PLASMA PHYSICS

III. FLUID DESCRIPTION OF A PLASMA

Fluid mechanics

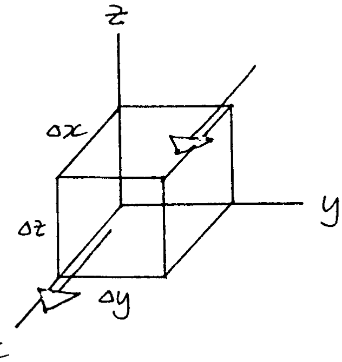
Liquids and gases can be characterized by the physical quantities; density ρ, pressure p, velocity $\mathbf{v}$ and temperature T.

In Fluid Mechanics the fluid is treated as a continuous medium. We look at what happens to large numbers of molecules. This is a *macroscopic* approach as distinct from the *microscopic* approach in Chapter II.

The key equations deal with mass, momentum and energy.

We obtain the equations either by Method 1 where we consider a *fluid particle,* or by Method 2 where we look at some property carried along by the fluid through a fixed volume.

Mass



Here we use Method 2.

mass flowing into the infinitesmal volume in Δt

$$= \rho v_x \Delta t\ \Delta y \Delta z - \left(\rho v_x + \frac{\partial}{\partial x}(\rho v_x)\Delta x\right)\Delta t\ \Delta y \Delta z + \ldots$$

$$= -\nabla \cdot (\rho \mathbf{v})\ \Delta x \Delta y \Delta z\ \Delta t$$

increase in mass in Δt

$$= \frac{\partial}{\partial t}(\rho\ \Delta x \Delta y \Delta z)\ \Delta t$$

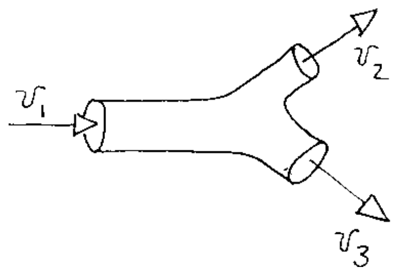
Since mass is conserved, these are equal, so

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 .$$

This is the equation of conservation of mass or the *Equation of continuity*.

(For liquids and often for gases we make the approximation that the fluid is incompressible i.e., ρ does not change and this equation reduces to $\nabla \cdot \mathbf{v} = 0$.

We usually express this equation in integral form and use it to solve problems like:



$$v_1 A_1 = v_2 A_2 + v_3 A_3\,)$$

Momentum

This time we use Method 1.

First some mathematics. How does the scalar property T of a fluid particle change as the fluid goes from point P at (x, y, z) to point P′ at (x+Δx, y+Δy, z+Δz) in time Δt?

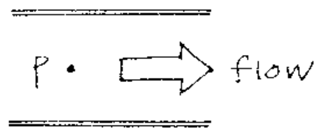
$$T = T(x, y, z, t)$$
$$\Delta T = \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y + \frac{\partial T}{\partial z}\Delta z + \frac{\partial T}{\partial t}\Delta t$$

Divide throughout by Δt and let $\Delta t \to 0$.

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T$$

dT/dt is called the time derivative following the motion. It is the total change in T as the fluid particle passes P. It is equal to (i) the change in T because T at P is a function of time plus (ii) the change in T because the particle is moving past P at velocity **v** and T varies with position. (This last change is known as the *convective* change).

Simple example. A river. Velocity of water is 10 km/day.



T is the temperature. At P, a fixed thermometer indicates T increases 0.2°C / day. Near P, T increases in the direction of the flow 0.07°C / km.

Consider a thermometer drifting with the water. It measures the total rate of change in T.

$$\begin{aligned} \frac{dT}{dt} &= \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} \\ &= 0.2^\circ\mathrm{C\ /\ day} + 10\mathrm{km\ /\ day} \times 0.07^\circ\mathrm{C\ /\ km} \\ &= 0.9^\circ\mathrm{C\ /\ day.} \end{aligned}$$

Momentum equation

Let $T = v_x$.

$$\frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + \mathbf{v} \cdot \nabla v_x$$

Now dv_x/dt is the acceleration a_x, and by Newton's 2nd Law $F_x = ma_x$.
Write $m = \rho\Delta x\Delta y\Delta z$.
Define force per unit volume

$$f_x = \frac{F_x}{\Delta x \Delta y \Delta z}$$

Then

$$\rho \frac{\partial v_x}{\partial t} + \rho \mathbf{v} \cdot \nabla v_x = f_x \text{, or}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{f}$$

In fluid mechanics, **f** is separated into three parts:

(i) force due to pressure $f_{\text{pressure}} = -\nabla p$

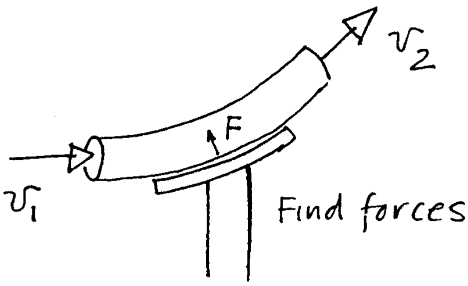
(ii) force due to shear stresses. This gives a viscosity term.

(iii) body forces e.g., gravity $f_{\text{gravity}} = \rho \mathbf{g}$

This leads to the *Navier-Stokes equation*

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \,.$$

(We usually express this equation in integral form and use it to solve problems like:

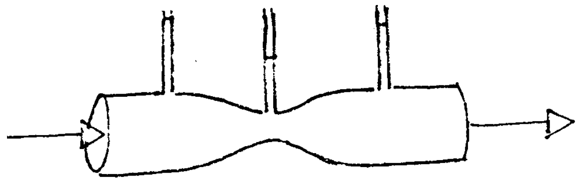


$$-\rho \mathbf{v}_1 v_1 A_1 + \rho \mathbf{v}_2 v_2 A_2 = \mathbf{F} \,)$$

Energy

We can take $\mathbf{v} \cdot$ of the momentum equation, integrate and get *Bernouilli's equation* which is just a form of the conservation of mechanical energy equation in a form suitable for fluid mechanics.

(We usually express this equation in integral form and use it to solve problems like:



$$\frac{1}{2} \rho v^2 + p + \rho g h = \text{constant})$$

Plasma

A plasma is an electrical fluid. We must add to the list of physical quantities; charge density σ, current density **j**, electric field **E** and magnetic field **B**.

Continuity and momentum

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 .$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = nq(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \nabla p$$

p here is isotropic. More generally, the pressure term is $\nabla \cdot \underline{\underline{\mathbf{P}}}$ where $\underline{\underline{\mathbf{P}}}$ is a pressure tensor. The off-diagonal terms would be associated with viscosity.

Suppose we have a fully-ionized plasma of (a single variety of singly-charged, to keep things simple) ions and electrons. It is a mixture of two fluids. An ion fluid and an electron fluid.

For ions, the momentum equation is

$$\rho_i \frac{\partial \mathbf{v}_i}{\partial t} + \rho_i \mathbf{v}_i \cdot \nabla \mathbf{v}_i = n_i e(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) - \nabla p_i - \rho_i \upsilon_{ie}(\mathbf{v}_i - \mathbf{v}_e)$$

and for electrons it is

$$\rho_e \frac{\partial \mathbf{v}_e}{\partial t} + \rho_e \mathbf{v}_e \cdot \nabla \mathbf{v}_e = -n_e e(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \nabla p_e - \rho_e \upsilon_{ei}(\mathbf{v}_e - \mathbf{v}_i)$$

A force term due to collisions between the ions and the electrons has been added.

The νs are called *collision frequencies for momentum transfer*. Since momentum must be conserved in collisions,

$$\rho_i \nu_{ie}(\mathbf{v}_i - \mathbf{v}_e) + \rho_e \nu_{ei}(\mathbf{v}_e - \mathbf{v}_i) = 0 \text{ or}$$

$$\rho_i \nu_{ie} = \rho_e \nu_{ei} .$$

In a less than fully-ionized plasma there would be equations for the neutrals as well.

In general,

$$\boxed{\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha) = 0}$$

$$\boxed{\rho_\alpha \frac{\partial \mathbf{v}_\alpha}{\partial t} + \rho_\alpha \mathbf{v}_\alpha \cdot \nabla \mathbf{v}_\alpha = n_\alpha q_\alpha (\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}) - \nabla p_\alpha - \sum_\beta \rho_\alpha \upsilon_{\alpha\beta}(\mathbf{v}_\alpha - \mathbf{v}_\beta)}$$

where $\alpha = i$, e or n.

The neutral fluid interacts with the others only via collisions. The ion and electron fluids interact via fields even in the absence of collisions.

Energy

It is difficult to include collisions. We normally use an equation of state instead.
If *isothermal* conditions apply, use

$$\boxed{p = nkT};$$

if *adiabatic* conditions apply use

$$\boxed{p\rho^{-\gamma} = \text{ constant}},$$

where $\gamma = \dfrac{c_p}{c_v} = \dfrac{5}{3}$ for a monotomic gas.

Both choices lead to $\nabla p = U^2 \nabla \rho$ where U is a sound speed.

> *You do*. Differentiate these expressions and obtain
>
> isothermal sound speed $U^2 = \dfrac{p}{\rho}$, and
>
> adiabatic sound speed $U^2 = \dfrac{\gamma p}{\rho}$.

We have a self-consistent problem here. E and B depend on the charges in the plasma and how they move; and how they move depends on E and B.

> We have more than enough equations at this point to solve for the 16 unknowns n_i, n_e, $\mathbf{v}_i$, $\mathbf{v}_e$, p_i, p_e, $\mathbf{E}$ and $\mathbf{B}$. We have continuity (1 for ions and 1 for electrons), momentum (3 components for ions and 3 for electrons), energy (1 and 1), Maxwells equations (8), a total of 18. We can drop two Maxwells equations $\nabla \cdot \mathbf{E}$ and $\nabla \cdot \mathbf{B}$. They can be obtained by taking $\nabla \cdot$ of the other Maxwells equations. This leaves 16 equations in 16 unknowns.

We usually let $n_i = n_e = n$ and avoid using $\nabla \cdot \mathbf{E} = \dfrac{\sigma}{\varepsilon_0}$. This is the so-called *plasma approximation*, closely related to the idea of quasineutrality.

E × B and diamagnetic drifts

Consider the ion momentum equation. Simplify it by ignoring time variations and collisions. The equation becomes

$$0 = n_i e(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) - \nabla p_i .$$

Take equation × **B**, use $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_{||}$ and rearrange

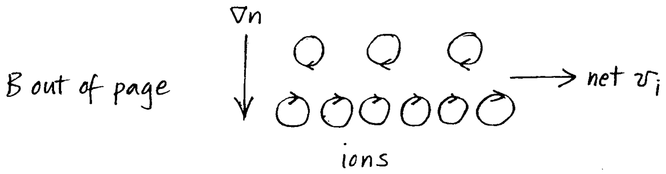
$$\mathbf{v}_{i\perp} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \frac{\nabla p_i \times \mathbf{B}}{n_i e B^2} .$$

You do. Show this.

The first term is the $E \times B$ drift of the guiding centres we obtained in the single particle approach.

The second term is called the *diamagnetic drift*. It does not involve any motion of the guiding centres. There is no drift equivalent to this in the single particle approach.

You can see how it arises from the sketch.

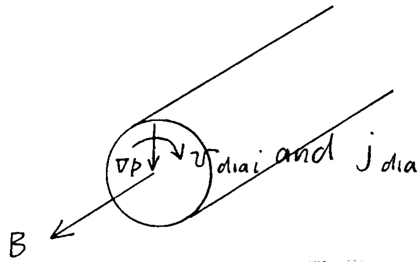


There will be a net $\mathbf{v}_i$ to the right.

The diamagnetic drift is in the opposite direction for electrons. The ion and the electron drifts combine to give a diamagnetic current, to the right in the sketch.

The direction of the current is such as to reduce the magnetic field - hence "diamagnetic".

e.g., in a cylindrical plasma with a density gradient radially inwards.



Plasma as a single fluid

We can define a density for the plasma as a whole,

$$\rho = \sum_{\alpha} \rho_{\alpha}$$

and a velocity for the plasma as a whole,

$$\mathbf{v} = \sum_{\alpha} \frac{\rho_{\alpha} \mathbf{v}_{\alpha}}{\rho}$$

Continuity equation

Add the equations of continuity of all the species

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0}.$$

Continuity of charge

Take $\dfrac{1}{m_i}\times$ the ion continuity equation $-\dfrac{1}{m_e}\times$ the electron continuity equation.

Use charge density $\boxed{\sigma = en_1 - en_e}$, current density $\boxed{\mathbf{j} = en_i\mathbf{v}_i - en_e\mathbf{v}_e}$.

$$\boxed{\frac{\partial\sigma}{\partial t} + \nabla\cdot\mathbf{j} = 0}.$$

Momentum

Add the momentum equations

But (i) *linearize*. This means neglect any quadratic terms in v. This is a considerable simplification.

(ii) use

total pressure $p = \sum_\alpha p_\alpha$

momentum is conserved so $\sum_\alpha\sum_\beta \rho_\alpha \upsilon_{\alpha\beta}\left(\mathbf{v}_\alpha - \mathbf{v}_\beta\right) = 0$.

$$\rho\frac{\partial\mathbf{v}}{\partial t} = \sigma\mathbf{E} + \mathbf{j}\times\mathbf{B} - \nabla p$$

At this point let $n_i = n_e = n$. The Continuity of charge equation is no longer required and $\sigma\mathbf{E}$ in the Momentum equation can be dropped.

$$\boxed{\rho\frac{\partial\mathbf{v}}{\partial t} = \mathbf{j}\times\mathbf{B} - \nabla p}$$

Generalized Ohm's Law

For an ordinary conductor, $\mathbf{E} = \eta\mathbf{j}$, where η is the resistivity. (Recall $V = RI$, hence $El = RjA$ so $\eta = RA/l$.)

For a plasma, take $\dfrac{1}{m_i}\times$ ion momentum equation $-\dfrac{1}{m_e}\times$ electron momentum equation.

But (i) linearize

(ii) use

$n_i = n_e = n$

$m_e << m_i$

Some useful manipulations include showing that $\rho\mathbf{v} = \rho_i\mathbf{v}_i + \rho_e\mathbf{v}_e$ and $\mathbf{j} = en(\mathbf{v}_i - \mathbf{v}_e)$ lead to $\mathbf{v}_i = \mathbf{v} + \dfrac{m_e}{e\rho}\mathbf{j}$ and $\mathbf{v}_e = \mathbf{v} - \dfrac{m_i}{e\rho}\mathbf{j}$,

$\nu_{ie} << \nu_{ei}$ leads to a collision term $n\nu_{ei}(\mathbf{v}_e - \mathbf{v}_i)$ which we write as $-\dfrac{ne}{m_e}\eta\mathbf{j}$. Here $\eta = \dfrac{m_e\nu_{ei}}{ne^2}$ is a constant of proportionality.

$$\boxed{\frac{m_e}{ne^2}\frac{\partial \mathbf{j}}{\partial t} = \mathbf{E} + \mathbf{v}\times\mathbf{B} - \frac{1}{ne}\mathbf{j}\times\mathbf{B} + \frac{1}{ne}\nabla p_e - \eta\mathbf{j}}$$

The $\mathbf{j} \times \mathbf{B}$ term is known as the *Hall term*.
You do. Refer to a textbook on the Hall effect in a solid material to see the connection.

Approximations

Various approximations are generally made.

1. *Quasineutral approximation.* $n_i = n_e = n$

2. *Steady-state or very slow time variations.* $\frac{\partial \mathbf{v}}{\partial t}$, $\frac{\partial \mathbf{j}}{\partial t}$ are negligible.

3. *Cold plasma.* ∇p_i, ∇p_e are negligible.

Under these approximations, Maxwell's $\nabla\times\mathbf{B}$ equation becomes

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j},$$

the momentum equation reduces to

$$\mathbf{j}\times\mathbf{B} = \mathbf{0},$$

and the generalized Ohms law reduces to

$$\mathbf{0} = \mathbf{E} + \mathbf{v}\times\mathbf{B} - \eta\mathbf{j}.$$

The set of equations including this approximation is known as the *magnetohydrodynamic or MHD equations*. These are standard tools for treating large scale plasma motion.

4. *Infinite conductivity.* Resistivity can be neglected and

$$\mathbf{0} = \mathbf{E} + \mathbf{v}\times\mathbf{B}.$$

When this is true the magnetic field lines can be regarded as being frozen into the plasma. We talk about the *ideal MHD equations*.

Confinement of a plasma

There are two ways in which we might discuss plasma confinement.

The first way is in terms of forces.

Consider a plasma in a cylinder where the plasma density falls off from a maximum at the centre to zero at the wall.

The momentum equation

$$\mathbf{0} = \mathbf{j}\times\mathbf{B} - \nabla p$$

says that if there is equilibrium the pressure forces and the $\mathbf{j}\times\mathbf{B}$ forces balance.

The second more picturesque way is in terms of magnetic pressures and magnetic tensions.

Substitute for $\mathbf{j}$ using $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, rearrange to obtain

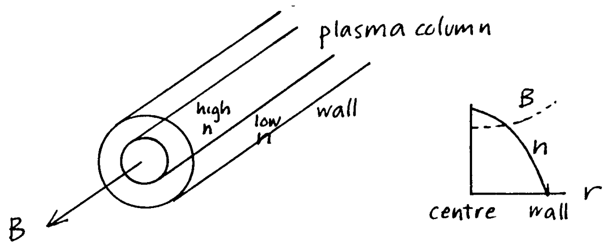
$$\nabla\left(p + \frac{B^2}{2\mu_0}\right) = \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{\mu_0}.$$

If we have a straight cylinder of plasma, the rhs is zero and

$$p + \frac{B^2}{2\mu_0} = \text{ constant}.$$

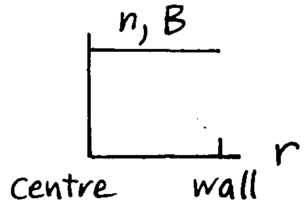
p is the particle pressure, $\frac{B^2}{2\mu_0}$ is the *magnetic pressure*.

The *magnetic tension* $= \frac{B^2}{\mu_0}$ N m^{-2}. If the field lines are straight, this is in the direction of the field lines, if curved there is a $\perp$ component. The term on the rhs is related to this component.

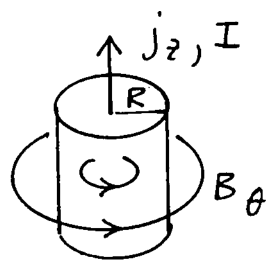


Near the centre n is larger, so p is larger and B is smaller (this is *plasma diamagnetism*); near the wall n is smaller, so p is smaller and B is larger.

Compare this with a gas filling a cylindrical vessel.



Pinch effect



Axial current j_z heats plasma and azimuthal B_θ due to this current confines the plasma. Nice idea as a way of achieving fusion conditions but extremely unstable.

In this analysis you will be working in cylindrical coordinates and will make the assumption that there are only radial variations.
The plasma has radius R. The total current is I.
(a) Suppose the current is distributed uniformly throughout the plasma.
Show $I = j_z \pi R^2$.

(b) Use $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ and show $j_\theta(r) = -\frac{1}{\mu_0}\frac{dB_z}{dr}$ and $B_\theta(r) = \frac{\mu_0 I r}{2\pi R^2}$.

(c) Write the radial component of the momentum equation $\mathbf{0} = \mathbf{j} \times \mathbf{B} - \nabla p$, substitute. Show

$$\frac{d}{dr}\left(p + \frac{B_z^{\ 2}}{2\mu_0}\right) = -\frac{\mu_0 I^2 r}{2\pi^2 R^4}.$$

(d) Finally, integrate over the range 0 to R and show

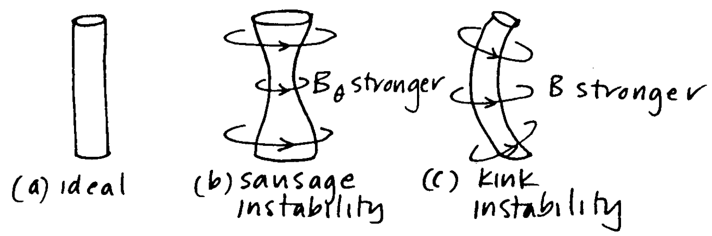
$$\frac{B_z^2(R)}{2\mu_0} = p(0) + \frac{B_z^2(0)}{2\mu_0} - \frac{\mu_0 I^2}{4\pi^2 R^2}.$$

$I = 0$. This is like the case discussed above.
If I is large enough $B_z(R) < B_z(0)$ and plasma is squashed. (This is similar to the effect we see in the force between parallel currents demonstration and the squashing of a lightning conductor.)

Problems with the pinch

(i) Linear pinch with electrodes at the ends. There will be cooling at the ends.
To overcome this problem:
Bend the cylinder into a torus so there are no ends.

(ii) There are instabilities.



$B_z = 0$. There is the sausage instability (b) and the kink instability in (c). The regions where B_θ is stronger are shown. In these regions the magnetic pressure on the plasma is larger and the instability becomes worse.

To overcome these instabilities:

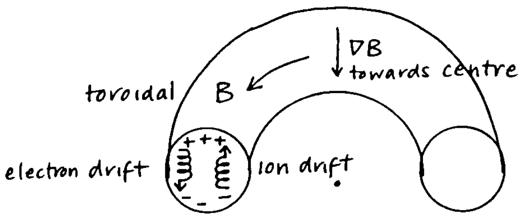
(i) Apply a B_z. The field lines are frozen-in. In (b) the magnetic pressure is increased where the plasma is being squeezed thus opposing the squeezing. In (c) the magnetic tension is increased tending to straighten out the kink.

(ii) Use a vessel with conducting walls. If the field is applied suddenly, the field lines are squashed against the wall increasing the magnetic pressure there.

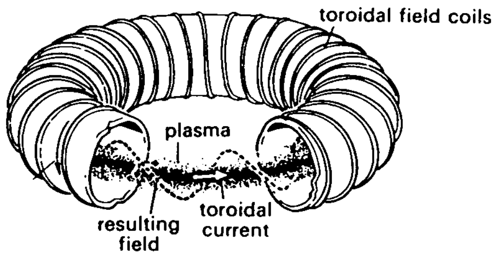
The sausage instability is an example of an interchange instability. An example from fluid mechanics is the Rayleigh-Taylor instability where one fluid is floating on a second, less dense fluid. Another example from plasma physics is the flute instability.

Tokamaks and stellarators

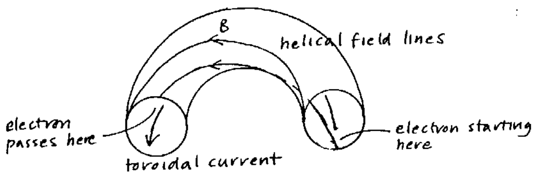
The toroidal coil windings provide a toroidal B. Recall that B is larger the closer you get to the centre of the torus and this ∇B causes drifts of the ions and electrons. The drifts are in opposite directions so an E builds up causing an $E \times B$ drift. The plasma moves out to the walls and cools.



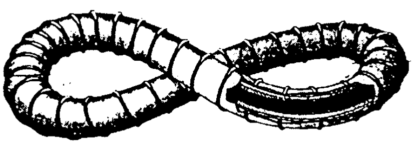
In a tokamak, there is a plasma current around the torus so the lines of B are twisted helically.



The E is shorted out. In a tokamak, this current also provides some heating.



In a stellarator, there is a helical coil winding to provide the twist. There is no need for a current in the plasma, except perhaps to heat it.



no helical winding required