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## Lecture 7

### *More on Energy Eigenstates*

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#### Assigned Reading:

E&R	$3_{all}, 5_{1,3,4,6}$
Li.	$2_{5-8}, 3_{1-3}$
Ga.	$2_{all \neq 4}$
Sh.	$3, 4$

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Suppose someone hands you a potential and asks, "What do the energy eigenfunctions look like?" As we will see throughout 8.04, there is a lot of physics in the form of these eigenfunctions!

We know that

$$\hat{E}\phi(x; E) = E \cdot \phi(x; E) \quad (0.1)$$

where

$$\hat{E} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \quad (0.2)$$

Knowing that

$$\hat{p}\phi(x) = -i\hbar \frac{\partial \phi}{\partial x}$$

and

$$\hat{x}\phi(x) = x\phi(x),$$

then the energy eigenvalue equation becomes

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \phi(x; E)}{\partial x^2} + V(x)\phi(x; E) = E\phi(x; E). \quad (0.3)$$

Using the simplification

$$k^2(x) = \frac{2m}{\hbar^2} (E - V(x)) \quad (0.4)$$

yields

$$\frac{\partial^2 \phi(x; E)}{\partial x^2} + k^2(x)\phi(x; E) = 0 \quad (0.5)$$

for the energy eigenvalue equation. Denoting  $f''(x) = \frac{\partial^2 f(x)}{\partial x^2}$  yields

$$\frac{\phi''(x; E)}{\phi(x; E)} = -k^2(x) \quad (0.6)$$

and this form will help us shortly.

If  $E > V(x)$ , then  $k^2(x) > 0$ , and

$$\frac{\phi''(x; E)}{\phi(x; E)} < 0$$

which implies *oscillatory* behavior of the eigenfunction in that region. If  $E < V(x)$ , then  $k^2(x) < 0$ , and

$$\frac{\phi''(x; E)}{\phi(x; E)} > 0$$

which implies *exponential* behavior of the eigenfunction in that region.

To reiterate, in classically allowed regions, the energy eigenfunctions are oscillatory, while in classically forbidden regions, the energy eigenfunctions are exponential.

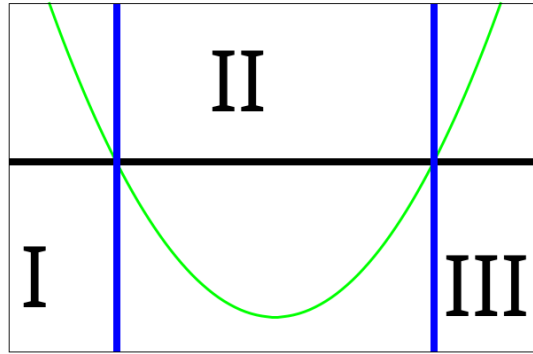


Figure 1: For the energy denoted by the black line for the potential denoted by the green curve, regions (I) and (III) are classically forbidden, so the energy eigenfunction exhibits exponential behavior, while region (II) is classically allowed, so the energy eigenfunction exhibits oscillatory behavior

Can we get more information about what these energy eigenfunctions look like?

1. They need to be normalizable. This means that

$$\lim_{x \rightarrow \pm\infty} \phi(x; E) = 0. \quad (0.7)$$

2. There needs to be continuity in both the energy eigenfunction and its first spatial derivative. The second spatial derivative of the energy eigenfunction depends through the energy eigenvalue equation on the potential, which may or may not be continuous depending on the situation.
3. In classically allowed regions, the energy eigenfunction oscillates in space.
4. In classically forbidden regions, the energy eigenfunction exhibits spatially exponential behavior. It *is not identically zero!* This also implies that the probability density in a

classically forbidden region *is not identically zero!* This is a feature that sets quantum mechanics apart from classical mechanics!

5. The energy eigenfunctions can always be taken as real functions even though general wavefunctions evolving in time are complex, and this can easily be shown.

Let us return to the infinite square well. What are the allowed energy eigenvalues? We can use the “shooting” method to answer this. We start at  $x = 0$ , pick a trial  $E$  with  $p = \sqrt{2mE}$ , and integrate the energy eigenvalue equation  $\phi''(x; E) + k^2(x) \phi(x; E) = 0$ . Note that as the energy eigenvalue equation involves the second spatial derivative of the energy eigenfunction, the energy eigenvalue controls its curvature!

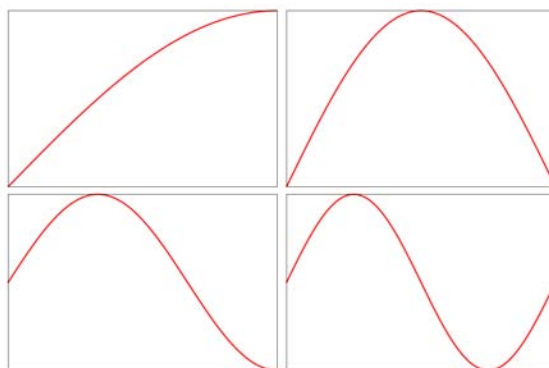


Figure 2: For  $E < E_0$  (top-left), the wavefunction overshoots to the right, and for  $E_0 < E < E_1$  (bottom-left), the wavefunction undershoots to the right, while for  $E = E_0$  (top-right) and  $E = E_1$  (bottom-right), the wavefunction reaches the boundary properly

Our revisiting the infinite square well leads us to two facts. One is that *bound* states have *discrete* energies, while *unbound* states have *continuous* energies. From this comes the *node theorem* which works for quantum mechanics in one spatial dimension. It states that as bound states have discrete energies, the  $n$ th spatial energy eigenfunction (starting from  $n = 0$ ) has  $n$  nodes. For example, the ground state has no nodes, the first excited state has one node, et cetera.

(picture of weird well potential and eigenfunctions?)

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## 8.04 Quantum Physics I

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