for a weakly-ionized plasma. It is  $\propto \frac{1}{B^2}$ , but is  $\propto n$  as well, decreases with  $T$ , and is automatically ambipolar.

*Comment*

Laboratory verification of the  $1/B^2$  dependence proved elusive. The experimental results were better described by the empirical formula

$$D_{\perp} = \frac{1}{16} \frac{k T_e}{e B}.$$

This was called *Bohm Diffusion*, usually many orders of magnitude larger. Anomalous losses due to oscillations and asymmetries are responsible.