PLASMA PHYSICS

VI. WAVES IN PLASMAS

A plasma has a characteristic length and a characteristic time. We study the characteristics length (Debye length) in the next chapter, which looks at the plasmas interaction with electrodes and surfaces. Here we look at the characteristic time.

Plasma oscillations - This leads to the *plasma frequency* ω_{pe}.

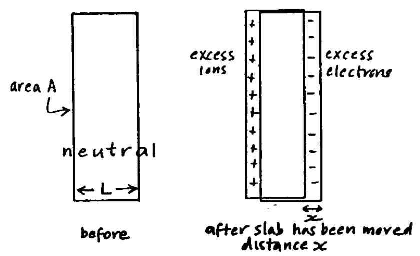
What would happen if electrons were displaced from their equilibrium positions? The electrostatic force due to the ions would pull them back, but the electrons would overshoot and oscillations would ensue. These are known as *plasma oscillations*.

This is a very fast oscillation, so fast that the massive ions do not have time to respond.

A simple calculation of the frequency follows.

Consider an infinite plasma. We will ignore thermal motions. We will treat the massive ions as not moving.

Suppose a slab of electrons is displaced a small distance x (so we are dealing with a 1-dimensional problem). The slab has thickness L. Consider an area A.



Equation of motion for the slab of electrons is $F = m\dfrac{d^2x}{dt^2}$

mass of electrons in slab $m = m_e n_e LA$

What is the force on the electrons? We have two oppositely charged sheets facing each other. The electric field between them is $E = \dfrac{\sigma_s}{\varepsilon_0}$ where σ_s is the surface charge density or charge /area.

$\sigma_s = en_e xA/A$ so the restoring force is equal to charge of electrons in slab × electric field, so force on electrons $= -n_e LAe\dfrac{en_e x}{\varepsilon_0}$.

Equation of motion becomes $\quad n_e LAm_e\dfrac{d^2x}{dt^2} = -n_e LAe\dfrac{en_e x}{\varepsilon_0}$

so that $\dfrac{d^2x}{dt^2} = -\dfrac{e^2 n_e}{\varepsilon_0 m_e}x$ with solution $x = A\cos(\omega_{pe}t)$,

where the (electron) plasma frequency is

$$\boxed{\omega_{pe} = \sqrt{\frac{n_e e^2}{\varepsilon_0 m_e}}}.$$

This is a 'natural' frequency for the plasma.

$$\boxed{f_{pe} = 8.98\sqrt{n_e}\,\mathrm{Hz}\ (n_e \text{ in m}^{-3})}.$$

Wave equation

Start with Maxwell's curl equations,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

The wave fields, which we shall write as $\mathbf{E}^1$, $\mathbf{B}^1$ to show that we are treating them as first order quantities, and their (first order) effects on particle densities and particle velocities all show an $\mathrm{e}^{j(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ variation. $\mathbf{k}$ is the wave vector and ω is the (angular) frequency.

Maxwell's equations can then be written,

$$j\mathbf{k} \times \mathbf{E}^1 = j\omega \mathbf{B}^1$$

$$j\mathbf{k} \times \mathbf{B}^1 = \mu_0 \mathbf{j}^1 - j\frac{\omega}{c^2}\mathbf{E}^1.$$

Waves in a vacuum

You do. Show that for a vacuum the only solution describes
(i) electromagnetic waves,
(ii) with their fields perpendicular to **k**, i.e., transverse waves,
(ii) with $k = \frac{\omega}{c}$.
(There are no currents.)

One approach is to take **k** along *z*, say. Write out the component Maxwell's equations. Eliminate $\mathbf{B}^1$, leaving equations in $\mathbf{E}^1$.

Phase velocity

$$v_{ph} \equiv \frac{\omega}{k} = c\,, \qquad \text{where } c \text{ is the velocity of light in a vacuum.}$$

Waves in a plasma, no magnetic field

The ion and electron momentum equations are

$$\rho_i \frac{\partial \mathbf{v}_i}{\partial t} + \rho_i \mathbf{v}_i \cdot \nabla \mathbf{v}_i = n_i e(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) - \nabla p_i - \rho_i \upsilon_{ie}(\mathbf{v}_i - \mathbf{v}_e)$$

and

$$\rho_e \frac{\partial \mathbf{v}_e}{\partial t} + \rho_e \mathbf{v}_e \cdot \nabla \mathbf{v}_e = -n_e e\left(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}\right) - \nabla p_e - \rho_e \upsilon_{ei}\left(\mathbf{v}_e - \mathbf{v}_i\right).$$

Write $n = n^0 + n^1$ and $\mathbf{v} = \mathbf{v}^0 + \mathbf{v}^1$, where n^1 and $\mathbf{v}^1$ show the $\mathrm{e}^{j(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ variation. Substitute and consider only terms to first order. But before we do we will make some simplifications.

(i) an infinite, uniform plasma,
(ii) no drifts, i.e., $\mathbf{v}^0 = \mathbf{0}$. So to first order terms like $\mathbf{v}.\nabla\mathbf{v} = \mathbf{0}$,
(iii) cold plasma, so terms like $\nabla p = \mathbf{0}$,
(iv) no collisions so last terms are zero,
(v) no steady magnetic field, so $\mathbf{B}^0 = \mathbf{0}$.

Then to first order the momentum equations are

$$n^0 m_i\left(-j\omega\mathbf{v}_i^1\right) = n^0 e\mathbf{E}^1$$

$$n^0 m_e\left(-j\omega\mathbf{v}_e^1\right) = -n^0 e\mathbf{E}^1.$$

It follows that $\mathbf{v}_i^1 = -\frac{m_e}{m_i}\mathbf{v}_e^1$ so ion motions are small. We shall neglect them.

$$\mathbf{v}_i^1 \cong \mathbf{0}$$

$$\mathbf{v}_e^1 = -j\frac{e}{\omega m_e}\mathbf{E}^1.$$

The electrons move but the ions remain at rest in the background.

Using the definition of current density, we have

$$\mathbf{j}^1 = en^0\left(\mathbf{v}_i^1 - \mathbf{v}_e^1\right) \cong j\frac{n^0 e^2}{\omega m_e}\mathbf{E}^1.$$

Take $\mathbf{k}$ along z, say. Write out the component Maxwell's equations. Eliminate $\mathbf{B}^1$, $\mathbf{j}^1$, leaving equations in $\mathbf{E}^1$.

The $\mathbf{E}_z^1$ equation gives

$$\omega^2 = \frac{n^0 e^2}{\varepsilon_0 m_e}. \quad (1)$$

The other equations give

$$k^2 = \frac{\omega^2}{c^2} - \frac{n^0 e^2}{\varepsilon_0 m_e c^2}. \quad (2)$$

Plasma oscillations

Equation (1) describes oscillations, not waves. These are called *plasma oscillations*. Recall $\omega_{pe}^2 = \frac{n^0 e^2}{\varepsilon_0 m_e}$. So the oscillation frequency is just the plasma frequency

$$\boxed{\omega = \omega_{pe}}.$$

Transverse electromagnetic waves

Equation (2) describes electromagnetic waves where the fields are perpendicular to the direction of propagation, i.e., *transverse electromagnetic waves*. An equation like this relating k and ω is called a *dispersion relation*.

$$\boxed{k^2 = \frac{\omega^2 - \omega_{pe}^2}{c^2}}$$

This will describe propagating waves as long as the rhs is > 0; i.e., $\omega > \omega_{pe}$. If $\omega < \omega_{pe}$ we say the wave is *cutoff.*

Critical density

Suppose we are carrying out a wave propagation experiment at a fixed frequency ω. If n is too high then, because $\omega_{pe}^2 = \dfrac{ne^2}{\varepsilon_0 m_e}$, ω_{pe}^2 will be too large and the wave will be cutoff.

Define *critical density* as

$$n_c = \frac{\varepsilon_0 m_e \omega^2}{e^2}.$$

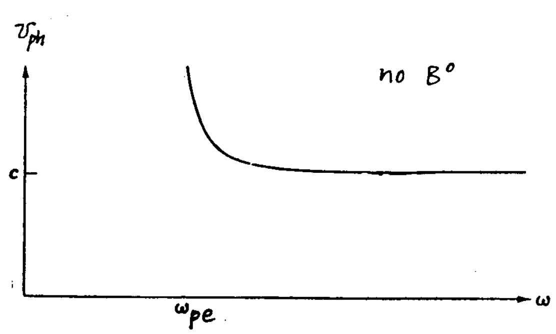
If $n > n_c$ the wave is cutoff.

Suppose we have a plane wave normally incident on a slab of plasma. We can calculate the powers transmitted and reflected at the boundary as a function of n. If $n > n_c$ the wave does not penetrate; all the power is reflected.

Alternatively, we can define a *cutoff frequency* $f_c = 8.98\sqrt{n}$.

Phase velocity

We can plot v_{ph} as a function of ω. $v_{ph} = \dfrac{c}{\sqrt{1 - \dfrac{\omega_{pe}^2}{\omega^2}}}$



At a cutoff $v_{ph} = \infty$.

Note $v_{ph} > c$! Is this a problem? No, recall that energy is transmitted at the *group velocity*

$$v_g = \frac{d\omega}{dk}.$$

You do. Show that in this case, $v_g = \dfrac{c^2}{v_{ph}}$ which is less than c.

We can think of a plasma as being like a *dielectric medium* with a *refractive index*

$$N = \frac{c}{v_{ph}}.$$

You do. Show that in this case

$$N^2 = 1 - \frac{\omega_{pe}^2}{\omega^2} = 1 - \frac{n}{n_c}.$$

Relax some of the simplifications made earlier, one by one.

(i) Allow a *collision frequency*.

Waves in a plasma, no magnetic field, with collisions

$\mathbf{v}_i$ is still small so the collision term in the electron momentum equation becomes

$$-n^0 m_e \nu_{ei} \mathbf{v}_e^1$$

the expression for $\mathbf{j}^1$ becomes

$$\mathbf{j}^1 = j \frac{n^0 e^2}{m_e(\omega + j\nu_{ei})} \mathbf{E}^1$$

and

$$k^2 = \frac{\omega(\omega + j\nu_{ei}) - \omega_{pe}^2}{c^2}.$$

The wave is attenuated.

(ii) Allow a *finite electron temperature*.

Waves in a plasma, no magnetic field, with finite electron temperature

$$\nabla p_e = U_e^2 \nabla \rho_e = U_e^2 j n_e^1 m_e \mathbf{k}$$

$\nabla \rho_e = m_e \nabla n_e$ and it is easy to show that $\nabla n_e = j n_e^1 \mathbf{k}$.

The expression for $\mathbf{j}^1$ becomes

$$\mathbf{j}^1 = j \frac{n^0 e^2}{m_e \omega} \mathbf{E}^1 - \frac{e U_e^2}{\omega} n_e^1 \mathbf{k}$$

instead of electron oscillations, now have a wave with dispersion relation

$$\omega^2 - \omega_{pe}^2 - k^2 U_e^2 = 0.$$

(iii) Allow a *steady magnetic field*.

Waves in a plasma, with magnetic field

We will suppose the steady magnetic field is in the z-direction, $B^0\hat{\mathbf{z}}$.

The electron momentum equation, to first order yields the following three component equations for $\mathbf{v}_e^1$.

$$v_{ex}^1 = -j\frac{e}{\omega m_e}E_x^1 - j\frac{\omega_{ce}}{\omega}v_{ey}^1$$

$$v_{ey}^1 = -j\frac{e}{\omega m_e}E_y^1 + j\frac{\omega_{ce}}{\omega}v_{ex}^1$$

$$v_{ez}^1 = -j\frac{e}{\omega m_e}E_z^1$$

where $\omega_{ce} = \dfrac{eB^0}{m_e}$ is the electron cyclotron frequency.

From these, we obtain the three components for $\mathbf{j}^1$

$$j_x^1 = j\frac{e^2n^0}{\omega m_e\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)}E_x^1 + \frac{\omega_{ce}e^2n^0}{\omega^2 m_e\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)}E_y^1$$

$$j_y^1 = j\frac{e^2n^0}{\omega m_e\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)}E_y^1 - \frac{\omega_{ce}e^2n^0}{\omega^2 m_e\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)}E_x^1$$

$$j_z^1 = j\frac{e^2n^0}{\omega m_e}E_z^1$$

We can think of the plasma as being like a *conducting fluid*.
$\mathbf{j}^1 = \underline{\underline{\sigma}}\cdot\mathbf{E}^1$ where $\underline{\underline{\sigma}}$ is the conductivity tensor defined as

$$\underline{\underline{\sigma}} = \begin{pmatrix} j\dfrac{\varepsilon_0\omega_{pe}^2}{\omega\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)} & \dfrac{\varepsilon_0\omega_{ce}\omega_{pe}^2}{\omega^2\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)} & 0 \\ -\dfrac{\varepsilon_0\omega_{ce}\omega_{pe}^2}{\omega^2\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)} & j\dfrac{\varepsilon_0\omega_{pe}^2}{\omega\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)} & 0 \\ 0 & 0 & j\dfrac{\varepsilon_0\omega_{pe}^2}{\omega} \end{pmatrix}$$

But, as we said above, we can think of a plasma as a *dielectric medium*. What is the connection between these two pictures?

Recall $\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \frac{\partial \mathbf{D}}{\partial t}$ and $\mathbf{D} = \underline{\underline{\varepsilon}} \cdot \mathbf{E}$.
Substitute for $\mathbf{j}^1$ and show

$$\underline{\underline{\varepsilon}} = \varepsilon_0 \left(\underline{\underline{\mathrm{I}}} + \frac{j}{\omega \varepsilon_0} \underline{\underline{\sigma}} \right).$$

Instead of deriving dispersion relations for waves in any arbitrary direction we will find them for the two simplest cases: propagation along $\mathbf{B}^0$ and perpendicular to $\mathbf{B}^0$.

Waves in a plasma. Propagation along $\mathbf{B}^0$

Take $\mathbf{k}$ along z, the direction of $\mathbf{B}^0$. Write out the component Maxwell's equations. Eliminate $\mathbf{B}^1$, $\mathbf{j}^1$, leaving equations in $\mathbf{E}^1$.

$$k^2 E_x = \frac{\omega^2}{c^2} E_x + \frac{\omega_{pe}^2}{c^2\left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)} \left(-E_x + j\frac{\omega_{ce}}{\omega} E_y \right) \qquad (1)$$

$$k^2 E_y = \frac{\omega^2}{c^2} E_y + \frac{\omega_{pe}^2}{c^2\left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)} \left(-E_y - j\frac{\omega_{ce}}{\omega} E_x \right) \qquad (2)$$

$$0 = \frac{\omega^2}{c^2} E_z + \frac{\omega_{pe}^2}{c^2} \left(-E_z \right) \qquad (3)$$

Equation (3) describes plasma oscillations at frequency $\omega = \omega_{pe}$ with $\mathbf{E}^1$ along $\mathbf{k}$ and $\mathbf{B}^0$.

Equations (1) and (2) have a solution if their determinant is zero, i.e., if

$$\begin{vmatrix} k^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2}{c^2\left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)} & j\frac{\omega_{pe}^2}{c^2\left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)}\frac{\omega_{ce}}{\omega} \\ -j\frac{\omega_{pe}^2}{c^2\left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)}\frac{\omega_{ce}}{\omega} & k^2 - \frac{\omega^2}{c^2} + \frac{\omega_{pe}^2}{c^2\left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)} \end{vmatrix} = 0$$

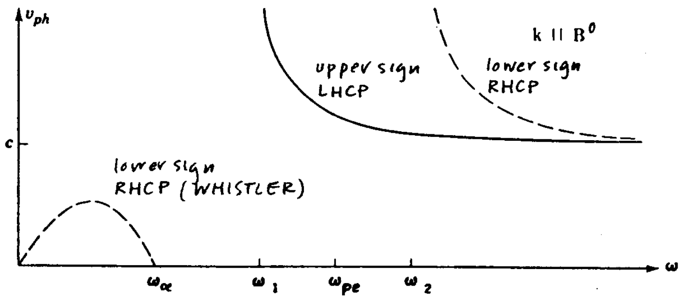
Multiply this out

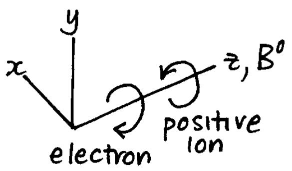
$$\boxed{k^2 = \frac{\omega^2 - \dfrac{\omega_{pe}^2}{1 \pm \dfrac{\omega_{ce}}{\omega}}}{c^2}}$$

This describes two waves with $\mathbf{E}^1$ perpendicular to $\mathbf{k}$ and $\mathbf{B}^0$.

Phase velocity

We can plot v_{ph} as a function of ω.





You do. Find expressions for ω_1 and ω_2.

It is easy to show that $E_y^1 = \mp jE_x^1$ so the waves are *circularly-polarized*.
The wave corresponding to the upper sign has its E vector rotating *opposite* to the way the electrons gyrate. This is called the *left-hand circularly-polarized (LHCP)* wave; the wave corresponding to the lower sign has its E vector rotating in the *same* way the electrons gyrate. This is called the *right-hand circularly-polarized (RHCP)* wave. Note that these definitions of handedness are different from those used in Optics. The *LHCP* and *RHCP* waves are the *characteristic waves* for propagation along the magnetic field.

Resonances

Resonances occur when $v_{ph} = 0$.

At a resonance, the present approach breaks down. This is easy to see. Consider the term $\mathbf{E} + \mathbf{v} \times \mathbf{B}$. This could be written

$$\mathbf{E}^1 + \mathbf{v}^1 \times \left(\mathbf{B}^0 + \mathbf{B}^1\right) = \mathbf{E}^1 + \mathbf{v}^1 \times \left(\mathbf{B}^0 + \frac{\mathbf{k} \times \mathbf{E}^1}{\omega}\right)$$

To first order, we have been ignoring the last part and writing $\mathbf{E}^1 + \mathbf{v}^1 \times \mathbf{B}^0$. But at a resonance, $v_{ph} = \dfrac{\omega}{k}$ is very small so this simplification may break down.

For propagation along $\mathbf{B}^0$ there is a resonance for the *RHCP* wave when $\omega = \omega_{ce}$. Note that if the magnetic field is high, the *RHCP* wave always propagates. In this regime it is known as the *whistler wave*.

The refractive index is

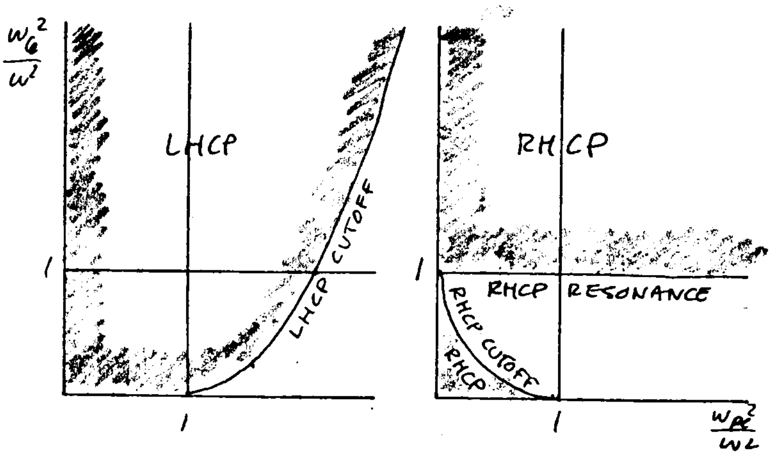
$$\mu^2_{\substack{L\\R}} = 1 - \frac{\dfrac{\omega_{pe}^2}{\omega^2}}{1 \pm \dfrac{\omega_{ce}}{\omega}}.$$

If we let $X = \frac{\omega_{pe}^2}{\omega^2}$ and $Y = \frac{\omega_{ce}}{\omega}$ this can be written more compactly as

$$\mu_{\substack{L\\R}}^2 = 1 - \frac{X}{1 \pm Y}.$$

CMA diagram

You have seen plots of v_{ph} vs. ω. The CMA diagram is another graphical representation of the same information.



Note the relation between the v_{ph} vs. ω graph and the CMA diagram.

Waves in a plasma. Propagation perpendicular to $\mathbf{B}^0$.

This time take **k** along x. $\mathbf{B}^0$ is still along z. Write out the component Maxwell's equations. Eliminate $\mathbf{B}^1$, $\mathbf{j}^1$, leaving equations in $\mathbf{E}^1$.

Proceeding as before

$$0 = \frac{\omega^2}{c^2}E_x + \frac{\omega_{pe}^2}{c^2\left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)}\left(-E_x + j\frac{\omega_{ce}}{\omega}E_y\right) \qquad (1)$$

$$k^2 E_y = \frac{\omega^2}{c^2}E_y + \frac{\omega_{pe}^2}{c^2\left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)}\left(-E_y - j\frac{\omega_{ce}}{\omega}E_x\right) \qquad (2)$$

$$k^2 E_z = \frac{\omega^2}{c^2}E_z + \frac{\omega_{pe}^2}{c^2}\left(-E_z\right) \qquad (3)$$

Equation (3) describes transverse electromagnetic waves with $\mathbf{E}^1$ along the z-direction, parallel to $\mathbf{B}^0$.

$$\boxed{k^2 = \frac{\omega^2 - \omega_{pe}^2}{c^2}}$$

This is the same as the dispersion relation for the wave when there is no magnetic field. This wave is called the *ordinary (O) wave*.

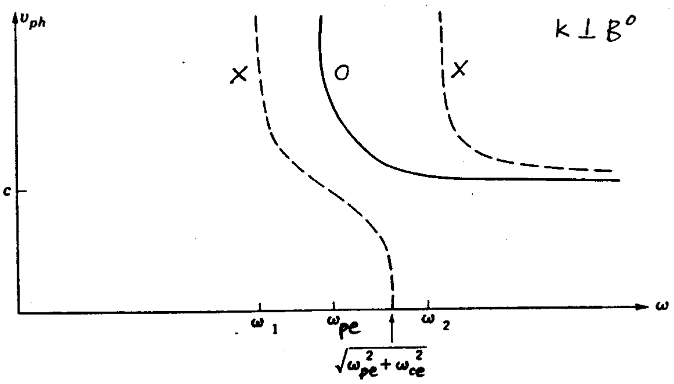
Equations (1) and (2) yield

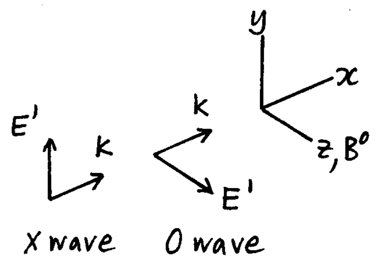
$$\boxed{k^2 = \frac{\omega^2 - \dfrac{\omega_{pe}^2}{1 - \dfrac{\omega_{ce}^2}{\omega^2 - \omega_{pe}^2}}}{c^2}}.$$

This wave is called the *extraordinary (X) wave*.

The *O* and *X* waves are the characteristic waves for propagation perpendicular to the magnetic field.

Phase velocity



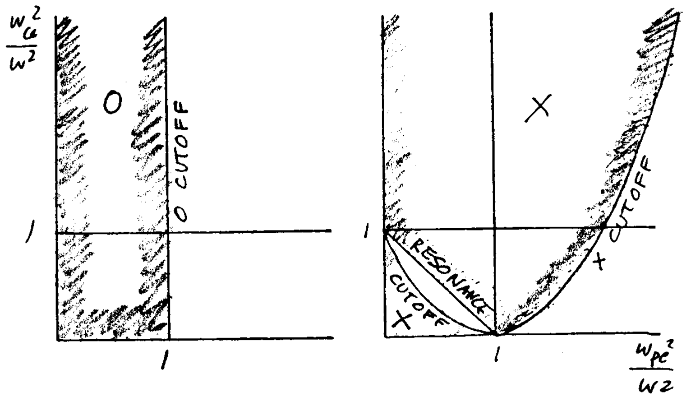


You do. Find expressions for ω_1 and ω_2.

The refractive indices, if we let $X = \dfrac{\omega_{pe}^2}{\omega^2}$ and $Y = \dfrac{\omega_{ce}}{\omega}$, are

$$N_O^2 = 1 - X \text{ and } N_X^2 = 1 - \frac{X}{1 - \dfrac{Y^2}{1-X}}.$$

CMA diagram



$\sqrt{\omega_{pe}^2 + \omega_{ce}^2}$ is known as the *upper hybrid frequency*.

Waves in a plasma. Propagation at an arbitrary angle to $\mathbf{B}^0$.

$$N^2 = 1 - \frac{X}{1 - \frac{Y^2 \sin^2\theta}{2(1-X)} \pm \left(\left(\frac{Y^2 \sin^2\theta}{2(1-X)} \right)^2 + Y^2 \cos^2\theta \right)^{\frac{1}{2}}}$$

where θ is the angle between **k** and $\mathbf{B}^0$. This is one of *Appleton's equations* for the case where collisions are neglected. The other equations describe the polarization of the waves. In general $\mathbf{E}^1$ will have a component parallel to **k**.

Ion motions

Is $\mathbf{v}_i^1$ still always negligible in the presence of a steady magnetic field?

The momentum equations are

$$n^0 m_i \left(-j\omega \mathbf{v}_i^1\right) = n^0 e\mathbf{E}^1 + n^0 e\mathbf{v}_i^1 \times \left(B^0 \hat{\mathbf{z}}\right)$$

$$n^0 m_e \left(-j\omega \mathbf{v}_e^1\right) = -n^0 e\mathbf{E}^1 - n^0 e\mathbf{v}_e^1 \times \left(B^0 \hat{\mathbf{z}}\right)$$

As an example, one of the components of $\mathbf{v}_i^1$ is

$$v_{ix}^1 = \frac{j \frac{e}{\omega m_i} E_x + \frac{\omega_{ci} e}{\omega^2 m_i} E_y}{1 - \frac{\omega_{ci}^2}{\omega^2}}.$$

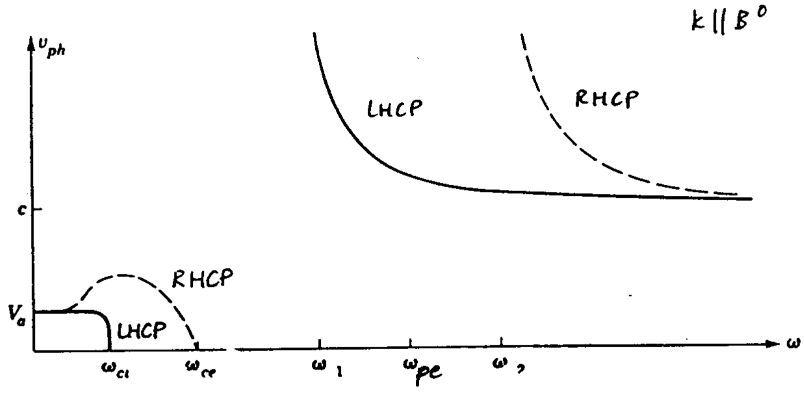
Clearly this becomes very important when $\omega \approx \omega_{ci}$.

e.g., for propagation along $\mathbf{B}^0$, the dispersion relation is

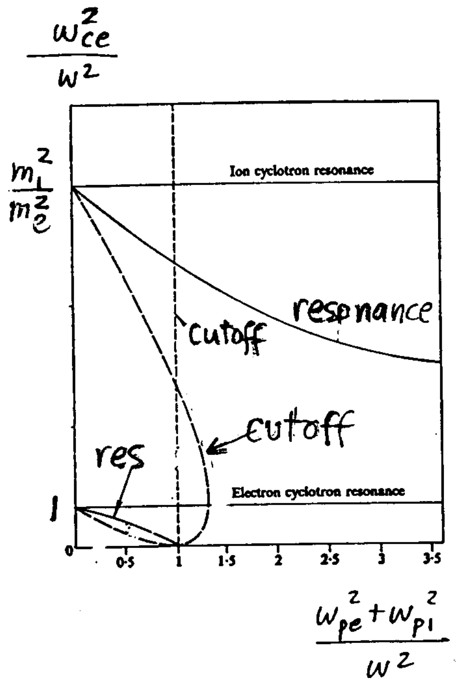
$$k^2 = \frac{\omega^2 - \omega_{pe}^2 \left(\frac{1}{1 \pm \frac{\omega_{ce}}{\omega}} + \frac{m_e}{m_i} \frac{1}{1 \mp \frac{\omega_{ci}}{\omega}} \right)}{c^2}$$

The effects can be seen by comparing the phase velocity and CMA diagrams below with those above.

Phase velocity



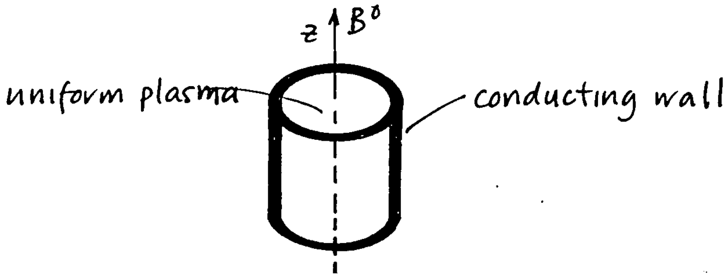
CMA diagram



Effects of geometry and boundaries

Helicon waves

Helicon waves have a role in industrial plasmas. They are whistler waves in a cylindrical plasma.



The method of calculating the properties of these waves is outlined below

Suppose that fields, etc., in a cylindrical geometry vary as $f(r)\mathrm{e}^{j(m\phi+k_z z-\omega t)}$, *m* integer.

Maxwell's equations

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{B}=\mu_0\mathbf{j}+\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$$

must be expanded using the cylindrical polar coordinate identities.

The electron momentum equation to first order is,

$$n^0 m_e\left(-j\omega\mathbf{v}_e^1\right)=-n^0 e\left(\mathbf{E}^1+\mathbf{v}_e^1\times\mathbf{B}^0\right)$$

Use $\omega_{pe}^2 >> \omega^2$ and $\omega_{ce} >> \omega$. Eliminate variables and finally arrive at a Bessel equation for B_z^1 which has a solution

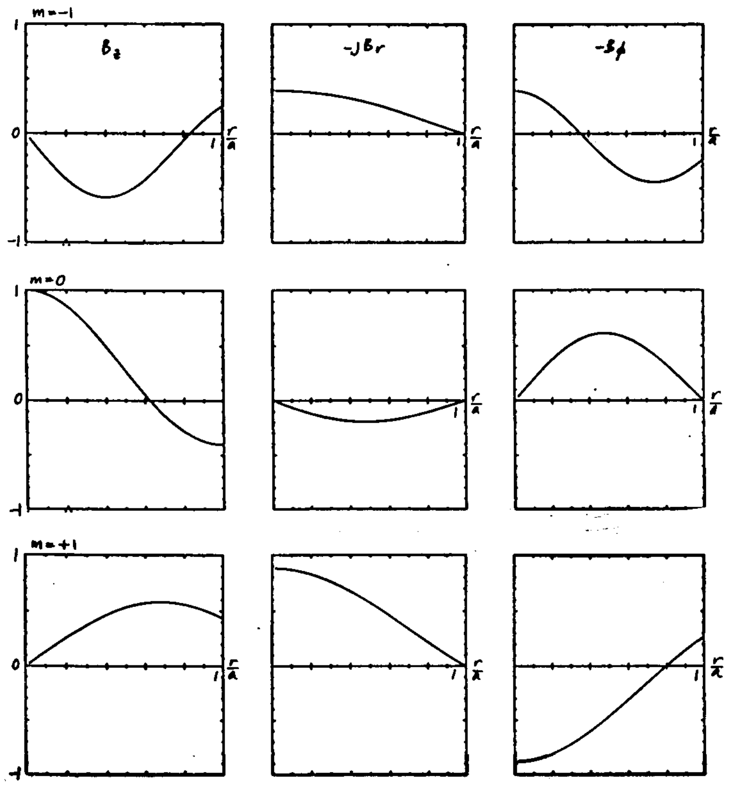
$$B_z^1 = A\mathrm{J}_m\left(k_\perp r\right).$$

Writing $k_\perp = k\sin\theta$ leads to the earlier dispersion equation for the whistler wave.

$$\frac{k^2c^2}{\omega^2}=\frac{\dfrac{\omega_{pe}^2}{\omega^2}}{\dfrac{\omega_{ce}}{\omega}\cos\theta}\quad\text{or}\quad \mu^2=\frac{X}{Y\cos\theta}.$$

Boundary condition. At the conducting wall, the tangential electric field must be zero, i.e., $E_\phi^1 = 0$. Now $E_\phi^1 \approx \left(k\dfrac{m\mathrm{J}_m\left(k_\perp r\right)}{k_\perp r}+k_z\mathrm{J'}_m\left(k_\perp r\right)\right)$ so $k_\perp a$ must be a zero of $\left(k\dfrac{m\mathrm{J}_m\left(k_\perp a\right)}{k_\perp a}+k_z\mathrm{J'}_m\left(k_\perp a\right)\right)$. We know the radius *a* so this limits the possible values of $k_\perp$.

For a uniform density profile radius a = 50 mm, $n = 3 \times 10^{18}$ m^{-3}, B = 500 gauss and f = 27.12 MHz, some representative wave magnetic field profiles are sketched below.



MHD Waves

This time work with the low-frequency Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

and the single fluid equations

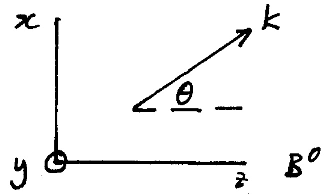
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\rho \frac{\partial \mathbf{v}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla p$$
$$\mathbf{0} = \mathbf{E} + \mathbf{v} \times \mathbf{B}\,.$$

To first order these can be written

$$j\mathbf{k}\times\mathbf{E}^1 = j\omega\mathbf{B}^1$$
$$j\mathbf{k}\times\mathbf{B}^1 = \mu_0\mathbf{j}^1$$
$$-j\omega\rho^1 + \rho^0 j\mathbf{k}\cdot\mathbf{v}^1 = 0$$
$$\rho^0\left(-j\omega\mathbf{v}^1\right) = \mathbf{j}^1\times\mathbf{B}^0 - U^2 j\mathbf{k}\rho^1$$
$$\mathbf{0} = \mathbf{E}^1 + \mathbf{v}^1\times\mathbf{B}^0 .$$

We will suppose the steady magnetic field is $B^0\hat{\mathbf{z}}$ and that $\mathbf{k} = k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}$. If the angle between the direction of propagation and the magnetic field is θ, then

$$k_x = k\sin\theta,\ k_z = k\cos\theta.$$



Eliminate ρ^1, $\mathbf{j}^1$, $\mathbf{E}^1$, $\mathbf{B}^1$ leaving $\mathbf{v}^1$ and use Alfven speed $V_A = \dfrac{B^0}{\sqrt{\mu_0\rho^0}}$.

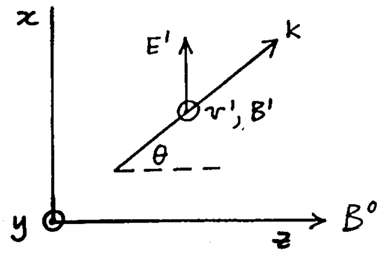
The component equations are

$$v_x^1 = \frac{1}{\omega^2}k^2V_A^2v_x^1 + \frac{1}{\omega^2}U^2\left(k_xv_x^1 + k_zv_z^1\right)k_x \qquad (1)$$
$$v_y^1 = \frac{1}{\omega^2}k_z^2V_A^2v_y^1 \qquad (2)$$
$$v_z^1 = \frac{1}{\omega^2}U^2\left(k_xv_x^1 + k_zv_z^1\right)k_z \qquad (3)$$

Alfven wave

Equation (2) gives a wave, the *Alfven wave*

$$\frac{\omega^2}{k^2} = V_A^2\cos^2\theta .$$

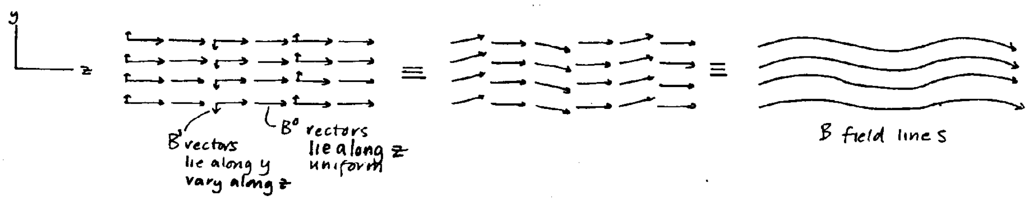


Properties of this wave

$\mathbf{v}^1$ is along y, from Equation (2)
$\mathbf{E}^1$ is along $\mathbf{v}^1\times\mathbf{B}^0$ so is along x
$\mathbf{B}^1$ is along $\mathbf{k}\times\mathbf{E}^1$ so is along y, i.e.,

$\mathbf{B}^1$ is perpendicular to the direction of propagation.
Phase velocity does not depend on frequency. It does depend on the direction of propagation.
Recall that for an electromagnetic wave in a vacuum, $B^1 \approx \dfrac{E^1}{c}$. Here $B^1 \approx \dfrac{E^1}{V_A}$. Since $V_A << c$, the wave magnetic field is particularly important.

Consider the special case $\theta = 0$.



The waves bend the field lines. This transverse perturbation of the field lines leads to the Alfven wave being called the *shear wave*. In a cylindrical geometry it is called the *torsional wave*.

This wave is analogous to the transverse wave on a stretched string.

velocity of a wave on a stretched string $v_{ph} = \sqrt{\frac{\tau}{\mu}}$, where τ is tension and μ is mass per unit length.

velocity of Alfven wave $v_{ph} = \frac{B^0}{\sqrt{\mu_0 \rho^0}}$.

$\mu = \rho A$, where A is area, so the two expressions are equivalent if $\tau = \frac{B^{02}}{\mu_0} A$.

Recall that magnetic tension $= \frac{B^{0\,2}}{\mu_0}$. This magnetic tension provides the restoring force.

Fast and slow MHD waves

Equations (1) and (3) give two waves. The upper sign gives the *fast MHD wave*; the lower sign the *slow MHD wave*.

$$\frac{\omega^2}{k^2} = \frac{\left(U^2 + V_A^2\right) \pm \sqrt{\left(U^2 + V_A^2\right)^2 - 4U^2 V_A^2 \cos^2\theta}}{2}$$

Note that in this limit the phase velocities do not depend on frequency.

Consider two special cases.

(i) Propagation along $\mathbf{B}^0$: ion acoustic wave and Alfven wave

$\theta = 0°$.

The two solutions are

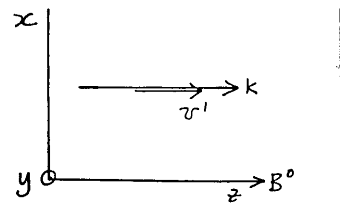
$$\frac{\omega^2}{k^2} = U^2 \qquad (1)$$

and

$$\frac{\omega^2}{k^2} = V_A^2 \qquad (2)$$

Equation (1) gives an acoustic wave, the *ion acoustic wave*.

Properties



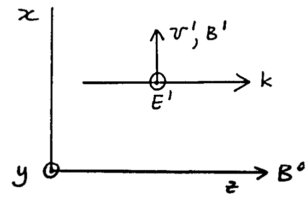
$\mathbf{v}^1$ is along z

$\mathbf{E}^1 = -\ \mathbf{v}^1 \times \mathbf{B}^0$. But $\mathbf{v}^1$ and $\mathbf{B}^0$ are parallel so there can be no $\mathbf{E}^1$.

$\mathbf{B}^1 \propto \mathbf{k} \times \mathbf{E}^1$ so there can be no $\mathbf{B}^1$.

So indeed it is an acoustic wave.

Equation (2) gives the *Alfven wave* again.



Properties

$\mathbf{v}^1$ is along x

$\mathbf{E}^1$ is along $\mathbf{v}^1 \times \mathbf{B}^0$ so is along y

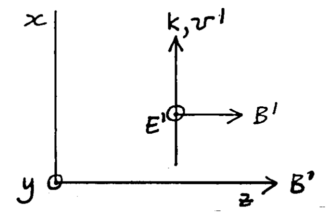
$\mathbf{B}^1$ is along $\mathbf{k} \times \mathbf{E}^1$ so is along x

so $\mathbf{B}^1$ is perpendicular to the direction of propagation.

(ii) Propagation perpendicular to $\mathbf{B}^0$: compressional wave

$\theta = 90°$.

$$\frac{\omega^2}{k^2} = U^2 + V_A^2$$



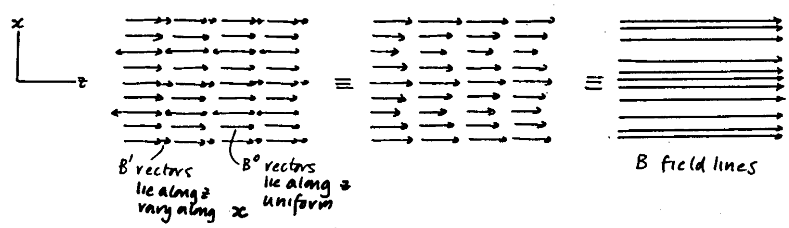
Properties

$\mathbf{v}^1$ is along x

$\mathbf{E}^1$ is along $\mathbf{v}^1 \times \mathbf{B}^0$ so is along y

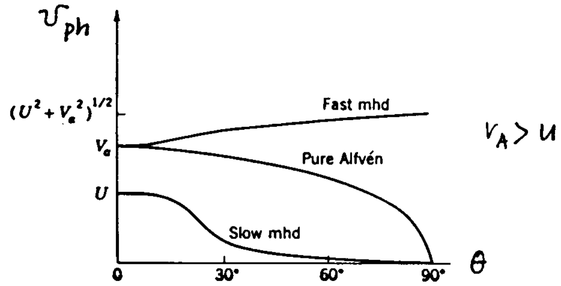
$\mathbf{B}^1$ is along $\mathbf{k} \times \mathbf{E}^1$ so is along z

so $\mathbf{B}^1$ is in the direction of $\mathbf{B}^0$ and perpendicular to the direction of propagation



Other names for this wave are the *fast wave*, and if U is not neglected, the *magnetosonic* or *magnetoacoustic wave*.

The waves are summarised on the phase velocity vs. angle of propagation diagram below.



In laboratory plasmas, $V_A >> U$; in space plasmas $V_A < U$.