FORMULAE FOR VECTOR ALGEBRA AND CALCULUS

1. Vector Algebra

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

$$\vec{A}.\vec{B}\times\vec{C} = \vec{B}.\vec{C}\times\vec{A} = \vec{C}.\vec{A}\times\vec{B} = -\vec{A}.\vec{C}\times\vec{B} \quad etc.$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$$

$$\vec{A}.\vec{B} = A_1B_1 + A_2B_2 + A_3B_3$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

2. Differential changes in unit vectors

$$\begin{aligned} d\vec{e}_r &= d\theta\vec{e}_\theta \\ d\vec{e}_\theta &= -d\theta\vec{e}_r \quad (cylindrical) \\ d\vec{e}_z &= 0 \end{aligned}$$

$$\begin{aligned} d\vec{e}_r &= d\theta\vec{e}_\theta + d\phi\sin\theta\vec{e}_\phi \\ d\vec{e}_\theta &= -d\theta\vec{e}_r + d\phi\cos\theta\vec{e}_\phi \qquad (spherical) \\ d\vec{e}_\phi &= -d\phi\sin\theta\vec{e}_r - d\phi\cos\theta\vec{e}_\theta \end{aligned}$$

3. Space Integrals

$$\oint_C \vec{D}.d\vec{s} = \textit{Circulation } \Gamma$$

$$\oint_S \vec{D}.\vec{n}dS = \textit{Net outflow of } \vec{D} \textit{ from } S$$

4. Differential operators.

$$grad\Phi = \frac{1}{d\tau}\oint_{\Delta S}\Phi\vec{n}dS \quad div\vec{A} = \frac{1}{d\tau}\oint_{\Delta S}\vec{A}.\vec{n}dS \quad curl\vec{A} = \frac{1}{d\tau}\oint_{\Delta S}-\vec{A}\times\vec{n}dS$$

$$\vec{e}_s.grad\Phi = \frac{\partial\Phi}{\partial s} \quad \vec{e}.curl\vec{A} = \frac{1}{d\sigma}\oint_{C_e}\vec{A}.d\vec{s}$$

$$grad\Phi = \nabla\Phi = \begin{cases} \vec{i}\frac{\partial\Phi}{\partial x} + \vec{j}\frac{\partial\Phi}{\partial y} + \vec{k}\frac{\partial\Phi}{\partial z} \\ \vec{e}_r\frac{\partial\Phi}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial\Phi}{\partial\theta} + \vec{e}_z\frac{\partial\Phi}{\partial z} \\ \vec{e}_r\frac{\partial\Phi}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial\Phi}{\partial\theta} + \frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial\Phi}{\partial\phi} \end{cases}$$

$$div\vec{A} = \nabla.\vec{A} = \begin{cases} \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ \frac{1}{r}\frac{\partial rA_r}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z} \\ \frac{1}{r^2}\frac{\partial r^2A_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \end{cases}$$

$$curl\vec{A} = \nabla\times A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & \vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\vec{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

5. Integral theorems.

$$\begin{array}{l} \textit{Gradient} \\ \textit{Divergence} \\ \textit{Curl} \end{array} \quad \oint_S \begin{pmatrix} \Phi \\ \vec{A}. \\ -\vec{A}\times \end{pmatrix} \vec{n}dS = \int_R \nabla \begin{pmatrix} \Phi \\ .\vec{A} \\ \times\vec{A} \end{pmatrix} d\tau$$

$$\textit{Stokes} \qquad \oint_C \vec{A}.d\vec{s} = \oint_S \nabla\times\vec{A}.\vec{n}dS$$

$$\textit{Green} \quad \begin{cases} 1\textit{st form} \quad \int_R (\psi\nabla^2\Phi + \nabla\psi.\nabla\Phi)d\tau = \oint_S \psi\nabla\phi.\vec{n}dS \\ 2\textit{nd form} \quad \int_R (\psi\nabla^2\Phi - \Phi\nabla^2\psi)d\tau = \oint_S (\psi\frac{\partial\Phi}{\partial n} - \Phi\frac{\partial\psi}{\partial n})dS \end{cases}$$

6. Second order relations, products and the convective operator.

$$\nabla.\nabla\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = \nabla^2\Phi \quad (\textit{the Laplacian})$$

(see also P 134 Karamcheti)

$$\nabla\times(\nabla\Phi) = 0 \quad (\textit{grad}\ \Phi\ \textit{is an irrotational field})$$

$$\nabla.(\nabla\times\vec{A}) = 0 \quad (\textit{curl}\ \vec{A}\ \textit{is a solenoidal field})$$

$$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla.\vec{A}) - \nabla^2\vec{A}$$

$$\nabla(\psi\Phi) = \psi\nabla\Phi + \Phi\nabla\psi$$

$$\nabla.(\Phi\vec{A}) = \Phi\nabla.\vec{A} + \nabla\Phi.\vec{A}$$

$$\nabla\times(\Phi\vec{A}) = \Phi\nabla\times\vec{A} + \nabla\Phi\times\vec{A}$$

$$\nabla(\vec{A}.\vec{B}) = (\vec{A}.\nabla)B + (\vec{B}.\nabla A) + \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A})$$

$$\nabla.(\vec{A}\times\vec{B}) = \vec{B}.\nabla\times\vec{A} - \vec{A}.\nabla\times\vec{B}$$

$$\nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla.\vec{B}) + (\vec{B}.\nabla)\vec{A} - \vec{B}(\nabla.\vec{A}) - (\vec{A}.\nabla)\vec{B}$$

$$(\vec{A}.\nabla)\Phi = \vec{A}.(\nabla\Phi)$$

$$(\vec{A}.\nabla)\vec{B} = \frac{1}{2}[\nabla(\vec{A}.\vec{B}) - \vec{A}\times(\nabla\times\vec{B}) - \vec{B}\times(\nabla\times\vec{A}) -$$

$$\nabla\times(\vec{A}\times\vec{B}) + \vec{A}(\nabla.B) - \vec{B}(\nabla.\vec{A})]$$

7. Substantial derivative.

$$\frac{D\vec{A}}{Dt} = \frac{\partial\vec{A}}{\partial t} + (\vec{V}.\nabla)\vec{A}$$

$$\frac{D\vec{V}}{Dt} = \frac{\partial\vec{V}}{\partial t} + (\vec{V}.\nabla)\vec{V}$$

$$= \frac{\partial\vec{V}}{\partial t} + \frac{\nabla V^2}{2} - \vec{V}\times(\nabla\times\vec{V})$$