Finite Difference Methods

"Research is to see what everybody else has seen, and think what nobody has thought." – Albert Szent-Gyorgyi

I. Introduction

Analytical methods may fail if:

1. The PDE is not linear and can't be linearized without seriously affecting the result.
2. The solution region is complex.
3. The boundary conditions are of mixed types.
4. The boundary conditions are time-dependent.
5. The medium is inhomogeneous or anisotropic.

The finite difference method (FDM) was first developed by A. Thom* in the 1920s under the title "the method of square" to solve nonlinear hydrodynamic equations.

*A. Thom an C. J. Apelt, *Field Computations in Engineering and Physics*. London: D. Van Nostrand, 1961.

The finite difference techniques are based upon the approximations that permit replacing **differential equations** by **finite difference equations**. These finite difference approximations are algebraic in form, and the solutions are related to grid points.

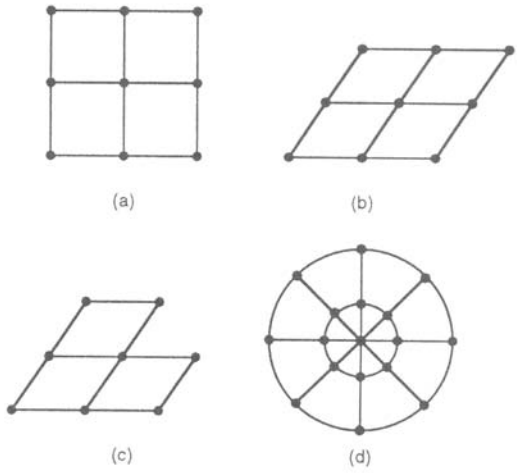


Fig. 1 Common two-dimensional grid patterns

Thus, a finite difference solution basically involves three steps:

1. Dividing the solution into grids of nodes.
2. Approximating the given differential equation by finite difference equivalence that relates the solutions to grid points.
3. Solving the difference equations subject to the prescribed boundary conditions and/or initial conditions.

II. Finite Difference Scheme

Differential equations → estimating derivatives numerically → finite difference equations

Given a function *f(x)* shown in Fig. 2, we can approximate its derivative, slope or tangent at **P** by the slope of the arcs **PB**, **PA**, or **AB**, for obtaining the forward-difference, backward-difference, and central-difference formulas respectively.

forward-difference formula

$$f'(x_o) \cong \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x}$$

backward-difference formula

$$f'(x_o) \cong \frac{f(x_o) - f(x_o - \Delta x)}{\Delta x}$$

central-difference formula

$$f'(x_o) \cong \frac{f(x_o + \Delta x) - f(x_o - \Delta x)}{2\Delta x}$$

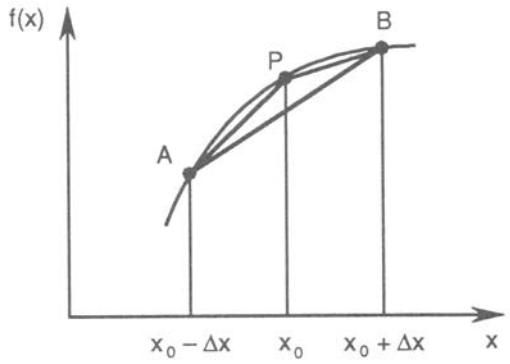


Fig. 2 Estimates for the derivative of f(x) at P by using forward, backward, and central differences.

The approach used for obtaining above finite difference equations is Taylor's series:

$$f(x_o + \Delta x) = f(x_o) + \Delta x f'(x_o) + \frac{1}{2!}(\Delta x)^2 f''(x_o) + \frac{1}{3!}(\Delta x)^3 f'''(x_o) + O(\Delta x)^4, \qquad (1)$$

and

$$f(x_o - \Delta x) = f(x_o) - \Delta x f'(x_o) + \frac{1}{2!}(\Delta x)^2 f''(x_o) - \frac{1}{3!}(\Delta x)^3 f'''(x_o) + O(\Delta x)^4, \qquad (2)$$

where $O(\Delta x)^4$ is the error introduced by truncating the series.

To subtract (1) by (2), we can obtain

$f(x_o + \Delta x) - f(x_o - \Delta x) = 2\Delta x f'(x_o) + O(\Delta x)^3$,

which could be re-written as

$f'(x_o) \cong \frac{f(x_o + \Delta x) - f(x_o - \Delta x)}{2\Delta x} + O(\Delta x)^2$, i.e. the central-difference formula. Note that the $O(\Delta x)^2$ means the truncation error is the order of $(\Delta x)^2$ for the central-difference.

The forward-difference and backward-difference formulas could be obtained by re-arranging (1) and (2) respectively, and we have

$f'(x_o) \cong \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x} + O(\Delta x)$, for forward difference,

and

$f'(x_o) \cong \frac{f(x_o) - f(x_o - \Delta x)}{\Delta x} + O(\Delta x)$, for backward difference. We can find the truncation errors of these two formulas are of order Δx.

Upon adding (1) and (2),

$f(x_o + \Delta x) - f(x_o - \Delta x) = 2f(x_o) + (\Delta x)^2 f''(x_o) + O(\Delta x)^4$,

and we have

$$f''(x_o) \cong \frac{f(x_o + \Delta x) - 2f(x_o) + f(x_o - \Delta x)}{(\Delta x)^2} + O(\Delta x)^2.$$

Higher order finite difference approximations can be obtained by taking more terms in Taylor series expansion.

Table 1 Finite Difference Approximations for Φ_x and Φ_{xx}, where FD = Forward Difference, BD = Backward Difference, and CD = Central Difference

Derivative	Finite Difference Approximation	Type	Error
Φ_x	$\frac{\Phi_{i+1} - \Phi_i}{\Delta x}$	FD	$O(\Delta x)$
	$\frac{\Phi_i - \Phi_{i+1}}{\Delta x}$	BD	$O(\Delta x)$
	$\frac{\Phi_{i+1} - \Phi_{i-1}}{\Delta x}$	CD	$O(\Delta x)^2$
	$\frac{-\Phi_{i+2} + 4\Phi_{i+1} - 3\Phi_i}{2\Delta x}$	FD	$O(\Delta x)^2$
	$\frac{3\Phi_i - 4\Phi_{i-1} + \Phi_{i-2}}{2\Delta x}$	BD	$O(\Delta x)^2$
	$\frac{-\Phi_{i+2} + 8\Phi_{i+1} - 8\Phi_{i-1} + \Phi_{i-2}}{12\Delta x}$	CD	$O(\Delta x)^4$
Φ_{xx}	$\frac{\Phi_{i+2} - 2\Phi_{i+1} + \Phi_i}{(\Delta x)^2}$	FD	$O(\Delta x)^2$
	$\frac{\Phi_i - 2\Phi_{i-1} + \Phi_{i-2}}{(\Delta x)^2}$	BD	$O(\Delta x)^2$
	$\frac{\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}}{(\Delta x)^2}$	CD	$O(\Delta x)^2$
	$\frac{-\Phi_{i+2} + 16\Phi_{i+1} - 30\Phi_i + 16\Phi_{i-1} - \Phi_{i-2}}{(\Delta x)^2}$	CD	$O(\Delta x)^4$

To apply the difference method to find the solution of a function $\Phi(x,t)$, we divide the solution region in x-t plane into equal retangles or meshes of sides Δx and Δt. The derivatives of Φ at the $(i,j)^{th}$ node are shown in the table, where

$x = i \bullet \Delta x$

$t = j \bullet \Delta t$.

$$\Phi_x|_{i,j} \simeq \frac{\Phi(i+1,j) - \Phi(i-1,j)}{2\Delta x},$$

$$\Phi_t|_{i,j} \simeq \frac{\Phi(i,j+1) - \Phi(i,j-1)}{2\Delta t},$$

$$\Phi_{xx}|_{i,j} \simeq \frac{\Phi(i+1,j) - 2\Phi(i,j) + \Phi(i-1,j)}{(\Delta x)^2},$$

$$\Phi_{tt}|_{i,j} \simeq \frac{\Phi(i,j+1) - 2\Phi(i,j) + \Phi(i,j-1)}{(\Delta t)^2}$$

Finite Differencing of Parabolic PDE's

Consider a simple example of a parabolic (or diffusion) partial differential equation with one spatial independent variable

$$k\frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial x^2}, \tag{3}$$

where k is a constant. The equivalent finite deference approximation is

$$k\frac{\Phi(i, j+1) - \Phi(i, j)}{\Delta t} = \frac{\Phi(i-1, j) - 2\Phi(i, j) + \Phi(i+1, j)}{(\Delta x)^2}. \tag{4}$$

where $x=i\Delta x$, $i=1,2,3,\ldots,n$, $t=j\Delta t$, $j=1,2,\ldots$. In (4), we use the forward difference formula for the derivative with respective to t and central difference formula for the with respect to x. If we let

$$r = \frac{\Delta t}{k(\Delta x)^2}, \tag{5}$$

Eq. (4) could be written as

$$\boxed{\Phi(i, j+1) = r\Phi(i+1, j) + (1-2r)\Phi(i, j) + r\Phi(i-1, j)}. \tag{6}$$

This ***explicit formula*** can be used to compute $\Phi(x,t+\Delta t)$ explicitly in terms of $\Phi(x,t)$.

Thus the values of Φ along the first time row (see Fig.3), $t=\Delta t$, can be calculated in terms of the boundary and initial conditions, then the values of Φ along the second row, $t=2\Delta t$, are calculated in terms of the first time row, and so on.

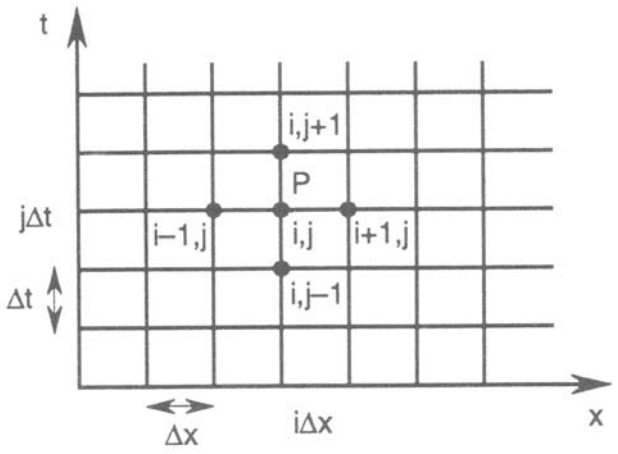


Fig. 3 Finite difference mesh for two independent variable *x* and *t*.

A graphic way of describing the difference equation (6) is through the computational molecule of fig. 4, where the square is used to represent the grid point where Φ is presumed known and a circle where Φ is unknown.

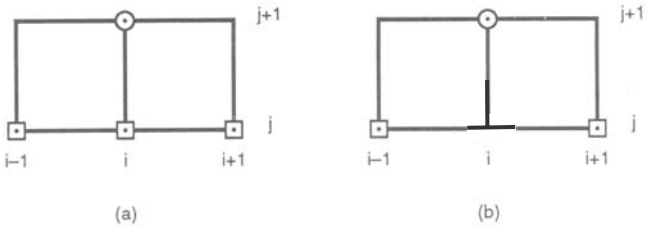


Fig. 4 Computational molecule for parabolic PDE: (a) for 0 < *r* < 1/2 (b) *r* = 1/2.

Stability Analysis of the explicit algorithm

Reducing the mesh size could increase accuracy, but the mesh size could not be infinitesimal.

Decreasing the truncation error by using a finer mesh may result in increasing the round-off error due to the increased number of arithmetic operations. A point is reached where minimum total error occurs for any particular algorithm using any given word length. This is illustrated in Fig. 5.

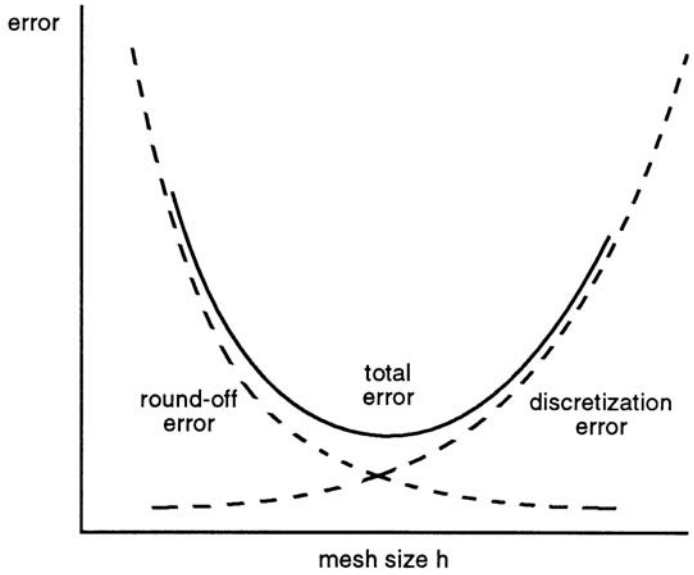


Fig. 5 Error as a function of the mesh size.

The concern about accuracy leads us to a question whether the finite difference solution can grow unbounded, or a property termed the ***instability*** of the difference scheme. A numerical algorithm is said to be ***stable*** if a small error at any stage produces a smaller cumulative error. Otherwise, it is ***unstable***.

To determine a whether a finite difference scheme is stable, we define an error, ε^n, which occurs at time step n, assuming there is one independent variable. We define the amplification of this error at time step n+1 as

$$\varepsilon^{n+1}=g\varepsilon^n, \tag{7}$$

where g is the ***amplification factor***. In more complicated situation, we have more independent variables, and (7) becomes

$$[\varepsilon]^{n+1}=[G][\varepsilon]^n, \tag{8}$$

where $[G]$ is ***amplification matrix***.

For the stability of the finite difference scheme, it is required that Eq. (7) satisfy

$$|\varepsilon^{n+1}| \le |\varepsilon^n|, \tag{9}$$

or

$$|g| \le 1. \tag{10}$$

For the case in (8), the determinant of $[G]$ must vanish, i.e.

$$\mathrm{Det}[G] = 0. \tag{11}$$

One useful method and simple method of finding a stability criterion for a finite difference scheme is to construct a Fourier analysis of the difference equation and thereby derive the amplification factor. The technique is known as von Neumann's method. The stability condition is called the von Neumann condition. (ref. To J. W. Thomas, Numerical Partial Differential Equations – Finite Difference Methods, Springer-Verlag New York, 1995)

Considering the explicit scheme of Eq. (6):

$$\Phi_i^{n+1} = (1-2r)\Phi_i^n + r(\Phi_{i+1}^n + \Phi_{i-1}^n), \tag{12}$$

where $r=\Delta t/k(\Delta x)^2$. We have changed our notation so that we can use $j=\sqrt{-1}$ in the Fourier series. Suppose the solution is

$$\Phi_i^n = \sum A^n(t)e^{j\kappa i\Delta x}, \ 0 \le \Delta x \le 1, \tag{13}$$

where κ is the wave number. Since the differential equation is linear, we need consider only one Fourier mode, i.e.

$$\Phi_i^n = A^n(t)e^{j\kappa i\Delta x}. \tag{14}$$

Substituting (14) into (12) gives

$$A^{n+1}e^{j\kappa i\Delta x} = (1-2r)A^n e^{j\kappa i\Delta x} + r(e^{j\kappa\Delta x} + e^{-j\kappa\Delta x})A^n e^{j\kappa i\Delta x} \tag{15}$$

or

$$A^{n+1} = A^n[1-2r+2r\cos\kappa x]. \tag{16}$$

Hence the amplification factor is

$$g = \frac{A^{n+1}}{A^n} = 1-2r+2r\cos\kappa\Delta x = 1-4r\sin^2\frac{\kappa x}{2}. \tag{17}$$

In order to satisfy (10)

$$\left|1-4r\sin^2\frac{\kappa x}{2}\right| \le 1. \tag{18}$$

Since this condition must hold for every wave number κ, we take the maximum value of the sine function so that

$$1-4r \ge -1 \text{ and } r \ge 0$$

or

$$r \ge ½ \text{ and } r \ge 0.$$

Of course, $r = 0$ implies $\Delta t = 0$, which is impractical. Thus we have

$$0 < r \le 1/2.$$

In order to ensure a stable solution or reduce errors, care must be exercised in selecting the value of r in (5) and (6). If we choose r=1/2, then (12) becomes

$$\Phi_i^{n+1} = \frac{1}{2}(\Phi_{i+1}^n + \Phi_{i-1}^n), \tag{19}$$

as shown in Fig. 4(b).

Although the formula is simple to implement, its computation is slow. An implicit formula, proposed by Crank and Nicholson in 1974, is valid for all finite values of *r*. We replace $\frac{\partial^2 \Phi}{\partial x^2}$ in Eq. (3) by the average of the central difference formulas on the jth and (j+1)th time rows such that

$$k\frac{\Phi(i,j+1)-\Phi(i,j)}{\Delta t}=\frac{1}{2}\left[\frac{\Phi(i-1,j)-2\Phi(i,j)+\Phi(i+1,j)}{(\Delta x)^2}+\frac{\Phi(i-1,j+1)-2\Phi(i,j+1)+\Phi(i+1,j+1)}{(\Delta x)^2}\right], \tag{20}$$

which can be re-written as

$$\boxed{\begin{aligned}-r\Phi(i-1,j+1)+2(1+r)\Phi(i,j+1)-r\Phi(i+1,j+1)\\ =r\Phi(i-1,j)+2(1-r)\Phi(i,j)+r\Phi(i+1,j)\end{aligned}}. \tag{21}$$

Where r is defined in (5). The left hand side of (21) consists of three unknown values of Φ. This is illustrated in the computational molecule of Fig. 5(a). Thus if there are n nodes along each time row, then for *j*=0, applying (21) to nodes *i*=1,2,…,*n* results in *n* simultaneous equations with *n* unknown values of Φ and known initial and boundary values of Φ. Similarly, for *j*=1, we obtain *n* simultaneous equations for *n* unknown values of Φ in terms of the known values at *j*=0, and so on. The combination of accuracy and unconditional stability allows the use of a much larger time step with Crank-Nicholson method than is possible with the explicit formula. Although the method is valid for all finite values of *r*, a convenient choice of *r*=1 reduces (21) to

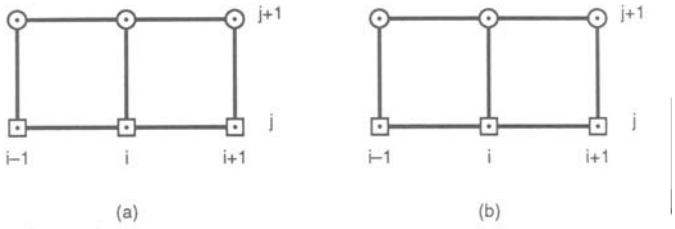


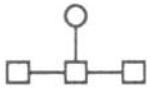
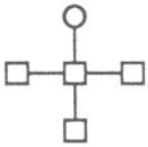
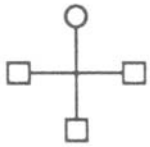
Fig. 5 Computational molecule for Crank-Nicholson method: (a) for finite *r* and (b) *r* = 1.

$$\boxed{-\Phi(i-1,j+1)+4\Phi(i,j+1)-\Phi(i+1,j+1)=\Phi(i-1,j)+\Phi(i+1,j)}, \tag{22}$$

with the computational molecule of Fig. 5(b).

<HW> Use the von Neumann approach to determine the stability condition of Eq. (21).

Table 3.2 Finite Difference Approximation to the Parabolic Equation: $\frac{\partial \Phi}{\partial t} = \frac{1}{k}\frac{\partial^2 \Phi}{\partial x^2}, \quad k > 0$

Method	Algorithm	Molecule
1. First order (Euler)	$\frac{\Phi_i^{j+1} - \Phi_i^j}{\Delta t} = \frac{\Phi_{i+1}^j - 2\Phi_i^j + \Phi_{i-1}^j}{k(\Delta x)^2}$ explicit, stable for $r = \Delta t / k(\Delta x)^2 \le 0.5$	
2. Crank-Nicholson	$\frac{\Phi_i^{j+1} - \Phi_i^{j+1}}{\Delta t} = \frac{\Phi_{i+1}^{j+1} - 2\Phi_i^{j+1} + \Phi_{i-1}^{j+1}}{k(\Delta x)^2} + \frac{\Phi_{i+1}^j - 2\Phi_i^j + \Phi_{i-1}^j}{k(\Delta x)^2}$ implicit, always stable	
3. Leapfrog	$\frac{\Phi_i^{j+1} - \Phi_i^{j-1}}{2\Delta t} = \frac{\Phi_{i+1}^j - 2\Phi_i^j + \Phi_{i-1}^j}{k(\Delta x)^2}$ explicit, always unstable	
4. Dufort-Frankel	$\frac{\Phi_i^{j+1} - \Phi_i^{j-1}}{2\Delta t} = \frac{\Phi_{i+1}^j - \Phi_i^{j+1} - \Phi_i^{j-1} + \Phi_{i-1}^j}{k(\Delta x)^2}$ explicit, unconditionally stable	

［**Example**］
Solve the diffusion equation

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial \Phi}{\partial t} \qquad 0 \le x \le 1$$

subject to the boundary conditions

$$\Phi(0,t) = 0, \ \Phi(1,t) = 0, \qquad t > 0$$

and initial condition

$$\Phi(x,0) = 100.$$

Solution
This problem may be regarded as a mathematical model of the temperature distribution in a rod of length L=1m with its end in contacts with ice blocks (or held at 0℃) and the rod initially at 100℃. With the physical interpretation, out problem is finding the internal temperature Φ as a function of position and time. We will solve this problem using both explicit and implicit methods.

(a)Analytical solution

$$\Phi(x,t) = \frac{\pi}{400} \sum_{k=0}^{\infty} \frac{1}{n} \sin(n\pi x) \exp(-n^2 \pi^2 t), \qquad n=2k+1$$

t \ Δx	0.00	0.10	0.20	0.30	0.40	0.50
0.0000	50.00	100.00	100.00	100.00	100.00	100.00
0.0050	0.00	68.27	95.45	99.73	99.99	100.00
0.0100	0.00	52.05	84.27	96.61	99.53	99.92
0.0150	0.00	43.63	75.18	91.67	97.85	99.22
0.0200	0.00	38.29	68.26	86.59	95.18	97.52
0.0250	0.00	34.52	62.86	81.85	91.91	94.93
0.0300	0.00	31.67	58.47	77.51	88.32	91.75
0.0350	0.00	29.39	54.78	73.50	84.61	88.24
0.0400	0.00	27.50	51.58	69.78	80.88	84.58
0.0450	0.00	25.87	48.74	66.31	77.21	80.88
0.0500	0.00	24.42	46.16	63.04	73.63	77.23
0.0550	0.00	23.12	43.79	59.96	70.18	73.67
0.0600	0.00	21.93	41.59	57.04	66.86	70.22
0.0650	0.00	20.82	39.53	54.27	63.68	66.90
0.0700	0.00	19.79	37.59	51.65	60.63	63.72
0.0750	0.00	18.81	35.75	49.15	57.73	60.68
0.0800	0.00	17.89	34.01	46.78	54.96	57.78
0.0850	0.00	17.02	32.37	44.52	52.32	55.00
0.0900	0.00	16.20	30.80	42.38	49.81	52.36
0.0950	0.00	15.41	29.31	40.34	47.41	49.85
0.1000	0.00	14.67	27.90	38.39	45.13	47.45

(b) Explicit Method
For simplicity, let us choose Δx=0.1, r=1/2 so that

$$\Delta t = \frac{r(\Delta x)^2}{k} = 0.005$$

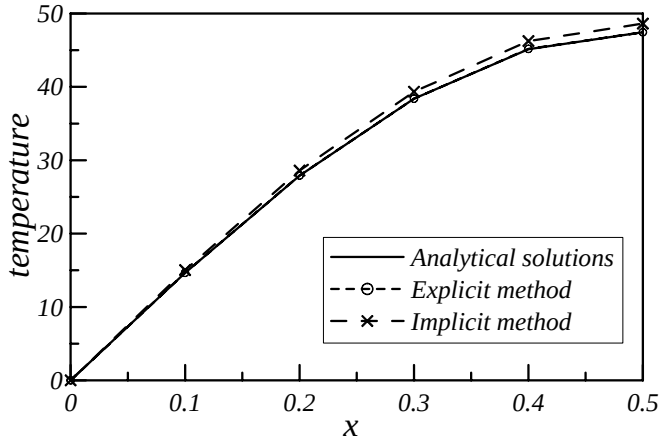
since k=1. We need the solution for only $0 \le x \le 0.5$ due to the fact that the problem is symmetric with respect to x=0.5. Notice that the value of Φ(0,0) and Φ(1,0) are taken as the average of 0 and 100.

t	Δx 0.00	0.10	0.20	0.30	0.40	0.50
0.0000	50.00	100.00	100.00	100.00	100.00	100.00
0.0050	0.00	75.00	100.00	100.00	100.00	100.00
0.0100	0.00	50.00	87.50	100.00	100.00	100.00
0.0150	0.00	43.75	75.00	93.75	100.00	100.00
0.0200	0.00	37.50	68.75	87.50	96.88	100.00
0.0250	0.00	34.38	62.50	82.81	93.75	96.88
0.0300	0.00	31.25	58.59	78.12	89.84	93.75
0.0350	0.00	29.30	54.69	74.22	85.94	89.84
0.0400	0.00	27.34	51.76	70.31	82.03	85.94
0.0450	0.00	25.88	48.83	66.89	78.12	82.03
0.0500	0.00	24.41	46.39	63.48	74.46	78.12
0.0550	0.00	23.19	43.95	60.42	70.80	74.46
0.0600	0.00	21.97	41.81	57.37	67.44	70.80
0.0650	0.00	20.90	39.67	54.63	64.09	67.44
0.0700	0.00	19.84	37.77	51.88	61.04	64.09
0.0750	0.00	18.88	35.86	49.40	57.98	61.04
0.0800	0.00	17.93	34.14	46.92	55.22	57.98
0.0850	0.00	17.07	32.42	44.68	52.45	55.22
0.0900	0.00	16.21	30.88	42.44	49.95	52.45
0.0950	0.00	15.44	29.33	40.41	47.45	49.95
0.1000	0.00	14.66	27.92	38.39	45.18	47.45

(c) Implicit Method

Let us choose Δx=0.1, r=1 so that Δt=0.01.

t	Δx 0.00	0.10	0.20	0.30	0.40	0.50
0.0000	50.00	100.00	100.00	100.00	100.00	100.00
0.0100	0.00	59.81	89.23	97.10	99.17	99.59
0.0200	0.00	40.18	71.50	88.92	95.77	97.47
0.0300	0.00	32.96	60.34	79.30	89.59	92.68
0.0400	0.00	28.33	52.97	71.30	82.31	85.95
0.0500	0.00	25.06	47.28	64.41	75.09	78.70
0.0600	0.00	22.46	42.56	58.30	68.27	71.68
0.0700	0.00	20.25	38.46	52.82	61.98	65.13
0.0800	0.00	18.32	34.82	47.87	56.23	59.11
0.0900	0.00	16.59	31.54	43.40	51.00	53.61
0.1000	0.00	15.03	28.59	39.34	46.24	48.62



The temperature profiles at Δt=0.1

Finite Differencing of Hyperbolic PDE's

Consider a simple example of a hyperbolic partial differential (or wave) equation with one spatial independent variable

$$u^2 \frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial^2 \Phi}{\partial t^2}, \tag{23}$$

where u is the speed of the wave. The equivalent finite deference approximation is

$$u^2 \frac{\Phi(i-1,j) - 2\Phi(i,j) + \Phi(i+1,j)}{(\Delta x)^2} = \frac{\Phi(i,j+1) - 2\Phi(i,j) + \Phi(i,j-1)}{(\Delta t)^2}. \tag{24}$$

where $x=i\Delta x$, $i=1,2,3,\ldots,n$, $t=j\Delta t$, $j=1,2,\ldots$. In (24), we use the central difference formula for the derivatives with respective to t as well as with respect to x. If we let

$$r = \left(\frac{u\Delta t}{\Delta x} \right)^2, \tag{25}$$

Eq. (24) could be written as

$$\boxed{\Phi(i,j+1) = 2(1-r)\Phi(i,j) + r[\Phi(i+1,j) + \Phi(i-1,j)] - \Phi(i,j-1)}. \tag{26}$$

If we choose r = 1, Eq. (26) becomes

$$\boxed{\Phi(i,j+1) = \Phi(i+1,j) + \Phi(i-1,j) - \Phi(i,j-1)}. \tag{27}$$

We assume a trial solution of the form

$$\Phi_i^n = A^n e^{jkix}. \tag{28}$$

Substituting this into Eq. (26) results in

$$A^{n+1} e^{jkix} = 2(1-r)A^n e^{jkix} + r(e^{jkx} + e^{-jkx})A^n e^{jkix} - A^{n-1} e^{jkix} \tag{29}$$

or

$$A^{n+1} = A^n \left[2(1-r) + 2r\cos kx \right] - A^{n-1}. \tag{30}$$

In terms of $g = A^{n+1}/A^n$, Eq. (30) becomes

$$g^2 - 2pg + 1 = 0 \tag{31}$$

where $p = 1 - 2r\sin^2 \frac{kx}{2}$. The quadratic equation (31) has solutions

$$g_1 = p + [p^2 - 1]^{1/2}, \qquad g_2 = p - [p^2 - 1]^{1/2}.$$

For $|g_i| \le 1$, where $i = 1,2$, p must lie between 1 and -1, i.e., $-1 \le p \le 1$ or

$$-1 \le 1 - 2r\sin^2 \frac{kx}{2} \le 1 \tag{32}$$

which implies that $r \le 1$ or $u\Delta t \le \Delta x$ for stability. This idea can be extended to show that the stability condition for two-dimensional wave equation is $u\Delta t/h < \frac{1}{\sqrt{2}}$, where $h = \Delta x = \Delta y$.

[Example]
Solve the wave equation

$$\Phi_{tt} = \Phi_{xx}, \qquad 0 < x < 1,\ t \geq 0$$

subject to the boundary conditions

$$\Phi(0,t) = 0 = \Phi(1,t),\ t \geq 0$$

and the initial conditions

$$\Phi(x,0) = \sin\ \pi x, \qquad 0 < x < 1,$$
$$\Phi_t(x,0) = 0, \qquad 0 < x < 1.$$

Solution

The analytical solution is easily obtained as

$$\Phi(x,t) = \sin \pi x\ \cos \pi t. \tag{33}$$

Using the explicit finite difference scheme of Eq. (26) with $r = 1$, we obtain

$$\Phi(i,j+1) = \Phi(i-1,j) + \Phi(i+1,j) - \Phi(i,j-1),\ j \geq 1. \tag{34}$$

For $j = 0$, substituting

$$\Phi_t = \frac{\Phi(i,1) - \Phi(i,-1)}{2\Delta t} = 0$$

or

$$\Phi(i,1) = \Phi(i,-1)$$

into Eq. (34) gives the starting formula

$$\Phi(i,1) = \frac{1}{2}[\Phi(i-1,0) + \Phi(i+1,0)]. \tag{35}$$

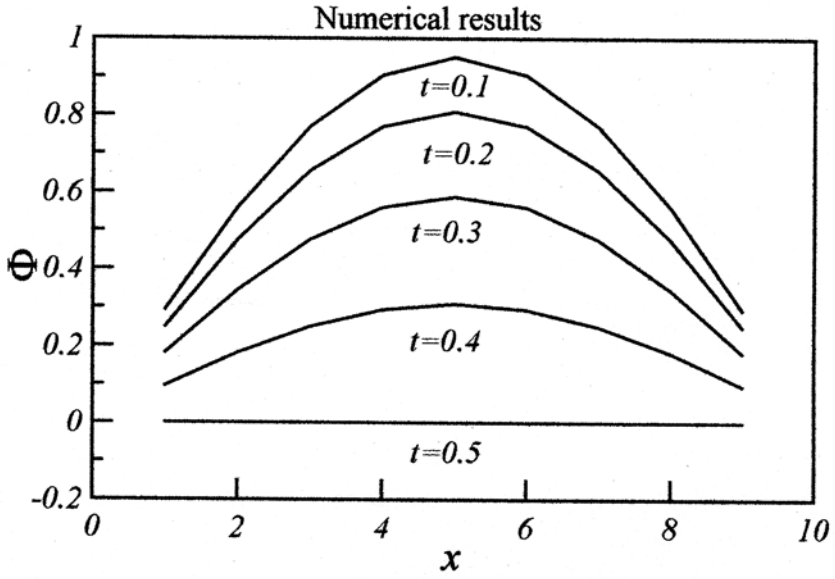
Since $u = 1$, and $r = 1$, $\Delta t = \Delta x$. Also, since the problem is symmetric with respect to $x = 0.5$, we solve for Φ using Eqs. (34) and (35) within $0 < x < 0.5$, $t \geq 0$. We can either calculate the values by hand or write a simple computer program. With the FORTRAN code in Fig. I , the result shown in Table I is obtained for $\Delta t = \Delta x = 0.1$. The finite difference solution agrees with the exact solution in Eq. (33) to six decimal places. The accuracy of the FD solution can be increased by choosing a smaller spatial increment Δx and a smaller time increment Δt.

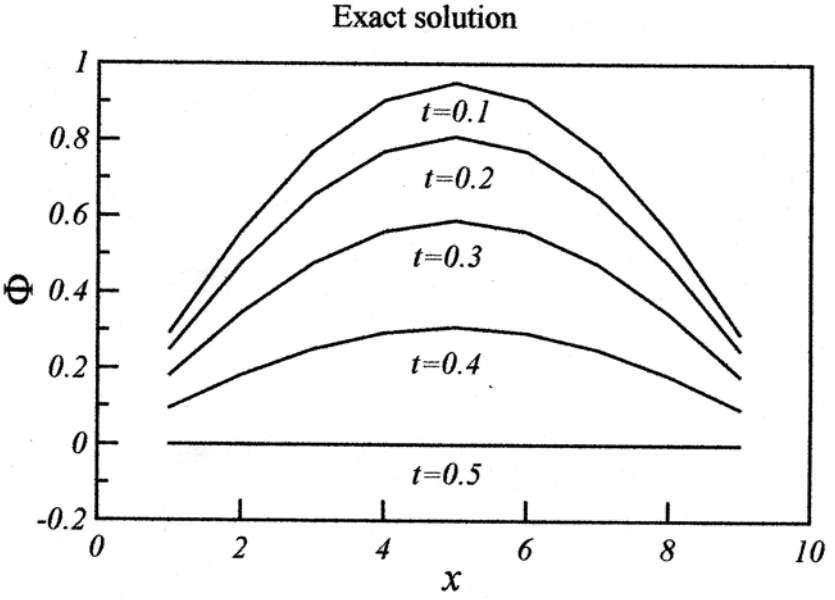
Table I

x / t	0	0.1	0.2	0.3	0.4	0.5	0.6	...
0.0	0	0.3090	0.5879	0.8990	0.9511	1.0	0.9511	
0.1	0	0.2939	0.5590	0.7694	0.9045	0.9511	0.9045	
0.2	0	0.2500	0.4755	0.6545	0.7694	0.8090	0.7694	
0.3	0	0.1816	0.3455	0.4755	0.5590	0.5878	0.5590	
0.4	0	0.0955	0.1816	0.2500	0.2939	0.3090	0.2939	
0.5	0	0	0	0	0	0	0	
0.6	0	-0.0955	-0.1816	-0.2500	-0.2939	-0.3090	- 0.2939	
0.7	0	-0.1816	-0.3455	-0.4755	-0.5590	-0.5878	-0.5590	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Figure I

```
0001
0002    C ***************************************************
0003    C  FORTRAN CODE FOR EXAMPLE 3.2
0004    C  ON ONE-DIMENSIONAL WAVE EQUATION
0005    C  SOLVED USING AN EXPLICIT FINITE DIFFERENCE SCHEME
0006    C ***************************************************
0007
0008          DIMENSION PHI(0:50,0:200), PHIEX(0:50,0:200)
0009          DATA  PIE/3.141592654/
0010    C   SET SPATIAL AND TIME INCREMENTS
0011          DX = 0.1
0012          DT = 0.1
0013    C   INITIALIZE - THIS ALSO TAKES CARE OF BOUNDARY CONDITIONS
0014          DO 10 I = 0,10
0015          DO 10 J = 0,20
0016          PHI(I,J) = 0.0
0017    10    CONTINUE
0018    C   INSERT THE INITIAL CONDITIONS
0019          DO 20 I = 0,10
0020          X = DX*FLOAT(I)
0021          PHI(I,0) = SIN(PIE*X)
0022    20    CONTINUE
0023        DO 30 I=1,9
0024          PHI(I,1) = ( PHI(I-1,0) + PHI(I+1,0) )/2.0
0025    30    CONTINUE
0026    C     NOW APPLY THE EXPLICIT FD SCHEME
0027          DO 40 J = 1,10
0028          DO 40 I = 1,9
0029            PHI(I,J+1) = PHI(I-1,J) + PHI(I+1,J) - PHI(I,J-1)
0030    40    CONTINUE
0031    C   CALCULATE EXACT RESULT
0032          DO 50 J = 0,10
0033          T = DT*FLOAT(J)
0034          CT = COS(PIE*T)
0035          DO 50 I=0,10
0036          X = DX*FLOAT(I)
0037          PHIEX(I,J) = SIN(PIE*X)*CT
0038    50    CONTINUE
0039    C   OUTPUT THE FD APPROXIMATE AND EXACT RESULTS
0040          DO 70 J = 0,7
0041          DO 70 I = 0,7
0042          WRITE(6,60) J,I,PHI(I,J),PHIEX(I,J)
0043    60    FORMAT(3X,'J=',I3,3X,'I=',I3,3X,2(F14.10,3X),/)
0044    70    CONTINUE
0045        STOP
0046      END
```


Numerical results
1
0.8
0.6
Φ 0.4
0.2
0
-0.2
0
2
4
6
8
10
x
t=0.1
t=0.2
t=0.3
t=0.4
t=0.5


Exact solution
1
0.8
0.6
Φ 0.4
0.2
0
-0.2
0
2
4
6
8
10
x
t=0.1
t=0.2
t=0.3
t=0.4
t=0.5

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