Lecture 4

Expectations, Momentum, and Uncertainty

Assigned Reading:

E&R	3_{all}, $5_{1,3,4,6}$
Li.	2_{5-8}, 3_{1-3}
Ga.	$2_{all\neq 4}$
Sh.	3, 4

Our job now is to properly define the uncertainties Δx and Δp.

As an aside, let us review the properties of discrete probability distributions.

a	N
14	1
15	1
16	3
20	2
21	4
22	5

Consider the number distribution N of ages a in a population. The probability of finding a person with a given age is $\mathbb{P}(a) = \frac{N(a)}{N_{\text{total}}}$, satisfying $\sum_a \mathbb{P}(a) = 1$.

What is the most likely age? In this case, that is 22.

What is the average age? In general, the weighted average

$$\langle a \rangle = \frac{\sum_a aN(a)}{N_{\text{total}}} = \sum_a a\mathbb{P}(a).$$

In this case, it is 19.4. Note that in general, as in this example, $\langle a \rangle$ does not have to be a measurable value of a!

What is the average of the squared age? In general,

$$\langle a^2 \rangle = \sum_a a^2 \mathbb{P}(a).$$

For a general function of the age,

$$\langle f(a) \rangle = \sum_a f(a)\mathbb{P}(a).$$

Is the average of the squared age equal to the square of the average age? In mathematical notation, is $\langle a^2 \rangle = \langle a \rangle^2$? No! If a represented a more general quantity rather than age, it could sometimes be positive or negative, and those terms might cancel out in the average. By contrast, a^2 would never be negative, so its average would satisfy that too.

How do we characterize the uncertainty? We could use $\Delta a = a - \langle a \rangle$, but the problem is that $\langle \Delta a \rangle = 0$ identically. Instead, we use the standard deviation defined by

$$(\Delta a)^2 = \langle (a - \langle a \rangle)^2 \rangle,$$

which also satisfies

$$(\Delta a)^2 = \langle a^2 \rangle - \langle a \rangle^2.$$

In this case, the standard deviation is about 2.8.

Similar expressions exist for continuous variables. Given that ψ has been discussed as a function of position x thus far, it makes sense to proceed in that way. Mathematically,

$$\langle f(x) \rangle = \int_{-\infty}^{\infty} f(x)\mathfrak{p}(x)\, dx \tag{0.1}$$

but $\mathfrak{p}(x) = \psi^\star(x)\psi(x)$. Hence, the way to find the expectation value of a function of position in a given quantum state is

$$\langle f(x) \rangle = \int_{-\infty}^{\infty} \psi^\star(x) f(x) \psi(x)\, dx. \tag{0.2}$$

In all this, the normalization $\int_{-\infty}^{\infty} \mathfrak{p}(x)\, dx = 1$ is assumed. From this, the uncertainty in position

$$\Delta x \equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \tag{0.3}$$

can be found.
Notice that expectation values $\langle f(x) \rangle$ depend on the state! This can be written as $\langle f(x) \rangle_\psi$, $\langle f(x) \rangle_{|\psi\rangle}$, or $\langle \psi | f(x) | \psi \rangle$.

For example, let us consider a wavefunction given by

$$\psi(x) = \{N \cdot (x^2 - l^2)^2 \text{ for } |x| \leq l, 0 \text{ otherwise}\}. \tag{0.4}$$

We need to figure out the normalization for this wavefunction by

$$\int_{-\infty}^{\infty} |\psi(x)|^2\, dx = 1 \tag{0.5}$$

which, when effected by nondimensionalization of the integral, yields $N = \sqrt{\frac{315}{256}} \frac{e^{i\varphi}}{\sqrt{l}}$.

After this, by noting that $|\psi(x)|^2$ is even while x is odd, then $\langle x \rangle = 0$. Also, $\langle x^2 \rangle = \frac{l^2}{11}$. Hence, $\Delta x = \frac{l}{\sqrt{11}}$.

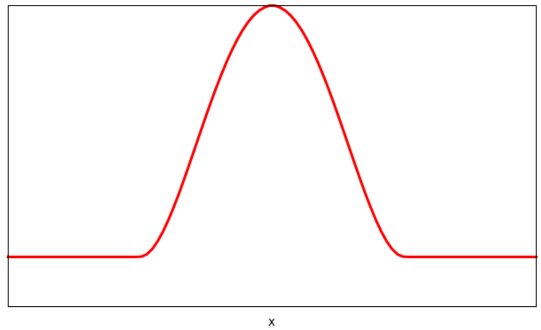

Figure 1: Plot of $\psi(x)$ in this case

After all of this, how do we find the momentum expectation value $\langle p \rangle$? Naïvely, we might say that $\langle p \rangle = \int_{-\infty}^{\infty} \psi^\star(x) p \psi(x)\, dx$. But how exactly are we to express p in an integral over functions of x? Clearly, this will not do!

Here's a hint: we know that a wave with

$$k = 2\pi\lambda^{-1}$$

is associated with a particle with

$$p = h\lambda^{-1} = \hbar k.$$

Disregarding normalization, the associated wavefunction is

$$\psi = e^{ikx}.$$

But note that

$$\frac{\partial e^{ikx}}{\partial x} = ike^{ikx}.$$

This means that

$$-i\hbar\frac{\partial e^{ikx}}{\partial x} = \hbar k e^{ikx}.$$

Thus

$$-i\hbar\frac{\partial e^{ikx}}{\partial x} = p \cdot e^{ikx},$$

and the units work out too! But what does momentum have to do with a derivative with respect to position anyway?

Here's another hint: Noether's theorem states that to every symmetry is associated a conserved quantity.

Symmetry	*Conservation*
$\mathbf{x} \to \mathbf{x} + \Delta\mathbf{x}$	$\mathbf{p}$
$t \to t + \Delta t$	E
$\mathbf{x} \to \overleftrightarrow{\mathbf{R}} \cdot \mathbf{x}$	$\mathbf{L}$

So momentum is associated with spatial translations!

Now consider how translations behave for functions:

$$f(x) \to f(x+l) = f(x) + \frac{l\partial f(x)}{\partial x} + \frac{l^2\partial^2 f(x)}{2\partial x^2} + \dots \quad (0.6)$$

$$= \sum_{n=0}^{\infty} (\frac{l\partial}{\partial x})^n f(x) \quad (0.7)$$

$$= e^{\frac{l\partial}{\partial x}} f(x). \quad (0.8)$$

Hence translations are *generated* by spatial derivatives $\frac{\partial}{\partial x}$. But we just said that translations are associated with p! This means that it is natural to associate p with $\frac{\partial}{\partial x}$ somehow. In a similar way, E would be associated with $\frac{\partial}{\partial t}$, and L_z with $\frac{\partial}{\partial \varphi}$.

That's enough for hints. We need to take a stand on this.

Momentum in quantum mechanics is realized by an *operator*

$$\boxed{\hat{p} = -i\hbar\frac{\partial}{\partial x}}. \quad (0.9)$$

This operator $\hat{p}$ is what we use to compute expectation values. More precisely,

$$\langle p^n \rangle = (-i\hbar)^n \int_{-\infty}^{\infty} \psi^\star(x)\frac{\partial^n \psi(x)}{\partial x^n}\, dx \quad (0.10)$$

and the uncertainty is then given by $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$.

Let us return to our previous example wavefunction given by

$$\psi(x) = \{N \cdot (x^2 - l^2)^2 \text{ for } |x| \leq l, 0 \text{ otherwise}\}. \quad (0.11)$$

Now we can find

$$\langle p \rangle = -i\hbar \int_{-\infty}^{\infty} \psi^\star(x)\frac{\partial \psi(x)}{\partial x}\, dx \quad (0.12)$$

$$= -i\hbar|N|^2 \int_{-\infty}^{\infty} (x^2 - l^2)^2 \cdot (2 \cdot 2x \cdot (x^2 - l^2))\, dx \quad (0.13)$$

$$= 0 \quad (0.14)$$

as the wavefunction is even while its spatial derivative is odd.
By a similar computation, $\langle p^2 \rangle = \frac{3\hbar^2}{l^2}$, which dimensionally makes sense as well. From this, we find that $\Delta p = \frac{\sqrt{3}\hbar}{l}$, and the uncertainty relation is satisfied as $\Delta x \Delta p = \sqrt{\frac{3}{11}}\hbar$.

But what does this new operator $\hat{p}$ have to do with having momentum $p = \hbar k$? Let us consider two states given by

$$\psi_k(x) = e^{ikx}$$

and

$$\psi_s(x) = e^{ikx} + e^{ik'x}.$$

The first has definite momentum $p = \hbar k$, while the second, being a superposition of states with definite momenta $p = \hbar k$ and $p' = \hbar k'$, is not itself a state of definite momentum. We can show this by acting on each state with the operator $\hat{p}$:

$$\hat{p}\psi_k(x) = \hbar k e^{ikx}$$

is simply proportional to $\psi_k(x)$, while

$$\hat{p}\psi_s(x) = \hbar \cdot (k e^{ikx} + k' e^{ik'x})$$

is not simply proportional to $\psi_s(x)$. We see that $\hat{p}$ is an operator which acts simply on wavefunctions corresponding to states with definite momenta, but not on arbitrary superpositions of momentum states. **This means that $\hat{p}$ is the operator whose eigenstates are states of definite momentum, and the corresponding eigenvalue is exactly the momentum of that state.**