A

Linear Algebra: Vectors

TABLE OF CONTENTS

§A.1. Motivation

Matrix notation was invented[1] primarily to express linear algebra relations in *compact form*. Compactness enhances visualization and understanding of essentials. To illustrate this point, consider the following set of m linear relations between one set of n quantities, x_1, x_2, ... x_n, and another set of m quantities, y_1, y_2, ..., y_m:

$$
\begin{array}{ccccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1j}x_j & + & \cdots & + & a_{1n}x_n & = & y_1 \\
a_{21}x_2 & + & a_{22}x_2 & + & \cdots & + & a_{2j}x_j & + & \cdots & + & a_{2n}x_n & = & y_2 \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
a_{i1}x_1 & + & a_{i2}x_2 & + & \cdots & + & a_{ij}x_j & + & \cdots & + & a_{in}x_n & = & y_i \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mj}x_j & + & \cdots & + & a_{mn}x_n & = & y_m
\end{array}. \tag{A.1}
$$

The subscripted a, x and y quantities that appear in this set of relations may be formally arranged as follows:

$$
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_m \end{bmatrix}. \tag{A.2}
$$

Two kinds of mathematical objects can be distinguished in (A.2). The two-dimensional array expression enclosed in brackets is a *matrix*, which we call $\mathbf{A}$. Matrices are defined and studied in Appendix B.

The one-dimensional array expressions in brackets are *column vectors* or simply *vectors*, which we call $\mathbf{x}$ and $\mathbf{y}$, respectively. The use of **boldface** uppercase and lowercase letters to denote matrices and vectors, respectively, follows conventional matrix-notation rules in engineering applications. See §A.2.1 for conevntions used in this book.

Replacing the expressions in (A.2) by these symbols we obtain the *matrix form* of (A.1):

$$
\mathbf{A}\,\mathbf{x} = \mathbf{y}. \tag{A.3}
$$

Putting $\mathbf{A}$ next to $\mathbf{x}$ means "matrix product of $\mathbf{A}$ times $\mathbf{x}$", which is a generalization of the ordinary scalar multiplication, and follows the rules explained later.

Clearly (A.3) is a more compact, "short hand" form of (A.1).

Another key practical advantage of the matrix notation is that it translates directly to the computer implementation of linear algebra processes in languages that offer array data structures.

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[1] Largely by Arthur Cayley (1821–1895) at Cambridge (UK) in 1858 [310] with substantial contributions from his colleague James Joseph Sylvester (1814–1897). See Appendix H for additional historical details related to the developmnent of FEM. Curiously, vectors for mathematical physics (that is, living in 3D space) were independently created by J. Willard Gibbs (1839–1903) and Oliver Heaviside (1850–1925) in the late XIX century [172]. The overlap between matrices and vectors in n dimensions was not established until the XX century.

§A.2. Vectors

We begin by defining a *vector*, a set of n numbers which we shall write in the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \tag{A.4}$$

This object is called a *column vector*. We shall see later that although a vector may be viewed as a special case of a matrix, it deserves treatment on its own. The symbol $\mathbf{x}$ is the name of the vector.

If n numbers are arranged in a horizontal array, as in

$$\mathbf{z} = [\, z_1 \quad z_2 \quad \ldots \quad z_n \,], \tag{A.5}$$

then $\mathbf{z}$ is called a *row vector*. If the term "vector" is used without a qualifier, it is understood to be a column vector such as (A.4).

§A.2.1. Notational Conventions

Typeset vectors will be designated by **bold** lowercase letters. For example:

$$\mathbf{a}, \quad \mathbf{b}, \quad \mathbf{c}, \quad \mathbf{x}, \quad \mathbf{y}, \quad \mathbf{z}. \tag{A.6}$$

On the other hand, *handwritten or typewritten* vectors, which are those written on paper or on the blackboard, are identified by putting a wiggle or bar underneath the letter. For example:

$$\underset{\sim}{a}. \tag{A.7}$$

Subscripted quantities such as x_1 in (A.4) are called the *entries* or *components*[2] of $\mathbf{x}$, while n is called the *order* of the vector $\mathbf{x}$. Vectors of order one ($n = 1$) are called *scalars*. These are the usual quantities of analysis.

For compactness one sometimes abbreviates the phrase "vector of order n" to just "n-vector." For example, $\mathbf{z}$ in equation (A.8) below is a 4-vector.

If the components are real numbers, the vector is called a *real vector*. If the components are complex numbers we have a *complex vector*. Linear algebra embraces complex vectors as easily as it does real ones; however, we rarely need complex vectors for most of our exposition. Consequently real vectors will be assumed unless otherwise noted.

[2] The term *component* is primarily used in mathematical treatments whereas *entry* is used more in conjunction with the computer implementation. The term *element* is also used in the literature but this will be avoided here (as well as in companion books) as it may lead to confusion with finite elements.

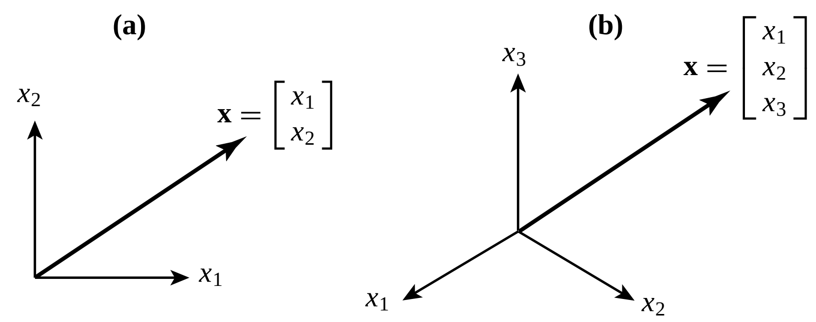

FIGURE A.1. Standard visualization of 2-vectors and 3-vectors as position vectors in 2-space and 3-space, respectively.

Example A.1.

$$\mathbf{x} = \begin{bmatrix} 4 \\ -2 \\ 3 \\ 0 \end{bmatrix} \tag{A.8}$$

is real column vector of order 4, or, briefly, a 4-vector.

Example A.2.

$$\mathbf{q} = [\,1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\,] \tag{A.9}$$

is a real row vector of order 6.

Occasionally we shall use the short-hand "component notation"

$$\mathbf{x} = [x_i], \tag{A.10}$$

for a generic vector. This comes handy when it is desirable to show the notational scheme obeyed by components.

§A.2.2. Visualization

To help visualization, a two-dimensional vector $\mathbf{x}$ ($n = 2$) can be depicted as a line segment, or arrow, directed from the chosen origin to a point on the Cartesian plane of the paper with coordinates (x_1, x_2). See Figure A.1(a). In mechanics this is called a *position vector* in 2-space.

One may resort to a similar geometrical interpretation in three-dimensional Cartesian space ($n = 3$). See Figure A.1(b). Drawing becomes a bit messier, however, although some knowledge of perspective and projective geometry helps.

The interpretation extends to Euclidean spaces of dimensions $n > 3$ but direct visualization is of course impaired.

§A.2.3. Special Vectors

The *null* vector, written $\mathbf{0}$, is the vector all of whose components are zero.

The *unit* vector, denoted by $\mathbf{e}_i$, is the vector all of whose components are zero, except the i^{th} component, which is one. After introducing matrices (Appendix B) a unit vector may be defined as the i^{th} column of the identity matrix.

The *unitary* vector, called $\mathbf{e}$, is the vector all of whose components are unity.

§A.3. Vector Operations

Operations on vectors in two-dimensional and three-dimensional space are extensively studied in courses on Mathematical Physics. Here we summarize operations on n-component vectors that are most useful from the standpoint of the development of finite elements.

§A.3.1. Transposition

The *transpose* of a column vector $\mathbf{x}$ is the row vector that has the same components, and is denoted by $\mathbf{x}^T$:

$$\mathbf{x}^T = [\, x_1 \quad x_2 \quad \ldots \quad x_n \,]. \tag{A.11}$$

Similarly, the transpose of a row vector is the column vector that has the same components. Transposing a vector twice yields the original vector: $(\mathbf{x}^T)^T = \mathbf{x}$.

Example A.3. The transpose of

$$\mathbf{a} = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} \qquad \text{is} \qquad \mathbf{b} = \mathbf{a}^T = [\, 3 \quad -1 \quad 6 \,]$$

and transposing $\mathbf{b}$ gives back $\mathbf{a}$.

§A.3.2. Equality

Two column vectors $\mathbf{x}$ and $\mathbf{y}$ of equal order n are said to be *equal* if and only if their components are equal, $x_i = y_i$, for all $i = 1, \ldots n$. We then write $\mathbf{x} = \mathbf{y}$. Similarly for row vectors.

Two vectors of different order cannot be compared for equality or inequality. A row vector cannot be directly compared to a column vector of the same order (unless their dimension is 1); one of the two has to be transposed before a comparison can be made.

§A.3.3. Addition and Subtraction

The simplest operation acting on two vectors is *addition*. The sum of two vectors of same order n, $\mathbf{x}$ and $\mathbf{y}$, is written $\mathbf{x} + \mathbf{y}$ and defined to be the vector of order n

$$\mathbf{x} + \mathbf{y} \stackrel{\text{def}}{=} \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}. \tag{A.12}$$

If $\mathbf{x}$ and $\mathbf{y}$ are not of the same order, the addition operation is undefined.

The operation is commutative: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$, and associative: $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$.

Strictly speaking, the plus sign connecting $\mathbf{x}$ and $\mathbf{y}$ is not the same as the sign connecting x_i and y_i. However, since it enjoys the same analytical properties, there is no harm in using the same symbol in both cases. The geometric interpretation of the vector addition operator for two- and three-dimensional vectors ($n = 2, 3$) is the well known paralellogram law. See Figure A.2. For $n = 1$ the usual scalar addition results.

For vector subtraction, replace $+$ by $-$ in the foregoing expressions.

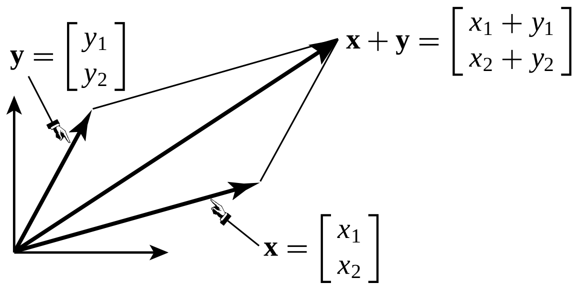

FIGURE A.2. In two and three-dimensional space, the vector addition operation is equivalent to the well known parallelogram composition law.

Example A.4. The sum of

$$\mathbf{a} = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} \qquad \text{is} \quad \mathbf{a}+\mathbf{b} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$$

§A.3.4. Multiplication and Division by Scalar

Multiplication of a vector $\mathbf{x}$ by a scalar c is defined by means of the relation

$$c\,\mathbf{x} \stackrel{\text{def}}{=} \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix} \tag{A.13}$$

This operation is often called *scaling* of a vector. The geometrical interpretation of scaling is: the scaled vector points the same way, but its magnitude is multiplied by c.

If $c = 0$, the result is the null vector. If $c < 0$ the direction of the vector is reversed. In paticular, if $c = -1$ the resulting operation $(-1)\mathbf{x} = -\mathbf{x}$ is called *reflexion about the origin* or simply *reflexion*.

Division of a vector by a scalar $c \neq 0$ is equivalent to multiplication by $1/c$. The operation is written

$$\frac{\mathbf{x}}{c} \equiv \mathbf{x}/c \stackrel{\text{def}}{=} (1/c)\mathbf{x} \tag{A.14}$$

The operation is not defined if $c = 0$.

§A.3.5. Span

Sometimes it is not just $\mathbf{x}$ which is of interest but the “line” determined by $\mathbf{x}$, namely the collection $c\mathbf{x}$ of all scalar multiples of $\mathbf{x}$, including $-\mathbf{x}$. We call this *Span*($\mathbf{x}$). Note that the span always includes the null vector.

The span of two vectors, $\mathbf{x}$ and $\mathbf{y}$ with common origin, is the set of vectors $c_1\,\mathbf{x} + c_2\,\mathbf{y}$ for arbitrary scaling coefficients c_1 and c_2. If the two vectors are not parallel, the geometric visualization of the span is that of the plane defined by the two vectors. Hence the statement: a plane is *spanned* by two noncoincident vectors, responds to this interpretation.

The span of an arbitrary number of vectors of common origin is studied in §A.3.10.

§A.3.6. Inner Product, Norm and Length

The *inner product* of two column vectors $\mathbf{x}$ and $\mathbf{y}$, of same order n, is a scalar function denoted by $(\mathbf{x}, \mathbf{y})$ — as well as two other forms shown below — and is defined by

$$(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} \stackrel{\text{def}}{=} \sum_{i=1}^{n} x_i y_i \stackrel{\text{sc}}{=} x_i y_i. \tag{A.15}$$

This operation is also called *dot product* and *interior product*. If the two vectors are not of the same order, the inner product is undefined. The two other notations shown in (A.15), namely $\mathbf{x}^T \mathbf{y}$ and $\mathbf{y}^T \mathbf{x}$, exhibit the inner product as a special case of the *matrix product* discussed in Appendix B.

The last expression in (A.15) applies the so-called *Einstein's summation convention*, which implies sum on repeated indices (in this case, i) and allows the $\sum$ operand to be dropped.

The inner product is commutative: $(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$ but not generally associative: $(\mathbf{x}, (\mathbf{y}, \mathbf{z})) \neq ((\mathbf{x}, \mathbf{y}), \mathbf{z})$. If $n = 1$ it reduces to the usual scalar product. The scalar product is of course associative.

Example A.5. The inner product of

$$\mathbf{a} = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} \quad \text{is} \quad (\mathbf{a}, \mathbf{b}) = \mathbf{a}^T \mathbf{b} = 3 \times 2 + (-1) \times 1 + 6 \times (-4) = -19.$$

Remark A.1. The inner product is not the only way of "multiplying" two vectors. There are other vector products, such as the cross or outer product, which are important in many applications such as fluid mechanics and nonlinear dynamics. They are not treated here because they are not defined when going from vectors to matrices. Furthermore they are not easily generalized for vectors of order beyond 3.

The *Euclidean norm* or *2-norm* of a real vector $\mathbf{x}$ is a scalar denoted by $\|\mathbf{x}\|$ that results by taking the inner product of the vector with itself:

$$\|\mathbf{x}\| \stackrel{\text{def}}{=} (\mathbf{x}, \mathbf{x}) = \sum_{i=1}^{n} x_i^2 \stackrel{\text{sc}}{=} x_i x_i. \tag{A.16}$$

Because the norm is a sum of squares, it is zero only if $\mathbf{x}$ is the null vector. It thus provides a "meter" (mathematically, a norm) on the vector magnitude.

The *Euclidean length* or simply *length* of a real vector, denoted by $|\mathbf{x}|$, is the positive square root of its Euclidean norm:

$$|\mathbf{x}| \stackrel{\text{def}}{=} +\sqrt{\|\mathbf{x}\|}. \tag{A.17}$$

This definition agrees with the intuitive concept of vector magnitude in two- and three-dimensional space ($n = 2, 3$). For $n = 1$ observe that the length of a scalar x is its absolute value $|x|$, hence the notation $|\mathbf{x}|$.

The Euclidean norm is not the only vector norm used in practice, but it will be sufficient for the present book.

Two important inequalities satisfied by vector norms (and not just the Euclidean norm) are the *Cauchy-Schwarz inequality*:

$$|(\mathbf{x}, \mathbf{y})| \leq |\mathbf{x}|\,|\mathbf{y}|, \tag{A.18}$$

and the *triangle inequality*:

$$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|. \tag{A.19}$$

Example A.6. The Euclidean norm of

$$\mathbf{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \tag{A.20}$$

is $\|x\| = 4^2 + 3^2 = 25$, and its length is $|x| = \sqrt{25} = 5$.

§A.3.7. Unit Vectors and Normalization

A vector of length one is called a *unit vector*. Any non-null vector $\mathbf{x}$ can be scaled to unit length by dividing all components by its original length:

$$\mathbf{x}/|\mathbf{x}| = \begin{bmatrix} x_1/|\mathbf{x}| \\ x_2/|\mathbf{x}| \\ \vdots \\ x_n/|\mathbf{x}| \end{bmatrix} \tag{A.21}$$

This particular scaling is called *normalization to unit length*. If $\mathbf{x}$ is the null vector, the normalization operation is undefined.

Example A.7. The unit-vector normalization of

$$\mathbf{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad \text{is} \qquad \mathbf{x}/|\mathbf{x}| = \mathbf{x}/5 = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix} \tag{A.22}$$

§A.3.8. Angles and Orthonormality

The *angle* in radians between two *unit* real vectors $\mathbf{x}$ and $\mathbf{y}$, written $\angle(\mathbf{x}, \mathbf{y})$, is the real number θ satisfying $0 \leq \theta \leq \pi$ (or $0^\circ \leq \theta \leq 180^\circ$ if measured in degrees). The value is defined by the cosine formula:

$$\cos\theta = (\mathbf{x}, \mathbf{y}). \tag{A.23}$$

If the vectors are not of unit length, they should be normalized to such, and so the general formula for two arbitrary vectors is

$$\cos\theta = \left(\frac{\mathbf{x}}{|\mathbf{x}|}, \frac{\mathbf{y}}{|\mathbf{y}|}\right) = \frac{(\mathbf{x}, \mathbf{y})}{|\mathbf{x}|\,|\mathbf{y}|}. \tag{A.24}$$

The angle θ formed by a non-null vector with itself is zero. This will always be the case for $n = 1$. If one of the vectors is null, the angle is undefined.

The definition (A.24) agrees with that of the angle formed by *oriented* lines in two- and three-dimensional Euclidean geometry ($n = 2, 3$). The generalization to n dimensions is a natural one.

For some applications the *acute* angle ϕ between the line on $\mathbf{x}$ and the line on $\mathbf{y}$ (i.e., between the vector spans) is more appropriate than θ. This angle between *Span*($\mathbf{x}$) and *Span*($\mathbf{y}$) is the real number ϕ satisfying $0 \le \phi \le \pi/2$ and

$$\cos\phi = \frac{|(\mathbf{x}, \mathbf{y})|}{|\mathbf{x}|\,|\mathbf{y}|} \tag{A.25}$$

Two vectors $\mathbf{x}$ and $\mathbf{y}$ connected by the relation

$$(\mathbf{x}, \mathbf{y}) = 0, \tag{A.26}$$

are said to be *orthogonal*. The acute angle formed by two orthogonal vectors is $\pi/2$ radians or 90°.

§A.3.9. Orthogonal Projection

The *orthogonal projection* of a vector $\mathbf{y}$ onto a vector $\mathbf{x}$ is the vector $\mathbf{p}$ that has the direction of $\mathbf{x}$ and is orthogonal to $\mathbf{y} - \mathbf{p}$. The condition can be stated as

$$(\mathbf{p}, \mathbf{y} - \mathbf{p}) = 0, \qquad \text{in which} \quad \mathbf{p} = c\,\frac{\mathbf{x}}{|\mathbf{x}|}. \tag{A.27}$$

It is easily shown that

$$c = (\mathbf{y}, \frac{\mathbf{x}}{|\mathbf{x}|}) = |\mathbf{y}|\,\cos\theta, \tag{A.28}$$

where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. If $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, the projection of any of them onto the other vanishes.

§A.3.10. Orthogonal Bases and Subspaces

Let $\mathbf{b}^k$, $k = 1, \ldots m$ be a set of n-dimensional *unit* vectors which are *mutually orthogonal*. Then if a particular n-dimensional vector $\mathbf{x}$ admits the representation

$$\mathbf{x} = \sum_{k=1}^{m} c_k \mathbf{b}^k \tag{A.29}$$

the coefficients c_k are given by the inner products

$$c_k = (\mathbf{x}, \mathbf{b}^k) = \sum_{i=1}^{n} x_i b_i^k \overset{\text{sc}}{=} x_i b_i^k. \tag{A.30}$$

We also have *Parseval's equality*

$$(\mathbf{x}, \mathbf{x}) = \sum_{k=1}^{m} c_k^2 \overset{\text{sc}}{=} c_k c_k. \tag{A.31}$$

If the representation (A.29) holds, the set $\mathbf{b}^k$ is called an *orthonormal basis* for the vector $\mathbf{x}$. The $\mathbf{b}^k$ are called the *base vectors*.

The set of all vectors $\mathbf{x}$ given by (A.30) forms a *subspace* of dimension m. The subspace is said to be *spanned* by the basis $\mathbf{b}^k$, and is called *Span* $(\mathbf{b}^k)$.[3] The numbers c_k are called the *coordinates* of $\mathbf{x}$ with respect to that basis.

If $m = n$, the set $\mathbf{b}^k$ forms a *complete orthonormal basis* for the n-dimensional space. (The qualifier "complete" means that all n-dimensional vectors are representable in terms of such a basis.)

The simplest complete orthonormal basis is of order n is the n-dimensional *Cartesian basis*

$$\mathbf{b}^1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \mathbf{b}^2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \qquad \cdots \qquad \mathbf{b}^n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{A.32}$$

In this case the *position coordinates* of $\mathbf{x}$ are simply its components, that is, $c_k \equiv x_k$.

[3] Note that if $m = 1$ we have the span of a single vector, which as noted in §A.3.5 is simply a line passing through the origin of coordinates.

Homework Exercises for Appendix A: Vectors

EXERCISE A.1 Given the four-dimensional vectors

$$\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ -8 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 1 \\ -5 \\ 7 \\ 5 \end{bmatrix} \tag{EA.1}$$

(a) compute the Euclidean norms and lengths of $\mathbf{x}$ and $\mathbf{y}$;

(b) compute the inner product $(\mathbf{x},\mathbf{y})$;

(c) verify that the inequalities (A.18) (A.19) hold;

(d) normalize $\mathbf{x}$ and $\mathbf{y}$ to unit length;

(e) compute the angles $\theta = \angle(\mathbf{x}, \mathbf{y})$ and ϕ given by (**?**) and (**?**).

(f) compute the projection of $\mathbf{y}$ onto $\mathbf{x}$ and verify that the orthogonality condition (A.28) holds.

EXERCISE A.2 Given the base vectors

$$\mathbf{b}^1 = \tfrac{1}{10} \begin{bmatrix} 1 \\ -7 \\ 1 \\ 7 \end{bmatrix}, \qquad \mathbf{b}^2 = \tfrac{1}{10} \begin{bmatrix} 5 \\ 5 \\ -5 \\ 5 \end{bmatrix}. \tag{EA.2}$$

(a) Check that $\mathbf{b}^1$ and $\mathbf{b}^2$ are orthonormal, i.e., orthogonal and of unit length.

(b) Compute the coefficients c_1 and c_2 in the following representation:

$$\mathbf{z} = \begin{bmatrix} 8 \\ -16 \\ -2 \\ 26 \end{bmatrix} = c_1 \mathbf{b}^1 + c_2 \mathbf{b}^2. \tag{EA.3}$$

(c) Using the values computed in (b), verify that the coordinate-expansion formulas and Parseval's equality are correct.

EXERCISE A.3 Prove that Parseval's equality holds in general.

EXERCISE A.4 What are the angles θ and ϕ formed by an arbitrary non-null vector $\mathbf{x}$ and the opposite vector $-\mathbf{x}$?

EXERCISE A.5 Show that

$$(\alpha\mathbf{x}, \beta\mathbf{y}) = \alpha\beta\,(\mathbf{x}, \mathbf{y}) \tag{EA.4}$$

where $\mathbf{x}$ and $\mathbf{y}$ are arbitrary vectors, and α and β are scalars.

Homework Exercises for Appendix A - Solutions

EXERCISE A.1

(a)

$$\|\mathbf{x}\| = 2^2 + 4^2 + 4^2 + (-8)^2 = 100, \quad |\mathbf{x}| = 10.$$
$$\|\mathbf{y}\| = 1^2 + (-5)^2 + 7^2 + 5^2 = 100, \quad |\mathbf{y}| = 10.$$

(b)

$$(\mathbf{x}, \mathbf{y}) = 2 - 20 + 28 - 40 = -30.$$

(c)

$$|(\mathbf{x}, \mathbf{y})| = 30 \leq |\mathbf{x}|\,|\mathbf{y}| = 100.$$
$$|\mathbf{x} + \mathbf{y}| = \sqrt{3^2 + (-1)^2 + (-11)^2 + (-3)^2} = \sqrt{140} \leq |\mathbf{x}| + |\mathbf{y}| = 10 + 10 = 20$$

(d)

$$\mathbf{x}^u = \frac{\mathbf{x}}{10} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \\ -0.8 \end{bmatrix}, \qquad \mathbf{y}^u = \frac{\mathbf{y}}{10} = \begin{bmatrix} 0.1 \\ -0.5 \\ 0.7 \\ 0.5 \end{bmatrix}$$

(e)

$$\cos\theta = (\mathbf{x}^u, \mathbf{y}^u) = -0.30, \qquad \theta = 107.46^\circ.$$
$$\cos\phi = |(\mathbf{x}^u, \mathbf{y}^u)| = 0.30, \qquad \phi = 72.54^\circ.$$

(f)

$$c = |\mathbf{y}|\cos\theta = -3.$$
$$\mathbf{p} = c\,\mathbf{x}^u = -3 \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \\ -0.8 \end{bmatrix} = \begin{bmatrix} -0.6 \\ -1.2 \\ -1.2 \\ 2.4 \end{bmatrix}.$$
$$\mathbf{y} - \mathbf{p} = \begin{bmatrix} 1.6 \\ -3.8 \\ 8.2 \\ 2.6 \end{bmatrix}.$$
$$(\mathbf{p}, \mathbf{y} - \mathbf{p}) = -0.96 + 4.56 - 9.84 + 6.24 = 0.$$

EXERCISE A.2

(a)

$$\|\mathbf{b}^1\| = 0.1^2 + (-0.7)^2 + 0.1^2 + 0.7^2 = 1, \qquad \mathbf{b}^1| = 1.$$
$$\|\mathbf{b}^2\| = 0.5^2 + 0.5^2 + (-0.5)^2 + 0.5^2 = 1, \qquad |\mathbf{b}^2| = 1.$$
$$(\mathbf{b}^1, \mathbf{b}^2) = 0.05 - 0.35 - 0.05 + 0.35 = 0.$$

(b)

$$c_1 = 30, \quad c_2 = 10.$$

(by inspection, or solving a system of 2 linear equations)

(c)

$$c_1 = (\mathbf{z}, \mathbf{b}^1) = 0.8 + 11.2 - 0.2 + 18.2 = 30.$$
$$c_2 = (\mathbf{z}, \mathbf{b}^2) = 4.0 - 8.0 + 1.0 + 13.0 = 10.$$
$$c_1^2 + c_2^2 = 30^2 + 10^2 = 1000 = \|\mathbf{z}\| = 8^2 + (-16)^2 + (-2)^2 + 26^2 = 1000.$$

EXERCISE A.3 See any linear algebra book, e.g. [699].

EXERCISE A.4

$$\theta = 180^\circ, \qquad \phi = 0^\circ.$$

EXERCISE A.5

$$(\alpha \mathbf{x}, \beta \mathbf{y}) = \alpha x_i \; \beta y_i = \alpha\beta \; x_i y_i = \alpha\beta \; (\mathbf{x}, \mathbf{y})$$

(summation convention used)