Finite Difference Method

MEL 807

Computational Heat Transfer (2-0-4)

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Discretization Methods

- Required to convert the general transport equation to set of algebraic equations
  - Finite difference method
  - Finite volume method
  - Finite element method

Introduction to Finite Difference

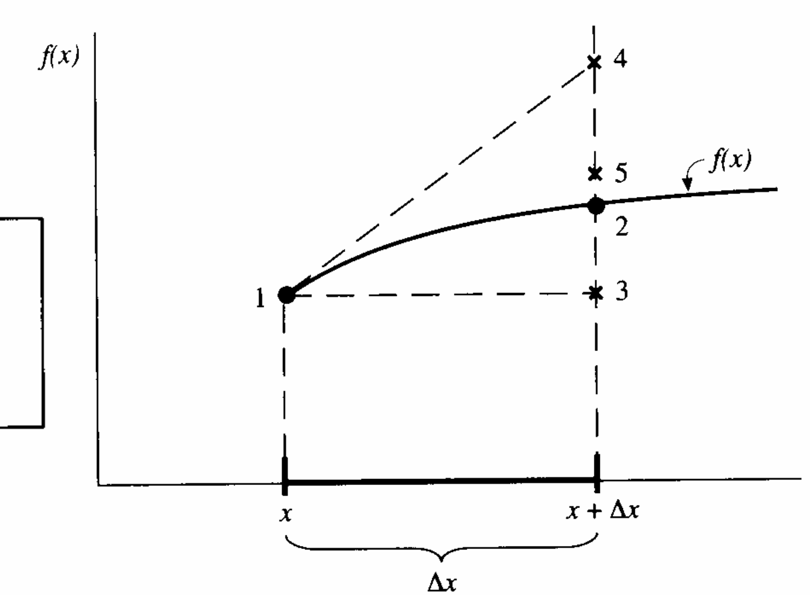
Taylor series expansion

$$f(x+\Delta x) = f(x) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial^2 f}{\partial x^2}\frac{(\Delta x)^2}{2} + ---- + \frac{\partial^n f}{\partial x^n}\frac{(\Delta x)^n}{n!} + --$$

gradient

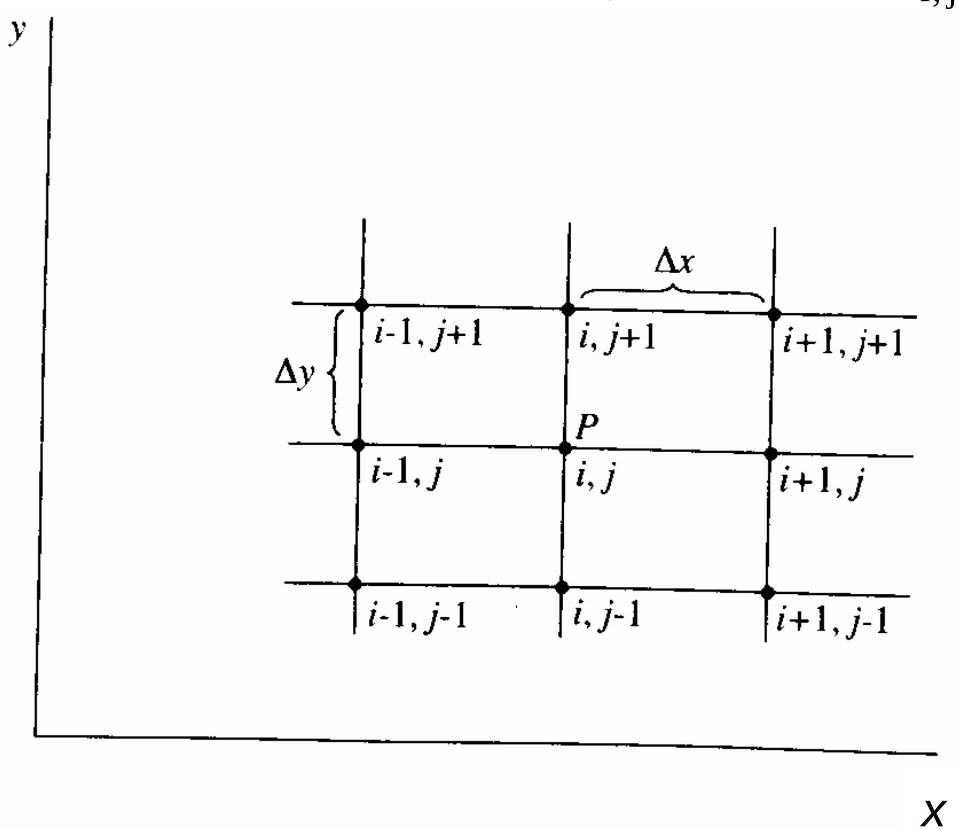
curvature

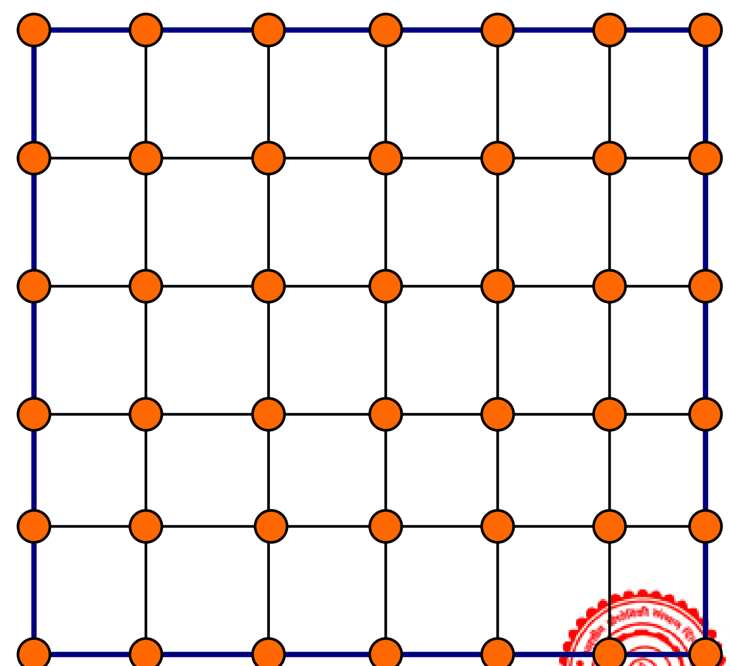
$$f(x+\Delta x) = \underbrace{f(x)}_{\text{First guess (not very good)}} + \underbrace{\frac{\partial f}{\partial x}\Delta x}_{\text{Add to capture slope}} + \underbrace{\frac{\partial^2 f}{\partial x^2}\frac{(\Delta x)^2}{2}}_{\text{Add to account for curvature}} + \cdots$$



Discretization

$$\phi_{i+1,j} = \phi_{i,j} + \left(\frac{\partial \phi}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 \phi}{\partial x^2}\right)_{i,j} \frac{(\Delta \phi)^2}{2} + \left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{6} - - -$$





Representation of a Derivative

$$\phi_{i+1,j} = \phi_{i,j} + \left(\frac{\partial \phi}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 \phi}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2} + \left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{6} - - -$$

$$\left(\frac{\partial \phi}{\partial x}\right)_{i,j} = \boxed{\frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x}} - \left(\frac{\partial^2 \phi}{\partial x^2}\right)_{i,j} \frac{(\Delta x)}{2} - \left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^2}{6} - - -$$

Finite difference
representation

Forward difference $\boxed{\left(\frac{\partial \phi}{\partial x}\right)_{i,j} = \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x}} + O(\Delta x)$

Backward Difference

$$\phi_{i-1,j} = \phi_{i,j} + \left(\frac{\partial \phi}{\partial x}\right)_{i,j} (-\Delta x) + \left(\frac{\partial^2 \phi}{\partial x^2}\right)_{i,j} \frac{(-\Delta x)^2}{2} + \left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i,j} \frac{(-\Delta x)^3}{6} - - -$$

$$\phi_{i-1,j} = \phi_{i,j} - \left(\frac{\partial \phi}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 \phi}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2} - \left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{6} - - -$$

Backward difference $$\left(\frac{\partial \phi}{\partial x}\right)_{i,j} = \frac{\phi_{i,j} - \phi_{i-1,j}}{\Delta x} + O(\Delta x)$$

Central difference

$$\phi_{i+1,j} = \phi_{i,j} + \left(\frac{\partial \phi}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 \phi}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2} + \left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{6} - - -$$

$$\phi_{i-1,j} = \phi_{i,j} - \left(\frac{\partial \phi}{\partial x}\right)_{i,j} \Delta x + \left(\frac{\partial^2 \phi}{\partial x^2}\right)_{i,j} \frac{(\Delta x)^2}{2} - \left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i,j} \frac{(\Delta x)^3}{6} - - -$$

Subtracting,

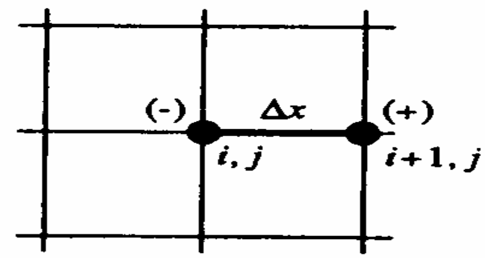
$$\left(\frac{\partial \phi}{\partial x}\right)_{i,j} = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x} + O(\Delta x)^2$$

Central difference

Forward, backward and central difference stencil

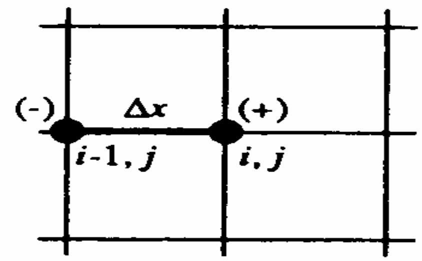
First-order forward difference with respect to x

$$\left(\frac{\partial\phi}{\partial x}\right)_{i,j} = \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x} + O(\Delta x)$$



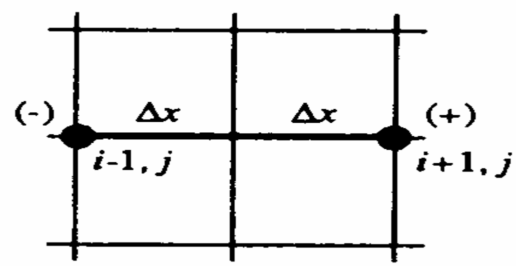
First-order rearward difference with respect to x

$$\left(\frac{\partial\phi}{\partial x}\right)_{i,j} = \frac{\phi_{i,j} - \phi_{i-1,j}}{\Delta x} + O(\Delta x)$$



Second-order central difference with respect to x

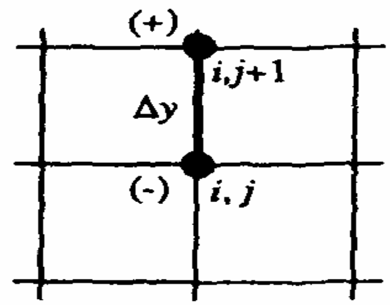
$$\left(\frac{\partial\phi}{\partial x}\right)_{i,j} = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x} + O(\Delta x)^2$$



Stencil in y direction

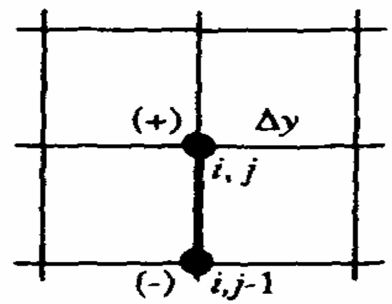
First-order forward difference with respect to y

$$\left(\frac{\partial\phi}{\partial y}\right)_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta y} + O(\Delta y)$$



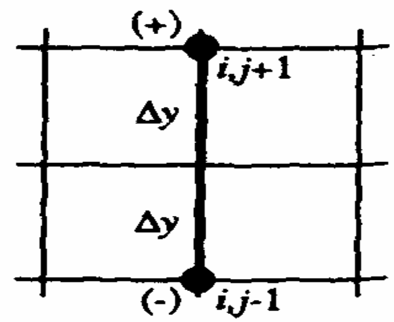
First-order rearward difference with respect to y

$$\left(\frac{\partial\phi}{\partial y}\right)_{i,j} = \frac{\phi_{i,j} - \phi_{i,j-1}}{\Delta y} + O(\Delta y)$$



Second-order central difference with respect to y

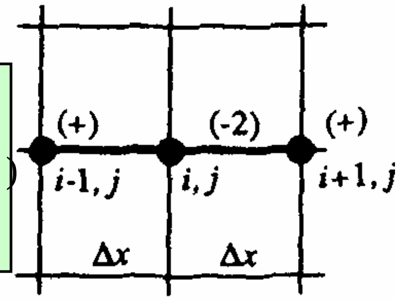
$$\left(\frac{\partial\phi}{\partial y}\right)_{i,j} = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta y} + O(\Delta y)^2$$



2nd Order and Mixed Derivative

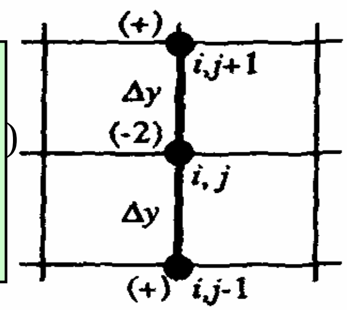
Second-order central second difference with respect to *x*

$$\left(\frac{\partial^2 \phi}{\partial x^2}\right)_{i,j} = \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{(\Delta x)^2} + O(\Delta x)^2)$$



Second-order central second difference with respect to *y*

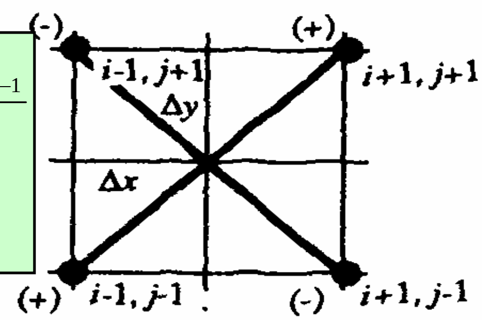
$$\left(\frac{\partial^2 \phi}{\partial y^2}\right)_{i,j} = \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{(\Delta y)^2} + O(\Delta y)^2)$$



Second-order central mixed difference with respect to *x* and *y*

$$\left(\frac{\partial^2 \phi}{\partial x \partial y}\right)_{i,j} = \frac{\phi_{i+1,j+1} + \phi_{i-1,j-1} - \phi_{i-1,j+1} - \phi_{i+1,j-1}}{4\Delta x \Delta y}$$

$$+ O[(\Delta x)^2, (\Delta y)^2]$$

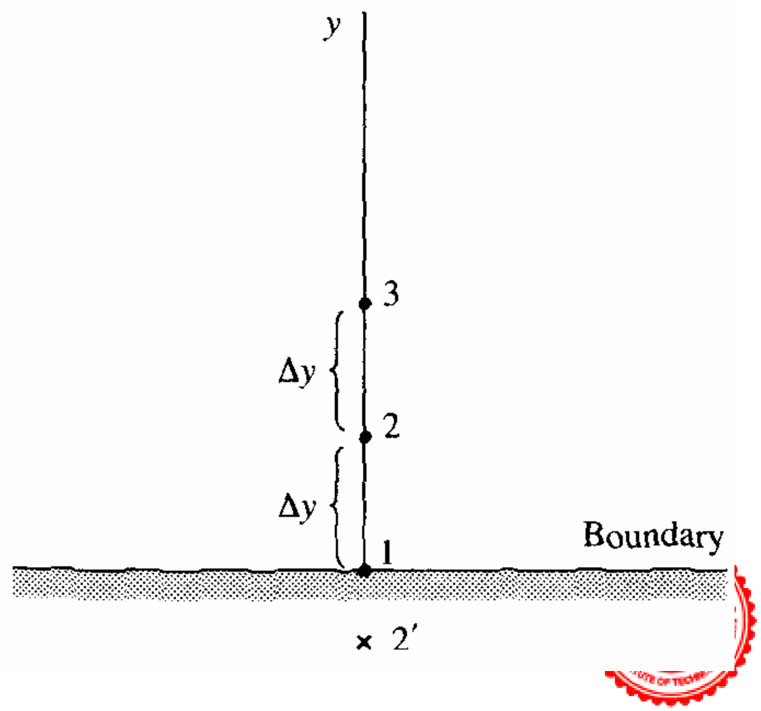


Boundary Consideration

- What kind of differencing scheme is possible at the boundaries?

$$\left(\frac{\partial \phi}{\partial y}\right)_1 = \frac{\phi_2 - \phi_1}{\Delta y} + O(\Delta y)$$

- Central difference approximation is not possible as point 2' is beneath the boundary
- How we can get a second order accurate scheme?
- Possibility: Polynomial approach



Polynomial Approach

- Assume $\phi = a + by + cy^2$

- At grid point 1, $\phi_1 = a$

- At grid point 2 where $y = \Delta y$,

$$\phi_2 = a + b\Delta y + c(\Delta y)^2$$

- At grid point 3 where $y = 2\Delta y$,

$$\phi_2 = a + b2\Delta y + c(2\Delta y)^2$$

Polynomial Approach (cont'd)

- Solving these equations, $b = \dfrac{-3\phi_1 + 4\phi_2 - \phi_3}{2\Delta y}$
- Differentiation of $\longrightarrow$ $\phi = a + by + cy^2$

  gives $\dfrac{\partial \phi}{\partial y} = b + 2cy$
- At point 1 (boundary), $\left(\dfrac{\partial \phi}{\partial y}\right)_1 = b$

$$\left(\frac{\partial \phi}{\partial y}\right)_1 = \frac{-3\phi_1 + 4\phi_2 - \phi_3}{2\Delta y}$$

- What is the order of approximation??

Order of Approximation

- Taylor series gives,

$$\phi(y) = \phi_1 + \left(\frac{\partial\phi}{\partial y}\right)_1 y + \left(\frac{\partial^2\phi}{\partial y^2}\right)_1 \frac{y^2}{2} + \left(\frac{\partial^3\phi}{\partial y^3}\right)_1 \frac{y^3}{6} + - - -$$

- Comparing with the polynomial expression , $\phi = a + by + cy^2$ we can say that our polynomial is of $O(\Delta y)^3$
- ϕ_1, ϕ_2, ϕ_3 can all be expressed in terms of the polynomial

$$\left(\frac{\partial\phi}{\partial y}\right)_1 = \frac{-3\phi_1 + 4\phi_2 - \phi_3}{2\Delta y} = \frac{O(\Delta y)^3}{\Delta y} = O(\Delta y)^2$$

- Represents one-sided difference of 2nd order accuracy

FDM for 1D diffusion

- Uses truncated Taylor series expansion to approximate the derivative of the DE
- Consider 1-D diffusion equation
- Expand ϕ in Taylor series about point 2

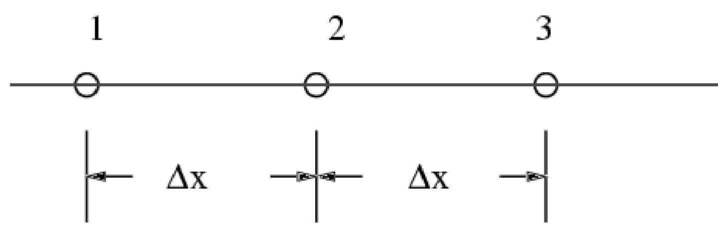
$$\Gamma \frac{d^2\phi}{dx^2} + S = 0$$

$$\phi_1 = \phi_2 - \Delta x \left(\frac{d\phi}{dx}\right)_2 + \frac{(\Delta x)^2}{2}\left(\frac{d^2\phi}{dx^2}\right)_2 + O\left((\Delta x)^3\right)$$

$$\phi_3 = \phi_2 + \Delta x \left(\frac{d\phi}{dx}\right)_2 + \frac{(\Delta x)^2}{2}\left(\frac{d^2\phi}{dx^2}\right)_2 + O\left((\Delta x)^3\right)$$

Subtracting these equations yields

$$\left(\frac{d\phi}{dx}\right)_2 = \frac{\phi_3 - \phi_1}{2\Delta x} + O((\Delta x)^2)$$



FDM (cont'd)

Adding the equations

$$\left(\frac{d^2\phi}{dx^2}\right)_2 = \frac{\phi_1 + \phi_3 - 2\phi_2}{(\Delta x)^2} + \mathbf{0}((\Delta x)^2)$$

Second order truncation error

Dropping the truncated terms

$$\Gamma\left(\frac{d^2\phi}{dx^2}\right)_2 = \Gamma\frac{\phi_1 + \phi_3 - 2\phi_2}{(\Delta x)^2}$$

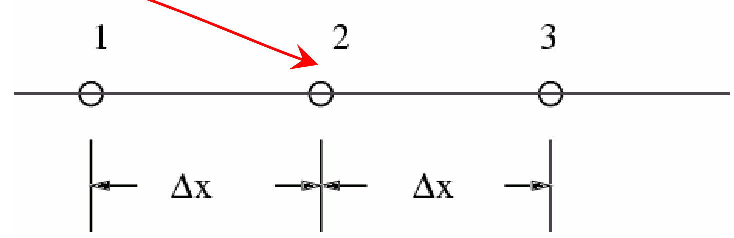
The final discretized equation

$S_2 = S(\phi_2)$

$$\frac{2\Gamma}{(\Delta x)^2}\phi_2 = \frac{\Gamma}{(\Delta x)^2}\phi_1 + \frac{\Gamma}{(\Delta x)^2}\phi_3 + S_2$$

FDM (cont'd)

$$\frac{2\Gamma}{(\Delta x)^2}\phi_2 = \frac{\Gamma}{(\Delta x)^2}\phi_1 + \frac{\Gamma}{(\Delta x)^2}\phi_3 + S_2$$

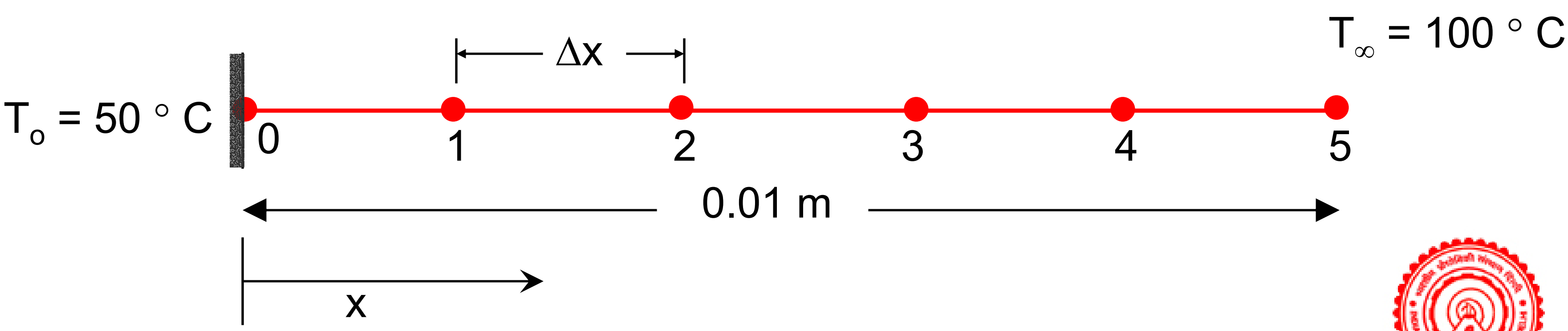


- We can write one such equation for each grid point
- Boundary conditions gives us boundary value
- Second order accurate
- Need to find a way to solve the couple algebraic equation set

1D Steady State Conduction

- Consider the steady state heat conduction in a slab of thickness L, in which energy is generated at a constant rate of S W/m^3. The boundary surface at x = 0 is maintained at a constant temperature $T_{o,}$ while the boundary surface at x = 0 dissipates heat by convection with a heat transfer coefficient h into an ambient at temperature T_∞ .

  Compute the temperature inside the slab for h = 200 W/(m^2.°C), k = 18 W/(m.°C), L = 0.01 m, T_∞ = 100°C, T_o = 50°C, and S = 7.2 x 10^7.



Solution

$$k\frac{d^2T}{dx^2}+S=0$$

B.C.

$$T(x)=T_o \qquad \text{at } x=0$$

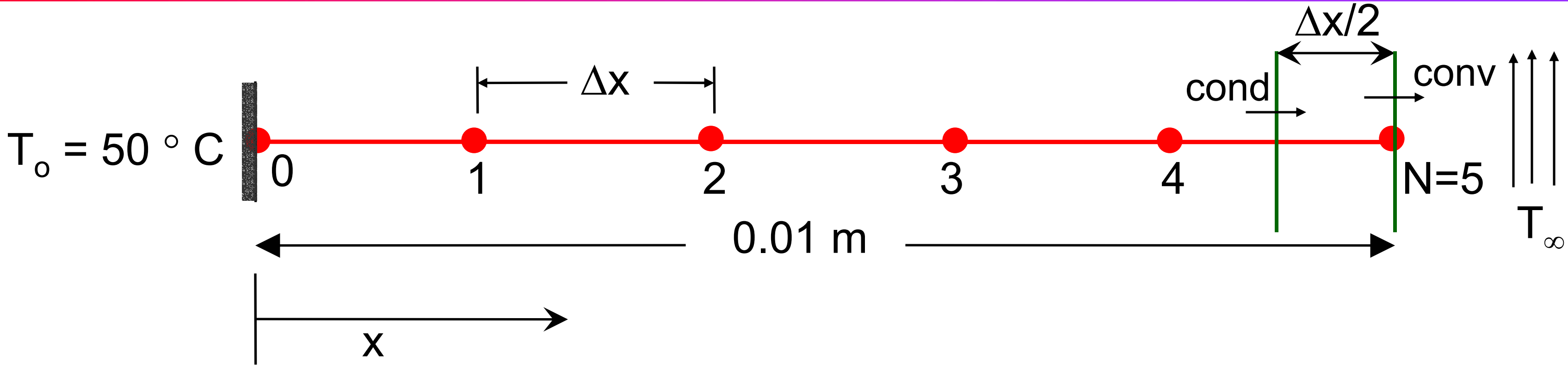
$$k\frac{dT(x)}{dx}+hT=hT_\infty \qquad \text{at } x=L=0.01\text{m}$$

$$\frac{2k}{(\Delta x)^2}T_i=\frac{k}{(\Delta x)^2}T_{i-1}+\frac{k}{(\Delta x)^2}T_{i+1}+S_i$$

$$\Rightarrow \frac{k}{(\Delta x)^2}T_{i-1}-\frac{2k}{(\Delta x)^2}T_i+\frac{k}{(\Delta x)^2}T_{i+1}=-S_i$$

$$\Rightarrow \boxed{T_{i-1}-2T_i+T_{i+1}=\frac{-(\Delta x)^2}{k}S_i}=-\frac{(L/5)^2}{k}S_i=-16$$

Treatment of Boundary Condition



$$T_{i-1} - 2T_i + T_{i+1} = -16$$

Applicable to node 1-4

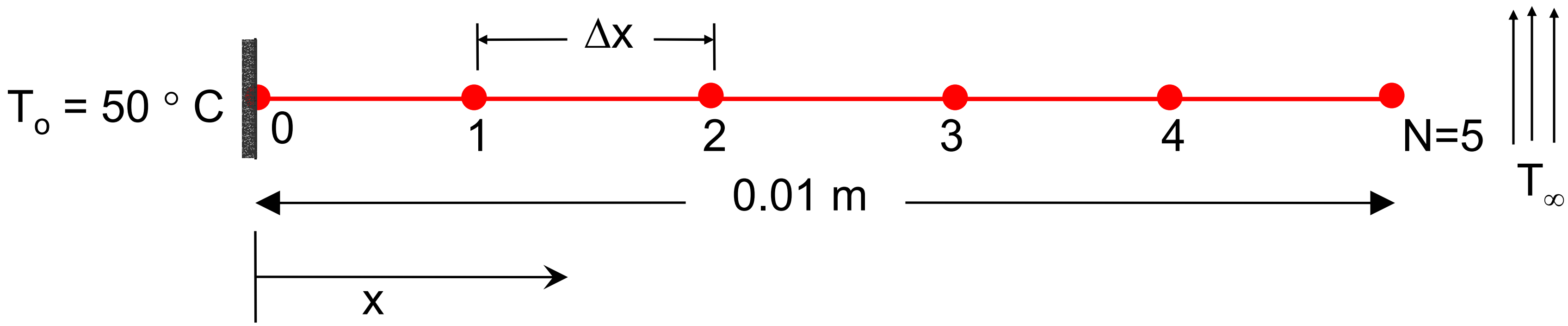
$$-k\frac{T_N - T_{N-1}}{\Delta x} - h(T_N - T_\infty) + S\frac{\Delta x}{2} = 0$$

$$\Rightarrow \boxed{2T_{N-1} - \left(2 + \frac{2\Delta x h}{k}\right)T_N + \frac{(\Delta x)^2 S_N}{k} + \frac{2\Delta x h}{k}T_\infty = 0}$$

Applying it to node 5

$$\Rightarrow 2T_4 - 2.044T_5 + 16 + 4.44 = 0$$

Treatment of Boundary Condition (Another Way)



$$T_{i-1} - 2T_i + T_{i+1} = -16$$

Apply to node 5 $$T_4 - 2T_5 + T_6 = -16$$

B.C. gives $$-k\frac{T_6 - T_4}{2\Delta x} - h(T_5 - T_\infty) = 0$$

$$\Rightarrow T_6 = -\frac{2\Delta x h}{k}(T_5 - T_\infty) + T_4$$

$$2T_4 - \left(2 + \frac{2h\Delta x}{k}\right)T_5 = -16 + \frac{2h\Delta x}{k}T_\infty$$

$$\Rightarrow 2T_4 - 2.044T_5 = -16 + 4.44$$

Algebraic Equations

$$T_0 - 2T_1 + T_2 = -16$$

$$-2T_1 + T_2 = -16 - 50$$

$$T_1 - 2T_2 + T_3 = -16$$

$$T_2 - 2T_3 + T_4 = -16$$

$$T_3 - 2T_4 + T_5 = -16$$

$$2T_4 - 2.044T_5 = -16 - 4.44$$

- Can be solved by Thomas' algorithm
- Matrix inversion as shown in next slide

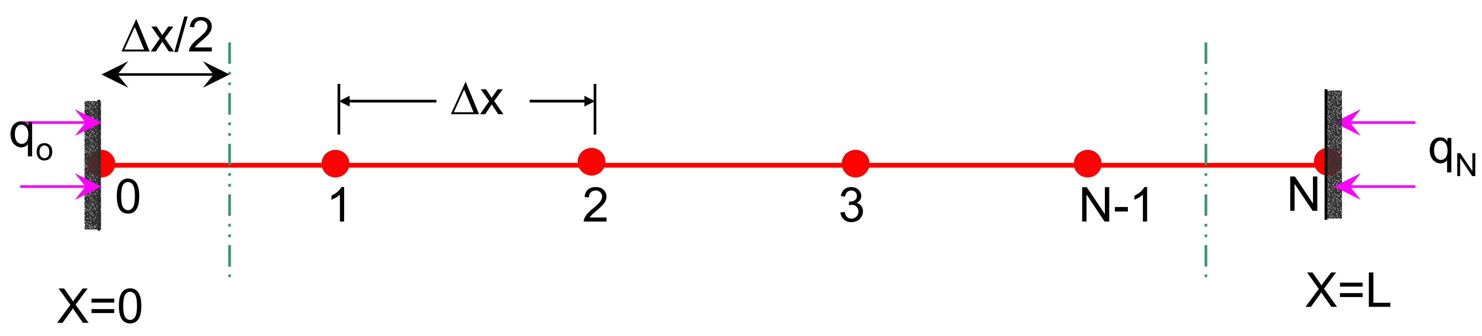
Matrix Form

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 2 & -2.044 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} -66 \\ -16 \\ -16 \\ -16 \\ -20.44 \end{bmatrix}$$

$$[A][x] = [B]$$

$$\Rightarrow [x] = [A]^{-1}[B]$$

Prescribed Heat Flux



Energy balance gives,

$$q_0 + k\frac{T_1 - T_0}{\Delta x} + \frac{1}{2}\Delta x S_0 = 0$$

$$q_N + k\frac{T_{N-1} - T_N}{\Delta x} + \frac{1}{2}\Delta x S_N = 0$$

FDM representation

Flux boundary,

$$2T_1 - 2T_0 + \frac{(\Delta x)^2 S_0}{k} + \frac{2\Delta x q_0}{k} = 0 \qquad \mathit{for} \quad i = 0$$

$$2T_{N-1} - 2T_N + \frac{(\Delta x)^2 S_N}{k} + \frac{2\Delta x q_N}{k} = 0 \qquad \mathit{for} \quad i = N$$

For insulated or symmetry boundary,

$$2T_1 - 2T_0 + \frac{(\Delta x)^2 S_0}{k} = 0 \qquad \mathit{for} \quad i = 0$$

$$2T_{N-1} - 2T_N + \frac{(\Delta x)^2 S_N}{k} = 0 \qquad \mathit{for} \quad i = N$$

Unsteady Heat Conduction with FDM

- Unsteady heat conduction in 1-D with constant thermal conductivity

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

- Expand the individual terms with Taylor series,

$$\left(\frac{\partial T}{\partial t}\right)_i^n = \frac{T_i^{n+1} - T_i^n}{\Delta t} - \left(\frac{\partial^2 T}{\partial t^2}\right)_i^n \frac{\Delta t}{2} + ----$$

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_i^n = \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} - \left(\frac{\partial^4 T}{\partial x^4}\right)_i^n \frac{(\Delta x)^2}{12} + ----$$

Unsteady Heat Conduction (cont'd)

$$\overbrace{\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2}}^{\text{Partial differential equation}} = 0 = \underbrace{\frac{T_i^{n+1} - T_i^n}{\Delta t} - \frac{\alpha (T_{i+1}^n - 2T_i^n + T_{i-1}^n)}{(\Delta x)^2}}_{\text{Difference equation}}$$

$$+ \underbrace{\left[- \left(\frac{\partial^2 T}{\partial t^2} \right)_i^n \frac{\Delta t}{2} + \alpha \left(\frac{\partial^4 T}{\partial x^4} \right)_i^n \frac{(\Delta x)^2}{12} + \cdots \right]}_{\text{Truncation error}}$$

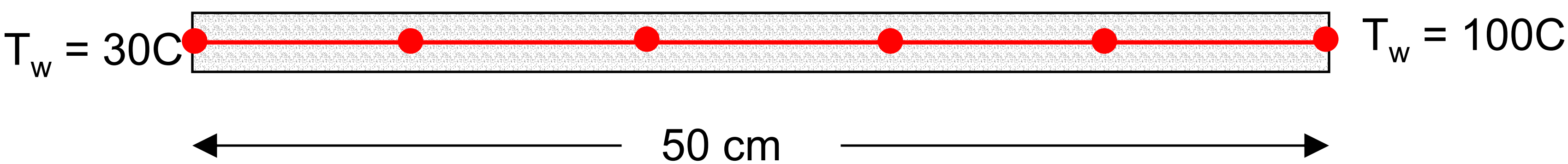
$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{\alpha (T_{i+1}^n - 2T_i^n + T_{i-1}^n)}{(\Delta x)^2}$$

Explicit Solutions

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{\alpha (T_{i+1}^n - 2T_i^n + T_{i-1}^n)}{(\Delta x)^2}$$

$$\Rightarrow T_i^{n+1} = T_i^n + \frac{\alpha \Delta t}{(\Delta x)^2}(T_{i+1}^n - 2T_i^n + T_{i-1}^n)$$

Find 1-D unsteady temperature distribution till steady state



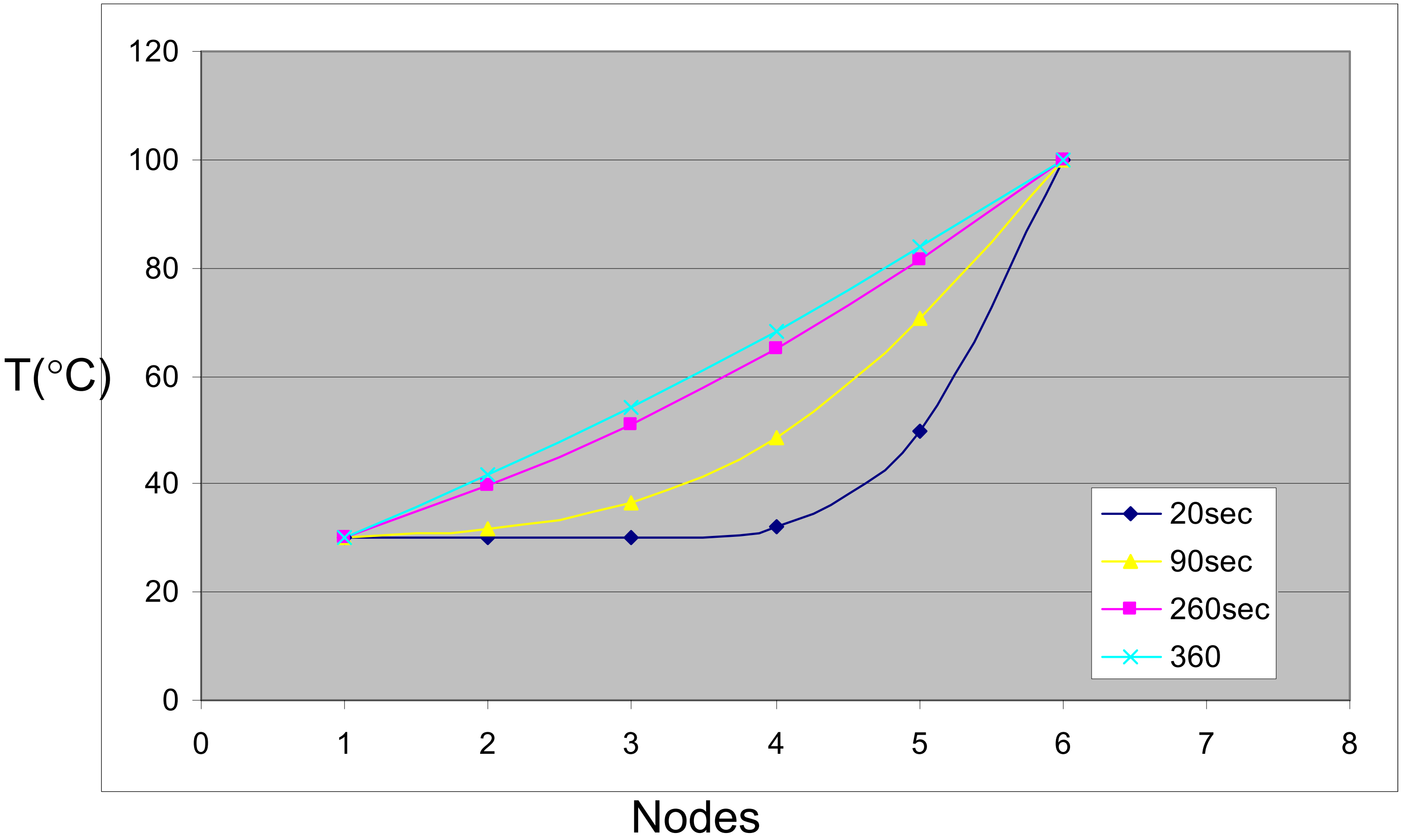
$$\Delta x = \frac{50}{5} = 10\text{cm} \qquad \alpha = 17\text{x}10^{-2}\,\text{cm}^2/\text{s}$$

Initial temp T_{in} = 30C, Δt = 10 sec

will talk later

$$\frac{\alpha \Delta t}{(\Delta x)^2} \le \frac{1}{2}$$

Results



120
100
80
60
40
20
0
T(°C)
0
1
2
3
4
5
6
7
8
Nodes
20sec
90sec
260sec
360

2D Steady State Heat Conduction

2D steady state with heat generation

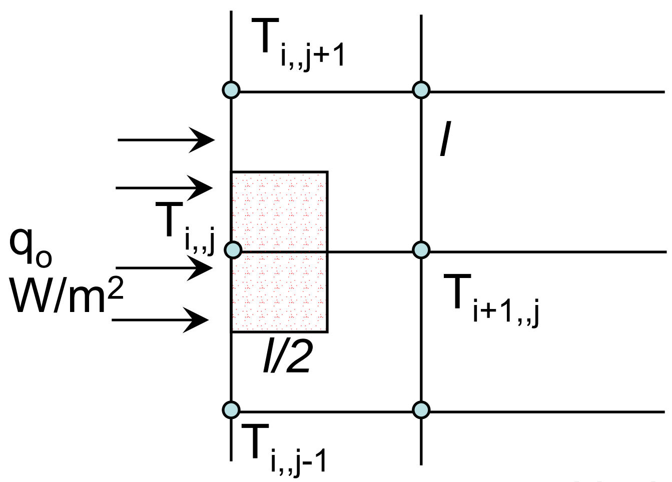
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{S(x,y)}{k} = 0$$

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} \qquad \left(\frac{\partial^2 T}{\partial y^2}\right)_{i,j} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

$$\boxed{T_{i-1,j} + T_{i+1,j} + T_{i,j-1} + T_{i,j+1} - 4T_{i,j} + \frac{S_{i,j} l^2}{k} = 0}$$

where $\Delta x = \Delta y = l$

Flux Boundary Condition



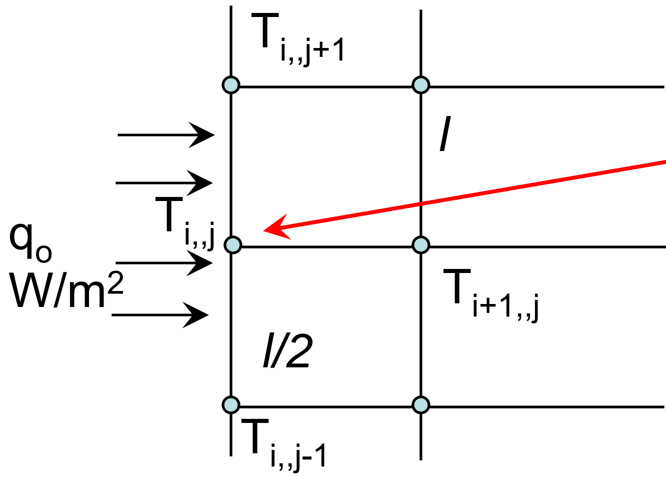
Nodes (i,j) on a prescribed heat flux boundary

$$q_0 l + k\frac{l}{2}\frac{T_{i,j+1}-T_{i,j}}{l} + kl\frac{T_{i+1,j}-T_{i,j}}{l} + k\frac{l}{2}\frac{T_{i,j-1}-T_{i,j}}{l} + \frac{1}{2}l^2 S_{i,j} = 0$$

After rearrangement

$$T_{i,j+1} + 2T_{i+1,j} + T_{i,j-1} - 4T_{i,j} + \frac{l^2 S_{i,j}}{k} + \frac{2lq_o}{k} = 0$$

Flux Boundary Condition
(another way)



Applying the finite difference equation at the boundary node (i, j)

$$T_{i-1,j} + T_{i+1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} + \frac{l^2 S_{i,j}}{k} = 0$$

B.C.

$$q_o = -k\frac{\partial T}{\partial x} = -k\frac{T_{i+1,j} - T_{i-1,j}}{2l}$$

$$\Rightarrow T_{i-1,j} = \frac{2lq_o}{k} + T_{i+1,j}$$

$$T_{i,j+1} + 2T_{i+1,j} + T_{i,j-1} - 4T_{i,j} + \frac{l^2 S_{i,j}}{k} + \frac{2lq_o}{k} = 0$$

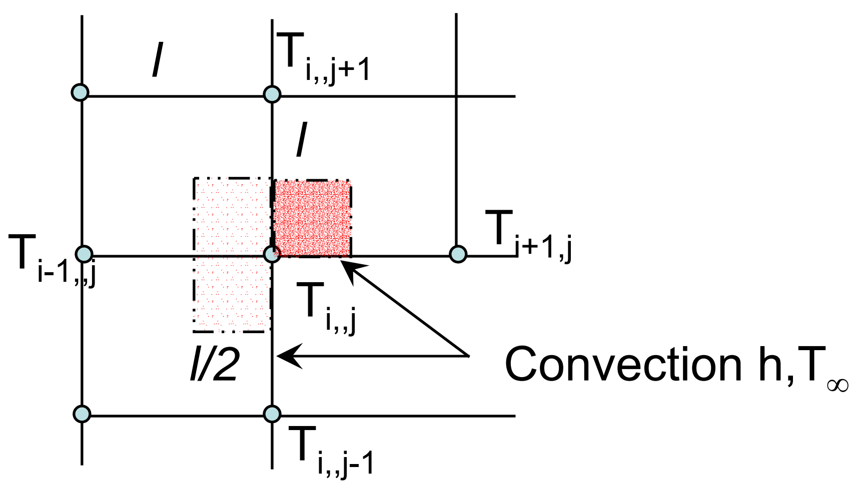
Convective Boundary Condition

Energy balance gives,

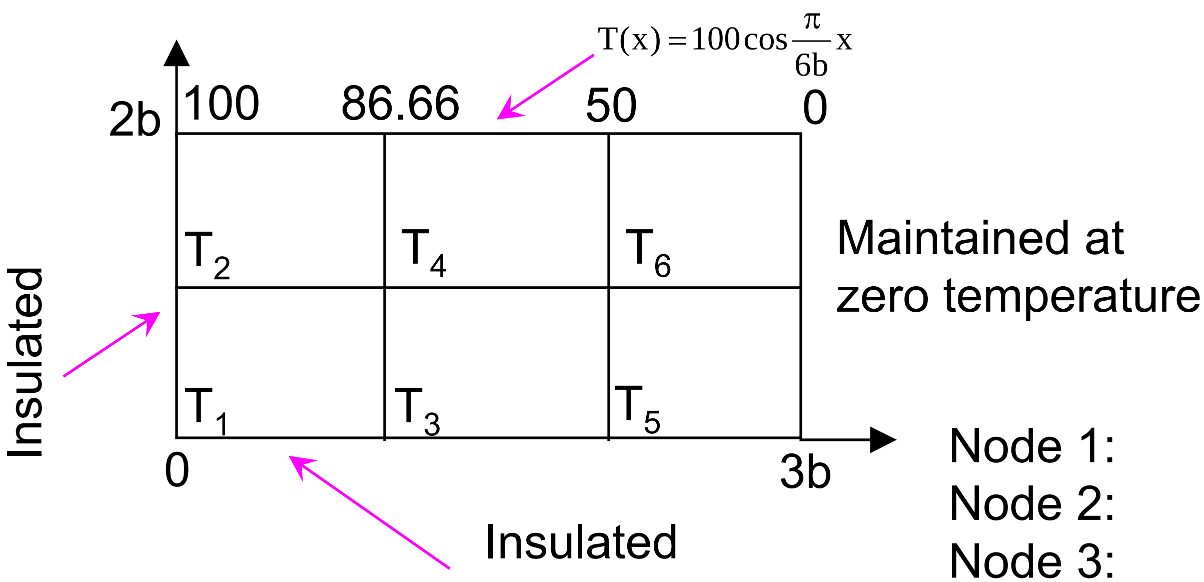
$$k\frac{l}{2}\frac{T_{i,j-1}-T_{i,j}}{l}+kl\frac{T_{i-1,j}-T_{i,j}}{l}+kl\frac{T_{i,j+1}-T_{i,j}}{l}+k\frac{l}{2}\frac{T_{i+1,j}-T_{i,j}}{l}+hl(T_\infty-T_{i,j})+\frac{3}{4}l^2S_{i,j}=0$$

After rearrangement,

$$T_{i,j-1}+2T_{i-1,j}+2T_{i,j+1}+T_{i+1,j}-\left(6+\frac{2hl}{k}\right)T_{i,j}+\frac{3}{2}\frac{l^2}{k}S_{i,j}+\frac{2hl}{k}T_\infty=0$$



Insulated Boundary



Node 1: $2T_2+2T_3-4T_1 = 0$

Node 2: $T_1+2T_4+100-4T_2 = 0$

Node 3: $T_1+2T_4+T_5-4T_3 = 0$

Node 4: $T_2+T_3+T_6+86.66-4T_4 = 0$

Node 5: $T_3+2T_6-4T_5 = 0$

Node 6: $T_4+T_5+50-4T_6 = 0$

Matrix Form

$$\begin{bmatrix} -4 & 2 & 2 & 0 & 0 & 0 \\ 1 & -4 & 0 & 2 & 0 & 0 \\ 1 & 0 & -4 & 2 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 & 1 \\ 0 & 0 & 1 & 0 & -4 & 2 \\ 0 & 0 & 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -100 \\ 0 \\ -86.66 \\ 0 \\ -50 \end{bmatrix}$$