PLASMA PHYSICS

II. MOTION OF IONS AND ELECTRONS IN E AND B FIELDS

We consider the paths of ions and electrons in E and B fields for some simple cases.

The Lorentz force on a point charge is $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$.

E is measured in V m^{-1}. B in T (often in gauss. 10000 gauss = 1 T)

1. E = *constant, uniform*

Suppose $\mathbf{E} = E\hat{\mathbf{x}}$. The Lorentz force equation becomes

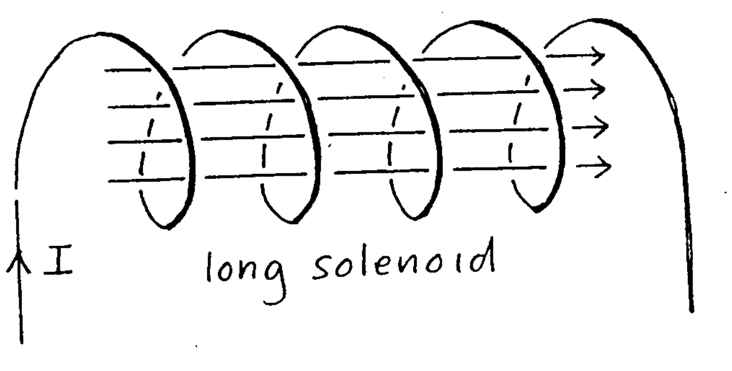
$$\frac{dv_x}{dt} = \frac{q}{m}E$$
$$\frac{dv_y}{dt} = 0$$
$$\frac{dv_z}{dt} = 0.$$

This describes a constant acceleration along x.

2. B = *constant, uniform*

Suppose $\mathbf{B} = B\hat{\mathbf{z}}$.

Here is how we might produce a uniform magnetic field.



$$\frac{dv_x}{dt} = \frac{q}{m}v_y B \qquad (1)$$
$$\frac{dv_y}{dt} = -\frac{q}{m}v_x B \qquad (2)$$
$$\frac{dv_z}{dt} = 0. \qquad (3)$$

Take $\frac{d}{dt}$ of (1), substitute using (2)

$$\frac{d^2 v_x}{dt^2} = \frac{q}{m}\frac{dv_y}{dt}B = \frac{q}{m}\left(\frac{q}{m}v_x B\right)B.$$

Write

$$\boxed{\omega_c = \frac{|q|B}{m}}$$

the (angular) *cyclotron frequency* or *gyrofrequency*. (Note the symbol Ω is often used.)

$$\frac{d^2 v_x}{dt^2} + \omega_c^2 v_x = 0.$$

Similarly,

$$\frac{d^2 v_y}{dt^2} + \omega_c^2 v_y = 0.$$

The solutions can be written as

$$v_x = -v_\perp \sin\omega_c t$$
$$v_y = \mp v_\perp \cos\omega_c t$$

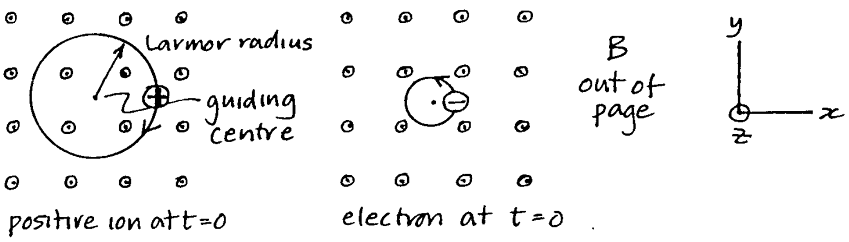
(The signs and phase angles have been chosen to match the sketches below. The upper sign is for a positive charge, the lower for a negative.)

Integrate again

$$x = \frac{v_\perp}{\omega_c}\cos\omega_c t = r_L \cos\omega_c t$$

$$y = \mp\frac{v_\perp}{\omega_c}\sin\omega_c t = \mp r_L \sin\omega_c t$$

$\boxed{r_L = \frac{v_\perp}{\omega_c}}$ is called the *Larmor radius*, *radius of gyration*, or *gyroradius.*

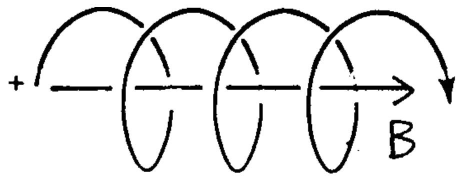


So a charge in a constant, uniform B moves in a circle with constant speed. Note that the cyclotron frequency does not depend on how fast the charge is moving.

You do. Check that the directions of motion are correct and that the equations above match the sketches.

You do. Calculate the cyclotron frequency (in Hz) for (a) hydrogen ions and (b) electrons in a magnetic field of 1 T.

If the charge has a v_z, this z-component of the motion is unchanged. The charge moves in a helical path.



3. **E** *constant, uniform.* **B** *constant, uniform.*

Suppose $\mathbf{B} = B\hat{\mathbf{z}}$.

$$\frac{dv_x}{dt} = \frac{q}{m}\left(E_x + v_y B\right) \qquad (1)$$

$$\frac{dv_y}{dt} = \frac{q}{m}\left(E_y - v_x B\right) \qquad (2)$$

$$\frac{dv_z}{dt} = \frac{q}{m}E_z. \qquad (3)$$

(3) gives constant acceleration along z.

You do. Suppose E has a z-component only. Describe the motion and sketch the path for this case.

(1) and (2) are manipulated as before, they give

$$\frac{d^2 v_x}{dt^2} + \omega_c{}^2 v_x = \omega_c{}^2 \frac{E_y}{B} \text{ and}$$

$$\frac{d^2 v_y}{dt^2} + \omega_c{}^2 v_y = -\omega_c{}^2 \frac{E_x}{B},$$

with solutions

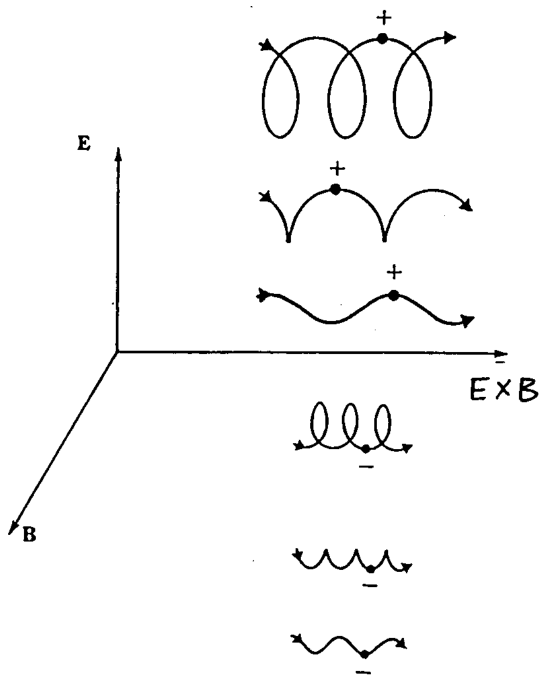
$$v_x = -v_\perp \sin\omega_c t + \frac{E_y}{B}$$

$$v_y = \mp v_\perp \cos\omega_c t - \frac{E_x}{B}.$$

The path of an electron is a combination of uniform circular motion plus a drift, called an $E \times B$ *drift.*

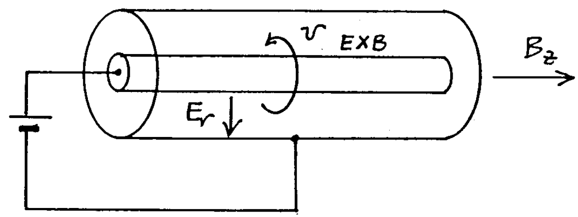
$$\mathbf{v}_{E\times B} = \frac{1}{B}\left(E_y\hat{\mathbf{x}} - E_x\hat{\mathbf{y}}\right) = \frac{\mathbf{E}\times\mathbf{B}}{B^2}.$$

Note that the drift term is independent of the charge and its sign, so all the charges will drift together. The paths are cycloids.



If $\mathbf{E} \perp \mathbf{B}$ then $v_{E\times B} = \dfrac{E}{B}$.

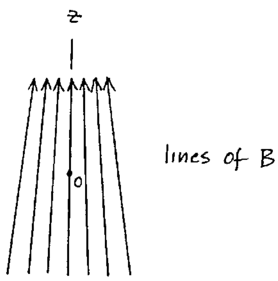
Here is an example where the $E \times B$ drift can cause a plasma to rotate.



4. **B** = ***constant, non-uniform.***

We will consider two distinct cases.

Case (a):



$$B_z = B_0(1+\alpha z)$$
$$B_x = -B_0\frac{\alpha}{2}x$$
$$B_y = -B_0\frac{\alpha}{2}y$$

where α is small.

You do. Show Maxwell's equations $\nabla.\mathbf{B} = 0$ and $\nabla \times \mathbf{B} = \mathbf{0}$ are satisfied as long as we include these small B_x and B_y terms.

The Lorentz force equation becomes

$$\frac{dv_x}{dt} = \frac{q}{m}\left(v_y B_z - v_z B_y\right) = \frac{q}{m}\left(v_y B_0 + v_y B_0 \alpha z + v_z B_0 \frac{\alpha}{2} y\right)$$
$$\frac{dv_y}{dt} = \frac{q}{m}\left(v_z B_x - v_x B_z\right) = \frac{q}{m}\left(-v_z B_0 \frac{\alpha}{2} x - v_x B_0 - v_x B_0 \alpha z\right)$$
$$\frac{dv_z}{dt} = \frac{q}{m}\left(v_x B_y - v_y B_x\right) = \frac{q}{m}\left(-v_x B_0 \frac{\alpha}{2} y + v_y B_0 \frac{\alpha}{2} x\right)$$

We will write $\mathbf{v} = \mathbf{v}^0 + \mathbf{v}^1$ where ^{0} indicates the uniform constant or zero-order part and ^{1} a small first-order correction, of the same order as α, and substitute in the equations. This is a standard approach and we will use it frequently.

The zero-order equations. If we write down the zero-order terms, i.e., the terms in $\mathbf{v}^0$ and those that do not contain α, the equations that remain describe motion in a constant, uniform *B***.** This was discussed earlier.

You do. Show this.

The first-order equations. We first solve the zero-order equations to obtain $v_x^0, v_y^0, v_z^0, x^0, y^0, z^0$; then we write down the first order terms, i.e., the terms in $\mathbf{v}^1$ and those containing α; then substitute for $v_x^0, v_y^0, v_z^0, x^0, y^0, z^0$.

This gives

$$\frac{dv_x^1}{dt} = \frac{q}{m}\left(v_y^1 B_0 + v_\perp^0 \cos\omega_c t\; B_0 \alpha v_z^0 t + v_z^0 B_0 \frac{\alpha}{2} r_L \sin\omega_c t\right)$$
$$\frac{dv_y^1}{dt} = \frac{q}{m}\left(-v_z^0 B_0 \frac{\alpha}{2} r_L \cos\omega_c t - v_x^1 B_0 - v_\perp^0 \sin\omega_c t\; B_0 \alpha v_z^0 t\right)$$
$$\frac{dv_z^1}{dt} = \frac{q}{m}\left(v_\perp^0 \sin\omega_c t\; B_0 \frac{\alpha}{2} r_L \sin\omega_c t + v_\perp^0 \cos\omega_c t\; B_0 \frac{\alpha}{2} r_L \cos\omega_c t\right)$$

which can be written as

$$\frac{dv_x^1}{dt} = \pm\omega_c v_y^1 - \alpha v_\perp^0 v_z^0 \omega_c t\ \cos\omega_c t - \frac{\alpha}{2} v_z^0 v_z^0\ \sin\omega_c t$$
$$\frac{dv_y^1}{dt} = \mp\frac{\alpha}{2} v_\perp^0 v_z^0\ \cos\omega_c t \mp \omega_c v_x^1 \pm \alpha v_\perp^0 v_z^0 \omega_c t\ \sin\omega_c t$$
$$\frac{dv_z^1}{dt} = -\frac{\alpha}{2} v_\perp^{0\,2}$$

upper sign ions, lower sign electrons, with solutions

$$v_x^1 = -\frac{\alpha}{2} v_\perp^0 v_z^0 \omega_c t^2\ \cos\omega_c t - \frac{\alpha}{2} v_\perp^0 v_z^0 t\ \sin\omega_c t$$
$$v_y^1 = \mp\frac{\alpha}{2} v_\perp^0 v_z^0 t\ \cos\omega_c t \pm \frac{\alpha}{2} v_\perp^0 v_z^0 \omega_c t^2\ \sin\omega_c t$$
$$v_z^1 = -\frac{\alpha}{2} v_\perp^{0\,2} t.$$

Adiabatic invariant

$\dfrac{\frac{1}{2} m v_\perp^2}{B}$ is a constant of the motion or *adiabatic invariant*. *Adiabatic* carries the idea of slowly-changing.

You do. Show this is true to first order in α. Start with $v_\perp^2 = \left(v_x^0 + v_x^1\right)^2 + \left(v_y^0 + v_y^1\right)^2$ and substitute using the solutions above. (Note. Chen p 31 gives an alternative derivation.)

You do. Use the definition of r_L and this result to show that the magnetic flux encircled by orbit $\Phi_M = BA$ is constant.

It follows that the *magnetic moment* of the gyrating charge is constant. The magnetic moment is defined as $\mu = iA$ where $i = \dfrac{q}{t}$, where q is the charge and t is the time for one gyration and A is the area encircled by the orbit.

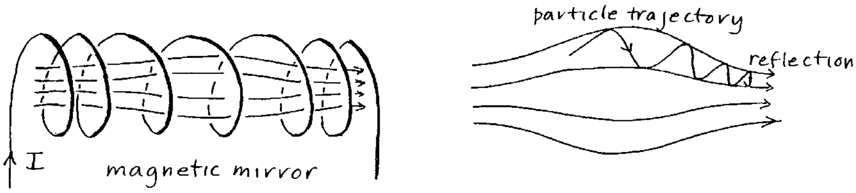
$$\mu = \frac{e}{\frac{2\pi}{\omega_c}} \pi r_L^2 = \frac{\frac{1}{2} m v_\perp^2}{B}.$$

So μ is constant.

Magnetic mirror

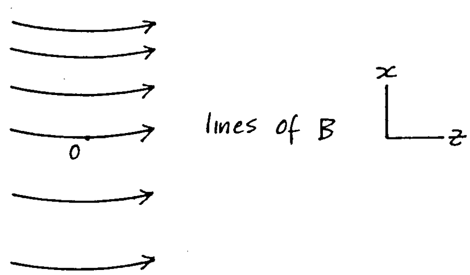
As an electron spirals into a higher B region, $v_\perp$ increases and r_L decreases. Since the total energy $\frac{1}{2} m v^2$ is a constant, v_z must decrease. Eventually $v_z = 0$ and the electron reverses direction. It has been reflected by a *magnetic mirror*.

e.g., magnetic mirror used to trap plasma in an experimental device.



To do a magnetic mirror calculation use $\dfrac{\frac{1}{2}mv_\perp^2}{B}$ = constant and conservation of energy $\frac{1}{2}mv_\perp^2 + \frac{1}{2}mv_z^2$ = constant.

Case (b):



$$B_z = B_0(1+\alpha x)$$
$$B_x = B_0 \alpha z$$

You do. Show Maxwell's equations are satisfied.

$B_0 \alpha x$ describes the *gradient.*

$B_0 \alpha z$ describes the *curvature*. In the derivations below, the curvature terms are underlined.

$$\frac{dv_x}{dt} = \frac{q}{m}\left(v_y B_0 + v_y B_0 \alpha x\right)$$
$$\frac{dv_y}{dt} = \frac{q}{m}\left(\underline{v_z B_0 \alpha z} - v_x B_0 - v_x B_0 \alpha x\right)$$
$$\frac{dv_z}{dt} = \underline{\frac{q}{m}\left(-v_y B_0 \alpha z\right)}$$

Proceed as before.

$$\frac{dv_x^1}{dt} = \pm\omega_c v_y^1 - \alpha v_\perp^{0\,2}\cos^2\omega_c t$$

$$\frac{dv_y^1}{dt} = \underline{\pm\,\alpha\omega_c v_z^{0\,2}\,t} \mp \omega_c v_x^1 \pm \alpha v_\perp^{0\,2}\sin\omega_c t\cos\omega_c t$$

$$\frac{dv_z^1}{dt} = \underline{\alpha\omega_c v_\perp^0 v_z^0 t\cos\omega_c t}$$

upper sign ions, lower sign electrons, with solutions

$$v_x^1 = -\frac{\alpha v_\perp^{0\,2}}{2\omega_c}\sin 2\omega_c t + \underline{\alpha v_z^{0\,2}\,t}$$

$$v_y^1 = \mp\frac{\alpha v_\perp^{0\,2}}{2\omega_c}\cos 2\omega_c t \pm \frac{\alpha v_\perp^{0\,2}}{2\omega_c} \underline{\pm \frac{\alpha v_z^{0\,2}}{\omega_c}}$$

$$v_z^1 = \underline{\frac{\alpha v_\perp^0 v_z^0}{\omega_c}\left(\cos\omega_c t - 1 - \omega_c t\sin\omega_c t\right)}$$

Consider v_y^1. There are constant drift terms.

$$\pm\frac{\alpha v_\perp^{0\,2}}{2\omega_c} \text{ due to the gradient}$$

$$\pm\frac{\alpha v_z^{0\,2}}{\omega_c} \text{ due to the curvature.}$$

They combine to give

$$v_d = \frac{\pm\alpha\left(\dfrac{v_\perp^{0\,2}}{2} + v_z^{0\,2}\right)}{\omega_c}.$$

This is the expression we will use.

It is perhaps unfortunate that these drifts are in the same direction. We cannot devise a B such that they cancel.

We can express α in terms of the gradient of the magnetic field or the radius of curvature R_c of the field lines.

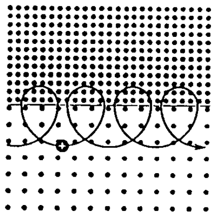
(i) From the equations for B above, $\alpha = \dfrac{\nabla B_z}{B_0}$.

(ii) From the sketch above, $\alpha = \dfrac{1}{R_c}$

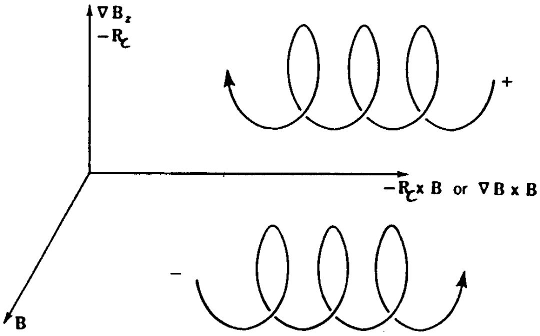
You do. Show this.

gradient drift

It is easy to see why a gradient gives rise to a drift. Consider the path of a charge where the B field is large above the line and small below it. Above, the Larmor radius is small and below, it is large.



We can sketch the drift.

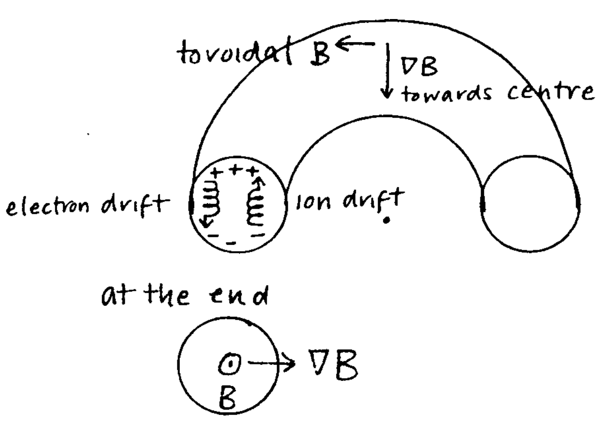


The drift, for positive ions, is in the direction of $-\nabla \mathbf{B} \times \mathbf{B}$ or $\mathbf{R}_c \times \mathbf{B}$.

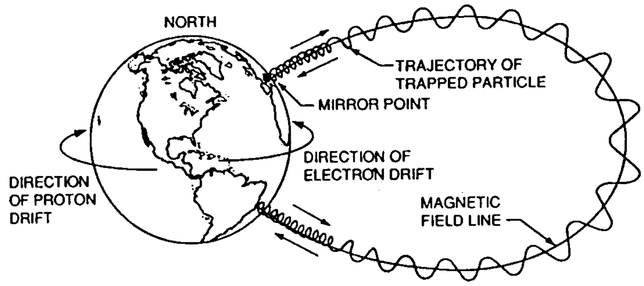
e.g. in a toroidal magnetic field

$$B = \frac{\mu_0 N i}{2\pi r}$$

so $B \approx \frac{1}{r}$ and $\nabla B \approx -\frac{1}{r^2}$. i.e., ∇B increases as you go radially in towards the axis.
In this case $\alpha = \frac{\nabla B}{B} = -\frac{1}{r}$.



e.g., radiation belts in the earth's magnetic field. This illustrates the magnetic mirror as well.



5. *Magnetic field with time variation*

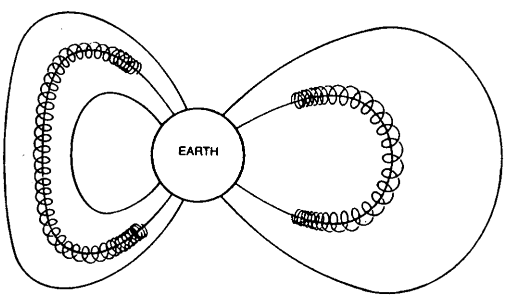
Drift and mirroring equations do not allow the long range prediction of trajectories, particularly if there is no symmetry. It is nice to have constants of the motion or invariants.

Again it can be shown that even when the magnetic field varies in time, the magnetic moment μ is constant. This is the *first adiabatic invariant*.

e.g. *Adiabatic compression* as a method of heating a plasma

Suppose a plasma is trapped by a magnetic field. If the magnetic field is increased then $v_{\perp}$ increases. Collisions will distribute this extra energy. The plasma is heated.

There are two other invariants. They are illustrated by the following example.



(1) μ = constant

(2*) Longitudinal (or second) adiabatic invariant*

$$J = \oint_{\substack{\text{over a path}\\ \text{back and forward}\\ \text{between mirrors}}} \mathbf{v} \cdot \mathbf{dl} = \text{constant}$$

So if the location of the mirrors changes slowly with time, due to the solar wind, this remains constant.

(3) *Third adiabatic invariant*

The guiding centre may precess going from one field line to another. But the field lines all lie on a flux surface - a barrel-shaped surface such that the enclosed flux is constant.