Chapter 3

Topology of the Real Line

In this chapter, we study the features of $\mathbb{R}$ which allow the notions of limits and continuity to be defined precisely. This is what is meant by topology. Though it is done here for the real line, similar notions also apply to more general spaces, called topological spaces. This branch of mathematics, in its modern form, dates back to the mid 1800s and the work of Bolzano (1781-1848), Cantor (1845-1918) and Weierstrass (1815, 1897).

3.1 Interior Points and Neighborhoods

Throughout this chapter, it will be useful to think of intervals (a, b) or $[a, b]$. Though the various notions studied (open, closed to name a few) apply to more general sets, understanding them in terms of intervals will be helpful.

3.1.1 Main Definitions

The notion of interior point tries to capture the idea of how "deep" within a set a point is. Intuitively, the interior of a set is the part of the set which is not on "the edge of the set". Of course these concepts need to be made much more precise mathematically.

Definition 228 (Interior Point) *Let $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.*

1. *x is said to be an **interior point** of S if there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq S$.*

2. *If the above is true, S is said to be a **neighborhood** of x.*

3. *Given a set $S \subseteq \mathbb{R}$, the **interior of** S, denoted $Int\,(S)$ or S°, is the set of interior points of S.*

Remark 229 *We will remark the following:*

1. *Intuitively speaking, a neighborhood of a point is a set containing the point, in which you can move the point a little without leaving the set.*

2. *A neighborhood of a point $x \in \mathbb{R}$ is any set which contains an interval of the form $(x-\delta, x+\delta)$ for some $\delta > 0$. This definition is a little bit more general that the definition some texts use.*

3. *$(x-\delta, x+\delta)$ is a neighborhood of x $\forall \delta > 0$. In fact, certain texts use this as a definition of a neighborhood. They say that a neighborhood of a point x is a set of the form $(x-\delta, x+\delta)$ for some $\delta > 0$. This means that for them, a neighborhood of a point x has x at its center. Our definition is a little bit more general.*

4. *It should be clear from the definition that an interior point of a set belongs to the set. We will state it as a result in the proposition below.*

Example 230 *Are* 1*,* 1.9*,* 2*,* 2.1 *interior points of* $(0, 2)$*?*

Example 231 *Same question for* $[0, 2]$.

Example 232 *Conjecture what* $Int\left((0,2)\right)$ *and* $Int\left([0,2]\right)$ *might be.*

Proposition 233 *The following results follow immediately from the definition.*

1. *If U is a neighborhood of x and $U \subseteq V$, then V is also a neighborhood of x.*

2. *$Int\left(S\right) \subseteq S$.*

   Proof. *We prove each part separately. Though these proofs are easy and could have been left to the reader, we will do them to illustrate how one works with these new concepts.*

   Part 1: *If U is a neighborhood of x then there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq U$. Since $U \subseteq V$, we also have $(x-\delta, x+\delta) \subseteq V$. Thus, V is also a neighborhood of x.*

   Part 2: *If $x \in Int\left(S\right)$ then there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq S$ and since $x \in (x-\delta, x+\delta)$, it follows that $x \in S$.*

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Example 234 *Every point of (a, b) with $a < b$ is an interior point of (a, b). Also, any point outside of (a, b) is not an interior point. Therefore, $Int\left((a,b)\right) = (a, b)$. This is easy to see. If $x \in (a, b)$, we let $\delta = \dfrac{\min\left\{|x-a|, |x-b|\right\}}{2}$. Then, $(x-\delta, x+\delta) \subseteq (a, b)$.*

Example 235 *a and b are not interior points of $[a, b]$ with $a \leq b$. Every other point of that interval is an interior point. Therefore, $Int\left([a,b]\right) = (a, b)$.*

Example 236 *Since every interval of positive length contains both a rational and an irrational, it follows that* $\mathbb{Q}$ *as well as* $\mathbb{R} \setminus \mathbb{Q}$ *have no interior points. If* x *were an interior point of* $\mathbb{Q}$ *then there would exist* $\delta > 0$ *such that* $(x - \delta, x + \delta) \subseteq \mathbb{Q}$*. But we know this can't happen since there is at least one irrational number between* $x - \delta$ *and* $x + \delta$*. The same argument works for* $\mathbb{R} \setminus \mathbb{Q}$*. Therefore,* $Int(\mathbb{Q}) = \varnothing$ *and* $Int(\mathbb{R} \setminus \mathbb{Q}) = \varnothing$*.*

Example 237 $Int(\mathbb{R}) = \mathbb{R}$*. To see this, we need to prove that every real number is an interior point of* $\mathbb{R}$ *that is we need to show that for every* $x \in \mathbb{R}$*, there is* $\delta > 0$ *such that* $(x - \delta, x + \delta) \subseteq \mathbb{R}$*. Let* $x \in \mathbb{R}$*. Then,* $(x - 1, x + 1) \subseteq \mathbb{R}$ *thus* x *is an interior point of* $\mathbb{R}$*.*

3.1.2 Properties

Theorem 238 *Let* $x \in \mathbb{R}$*, let* U_i *denote a family of neighborhoods of* x*.*

1. $\displaystyle\bigcap_{i=1}^{n} U_i$ *is a neighborhood of* x*.*
2. $\displaystyle\bigcap_{i=1}^{\infty} U_i$ *is not always a neighborhood of* x*.*
3. $\displaystyle\bigcup_{i=1}^{\infty} U_i$ *is a neighborhood of* x*.*

Proof. *We prove each part separately.*

1. *We need to prove that there exists* $\delta > 0$ *such that* $(x - \delta, x + \delta) \subseteq \displaystyle\bigcap_{i=1}^{n} U_i$*. Since each* U_i *is a neighborhood of* x*, for each* i *there exists* δ_i *such that* $(x - \delta_i, x + \delta_i) \subseteq U_i$*. Let* $\delta = \min\{\delta_1, \delta_2, ..., \delta_n\}$*. Then,* $(x - \delta, x + \delta) \subseteq U_i$ *for each* $i = 1, 2..., n$*. It follows that* $(x - \delta, x + \delta) \subseteq \displaystyle\bigcap_{i=1}^{n} U_i$ *thus proving that* $\displaystyle\bigcap_{i=1}^{n} U_i$ *is a neighborhood of* x*.*
2. *It is enough to find a counterexample. Let* $U_i = \left(\dfrac{-1}{i}, \dfrac{1}{i}\right)$*. Clearly, for each* i*,* U_i *is a neighborhood of* 0*. Yet,* $\displaystyle\bigcap_{i=1}^{\infty} U_i = \{0\}$ *which is not a neighborhood of* 0*. It remains to prove that* $\displaystyle\bigcap_{i=1}^{\infty} U_i = \{0\}$*(see problems).*
3. *See problems.*

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3.1.3 Techniques to Remember

- To prove x is an interior point of S, one has to find a δ such that $(x-\delta, x+\delta) \subseteq S$.
- Note the above also proves S is a neighborhood of x.
- To prove x is not an interior point of S, one needs to show that no δ works in other words $\forall \delta > 0$, $(x-\delta, x+\delta)$ contains points not in S.

3.1.4 Important Facts to Know and Remember

Some of the facts listed below were discussed in the notes, others are addressed in the exercises.

1. Definitions and theorems in this section.
2. Let $S = \{a\}$. What is $Int\,(S)$?
3. Let S be a finite set. What is $Int\,(S)$?
4. Let S be an non-empty bounded set. Is $\sup S \in Int\,(S)$?
5. Let a, b and c be three real numbers such that $a < b < c$. Let $S = (a, b) \cup \{c\}$. What is $Int\,(S)$? Same question for $S = [a, b] \cup \{c\}$.
6. What is the interior of known sets such as $\varnothing$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, (a, b), $[a, b]$?
7. If A and B are two sets, $int\,(A \cap B) = int\,(A) \cap int\,(B)$, but in general $int\,(A \cup B) \neq int\,(A) \cup int\,(B)$.
8. S is open $\iff Int\,(S) = S$.

3.1.5 Exercises

1. Using the definition of a neighborhood, complete the sentence: "A set S fails to be a neighborhood of x if ...".
2. Prove $\displaystyle\bigcap_{i=1}^{\infty} U_i = \{0\}$ where $U_i = \left(\dfrac{-1}{i}, \dfrac{1}{i}\right)$.
3. Prove part 3 of theorem 238.
4. Suppose that x is an interior point of U and $U \subseteq V$. Prove that x is also an interior point of V.
5. Let $x \in \mathbb{R}$ and $S \subseteq \mathbb{R}$. Prove that the two conditions below are equivalent:
   (a) S is a neighborhood of x.
   (b) There exists two numbers a and b such that $x \in (a, b) \subseteq S$.

6. Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Assume further that $x \neq y$. Prove that one can find a neighborhood S of x and a neighborhood T of y such that $S \cap T = \varnothing$.

7. Given that $S \subseteq \mathbb{R}$ and x is an upper bound of S, explain why S cannot be a neighborhood of x.

8. Given that $S \subseteq \mathbb{R}$, $S \neq \varnothing$, and S is bounded above, explain why neither S nor $\mathbb{R} \setminus S$ can be a neighborhood of $\sup S$.

9. Suppose that A and B are subsets of $\mathbb{R}$ and that x is an interior point of $A \cup B$. Is it true that x must be an interior point of A or B? Explain and justify your answer.

10. Suppose that A and B are subsets of $\mathbb{R}$ and that x is an interior point of both A and B. Is it true that x must be an interior point of $A \cap B$? Explain and justify your answer.

11. Let A and B be subsets of $\mathbb{R}$.

    (a) Prove that if $A \subseteq B$ then $Int\,(A) \subseteq Int\,(B)$.

    (b) Prove that $Int\,(A \cap B) = Int\,(A) \cap Int\,(B)$.

    (c) Is $Int\,(A \cup B) = Int\,(A) \cup Int\,(B)$?

12. Let $a \in \mathbb{R}$. What is $Int\,(\{a\})$?

13. Let S be a finite set. What is $Int\,(S)$?

14. What are $int\,(\varnothing)$, $int\,(\mathbb{N})$, $int\,(\mathbb{Z})$?

15. Let a, b and c be three real numbers such that $a < b < c$. Let $S = (a, b) \cup \{c\}$. What is $Int\,(S)$?

16. Let S be an non-empty bounded set. Is $\sup S \in Int\,(S)$?

17. Let $S \subseteq \mathbb{R}$. Prove that $int\,(int\,(S)) = int\,(S)$.