MODULE 3

Topology and the Real Number Line

If 6 turned out to be 9, I don't mind – Jimi Hendrix

1. Sets

Skip this section if you remember everything you learned about sets from high school. It's kinda boring if you already know the stuff. (It's also boring if you don't.)

It is often useful to collect objects we wish to study into separate containers. In mathematics these containers are called sets. A **set** is a collection of objects. The terms "set," "collection," and "family" are synonomous. If A is a set, then "$x \in A$" means that x is an **element** (or **member**) of A, or that x **belongs to** A). The notation $x \notin A$ indicates x is **not** an element of A. Sets A and B are **equal**, $A = B$, if and only if they have the same elements.

Curly braces are used for set description. Sets may be specified by listing, for example $\{1, 2, 3\}$ or $\{1, 2, 3, \dots\}$. If $P(x)$ is a proposition about x, then $\{x : P(x)\}$ is the set of exactly those x for which $P(x)$ is true. For example, $\{n : n \text{ is a positive integer}\} = \{1, 2, 3, \dots\}$.

The **empty set**, denoted by $\emptyset$, is the set with no elements. Generally, we use "set" to mean "nonempty set."

A set A is a **subset** of a set B if and only if each element of A is also an element of B. The notation $A \subseteq B$ means that A is a subset of B. The set of primes is a subset of the set of positive integers. It is important that you understand the difference between $\in$ ("belongs to") and $\subseteq$ ("is a subset of").

Unions of sets are indicated by "$\cup$." Thus $A \cup B$ is the set which contains the elements of A together with the elements of B. The union of the Senate of the United States and the House of Representatives is the Congress of the United States.

Intersections of sets are indicated by "$\cap$." $A \cap B$ is the set of elements which belong to both sets A and B. The intersection of the set of even integers and the set of prime integers is $\{2\}$, the set that contains the single number 2. We say that two sets are **disjoint** if their intersection is the empty set, otherwise we say that the two sets **overlap**. The set of primes and the set of even numbers overlap, while the set of primes and the set of squares $\{1, 4, 9, 16, \cdots\}$ are disjoint.

Frequently one speaks of complements of sets. The **complement** of set A, written A^c, is the set of objects not in A. We have to be careful to specify the complete set of objects under consideration, called the **universal set**. For example, suppose A is the set of prime numbers. It is silly to say that Lassie (the TV dog) is an element of A^c because Lassie is not an element of A. Since our universe of discussion is the set of positive integers, the complement of the set of primes is the set of composite positive integers together with the special number 1 (which is not a prime and not a composite).

2. Distance

Everyone knows what distance is, however, most people would be hard-pressed to give an adequate definition. Consider the following examples. The distance from the earth to the moon is 239,000 miles. The distance from San Francisco to New York is 3025 miles. The distance from Forty-second Street and Broadway to Fifty-seventh Street and Seventh Avenue in New York City is 11 blocks. We know from geometry class that the shortest distance between two points is measured along a straight line. Suppose we define distance to be the shortest distance and consider the previous three examples. In the statement about the earth and the moon, we certainly mean straight line distance. In the statement about San Francisco and New York, however, we don't mean staight line distance, since the straight line would pass through the earth, making travel along this line unfeasible, to say the least. Distance between two cities is measured along the earth's surface. Yet in the statement that it is 11 blocks from Forty-second and Broadway to Fifty-first and Seventh Avenue, we again don't mean straight line distance, nor do we mean the shortest distance on the earth's surface. We mean instead the shortest distance if we restrict ourselves to traveling along the network of streets. We do not need to restrict ourselves to points in the physical universe. The word "not," for example, is a distance of one letter away from the word "now." Thus, in everyday usage, the meaning of the word distance is not precise, but must be judged from context. Still, the various meanings of the word distance have certain things in common. The distance from A to B, for example, is certainly the same

as the distance from B to A. The method of modern mathematics is to abstract from the various meanings the essential properties they have in common and then to take these as the defining properties of distance.

Let X be a non-empty set. A distance function on X is a way of assigning to each pair of points x and y in the set X a real number, $d(x, y)$, which represents the **distance** between x and y. Over the years the basic properties of distance have been distilled down to the following four properties:

$$\begin{aligned}
(i)&\ \ d(a,b) \geq 0 \\
(ii)&\ \ d(a,b) = 0 \ \text{ if and only if } \ a = b \\
(iii)&\ \ d(a,b) = d(b,a) \ \ \text{[symmetry]} \\
(iv)&\ \ d(a,c) \leq d(a,b) + d(b,c) \ \ \text{[triangle inequality]}.
\end{aligned}$$

If X is any non-empty set and d is a distance function on X, then X together with the function d is called a **metric space**.

Notice how general this definition is. It could apply to any non-empty set X, not just to real numbers or points in the plane. Moreover, different functions d could be be distance functions for the same set X.

Example 3.1 (Standard distance on $\mathbb{R}$)**.** If X is the set of real numbers, then the standard distance between two numbers x and y is found by subtracting them. We cannot, however, say that this distance is $x - y$, since x may be smaller than y. The distance between 2 and 5 is not $2 - 5 = -3$. If only we could throw away the negative sign in -3. We can. The absolute value function was invented to do just that, that is, the *absolute value* of a number t, written $|t|$, is the magnitude of t, *disregarding its sign.* So the distance between 2 and 5 is $|2 - 5| = |-3| = 3$. In general, the standard distance between the real numbers x and y is

$$d(x, y) = |x - y|.$$

Example 3.2 (Standard distance on $\mathbb{R}^2$)**.** The usual definition of distance between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the plane is

$$d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

You might recall that this is the *distance formula* you were told in high school algebra.

Example 3.3 (The Taxicab Metric)**.** If we restrict ourselves to horizontal and vertical lines (streets and avenues), then the distance between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the plane is

$$d(P, Q) = |x_1 - x_2| + |y_1 - y_2|.$$

It is not difficult to show that this distance functions satisfies the four basic properties.

Example 3.4 (Distance on a sphere)**.** Let X be a sphere and suppose that we restrict travel to the surface of the sphere. Then the distance between two points P and Q on the sphere is measured along the great circle of the sphere which passes through the two points. A *great circle* is a circle on the sphere whose center is the center of the sphere. Longitudes are great circles. The distance, for example, between the North Pole and the South Pole on the sphere we call planet Earth is half the circumference of the earth, or roughly 12500 miles.

Example 3.5 (Discrete Metric)**.** If X is *any* set, the following function is a distance function:

$$d(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b. \end{cases}$$

This function is called the *discrete* metric on X.

Example 3.6 (Distance between People)**.** Let X be the set of all people in the world. (We might have to exclude some true hermits and totally isolated people.) Define the distance between two people A and B to be the smallest number n such that there is a chain of n people P_1, P_2, P_3, $\cdots$, P_n, where P_1 knows P_2, P_2 knows P_3, etc., and the first person P_1 is A and the last person P_n is B. The distance between a person and his or herself is defined to be 0. This function satisfies the four properties of distance. (To make symmetry work, we will assume that if P knows Q, then Q knows P.) What is the maximal distance between any two people on earth? Sociologists have performed experiments where they ask a person, say, in California, to mail a letter to (presumably) a total stranger, say, in Maine. The instructions are to mail the letter to someone closer, who can in turn, forward the letter along its way. Apparently most people are within 4 or 5 contacts of each other.

Example 3.7 (Hamming Distance)**.** Let X be the set of all binary "words" of length 7. There are $2^7 = 128$ choices from 0000000 (base 2) to 1111111 (base 2). Define the distance between two words a and b to be the number of places that their binary representations disagree. For example, $d(1010101, 1001101) = 2$. This distance function is called the Hamming distance between the two words. The Hamming distance is important in devising error correcting codes, where one bit may change to another during transmission.

3.2 Exercises

1. If X is the set of real numbers, determine whether or not each of the following functions is a distance function. Here a and b are real numbers.

(a) $d(a,b) = a^2 + b^2$

(b) $d(a,b) = |a-b|$

(c) $d(a,b) = \min(|b-a|, 1)$

(d) $d(a,b) = |a^2 - b^2|$

(e) $d(a,b) = 0 \quad$ for all a, b.

In each case, find $d(-1,2)$.

2. If X is the x-y plane, determine whether or not each of the following functions is a distance function. Here $a = (x_1, y_1)$ and $b = (x_2, y_2)$.

(a) $d(a,b) = |x_1 - x_2| + |y_1 - y_2|$

(b) $d(a,b) = |(x_1^2 + x_2^2) - (y_1^2 + y_2^2)|$

(c) $d(a,b) = \max(|x_1 - x_2|, |y_1 - y_2|)$

(d) $d(a,b) = \min(|x_1 - x_2|, |y_1 - y_2|)$.

In each case, find $d((-1,3),(2,1))$.

3. Show that the function d_1 below is a distance formula between two points $p = (x_1, y_1)$ to $q = (x_2, y_2)$ in the x-y plane:

$$d_1(p,q) = [2(x_1 - x_2)^2 + 3(y_1 - y_2)^2]^{1/2}.$$

4. For each of the first four examples, determine the set

$$\{a \in X : d(x,0) = 1\}.$$

5. Show that: For every a, b, c in X,

$$|d(a,b) - d(b,c)| \leq d(a,c).$$

3. Inside, Outside, and Boundary

Suppose we are given a set A inside a metric space X, with a distance function d. Our goal is to figure out how to define the inside, outside, and boundary of our set A. Intuitively

we know what it means to be inside the house as opposed to outside the house. A small child with a coloring book and crayons can certainly point to the part inside the balloon she has just colored. In fact this last example motivates the concept. We will say that a point a is inside the balloon if, after coloring the balloon green, we can draw a small circle about point a so that every point inside the circle is green. If a is near the boundary of the balloon, then our circle may have to be very tiny. In mathematics, it is common to use the Greek letter epsilon (ϵ) to represent a (possibly) small amount.

Definition 3.1 (ϵ-ball)**.** Let ϵ be a positive real number and let $a \in X$. Then the set

$$B_\epsilon(a) \overset{\text{def}}{=} \{x \in X : \; d(a,x) < \epsilon\}$$

is called an open **ϵ-ball** or just **ball** about a.

Definition 3.2 (interior point)**.** A point $a \in X$ is an **interior point** of A if and only if there exists an $\epsilon > 0$ such that $B_\epsilon(a) \subseteq A$. The set of all interior points of A is called the **interior** of A and is denoted by Int A.

Definition 3.3 (boundary point)**.** A point $b \in X$ is a **boundary point** of A if and only if for all $\epsilon > 0$, $B_\epsilon(b)$ contains points in A and points not in A. The set of all boundary points of A is called the **boundary** of A and is denoted by Bd A.

Definition 3.4 (exterior point)**.** A point $c \in X$ is an **exterior point** of A if and only if there exists an $\epsilon > 0$ such that $B_\epsilon(c) \subseteq A^{\mathrm{c}}$. The set of all exterior points of A is called the **exterior** of A and is denoted by Ext A.

Theorem 3.1. Let A be a given set and let $b \in X$.

(1) If b lies in A, then b is not an exterior point of A.
(2) If b does not lie in A, then b is not an interior point of A.
(3) $\operatorname{Bd} A$ does not overlap $\operatorname{Int} A$.
(4) $\operatorname{Bd} A$ does not overlap $\operatorname{Ext} A$.

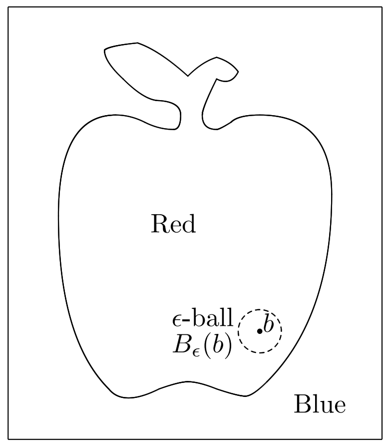
Let's play "mathematician" and try to prove the statement (1) of Theorem 3.1. First of all, since a statement is logically equivalent to its contrapositive, it may be easier to prove the contrapositive of (1):

If b is an exterior point of A, then $b \notin A$.

Suppose set A is an apple (A for Apple) in a child's coloring book and we have two crayons: red and blue. We will use red to color the points in set A (the apple) and blue to color the points not in A (the background). If b is an exterior point, then we can draw a

little round dot about b (like one of those small round bandages inside a box of band-aids) which lies entirely in a sea of blue points. Since the point b itself is covered by this dot, the color of b must be blue, meaning that b is not in A.

The Apple



We can beef up our coloring argument into a completely rigorous proof. Start by assuming that b is an exterior point of A. Then there exists an ϵ–ball $B_\epsilon(b)$ (our round band-aid) which lies entirely in A^c. Any point in the ϵ–ball $B_\epsilon(b)$ must also lie in the set A^c. The point b itself lies in $B_\epsilon(b)$ since $d(b, b)$ is 0 and 0 is less than ϵ no matter how small ϵ is. It follows that b lies in A^c, or equivalently, $b \notin A$.

Our argument proves the contrapositive of (1), which is logically equivalent to (1). $\square$

Exercise 1. Prove statement (2) of Theorem 3.1.

Moving to statement (3), we wish to prove that $\operatorname{Bd} A$ does not overlap $\operatorname{Int} A$. Let's give a "coloring" argument. Suppose we color the points of A red and the points of A^c blue. Pick any point p in the interior of A. By the definition of interior point, we know that there is some small disk (another round bandage) about p—call it B (for "band-aid")—which contains only red points.

We need to prove that p is *not* a boundary point. By definition, p is a boundary point if every disk centered about p contains both red and blue points. How do we negate this statement? Well, the negation of a "for every" statement is a "there exists" statement. Moreover, the negation of an "and" statement is an "or" statement. So p is *not* a boundary

point if there exists a small disk about p which either contains no red points or else contains no blue points. How are we going to find such a disk? Try using disk B in the paragraph above. Since it contains only red points, it cannot contain any blue points. So if you want a little disk about b which either contains no red points or else contains no blue points, then disk B does the job for you.

We're done. We have shown that if p is an interior point of A, then it cannot be a boundary point, justifying the statement in (3) that the interior and the boundary of A do not overlap. $\square$

Exercise 2. Try to beef up the above argument to a rigorous proof of statement (3). Hint: substitute (i) "ϵ–ball $B_\epsilon(b)$" for "disk B"; (ii) "$p \in A$" for "p is a red point"; and (iii) "$p \notin A$" for "p is a blue point."

Exercise 3. Prove statements (4) in Theorem 3.1.

Theorem 3.1 can be used to prove the following "intuitive" fact:

Theorem 3.2. Let A be a given set and b any point in X. Then exactly one of the following statements is true:
(1) b is an interior point of A;
(2) b is an exterior point of A;
(3) b is a boundary point of A.

Exercise 4. Prove Theorem 3.2.

The notion of interior and boundary points can be used to define the concept of open and closed sets.

Definition 3.5 (open set)**.** A set $A \subseteq X$ is an **open set** if and only if every point in A is an interior point of A.

Definition 3.6 (closed set)**.** A set $A \subseteq X$ is a **closed set** if and only if A contains all of its boundary points.

Different sports vary as to whether the boundary is considered "in-bounds," that is, part of the playing court. In tennis, for example, a ball hit on the line is considered "in play," so a tennis court includes its boundary and is therefore a closed set. On the other hand, in basketball, if your foot touches the line you will be whistled "out of bounds" by

the referee. So a basketball court does not include its boundary and is therefore an open set.

The collection of all open sets in a space is called the **topology** for the space.

Let's listen in on the Professor and his class.

Shelly: Up to this point the theorems have been very intuitive.

Terry: Yeah, what could be more obvious that saying you are either inside, outside, or on the boundary of a given set?

Professor: Don't get too comfortable thinking that topology involves formalizing obvious statements.

Ernest: I sense another weird example is coming.

Professor: As a matter of fact I do have an example for you to think about. Suppose our set A is a single point $A = \{a\}$. Is a a boundary point of A?

Terry: My first guess is "yes." Though I'm not sure why.

Shelly: To show that a is a boundary point of A, we must show that every ϵ–ball $B_\epsilon(a)$ contains points in A and points not in A. Pick any ϵ–ball $B_\epsilon(a)$ you like. It obviously contains one point of A, the point a itself. So we gotta show that $B_\epsilon(a)$ contains points not in A.

Diana: (interrupting) But every other point in $B_\epsilon(a)$ must be a point of A^c since a is the only point in A. So $B_\epsilon(a)$ must contains lots of points not in A.

Terry: Doesn't this prove that a is a boundary point?

Ernest: I'm convinced.

Professor: Something is wrong with the argument?

Try to find the flaw before reading further.

Professor: Diana is right. Any other point besides a in $B_\epsilon(a)$ must be in the complement of A. The problem is: how do we know that there actually are other points in $B_\epsilon(a)$ besides a itself? Remember that the ϵ–ball depends on our choice of the distance function d. Change d and you change $B_\epsilon(a)$. For example, what happens d is the discrete topology?

Terry: You mean where the distance between different points is always 1?

Professor: That's the discrete topology alright.

Shelly: I see what you're driving at. Suppose we make $\epsilon = \frac{1}{2}$. Then the only point inside the ϵ-ball $B_\epsilon(a)$ is a itself. You don't get any other points, because they're all at least one unit away from a.

Terry: Pretty tricky.

Diana: So a is not a boundary point?

Professor: It's not a boundary point when we use the discrete distance function. On the other hand, if our set is the x−y plane and we use the standard metric given by the distance formula you learned in high school, then Shelly and Diana's argument works perfectly and a is a boundary point. It all depends on what the distance function is.

Ernest: How can a subject that seems so obvious turn out to be so strange!

3.3 Exercises

The first four exercises are given in the text.

5. Describe intuitively the interior and boundary of the following sets: (a) Lake Michigan; (b) an apple sitting on a desk; (c) a birthday present, all wrapped up in fancy paper.

6. Describe $B_\epsilon(p)$ if
(a) $X = \mathbb{R}^2$, $d =$ the standard metric, $p = (0, 0)$, and $\epsilon = 1$.
(b) $X = \mathbb{R}^2$, $d =$ the taxicab metric, $p = (0, 0)$, and $\epsilon = 1$.
(c) $X =$ the set of code words of length 8, $d =$ the Hamming distance, $p = 10101010$, and $\epsilon = 1$.
(d) $X = \mathbb{R}$, $d =$ the discrete metric, $p = 0$, and $\epsilon = \frac{2}{3}$.

7. For the real number line with the usual distance function $d(x, y) = |x - y|$, determine the interior and boundary of the following sets:
(a) the interval (a, b)
(b) the intervals $(\frac{1}{2}, 1) \cup (\frac{1}{3}, \frac{1}{2}) \cup (\frac{1}{4}, \frac{1}{3}) \cup \cdots \cup (\frac{1}{n}, \frac{1}{n-1}) \cup \cdots$
(c) the intervals $(0, 1) \cup (1, 2) \cup (2, 3) \cup \cdots \cup (n - 1, n) \cup \cdots$
(d) the integers $\{n : n \text{ is an integer}\}$

(e) the Cantor Set
(f) the rational numbers.

8. For the $x - y$ plane with the usual topology, determine the interior and boundary of the following sets:
(a) the circle $x^2 + y^2 = 1$
(b) the unit disk $x^2 + y^2 \leq 1$

9. In three dimensional space $\mathbb{R}^3$, determine the interior and boundary of the following sets:
(a) the sphere $x^2 + y^2 + z^2 = 1$
(b) the ball $x^2 + y^2 + z^2 \leq 1$
(c) a solid doughnut

10. Given the set $X = \{a, b, c\}$ with the discrete topology, what are the open subsets of X? What are the closed subsets?

11. Give an example of a set A in a metric space X which is neither open nor closed.

12. Show that any ϵ-ball $B_\epsilon(a)$ is open.

13. Let A be a given set in a metric space X. Prove the follwoing statements:
(a) Int A is an open set.
(b) Ext A is an open set.
(c) Bd A is a closed set.

14. Must every singleton $\{a\}$ be a closed set?

15. Can a singleton $\{a\}$ ever be an open set?

16. Show that Bd $A =$ Bd A^c.

17. Show that Int $A =$ Ext A^c.

4. Connectedness

Our goal is to define what it means for a set to be connected. Our intuitive notion is that a set is connected if it is "in one piece." It turns out this notion is difficult to define precisely. It is easier to think negatively and concentrate on sets which are *not* connected.

A disconnected set A must consist of at least two pieces M and K which do not touch each other. Paint the points in set M blue and the points in set K yellow. If the two pieces M and K are disconnected, then we will see no green (= blue + yellow) points, that is, the sets M and K cannot overlap. But what if M and K share a boundary point? For example, suppose our space is the real number line and we consider the following sets. Which ones do we want to call disconnected?

1. $A = [0, 1]$
2. $B = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$
3. $C = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$
4. $D = [0, \frac{1}{2}] \cup (\frac{1}{2}, 1]$

It is clear that set A is connected while set B is not. But what about sets C and D? C consists of two pieces $M = [0, \frac{1}{2})$ and $K = (\frac{1}{2}, 1]$. These pieces have a common boundary point $b = \frac{1}{2}$, but this boundary point does not lie in either piece M or K. So set C is disconnected, the result of cutting the interval $[0, 1]$ in two pieces at the point $\frac{1}{2}$. On the other hand, set D consists of two pieces $M' = [0, \frac{1}{2}]$ and $K' = (\frac{1}{2}, 1]$ with the common boundary point $b = \frac{1}{2}$, but in this case, b belongs to M'. Here set D is connected since it equals set A, and A is connected.

Let's generalize. We say that a set A is disconnected if we can break A into two separated pieces M and K. By saying M and K are separated, we mean that if M and K touch at a common boundary point b, then b cannot lie in either piece M or K. Since open sets do not contain any of their boundary points, we can use them to isolate the two separated pieces M and K. The idea is to find an open set U which contains M and an open set V which contains K. If U and V are not empty, do not overlap, and completely cover A, then we say that A is disconnected. Here is the formal definition.

Definition 3.7. A set A is **disconnected** if and only if there exist open sets U and V with the properties:

$$(i)\ A \cap U \neq \emptyset \quad \text{and} \quad A \cap V \neq \emptyset$$
$$(ii)\ A \subseteq U \cup V$$
$$(iii)\ U \cap V = \emptyset.$$

A set A is **connected** if and only if it is not disconnected.

What does it mean for the whole space X to be disconnected? We can just apply our definition with $A = X$.

Theorem 3.3. The space X is **disconnected** if and only if there exist open sets U and V with the properties:

$$(i)\ U \neq \emptyset \quad \text{and} \quad V \neq \emptyset$$
$$(ii)\ U \cup V = X$$
$$(iii)\ U \cap V = \emptyset.$$

Definition 3.8. A **component** of a set A is a maximal connected set, i.e., a subset B such that B is connected, but if $B \subseteq C \subseteq A$ and $B \neq C$, then C is not connected, that is, no larger set than B lying inside A is connected.

It can be shown that the set X is connected if and only if X has exactly one component, namely X itself.

3.4 Exercises

1. Is a map of the United States connected?

2. Which of the following sets in $\mathbb{R}^2$ are connected?

a. The unit circle $\{(x, y) : x^2 + y^2 = 1\}$

b. The unit disk $\{(x, y) : x^2 + y^2 \leq 1\}$

c. Two concentric circles $\{(x, y) : x^2 + y^2 = 1\} \cup \{(x, y) : x^2 + y^2 = 2\}$

d. The "infinite ladder" which consists of the following pieces: (i) the y-axis $x = 0$, (ii) the vertical line $x = 1$, (iii) for each integer n the "rung" $\{(x, n) : 0 \leq x \leq 1\}$.

3. What are the components of the following sets in $\mathbb{R}^2$?

a. The unit circle $\{(x, y) : x^2 + y^2 = 1\}$

b. The integer lattice points $\{(m, n) : m \text{ and } n \text{ are integers}\}$

c. Two concentric circles $\{(x, y) : x^2 + y^2 = 1\} \cup \{(x, y) : x^2 + y^2 = 2\}$

d. The set which consists of infinitely many disks: the n^{th} piece is the disk of radius $\frac{1}{n}$ centered on the x-axis at the point $(n, 0)$.

4. Let X be a space with the discrete distance function $d(x, y) = 1$ if and only if $x \neq 1$. Is each singleton $\{x\}$ a component of X?

5. Topology of the Real Numbers

5.1. The Least Upper Bound Axiom. By the *real numbers* we mean the metric space $\mathbb{R}$ consisting of the set of real numbers together with the usual metric

$$d(a, b) = |a - b|.$$

The set of real numbers satisfies exactly the same properties for addition and multiplication as the set of rational numbers. In both systems, addition is commutative and associative:

$$x + y = y + x$$

and

$$(x + y) + z = x + (y + z).$$

The distributive law

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$

holds for both systems. So what distinguishes the real numbers from the rationals? Topology provides a way to answer this question. It turns out that the real numbers are connected, that is to say, there are enough real numbers to glue them all together in a line. The rational numbers, on the other hand, are not connected, as we shall shortly see.

In order to pursue questions about the connectedness of the real numbers, we need a few basic concepts. A set A is called **bounded above** if there is a real number M such that

$$a \leq M$$

for all elements a in A. The **least upper bound** (or LUB) of set A is an upper bound L with the property that if M is any upper bound of A, then $L \leq M$. For example, suppose A is the set of heights of all people on earth. Clearly an upper bound of A is 12 feet. The least upper bound of A is the height of the tallest person in the world.

We shall need the following axiom for the real numbers:

Least Upper Bound Axiom for the Real Numbers:

$$\boxed{\text{Every nonempty set } A \subseteq \mathbb{R} \text{ which has an upper bound has a least upper bound.}}$$

The statement inside the box is false if we replace $\mathbb{R}$ with $\mathbb{Q}$, the set of rational numbers.

3.5.1 Exercises

Find the least upper bound, if it exists, of the following sets in $\mathbb{R}$.

1. $\{x : x^2 < 2\}$
2. $\left\{\dfrac{n-1}{n} : n \text{ a positive integer}\right\}$
3. $(0, 1)$
4. $\{4.9, 4.99, 4.999, \dots\}$
5. $\left\{\displaystyle\sum_{k=1}^{n} \frac{1}{2^k} : n \text{ a positive integer}\right\}$
6. $\left\{\displaystyle\sum_{k=1}^{n} \frac{1}{k} : n \text{ a positive integer}\right\}$.

7. Show that the LUB axiom fails to hold for the rational numbers. Hint: Look at Exercise 1 above.

5.2. Connected subsets of $\mathbb{R}$.

Theorem 3.4. If $\emptyset \neq A \subseteq \mathbb{R}$ has an upper bound and $x_\circ = \text{LUB}\, A$, then $x_\circ \in \text{Bd}\ A$.

Theorem 3.4 can be used to prove the following important fact about the real number line:

Theorem 3.5. $\mathbb{R}$ is connected.

Let's write down a "coloring" proof. Suppose, to the contrary, that $\mathbb{R}$ is not connected. Then $R = U \cup V$, where U and V are nonempty, disjoint, open sets. Color the points of U red and the points of V blue. Since U and V are disjoint and since their union is the entire number line, every point in $\mathbb{R}$ has a unique color: red or blue. Since U is nonempty, there must be a red point—call it a, and likewise there must be a blue point—call it b. We may assume that $a < b$, otherwise we could just switch colors.

Let A be the set of all red points in the closed interval $[a, b]$. The set A is bounded by b, so by the LUB axiom, A has a least upper bound—call it $x_\circ$. Here's where the previous theorem comes in. By Theorem 3.4, $x_\circ$ is a boundary point of A. This means that every ϵ-ball $B_\epsilon(x_\circ)$ contains points in A (red points) and points not in A (blue points).

Here comes the killer question: What color is $x_\circ$?

If you answer "red," then set U has a boundary point $x_\circ$. But that ain't right; an open set does not contain any of its boundary points. If you answer "blue," then you run into the same problem with the open set V. Either way you run into a contradiction. This mess came about because we made the assumption that $\mathbb{R}$ was disconnected. We must have been wrong and the reals must, in fact, be connected. $\square$

Theorem 3.5 gives us a formal answer to our old question "What holds the real numbers together?" Indeed topology provides a way to formally define what we mean by saying that the real numbers are connected. Once we have a precise definition of "connected set," then proving that the reals are connected boils down to applying the least upper bound axiom.

If we restrict ourselves to just the rational numbers on the number line, then we lose connectivity.

Theorem 3.6. The space $\mathbb{Q}$ of rational numbers is disconnected.

It can be shown that the only connected subsets of $\mathbb{R}$ are finite intervals $[a, b]$, (a, b), $[a, b)$, $(a, b]$, rays $(-\infty, a]$, $(-\infty, a)$, $[b, \infty)$, (b, ∞), and $\mathbb{R}$ itself.

3.5.2 Exercises

1. Prove Theorem 3.4.

2. Prove Theorem 3.6.

5.3. Dense Sets.

Definition 3.9. A subset D is **dense** in X if and only if every open set U contains some element of D.

Exercise. Show that the set of rationals $\mathbb{Q}$ is a dense subset of the set of real numbers $\mathbb{R}$.

Hint: Given an interval (a, b) look at the decimal expansion of a and b.

This exercise is an understatement. It turns out that every open interval, no matter how small, contains infinitely many rational numbers. They're everywhere. If you stroll along the number line, for even an extemely short distance, you will pass millions, billions, zillions, infinitely many of them. The rationals are not in any danger of becoming extinct.

It is also true that the irrational numbers form a dense subset of the reals. The set of integers, on the other hand, are not dense. It is easy to write down an interval which does not contain an integer.

6. Topological Equivalence

Two sets A and B are **topologically equivalent** if there is a one–to–one correspondence between them which is continuous both ways. The standard inuitive notion is that if two sets are topologically equivalent, it is possible to deform one onto the other by "stretching, pulling, or bending—but without breaking, cutting, gluing, or tearing." For example, you can bend a line segment to make the letter L or you can curve it to make the letter S. So the letters L and S are topologically equivalent to the line segment $[0, 1]$. On the other hand a circle is *not* equivalent to a line segment, because you need to glue the two endpoints of the segment together in order to make a circle.

3.6 Exercises

1. Which of the following objects are topologically equivalent:

   (a) a cube

   (b) a theta curve θ

   (c) a circle

   (d) a square disk pierced by a segment

   (e) a ball

   (f) a hemisphere with a tangent line segment at the bottom

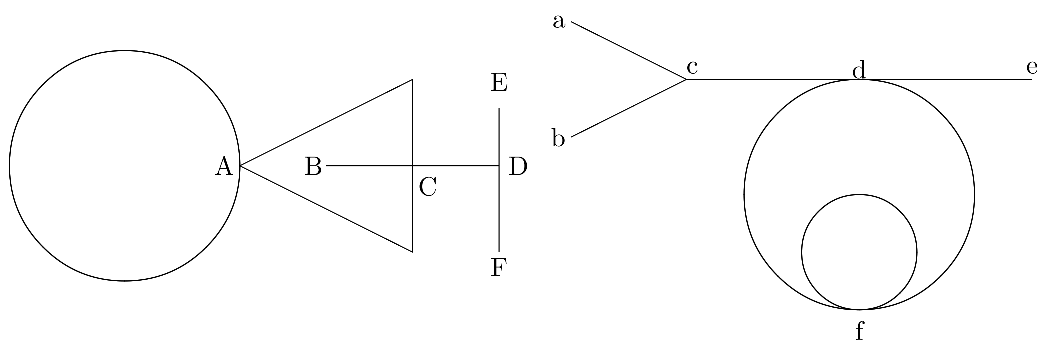
   (g) a figure 8

   (h) a capital letter B

   (i) a square

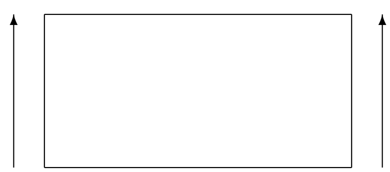
2. Write the capital letters of the alphabet (without serifs). Group the letters so that letters in the same group are topolgically equivalent. For example, S and L are in the same group.

3. Show that the following figures are topologically equivalent. Which points in the second figure correspond to the points A, B, C, D, E, F in the first figure?

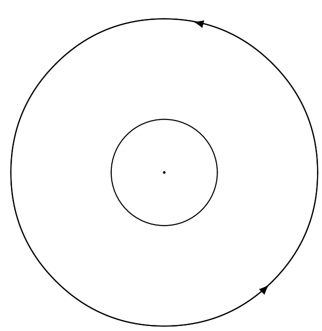


7. Surfaces

Suppose we take a rectangular sheet of paper and glue the left and right edges together. What do we get?

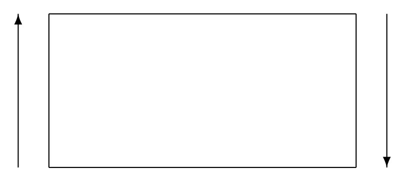


The answer is: a cylinder (or tin can without lids). If we flatten the cylinder we get an annulus, the region between two concentric circles.



Problem 3.7. Explain how a cylinder is topologically equivalent to an annulus.

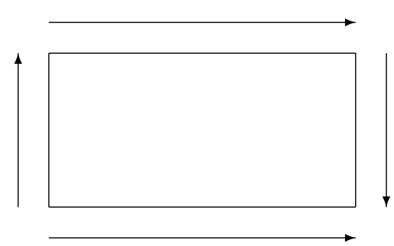
Now suppose we make a half twist before glueing the left and right edges together, so that the sides are connected as indicated by the arrows in the following diagram:



The resulting band is called a Möbius strip. It is a one sided surface. To see this, trying drawing on one side "around the band" and you will discover that "both sides" of the original rectangle are marked.

Problem 3.8. Make your own Möbius strip by taping the edges of a rectangular band with a half twist. Now draw a line in the middle of the Möbius strip. Cut along this middle line with a pair of scissors. What happens? Can you explain why?

If we tape the top and bottom sides of a rectangle, as well as the left and right hand sides, we get a doughnut or **torus**.



Problem 3.9. How many sides does a torus have? What do we get if we cut a torus in half lengthwise, that is, by cutting along a dotted line marked on the middle of the original rectangle? Does it matter if the dotted line is horizontal or vertical?