Chapter 1
Vector Algebra

1.1 Terminology and Notation

Scalars are mathematics quantities that can be fully defined by specifying their magnitude in suitable units of measure. Mass is a scalar quantity and can be expressed in kilograms, time is a scalar and can be expressed in seconds, and temperature is a scalar quantity that can be expressed in degrees Celsius.

Vectors are quantities that require the specification of magnitude, orientation, and sense. The characteristics of a vector are the magnitude, the orientation, and the sense.

The *magnitude* of a vector is specified by a positive number and a unit having appropriate dimensions. No unit is stated if the dimensions are those of a pure number.

The *orientation* of a vector is specified by the relationship between the vector and given reference lines and/or planes.

The *sense* of a vector is specified by the order of two points on a line parallel to the vector.

Orientation and sense together determine the *direction* of a vector.

The *line of action* of a vector is a hypothetical infinite straight line collinear with the vector.

Displacement, velocity, and force are examples of vectors quantities.

To distinguish vectors from scalars it is customary to denote vectors by boldface letters Thus, the displacement vector from point A to point B could be denoted as $\mathbf{r}$ or $\mathbf{r}_{AB}$. The symbol $|\mathbf{r}| = r$ represents the magnitude (or module, norm, or absolute value) of the vector $\mathbf{r}$. In handwritten work a distinguishing mark is used for vectors, such as an arrow over the symbol, $\overrightarrow{r}$ or $\overrightarrow{AB}$, a line over the symbol, $\bar{r}$, or an underline, $\underline{r}$.

The vectors are most frequently depicted by straight arrows. A vector represented by a straight arrow has the direction indicated by the arrow. The displacement vector from point A to point B is depicted in Fig. 1.1(a) as a straight arrow. In some cases it is necessary to depict a vector whose direction is perpendicular to the surface

in which the representation will be drawn. Under this circumstance the use of a portion of a circle with a direction arrow is useful. The orientation of the vector is perpendicular to the plane containing the circle and the sense of the vector is the same as the direction in which a right-handed screw moves when the axis of the screw is normal to the plane in which the arrow is drawn and the screw is rotated as indicated by the arrow. Figure 1.1(b) uses this representation to depict a vector directed out of the reading surface toward the reader.

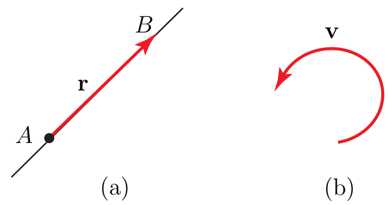


Fig. 1.1 Representations of vectors

A *bound* vector is a vector associated with a particular point P in space (Fig. 1.2). The point P is the *point of application* of the vector, and the line passing through P and parallel to the vector is the line of action of the vector. The point of application may be represented as the tail, Fig. 1.2(a), or the head of the vector arrow, Fig. 1.2(b). A *free* vector is not associated with any particular point in space. A *transmissible* (or *sliding*) vector is a vector that can be moved along its line of action without change of meaning.

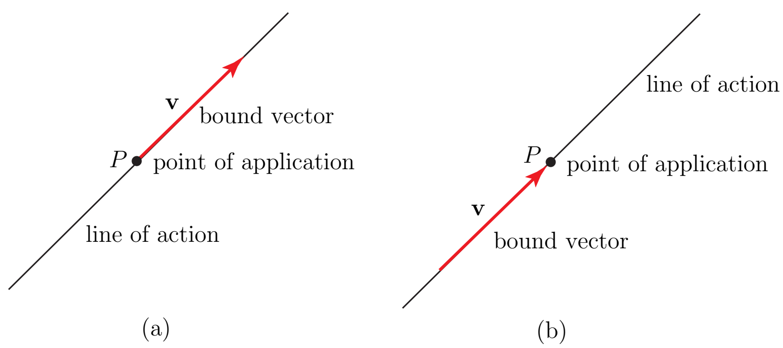


Fig. 1.2 Bound or fixed vector: (a) point of application represented as the tail of the vector arrow and (b) point of application represented as the head of the vector arrow

To move the rigid body in Fig. 1.3 the force vector $\mathbf{F}$ can be applied anywhere along the line Δ or may be applied at specific points A, B and C. The force vector $\mathbf{F}$ is a transmissible vector because the resulting motion is the same in all cases.

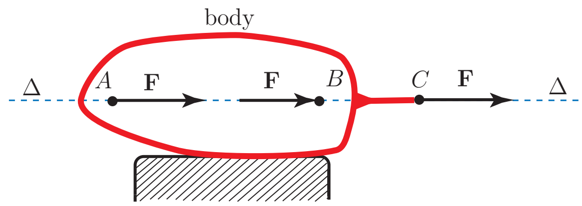


Fig. 1.3 Transmissible vector: the force vector **F** can be applied anywhere along the line *Δ*

If the body is not rigid, the force **F** applied at A will cause a different deformation of the body than **F** applied at a different point B. If one is interested in the deformation of the body, the force **F** positioned at C is a bound vector.

The operations of vector analysis deal only with the characteristics of vectors and apply, therefore, to bound, free, and transmissible vectors.

Equality
Two vectors **a** and **b** are said to be equal to each other when they have the same characteristics. One then writes

$$\mathbf{a} = \mathbf{b}. \tag{1.1}$$

Equality does not imply physical equivalence. For instance, two forces represented by equal vectors do not necessarily cause identical motions of a body on which they act.

Product of a Vector and a Scalar
The product of a vector **v** and a scalar s, $s\mathbf{v}$ or $\mathbf{v}s$, is a vector having the following characteristics:

1. Magnitude. $|s\mathbf{v}| \equiv |\mathbf{v}s| = |s||\mathbf{v}|$, where $|s| = s$ denotes the absolute value (or magnitude, or module) of the scalar s.
2. Orientation. $s\mathbf{v}$ is parallel to **v**. If $s = 0$, no definite orientation is attributed to $s\mathbf{v}$.
3. Sense. If $s > 0$, the sense of $s\mathbf{v}$ is the same as that of **v**. If $s < 0$, the sense of $s\mathbf{v}$ is opposite to that of **v**. If $s = 0$, no definite sense is attributed to $s\mathbf{v}$.

Zero Vector
A *zero vector* is a vector that does not have a definite direction and whose magnitude is equal to zero. The symbol used to denote a zero vector is **0**.

Unit Vector
A *unit vector* is a vector with magnitude equal to 1. Given a vector **v**, a unit vector **u** having the same direction as **v** is obtained by forming the product of **v** with the reciprocal of the magnitude of **v**

$$\mathbf{u} = \mathbf{v}\,\frac{1}{|\mathbf{v}|} = \frac{\mathbf{v}}{|\mathbf{v}|}\ . \tag{1.2}$$

Vector Addition
The sum of a vector $\mathbf{v}_1$ and a vector $\mathbf{v}_2$: $\mathbf{v}_1 + \mathbf{v}_2$ or $\mathbf{v}_2 + \mathbf{v}_1$ is a vector whose characteristics can be found by either graphical or analytical processes. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ add according to the parallelogram law: the vector $\mathbf{v}_1 + \mathbf{v}_2$ is represented by the diagonal of a parallelogram formed by the graphical representation of the vectors, see Fig. 1.4(a).

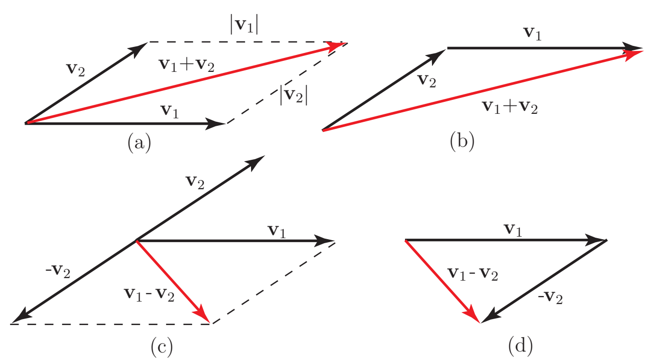


Fig. 1.4 Vector addition: (a) parallelogram law, (b) moving the vectors successively to parallel positions. Vector difference: (c) parallelogram law, (d) moving the vectors successively to parallel positions

The vector $\mathbf{v}_1 + \mathbf{v}_2$ is called the *resultant* of $\mathbf{v}_1$ and $\mathbf{v}_2$. The vectors can be added by moving them successively to parallel positions so that the head of one vector connects to the tail of the next vector. The resultant is the vector whose tail connects to the tail of the first vector, and whose head connects to the head of the last vector, see Fig. 1.4(b).

The sum $\mathbf{v}_1 + (-\mathbf{v}_2)$ is called the *difference* of $\mathbf{v}_1$ and $\mathbf{v}_2$ and is denoted by $\mathbf{v}_1 - \mathbf{v}_2$, see Figs. 1.4(c) and 1.4(d). The sum of n vectors $\mathbf{v}_i$, $i = 1, \ldots, n$,

$$\sum_{i=1}^{n} \mathbf{v}_i \text{ or } \mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_n$$

is called the *resultant* of the vectors $\mathbf{v}_i$, $i = 1, \ldots n$.

Vector addition is:

1. commutative, that is, the characteristics of the resultant are independent of the order in which the vectors are added (commutativity law for addition)

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1.$$

2. associative, that is, the characteristics of the resultant are not affected by the manner in which the vectors are grouped (associativity law for addition)

$$\mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3) = (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3.$$

3. distributive, that is, the vector addition obeys the following laws of distributivity

$$(s_1 + s_2)\mathbf{v} = s_1\mathbf{v} + s_2\mathbf{v} \quad \text{and} \quad s(\mathbf{v}_1 + \mathbf{v}_2) = s\mathbf{v}_1 + s\mathbf{v}_2,$$

or equivalent (for the general case)

$$\mathbf{v}\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} (\mathbf{v}s_i) \quad \text{and} \quad s\sum_{i=1}^{n} \mathbf{v}_i = \sum_{i=1}^{n} (s\mathbf{v}_i).$$

Moreover, the characteristics of the resultant is not affected by the manner in which the vector is multiplied with scalars (associativity law for multiplication)

$$s_1\,(s_2\mathbf{v}) = (s_1 s_2)\,\mathbf{v}.$$

Every vector can be regarded as the sum of n vectors ($n = 2, 3, \ldots$) of which all but one can be selected arbitrarily.

Linear Independence

If $\mathbf{v}_i$, $i = 1, \ldots n$ are vectors and s_i, $i = 1, \ldots n$ are scalars, then a *linear combination* of the vectors with the scalars as coefficients is defined as $\sum_{i=1}^{n} s_i\,\mathbf{v}_i = s_1\,\mathbf{v}_1 + \ldots + s_n\,\mathbf{v}_n$.

A collection of non-zero vectors is said to be *linearly independent* if no vector in the set can be written as a linear combination of the remaining vectors in the set. The dimension of the space is equal to the maximum number of non-zero vectors that can be included in a linearly independent set of vectors Thus for a three-dimensional space the maximum number of non-zero vectors in a linearly independent collection is three. Given a set of three linearly independent vectors, any other vector can be constructed as a resultant of scalar multiplication of the three vectors. Such a set of vectors is called a basis set. A set of vectors which is not linearly independent is called linearly dependent.

Resolution of Vectors and Components

Let $\mathbf{\imath}_1$, $\mathbf{\imath}_2$, $\mathbf{\imath}_3$ be three linearly independent unit vectors as a basis set:

$$|\mathbf{\imath}_1| = |\mathbf{\imath}_2| = |\mathbf{\imath}_3| = 1.$$

For a given vector $\mathbf{v}$ (Fig. 1.5), there exist three unique scalars v_1, v_1, v_3, such that $\mathbf{v}$ can be expressed as

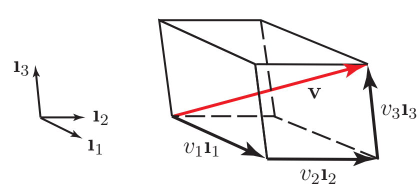


Fig. 1.5 Resolution of a vector **v** and components

$$\mathbf{v} = v_1 \mathbf{\imath}_1 + v_2 \mathbf{\imath}_2 + v_3 \mathbf{\imath}_3. \tag{1.3}$$

The opposite action of addition of vectors is the *resolution* of vectors. Thus, for the given vector $\mathbf{v}$ the vectors $v_1\mathbf{\imath}_1$, $v_2\mathbf{\imath}_2$, and $v_3\mathbf{\imath}_3$ sum to the original vector. The vector $v_k\mathbf{\imath}_k$ is called the $\mathbf{\imath}_k$ *component* of $\mathbf{v}$ relative to the given basis set and v_k is called the $\mathbf{\imath}_k$ *scalar component* of $\mathbf{v}$ relative to the given basis set, where $k = 1, 2, 3$. A vector is often replaced by its components since the components are equivalent to the original vector.

Frequently a vector will be given and its components relative to a particular basis set need to be calculated. A trivial example of this situation occurs when the vector to be resolved is the zero vector. Then each of its components are zero. Thus, under these circumstances every vector equation $\mathbf{v} = \mathbf{0}$, where $\mathbf{v} = v_1\mathbf{\imath}_1 + v_2\mathbf{\imath}_2 + v_3\mathbf{\imath}_3$, is equivalent to three scalar equations $v_1 = 0,\ \ v_2 = 0,\ \ v_3 = 0$. Note that the zero vector $\mathbf{0}$ is not the number zero.

If the unit vectors $\mathbf{\imath}_1$, $\mathbf{\imath}_2$, $\mathbf{\imath}_3$ are mutually perpendicular they form a *cartesian basis* or a *cartesian reference frame*. For a cartesian reference frame the following notation is used (Fig. 1.6)

$$\mathbf{\imath}_1 \equiv \mathbf{\imath},\ \mathbf{\imath}_2 \equiv \mathbf{\jmath},\ \mathbf{\imath}_3 \equiv \mathbf{k} \quad \text{and} \quad \mathbf{\imath} \perp \mathbf{\jmath},\ \mathbf{\imath} \perp \mathbf{k},\ \mathbf{\jmath} \perp \mathbf{k}.$$

The symbol $\perp$ denotes perpendicular. When a vector $\mathbf{v}$ is expressed in the form

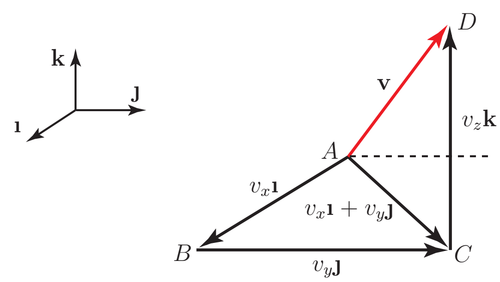


Fig. 1.6 Cartesian reference frame and the orthogonal scalar components v_x, v_y, v_z

$\mathbf{v} = v_x\mathbf{\imath} + v_y\mathbf{\jmath} + v_z\mathbf{k}$ where $\mathbf{\imath}$, $\mathbf{\jmath}$, $\mathbf{k}$ are mutually perpendicular unit vectors (cartesian reference frame or orthogonal reference frame), the magnitude of $\mathbf{v}$ is given by

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}. \tag{1.4}$$

The vectors $\mathbf{v}_x = v_x\mathbf{\imath}$, $\mathbf{v}_y = v_y\mathbf{\jmath}$, and $\mathbf{v}_z = v_y\mathbf{k}$ are the *orthogonal* or *rectangular component vectors* of the vector $\mathbf{v}$. The measures v_x, v_y, v_z are the *orthogonal* or *rectangular scalar components* of the vector $\mathbf{v}$.

The resolution of a vector into components frequently facilitate the valuation of a vector equation. If $\mathbf{v}_1 = v_{1x}\mathbf{\imath} + v_{1y}\mathbf{\jmath} + v_{1z}\mathbf{k}$ and $\mathbf{v}_2 = v_{2x}\mathbf{\imath} + v_{2y}\mathbf{\jmath} + v_{2z}\mathbf{k}$, then the sum of the vectors is

$$\mathbf{v}_1 + \mathbf{v}_2 = (v_{1x} + v_{2x})\,\mathbf{\imath} + (v_{1y} + v_{2y})\,\mathbf{\jmath} + (v_{1z} + v_{2z})\,\mathbf{k}.$$

Similarly,

$$\mathbf{v}_1 - \mathbf{v}_2 = (v_{1x} - v_{2x})\,\mathbf{\imath} + (v_{1y} - v_{2y})\,\mathbf{\jmath} + (v_{1z} - v_{2z})\,\mathbf{k}.$$

In the MATLAB® environment, a three-dimensional row vector `v` is written as a list of variables `v = [ v_x v_y v_z ]` or `v = [ v_x, v_y, v_z ]` where `v_x`, `v_y`, and `v_z` are the spatial coordinates of the vector `v`. The elements of a row are separated with blanks or commas. The list of elements are surrounded with square brackets, `[ ]`. The first component of the vector `v` is `v_x=v(1)`, the second component is `v_y=v(2)`, and the third component is `v_z=v(3)`. The semicolon `;` is used to separate the end of each row for a column vector. To create a numerical vector the following statement is used:

```
p = [ 1 2 3 ]
```

where 1, 2, and 3 are the numerical components of the row vector `p`. When a variable name is assigned to data, the data is immediately displayed, along with its name. The display of the data can be suppressed by using the semicolon, `;`, at the end of a statement.

Based on Maple kernel, symbolic MATLAB Toolbox can perform symbolical calculation and a vector `v` can be expressed in MATLAB in a symbolical fashion. In MATLAB the `sym` command constructs symbolic variables and expressions. The commands:

```
v_x = sym('v_x','real');
v_y = sym('v_y','real');
v_z = sym('v_z','real');
```

create a symbolic variables `v_x`, `v_y`, and `v_z` and also assume that the variables are real numbers. The symbolic variables can then be treated as mathematical variables. One can use the statement `syms` for generating a shortcut for constructing symbolic objects:

```
syms v_x v_y v_z real
v = [ v_x v_y v_z ];
```

where `v` is a symbolic vector. The same symbolic vector can be created with:

```
 v = sym('[v_x v_y v_z]');
```

In MATLAB a vector is defined as a matrix with either one row or one column. To make distinction between row vectors and column vectors is essential, especially when operations with vectors are required. Many errors are caused by using a row vector instead a column vector, or vice versa. The command `zeros(m,n)` or `zeros([m n])` returns an `m`-by-`n` matrix of zeros. A zero row vector `[ 0 0 0 ]` is generated with `zeros(1,3)` and a zero column vector is generated with `zeros(3,1)`. The command `ones(m,n)` or `ones([m n])` returns an `m`-by-`n` matrix of ones. In MATLAB two vectors `u` and `v` of the same size (defined either as column vectors or row vectors) can be added together using the next command:

```
u + v
```

Vectors addition in MATLAB must follow strict rules. The vectors should be either column vectors or row vectors in order to be added and should have the same dimension. It is not possible to add a row vector to a column vector. To subtract one vector from another of the same size, use a minus (-) sign. The subtraction applied to the vectors `u` and `v` can be written in MATLAB as:

```
u - v
```

or

```
v - u
```

The magnitude of the vector `p` can be found using the next MATLAB command:

```
norm(p)
```

The MATLAB command `norm(p)` does not work if the components of the vector `p` are given symbolically. Thus, a more general MATLAB function is created for the magnitude of the vector, `v`, with the components `v(1)`, `v(2)`, and `v(3)`. A MATLAB function is a program that performs an action and returns a result. The MATLAB function `magn` calculates the magnitude of the vector, `v`, in a symbolical or numerical fashion:

```
function val = magn(v)
% The symbolic magnitude function of a vector
%   v = [v(1) v(2) v(3)]
% The function accepts sym as the input argument
val = sqrt(v(1)*v(1)+v(2)*v(2)+v(3)*v(3));
```

The MATLAB statement `sqrt(x)` is the square root of the elements of `x`. The power of MATLAB comes into play when one can add new functions to enhance the language. The m-file function file starts with a line declaring the function, the arguments and the outputs. Next the statements required to produce the outputs from the inputs (arguments) are presented. It is important to note that the argument and output names used in a function file are strictly local variables that exist only within the function itself. The function returns information via the output. To calculate the magnitude of the vector `v = [v_x v_y v_z]` using the `magn` function the following MATLAB command is used:

```
mv = magn(v)
```

and the output is:

```
mv =
(v_x^2+v_y^2+v_z^2)^(1/2)
```

To create a unit vector in the direction of the vector `v` the following command is used `p/norm(p)` or `v/magn(v)` where the division symbol (/) divides all the elements in the vector by the magnitude of the vector, producing a vector of the same size and direction.

Vector transposition is as easy as adding an apostrophe, `'`, (prime) to the name of the vector. Thus if `v = [v_x v_y v_z]` then `v'` is:

```
v_x
v_y
v_z
```

The mutually perpendicular unit vectors $\imath$, $\jmath$, and $\mathbf{k}$ are defined in MATLAB by:

```
i=[1 0 0]; j=[0 1 0]; k=[0 0 1];
```

Angle Between Two Vectors
The angle between two vectors can be determined by moving either or both vectors parallel to themselves (leaving the sense unaltered) until their initial points (tails) coincide. This angle will always be in the range between 0○ and 180○ inclusive. Four possible situation are shown in Fig. 1.7 where the two vectors are denoted **a** and **b**. The *angle* between **a** and **b** is the angle θ in Figs. 1.7(a) and 1.7(b). The angle

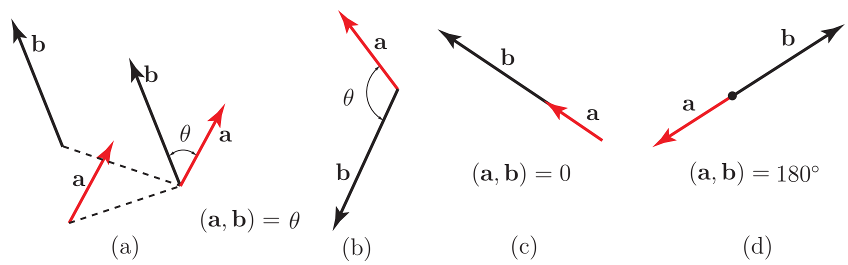


Fig. 1.7 The angle θ between the vectors **a** and **b**: (a) $0 < \theta < 90^\circ$, (b) $90^\circ < \theta < 180^\circ$, and (c) $\theta = 0^\circ$, and (d) $\theta = 180^\circ$

between **a** and **b** is denoted by the symbols $(\mathbf{a},\mathbf{b})$ or $(\mathbf{b},\mathbf{a})$. Figure 1.7(c) represents the case $(\mathbf{a},\mathbf{b}) = 0$, and Fig. 1.7(d) represents the case $(\mathbf{a},\mathbf{b}) = 180^\circ$.

The direction of a vector $\mathbf{v} = v_x\imath + v_y\jmath + v_z\mathbf{k}$ and relative to a cartesian reference, $\imath$, $\jmath$, $\mathbf{k}$, is given by the cosines of the angles formed by the vector and the respective unit vectors. These are called *direction cosines* and are denoted as (Fig. 1.8)

$$\cos(\mathbf{v},\imath) = \cos\alpha = \cos\theta_x = l,\ \cos(\mathbf{v},\jmath) = \cos\beta = \cos\theta_y = m, \text{ and}$$
$$\cos(\mathbf{v},\mathbf{k}) = \cos\gamma = \cos\theta_z = n. \tag{1.5}$$

The following relations exist: $v_x = |\mathbf{v}|\cos\alpha;\ v_y = |\mathbf{v}|\cos\beta;\ v_z = |\mathbf{v}|\cos\gamma$.
From these definitions, it follows that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad \text{or} \quad l^2 + m^2 + n^2 = 1. \tag{1.6}$$

Equation (1.6) is proved using the MATLAB commands:

```
syms v_x v_y v_z
v = [v_x v_y v_z];
```

```
mv = magn(v);
l = v_x/mv;
m = v_y/mv;
n = v_z/mv;
simplify(l^2+m^2+n^2)
```

The MATLAB statement `simplify(x)` simplifies each element of the symbolic matrix `x`.

Recall, the formula for the unit vector of the vector $\mathbf{v}$ is

$$\mathbf{u_v} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{v} = \frac{v_x}{v}\,\mathbf{\imath} + \frac{v_y}{v}\,\mathbf{\jmath} + \frac{v_x}{v}\,\mathbf{k},$$

or written another way

$$\mathbf{u_v} = \cos\alpha\,\mathbf{\imath} + \cos\beta\,\mathbf{\jmath} + \cos\gamma\,\mathbf{k}. \tag{1.7}$$

1.2 Position Vector

The position vector of a point P relative to a point O is a vector $\mathbf{r}_{OP}=\overrightarrow{OP}$ having the following characteristics:

1. magnitude the length of line OP;
2. orientation parallel to line OP;
3. sense OP (from point O to point P).

The vector $\mathbf{r}_{OP}$ is shown as an arrow connecting O to P, as depicted in Fig. 1.9(a). The position of a point P relative to P is a zero vector.

Let $\mathbf{\imath}$, $\mathbf{\jmath}$, $\mathbf{k}$ be mutually perpendicular unit vectors (cartesian reference frame) with the origin at O, as shown in Fig. 1.9(b). The axes of the cartesian reference frame are x, y, z. The unit vectors $\mathbf{\imath}$, $\mathbf{\jmath}$, $\mathbf{k}$ are parallel to x, y, z, and they have the senses

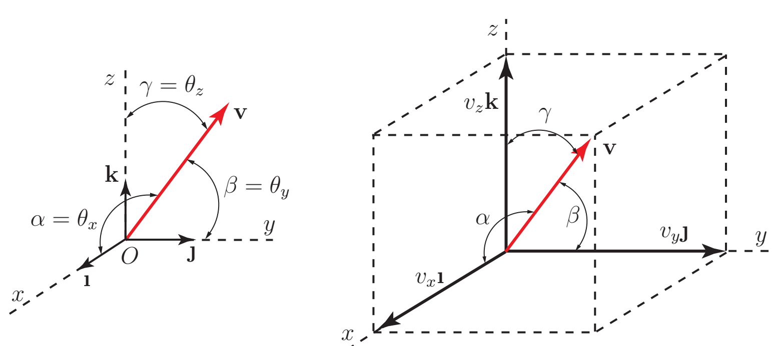


Fig. 1.8 Direction cosines

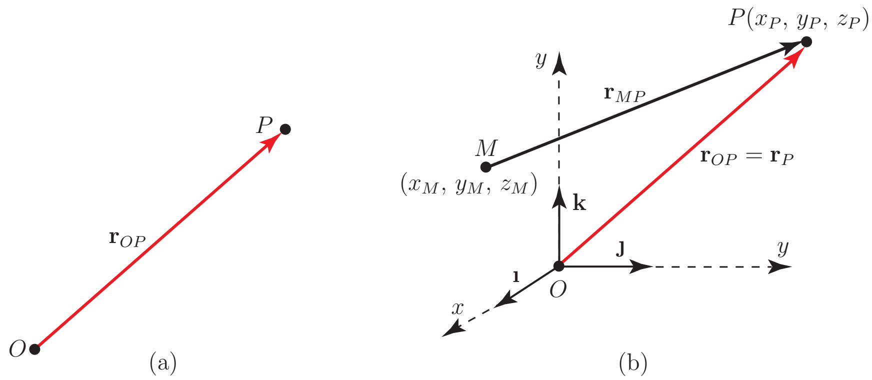


Fig. 1.9 Position vector

of the positive x, y, z axes. The coordinates of the origin O are $x = y = z = 0$, i.e., $O(0, 0, 0)$. The coordinates of a point P are $x = x_P$, $y = y_P$, $z = z_P$, i.e., $P(x_P, y_P, z_P)$. The position vector of P relative to the origin O is

$$\mathbf{r}_{OP} = \mathbf{r}_P = \overrightarrow{OP} = x_P\,\boldsymbol{\imath} + y_P\,\boldsymbol{\jmath} + z_P\,\mathbf{k}. \tag{1.8}$$

The position vector of the point P relative to a point M, $M \neq O$ of coordinates (x_M, y_M, z_M) is

$$\mathbf{r}_{MP} = \overrightarrow{MP} = (x_P - x_M)\,\boldsymbol{\imath} + (y_P - y_M)\,\boldsymbol{\jmath} + (z_P - z_M)\,\mathbf{k}. \tag{1.9}$$

The distance d between P and M is given by

$$d = |\mathbf{r}_P - \mathbf{r}_M| = |\mathbf{r}_{MP}| = |\overrightarrow{MP}| = \sqrt{(x_P - x_M)^2 + (y_P - y_M)^2 + (z_P - z_M)^2}. \tag{1.10}$$

1.3 Scalar (Dot) Product of Vectors

Definition. The scalar (dot) product of a vector **a** and a vector **b** is

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos(\mathbf{a}, \mathbf{b}). \tag{1.11}$$

For the scalar (dot) product the following rules apply:

1. for any vectors **a** and **b** one can write the commutative law for scalar product

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$$

2. for any two vectors **a** and **b** and any scalar s the following relation is written

$$(s\,\mathbf{a})\cdot\mathbf{b} = s(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a}\cdot(s\,\mathbf{b}) = s\,\mathbf{a} \cdot \mathbf{b}.$$

3. for any vectors **a**, **b**, and **c** the distributive law in the first argument is

$$(\mathbf{a}+\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c},$$

and the distributive law in the second argument is

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}.$$

It can be shown that the dot product is distributive and the following relation can be written

$$s_a\,\mathbf{a}\cdot(s_b\,\mathbf{b}+s_c\,\mathbf{c})=s_a\,s_b\,\mathbf{a}\cdot\mathbf{b}+s_a\,s_c\,\mathbf{a}\cdot\mathbf{c}.$$

If

$$\mathbf{a}=a_x\boldsymbol{\imath}+a_y\boldsymbol{\jmath}+a_z\mathbf{k} \quad\text{and}\quad \mathbf{b}=b_x\boldsymbol{\imath}+b_y\boldsymbol{\jmath}+b_z\mathbf{k},$$

where $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ are mutually perpendicular unit vectors, then

$$\mathbf{a}\cdot\mathbf{b}=a_xb_x+a_yb_y+a_zb_z\,. \tag{1.12}$$

The following relationships exist

$$\boldsymbol{\imath}\cdot\boldsymbol{\imath}=\boldsymbol{\jmath}\cdot\boldsymbol{\jmath}=\mathbf{k}\cdot\mathbf{k}=1,$$
$$\boldsymbol{\imath}\cdot\boldsymbol{\jmath}=\boldsymbol{\jmath}\cdot\boldsymbol{\imath}=\ \boldsymbol{\jmath}\cdot\mathbf{k}=\mathbf{k}\cdot\boldsymbol{\jmath}=\mathbf{k}\cdot\boldsymbol{\imath}=\boldsymbol{\imath}\cdot\mathbf{k}=0.$$

Every vector **v** can be expressed in the form

$$\mathbf{v}=\boldsymbol{\imath}\cdot\mathbf{v}\boldsymbol{\imath}+\boldsymbol{\jmath}\cdot\mathbf{v}\boldsymbol{\jmath}+\mathbf{k}\cdot\mathbf{v}\mathbf{k}. \tag{1.13}$$

Proof. The vector **v** can always be expressed as

$$\mathbf{v}=v_x\boldsymbol{\imath}+v_y\boldsymbol{\jmath}+v_z\mathbf{k}.$$

Dot multiply both sides by $\boldsymbol{\imath}$

$$\boldsymbol{\imath}\cdot\mathbf{v}=v_x\boldsymbol{\imath}\cdot\boldsymbol{\imath}+v_y\boldsymbol{\imath}\cdot\boldsymbol{\jmath}+v_z\boldsymbol{\imath}\cdot\mathbf{k}.$$

But,

$$\boldsymbol{\imath}\cdot\boldsymbol{\imath}=1,\ \text{and}\ \boldsymbol{\imath}\cdot\boldsymbol{\jmath}=\boldsymbol{\imath}\cdot\mathbf{k}=0.$$

Hence, $\boldsymbol{\imath}\cdot\mathbf{v}=v_x$. Similarly, $\boldsymbol{\jmath}\cdot\mathbf{v}=v_y$ and $\mathbf{k}\cdot\mathbf{v}=v_z$.

The MATLAB command `dot(v, u)` calculates the scalar product (or vector dot product) of the vectors `v` and `u`. The dot product of two vectors `v` and `u` can be expressed as:

```
sum(v.*u)
```

The command `sum(x)` with `x` defined as a vector, returns the sum of its elements. The MATLAB command `.*`, named *array multiplication* is the element-by-element

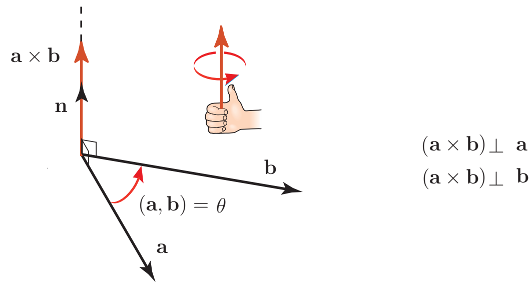


Fig. 1.10 Vector (cross) product of the vector **a** and the vector **b**

product of the associated arrays, i.e., `v.*u`, and the arrays must have the same size, unless one of them is a scalar. To indicate an array (element-by-element) operation, the standard operator is preceded with a period (dot). Thus `v.*u` is:

```
[ v_x*u_x, v_y*u_y, v_z*u_z ]
```

1.4 Vector (Cross) Product of Vectors

Definition. The vector (cross) product of a vector **a** and a vector **b** is the vector (Fig. 1.10)

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin(\mathbf{a},\mathbf{b})\mathbf{n} \tag{1.14}$$

where **n** is a unit vector whose direction is the same as the direction of advance of a right-handed screw rotated from **a** toward **b**, through the angle $(\mathbf{a},\mathbf{b})$, when the axis of the screw is perpendicular to both **a** and **b**. The magnitude of $\mathbf{a}\times\mathbf{b}$ is given by

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin(\mathbf{a},\mathbf{b}).$$

If **a** is parallel to **b**, $\mathbf{a}||\mathbf{b}$, then $\mathbf{a}\times\mathbf{b}=0$. The symbol $||$ denotes parallel. The relation $\mathbf{a}\times\mathbf{b}=\mathbf{0}$ implies only that the product $|\mathbf{a}|\,|\mathbf{b}|\sin(\mathbf{a},\mathbf{b})$ is equal to zero, and this is the case whenever $|\mathbf{a}|=0$, or $|\mathbf{b}|=0$, or $\sin(\mathbf{a},\mathbf{b})=0$.

For any two vectors **a** and **b** and any real scalar s the following relation can be written

$$(s\mathbf{a})\times\mathbf{b} = s(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(s\mathbf{b}) = s\,\mathbf{a}\times\mathbf{b}.$$

The sense of the unit vector **n** which appears in the definition of $\mathbf{a}\times\mathbf{b}$ depends on the order of the factors **a** and **b** in such a way that (cross product is not commutative)

$$\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}. \tag{1.15}$$

The cross product distributive law for the first argument can be written as

$$(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c},$$

while the distributive law for the second argument is

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}.$$

Vector multiplication obeys the following law of distributivity (Varignon theorem)

$$\mathbf{a}\times\sum_{i=1}^{n}\mathbf{v}_i=\sum_{i=1}^{n}(\mathbf{a}\times\mathbf{v}_i).$$

A set of mutually perpendicular unit vectors $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ is called *right-handed* if $\boldsymbol{\imath}\times\boldsymbol{\jmath}=\mathbf{k}$ (Fig. 1.11). A set of mutually perpendicular unit vectors $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ is called *left-handed* if $\boldsymbol{\imath}\times\boldsymbol{\jmath}=-\mathbf{k}$.

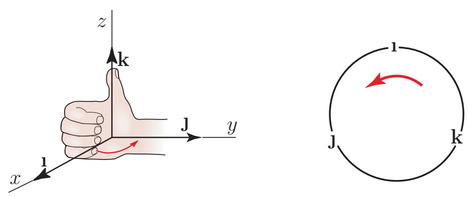


Fig. 1.11 Cartesian right-handed reference set. The cross product of two unit vectors in a counterclockwise sense around the circle yields the positive third unit vector

If $\mathbf{a}=a_x\boldsymbol{\imath}+a_y\boldsymbol{\jmath}+a_z\mathbf{k}$, and $\mathbf{b}=b_x\boldsymbol{\imath}+b_y\boldsymbol{\jmath}+b_z\mathbf{k}$, where $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ are right-handed mutually perpendicular unit vectors, then $\mathbf{a}\times\mathbf{b}$ can be expressed in the following determinant form

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}. \tag{1.16}$$

The determinant can be expanded by minors of the elements of the first row

$$\begin{aligned}\begin{vmatrix}\boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}&=\boldsymbol{\imath}\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}-\boldsymbol{\jmath}\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}+\mathbf{k}\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\\ &=\boldsymbol{\imath}(a_y b_z-a_z b_y)-\boldsymbol{\jmath}(a_x b_z-a_z b_x)+\mathbf{k}(a_x b_y-a_y b_x)\\ &=(a_y b_z-a_z b_y)\boldsymbol{\imath}+(a_z b_x-a_x b_z)\boldsymbol{\jmath}+(a_x b_y-a_y b_x)\mathbf{k}.\end{aligned} \tag{1.17}$$

As a general rule a third order determinant can be expanded by diagonal multiplication, i.e., repeating the first two columns on the right side of the determinant, and adding the signed diagonal products of the diagonal elements as

$$
\begin{array}{ccc|cc}
c_x & c_y & c_z & c_x & c_y \\
a_x & a_y & a_z & a_x & a_y \\
b_x & b_y & b_z & b_x & b_y \\
- & - & - & + \;\; + & +
\end{array}
$$

The determinant in Eq. (1.16) can be expanded using the general rule as

$$
\begin{aligned}
\begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} &= -\mathbf{k}\, a_y\, b_x - \boldsymbol{\imath}\, a_z\, b_y - \boldsymbol{\jmath}\, a_x\, b_z + \boldsymbol{\imath}\, a_y\, b_z + \boldsymbol{\jmath}\, a_z\, b_x + \mathbf{k}\, a_x b_y \\
&= (a_y\, b_z - a_z\, b_y)\boldsymbol{\imath} + (a_z\, b_x - a_x\, b_z)\boldsymbol{\jmath} + (a_x\, b_y - a_y\, b_x)\mathbf{k}.
\end{aligned}
$$

The MATLAB command `cross(a, b)` calculates the cross product of the vectors `a` and `b`. Next a MATLAB function that calculates the cross product of two vectors is presented:

```
function val = crossproduct(a,b)
% symbolic cross product function of a vector a x b
a = a(:);
% a(:) represents all elements of a,
%      regarded as a single column
b = b(:);
% b(:) represents all elements of b,
%      regarded as a single column
val = [a(2,:).*b(3,:)-a(3,:).*b(2,:) ...
       a(3,:).*b(1,:)-a(1,:).*b(3,:) ...
       a(1,:).*b(2,:)-a(2,:).*b(1,:)];
```

In the previous MATLAB function, the general MATLAB command colon (`:`), i.e., `a(:)`, has been used. The colon (`:`) is one of the most useful operators in MATLAB. It can create vectors, subscript arrays, and specify `for` iterations. The ellipses (`...`) after the command are used to execute the commands together.

1.5 Scalar Triple Product of Three Vectors

Definition. The scalar triple product of three vectors **a**, **b**, and **c** is defines as

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}. \tag{1.18}$$

The MATLAB commands for the scalar triple product of three vectors `a`, `b`, and `c` is:

```
syms a_x a_y a_z b_x b_y b_z c_x c_y c_z real
a=[a_x a_y a_z]; b=[b_x b_y b_z]; c=[c_x c_y c_z];
```

```
% [a,b,c] = a.(b x c)
abc = dot(a, cross(b, c));
```

It does not matter whether the dot is placed between **a** and **b**, and the cross between **b** and **c**, or vice versa, that is,

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = \mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{b}\cdot\mathbf{c}. \tag{1.19}$$

The relation given by Eq. (1.19) is demonstrated using the MATLAB commands:

```
% [a,b,c] = a.(b x c)
abxc = simplify(dot(a, cross(b, c)));
% [a,b,c] = (a x b).c
axbc = simplify(dot(cross(a, b), c));
% a.(b x c)==(a x b).c
abxc == axbc
```

The MATLAB relational operator == or `eq` is used to compare each element of array for equality. The statement `LHS == RHS` or `eq(LHS, RHS)` compares each element of the array `LHS` for equality with the corresponding element of the array `RHS`, and returns an array with elements set to logical 1 (true) if `LHS` and `RHS` are equal, or logical 0 (false) where they are not equal.

A change in the order of the factors appearing in a scalar triple product at most changes the sign of the product, that is,

$$[\mathbf{b},\mathbf{a},\mathbf{c}] = -[\mathbf{a},\mathbf{b},\mathbf{c}] \quad \text{and} \quad [\mathbf{b},\mathbf{c},\mathbf{a}] = [\mathbf{a},\mathbf{b},\mathbf{c}].$$

If **a**, **b**, **c** are parallel to the same plane, or if any two of the vectors **a**, **b**, **c** are parallel to each other, then $[\mathbf{a},\mathbf{b},\mathbf{c}] = 0$.

The scalar triple product $[\mathbf{a},\mathbf{b},\mathbf{c}]$ can be expressed in the following determinant form

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{1.20}$$

In MATLAB the scalar triple product of three vectors `a`, `b`, and `c` is expressed as:

```
det([a; b; c])
```

where `det(x)` is the determinant of the square matrix `x`. To verify Eq. (1.20) the following MATLAB command is used:

```
det([a; b; c]) == simplify(dot(a, cross(b, c)))
```

Exercise: Volume of a Parallelepiped
Figure 1.12 depicts three vectors **a**, **b**, and **c** that form a parallelepiped. Show that the scalar $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ represents the volume of the parallelepiped with the sides **a**, **b** and **c**.

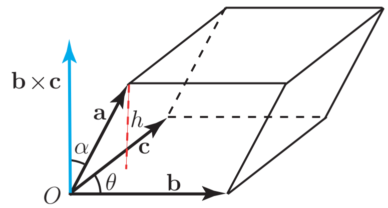


Fig. 1.12 Parallelepiped with the sides **a**, **b** and **c**

Solution
The scalar scalar triple product is $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = |\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|\sin\theta\cos\alpha = hA$, where $h = |\mathbf{a}|\cos\alpha$ represents the height of the parallelepiped and $A = |\mathbf{b}|\,|\mathbf{c}|\sin\theta$ represents the area of the parallelogram with the sides **b** and **c**. The product between h and A represents the volume of a parallelepiped, $V = hA$, so the scalar $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ represents the volume of the parallelepiped with the sides formed by the vectors **a**, **b**, and **c**.

Exercise: Vector Expressed in a Base
Let **a**, **b**, **c**, and **w** be non-zero vectors and $[\mathbf{a},\mathbf{b},\mathbf{c}] \neq 0$. The vectors **a**, **b**, **c**, and **w** are given vectors. The vectors **a**, **b**, and **c** are free vectors and can be moved in a given point. The vectors **a**, **b**, and **c** form the edges of a parallelepiped of non-zero volume. Then the scalars s_a, s_b, and s_c exist such as the vector **w** can be represented as a linear combination of the vectors **a**, **b**, **c**: $\mathbf{w} = s_a\,\mathbf{a} + s_b\,\mathbf{b} + s_c\,\mathbf{c}$. Show that the scalars s_a, s_b, and s_c are given by

$$s_a = \frac{[\mathbf{w},\mathbf{b},\mathbf{c}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]},\quad s_b = \frac{[\mathbf{a},\mathbf{w},\mathbf{c}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]},\quad \text{and} \quad s_c = \frac{[\mathbf{a},\mathbf{b},\mathbf{w}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}.$$

Solution
The components of the vectors **a**, **b**, **c** and the scalars s_a, s_b, and s_c are introduced as symbolic variables using MATLAB:

```
syms a_x a_y a_z b_x b_y b_z c_x c_y c_z  real
syms s_a s_b s_c real
```

The vectors `a`, `b`, and `c` are:

```
a =  [ a_x a_y a_z ];
b =  [ b_x b_y b_z ];
c =  [ c_x c_y c_z ];
```

and the vector `w` is:

```
w = s_a*a + s_b*b + s_c*c;
```

The scalar triple products $[\mathbf{a},\mathbf{b},\mathbf{c}]$, $[\mathbf{w},\mathbf{b},\mathbf{c}]$, $[\mathbf{a},\mathbf{w},\mathbf{c}]$, and $[\mathbf{a},\mathbf{b},\mathbf{w}]$ are:

```
abc = det([a; b; c]);
```

```
wbc = det([w; b; c]);
awc = det([a; w; c]);
abw = det([a; b; w]);
```

The scalars `s_a`, `s_b`, and `s_c` are obtained from: $\dfrac{[\mathbf{w},\mathbf{b},\mathbf{c}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$, $\dfrac{[\mathbf{a},\mathbf{w},\mathbf{c}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$, and $\dfrac{[\mathbf{a},\mathbf{b},\mathbf{w}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$ or:

```
simplify(wbc/abc)
simplify(awc/abc)
simplify(abw/abc)
```

1.6 Vector Triple Product of Three Vector

Definition. The vector triple product of three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is the vector $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. The parentheses are essential because $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is not, in general, equal to $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$. For any three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}. \tag{1.21}$$

The previous relation given by Eq. (1.21) can be expalined using the MATLAB statements:

```
% a x (b x c)
axbxc = cross(a, cross(b, c));
% (a.c)b - (a.b)c
RHS = dot(a, c)*b - dot(a, b)*c;
% a x (b x c) - (a.c)b + (a.b)c = [0, 0, 0]
simplify(axbxc-RHS)
```

1.7 Derivative of a Vector Function

The derivative of a vector function is defined in exactly the same way as is the derivative of a scalar function. Thus

$$\frac{d}{dt}\mathbf{a} = \lim_{\Delta t\to 0}\frac{\mathbf{a}(t+\Delta t)-\mathbf{a}(t)}{\Delta t}.$$

The derivative of a vector has some of the properties of the derivative of a scalar function. The derivative of the sum of two vector functions $\mathbf{a}$ and $\mathbf{b}$ is

$$\frac{d}{dt}(\mathbf{a}+\mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}. \tag{1.22}$$

The components of the vectors a and b are functions of time, t, and are introduced in MATLAB with:

```
syms t real
a_x = sym('a_x(t)');
a_y = sym('a_y(t)');
a_z = sym('a_z(t)');
b_x = sym('b_x(t)');
b_y = sym('b_y(t)');
b_z = sym('b_z(t)');
a = [a_x a_y a_z];
b = [b_x b_y b_z];
```

To calculate symbolically the derivative of a vector using the MATLAB the command `diff(p,t)` is used, which gives the derivative of p with respect to t. The relation given by Eq. (1.22) can be demonstrated using the MATLAB command:

```
diff(a+b, t) == diff(a, t) + diff(b, t)
```

The time derivative of the product of a scalar function f and a vector function $\mathbf{a}$ is

$$\frac{d(f\mathbf{a})}{dt} = \frac{df}{dt}\mathbf{a} + f\frac{d\mathbf{a}}{dt}. \tag{1.23}$$

Equation (1.23) is verified using the MATLAB command:

```
syms f real
diff(f*a, t) == diff(f, t)*a + f*diff(a, t)
```

Combining the previous results one can conclude

$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) = \frac{d\mathbf{a}}{dt}\cdot\mathbf{b} + \mathbf{a}\cdot\frac{d\mathbf{b}}{dt} \quad \text{and} \quad \frac{d}{dt}(\mathbf{a}\times\mathbf{b}) = \frac{d\mathbf{a}}{dt}\times\mathbf{b} + \mathbf{a}\times\frac{d\mathbf{b}}{dt}. \tag{1.24}$$

Equation (1.24) is demonstrated with the MATLAB commands:

```
diff(a*b.', t) == diff(a, t)*b.' + a*diff(b, t).'
diff(cross(a, b), t) == cross(diff(a, t), b) ...
                        + cross(a, diff(b, t))
```

where `A.'` is the array transpose of A.

The general derivative a vector is

$$\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v_x\mathbf{\imath} + v_y\mathbf{\jmath} + v_z\mathbf{k},) = \frac{dv_x}{dt}\mathbf{\imath} + v_x\frac{d\mathbf{\imath}}{dt} + \frac{dv_y}{dt}\mathbf{\jmath} + v_y\frac{d\mathbf{\jmath}}{dt} + \frac{dv_z}{dt}\mathbf{k} + v_z\frac{d\mathbf{k}}{dt},$$

and if the reference basis or reference frame $[\mathbf{\imath},\mathbf{\jmath},\mathbf{k}]$ is unchanging then

$$\frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\mathbf{\imath} + \frac{dv_y}{dt}\mathbf{\jmath} + \frac{dv_z}{dt}\mathbf{k}.$$

1.8 Cauchy's Inequality, Lagrange's Identity, and Triangle Inequality

The vectors **a** and **b** are non-zero vectors. The *Cauchy's inequality* can be written in vector form as

$$(\mathbf{a}\cdot\mathbf{b})^2 \le a^2\, b^2, \tag{1.25}$$

where $a^2 = |\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{a}$ and $b^2 = |\mathbf{b}|^2 = \mathbf{b}\cdot\mathbf{b}$. If **a** and **b** are parallel vectors then

$$(\mathbf{a}\cdot\mathbf{b})^2 = a^2\, b^2.$$

The vector derivation of the inequality is

$$(\mathbf{a}\cdot\mathbf{b})^2 = a^2 b^2 \cos^2\theta \le a^2 b^2.$$

The *Lagrange's identity* in vector form is

$$(\mathbf{a}\cdot\mathbf{b})^2 = a^2\, b^2 - (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})\,. \tag{1.26}$$

The vectors **a** and **b** are non-zero vectors and the vectorial product between **a** and **b** is

$$\mathbf{a}\times\mathbf{b} = (a_y b_z - a_z b_y)\,\mathbf{\imath} + (a_z b_x - a_x b_z)\,\mathbf{\jmath} + (a_x b_y - a_y b_x)\,\mathbf{k},$$

where a_x, a_y, a_z and b_x, b_y, b_z are the Cartesian components of the vectors **a** and **b**. One can compute

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) = (a_y b_z - a_z b_y)^2 + (a_z b_x - a_z b_z)^2 + (a_z b_y - a_y b_x)^2\,. \tag{1.27}$$

The scalar product definition gives

$$(\mathbf{a}\cdot\mathbf{b})^2 = [(a_x\mathbf{\imath} + a_y\mathbf{\jmath} + a_z\mathbf{k})\cdot(b_x\mathbf{\imath} - b_y\mathbf{\jmath} + b_z\mathbf{k})]^2 = (a_x b_x + a_y b_y + a_z b_z)^2\,, \tag{1.28}$$

and

$$\begin{aligned} a^2\, b^2 &= |\mathbf{a}|^2\,|\mathbf{b}|^2 = \left|a_x\,\mathbf{\imath} + a_y\,\mathbf{\jmath} + a_z\,\mathbf{k}\right|^2\,\left|b_x\,\mathbf{\imath} - b_y\,\mathbf{\jmath} + b_z\,\mathbf{k}\right|^2 \\ &= \left(a_x^2 + a_y^2 + a_z^2\right)\left(b_x^2 + b_y^2 + b_z^2\right). \end{aligned} \tag{1.29}$$

Using Eqs. (1.27), (1.28) and (1.29) it results

$$\left(a_x^2 + a_y^2 + a_z^2\right)\left(b_x^2 + b_y^2 + b_z^2\right) - (a_x b_x + a_y b_y + a_z b_z)^2$$

$$= (a_y b_z - a_z b_y)^2 + (a_z b_x - a_z b_z)^2 + (a_z b_y - a_y b_x)^2\,,$$

or

$$a^2\, b^2 - (\mathbf{a}\cdot\mathbf{b})^2 = (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})\,.$$

The previous equation can be written as the identity given by Eq. (1.26).

The MATLAB proof for Lagrange's identity is

```
syms a_x a_y a_z b_x b_y b_z real
a = [ a_x a_y a_z ];
b = [ b_x b_y b_z ];
% LHS = (a.b)^2
% RHS = (a.a)*(b.b) - (a x b).(a x b)
LHS = (dot(a,b))^2;
RHS = dot(a,a)*dot(b,b)-dot(cross(a,b),cross(a,b));
expand(LHS)==expand(RHS)
```

If **a** and **b** are non-zero vectors the following relation can be obtain

$$|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}| . \tag{1.30}$$

The inequality given by Eq. (1.30) is known as *triangle inequality*.

Proof: It is obvious that

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b} = |\mathbf{a}|^2+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+|\mathbf{b}|^2. \tag{1.31}$$

The following relation exists

$$\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a} \le 2\,|\mathbf{a}\cdot\mathbf{b}| \le 2\,|\mathbf{a}|\,|\mathbf{b}| . \tag{1.32}$$

Equations. (1.31) and (1.32) give

$$|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) \le |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\,|\mathbf{a}|\,|\mathbf{b}| = (|\mathbf{a}|+|\mathbf{b}|)^2 .$$

Moreover one can prove that

$$|\mathbf{a}| - |\mathbf{b}| \le |\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}| , \text{ for } |\mathbf{a}| > |\mathbf{b}| ,$$

$$|\mathbf{b}| - |\mathbf{a}| \le |\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}| , \text{ for } |\mathbf{a}| < |\mathbf{b}| .$$

For this let $\mathbf{a} = (\mathbf{a}+\mathbf{b}) - \mathbf{b}$ and applying the inequality given by Eq. (1.30) for $\mathbf{a}+\mathbf{b}$ and $-\mathbf{b}$, it results

$$|\mathbf{a}| = |(\mathbf{a}+\mathbf{b}) + (-\mathbf{b})| \le |\mathbf{a}+\mathbf{b}| + |-\mathbf{b}| ,$$

or

$$|\mathbf{a}+\mathbf{b}| \ge |\mathbf{a}| - |-\mathbf{b}| = |\mathbf{a}| - |\mathbf{b}| . \tag{1.33}$$

Using Eqs. (1.30) and (1.33) the following relations can be written

$$|\mathbf{a}| - |\mathbf{b}| \le |\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}| , \text{ for } |\mathbf{a}| > |\mathbf{b}| ,$$

$$|\mathbf{b}| - |\mathbf{a}| \le |\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}| , \text{ for } |\mathbf{a}| < |\mathbf{b}| .$$

1.9 Examples

Example 1.1
In Fig. E1.1(a) the rectangular component of the vector **F** on the *OA* direction is **f**, with the magnitude $|\mathbf{f}| = f$. The vector **F** acts at an angle β with the positive direction of the x-axis. Find the magnitude $|\mathbf{F}| = F$ of the vector **F**. Numerical application: f =20, $\alpha = 30°$, and $\beta = 60°$.

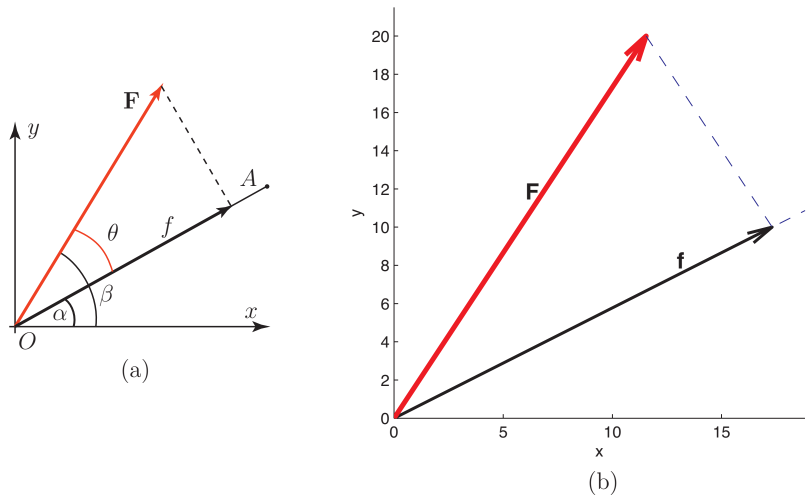


Fig. E1.1 Example 1.1

Solution
The component of **F** on the *OA* direction is $|\mathbf{F}|\cos\theta = f$. From Fig. E1.1 the angle θ of the vector **F** with the *OA* direction is $\theta = \beta - \alpha = 60° - 30° = 30°$. The magnitude F is calculated from the equation

$$|\mathbf{F}|\cos\theta = f \Leftrightarrow |\mathbf{F}|\cos 30° = 20 \Rightarrow F = |\mathbf{F}| = \frac{f}{\cos\theta} = \frac{20}{\cos 30°} \text{ or } F = 23.094.$$

The MATLAB program starts with the statements:

```
clear all
% clears all the objects in the MATLAB workspace and
% resets the default MuPAD symbolic engine
clc  % clears the command window and homes the cursor
close all  % closes all the open figure windows
```

The MATLAB commands for the input data are:

```
f = 20;
alpha = pi/6;
beta = pi/3;
```

The angle θ and the magnitude of the vector **F** are calculated with:

```
theta = beta-alpha;
F = f/cos(theta);
```

The statement `cos(s)` is the cosine of the argument `s` in radians. The numerical solution for *F* is printed using the statement:

```
fprintf('F = %f',F)
```

The statement `fprintf(f,format,s)` writes data in the real part of array `s` to the file `f`. The data is formated under control of the specified `format` string and contains ordinary characters and/or C language conversion specifications. The conversion character `%f` specifies the output as fixed-point notation. For more details, about `fprintf` see online help.

Next the two vectors **f** and **F** will be plotted. The *x* and *y* components of the vectors **f** and **F** are:

```
% components of vector f
f_x=f*cos(alpha);
f_y=f*sin(alpha);

% components of vector F
F_x=F*cos(beta);
F_y=F*sin(beta);
```

The following MATLAB commands are used to introduce the plotting of the vectors:

```
hold on
s = 1.5; % scale factor
axis([0 f_x+s 0 F_y+s])
axis square
```

The MATLAB command `hold on` retains the current graph and all axis properties so that succeeding plot commands add to the existing graph. The MATLAB command `axis(([xMIN xMAX yMIN yMAX])` sets scaling for the *x* and *y* axes on the current plot and the statement `axis square` makes the current axis box square in size. The direction of vector `f` is represented with:

```
line([0 s*f_x],[0 s*f_y],'LineStyle','--')
```

where the command `line(x,y)` creates the line in vectors `x` and `y` to the current axes The `LineStyle` specifies the line style: `'-'` solid line (default), `'--'` dashed line, `':'` dotted line, and `'?.'` dash-dot line.

The vector `f` is represented with:

```
quiver(0,0,f_x,f_y,0,'Color','k','LineWidth',1.5)
```

The MATLAB command `quiver(x,y,u,v,s,LineSpec)` draws vectors specified by `u` and `v` at the coordinates `x` and `y`. The parameter `s` automatically scales the arrows to fit within the grid: `s = 2` doubles the relative length, `s = 0.5` halves the length, and `s = 0` plots the vectors without automatic scaling. The parameter `LineSpec` specifies line style, marker symbol, and the `'Color'` specifiers are `'r'` red, `'g'` green, `'b'` blue, `'y'` yellow, and `'k'` black. The `'LineWidth'` creates the width of the line in points (1 point = 1/72 inch) and by default the line width is 0.5 point. The vector `f` is denoted with the MATLAB command:

```
text(f_x/s+s,f_y/s+s,'f',...
     'fontsize',14,'fontweight','b')
```

where `text(x,y,'text')` adds the text in the quotes to location `(x,y)`. The fontsize for the vector is 14 and the font is bold, `'fontweight','b'`. The ellipses (`...`) after the command was used to execute the statements together. The vector `F` is plotted and denoted with the MATLAB commands:

```
quiver(0,0,F_x,F_y,0,'Color','r','LineWidth',2.5)
text(F_x/s-s,F_y/s-s,'F',...
     'fontsize',14,'fontweight','b')
```

The line that connects the end of the vector `F` with the end of the vector `f` is represented in MATLAB with:

```
line([F_x f_x],[F_y f_y],'LineStyle','--')
```

The labels for the x and y axes are added with:

```
xlabel('x')
ylabel('y')
```

The MATLAB figure of the vectors is shown in Fig. E1.1(b).

Example 1.2
The coordinates of two points A and B relative to the origin $O(0,0,0)$ are given by: $A(x_A = 1, y_A = 2, z_A = 3)$ and $B(x_B = 3, y_B = 3, z_B = 3)$. Determine the unit vector of the line Δ that starts at point $A(x_A, y_A, z_A)$ and passes through the point $B(x_B, y_B, z_B)$.

Solution
The position vectors of the points $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ with respect to the origin $O(0,0,0)$ are

$$\overrightarrow{OA} = \mathbf{r}_A = x_A\,\boldsymbol{\imath} + y_A\,\boldsymbol{\jmath} + z_A\,\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = \mathbf{r}_B = x_B\,\boldsymbol{\imath} + y_B\,\boldsymbol{\jmath} + z_B\,\mathbf{k}.$$

The symbolic expressions of the vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ are introduced in MATLAB as:

```
syms x_A y_A z_A x_B y_B z_B real
r_A = [ x_A y_A z_A ];
r_B = [ x_B y_B z_B ];
```

The vector $\overrightarrow{AB} = \mathbf{r}_{AB}$ is defined as

$$\overrightarrow{AB} = \mathbf{r}_{AB} = \mathbf{r}_B - \mathbf{r}_A = (x_B - x_A)\,\imath + (y_B - y_A)\,\jmath + (z_B - z_A)\,\mathbf{k},$$

or in MATLAB:

```
r_AB = r_B - r_A;
```

The magnitude of the vector $\mathbf{r}_{AB}$ is

$$|\mathbf{r}_{AB}| = r_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

or in MATLAB:

```
mr_AB = sqrt(dot(r_AB,r_AB));
```

The unit vector, $\mathbf{u}_\Delta$, of the line Δ (line AB) is

$$\begin{aligned}
\mathbf{u}_\Delta &= \frac{\mathbf{r}_{AB}}{|\mathbf{r}_{AB}|} = \frac{(x_B - x_A)\,\imath + (y_B - y_A)\,\jmath + (z_B - z_A)\,\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}} \\
&= \frac{x_B - x_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\,\imath \\
&\quad + \frac{y_B - y_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\,\jmath \\
&\quad + \frac{z_B - z_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\,\mathbf{k}.
\end{aligned}$$

Using MATLAB the unit vector is:

```
u_AB = r_AB/mr_AB;
```

The numerical values of the components of the unit vector $\mathbf{u}_\Delta = u_x\,\imath + u_y\,\jmath + u_z\,\mathbf{k}$ are obtained replacing the symbolic expressions with their numerical values

$$\begin{aligned}
u_x &= \frac{x_B - x_A}{r_{AB}} = \frac{3-1}{\sqrt{(3-1)^2 + (3-2)^2 + (3-3)^2}} = \frac{2}{2.2361} = 0.894, \\
u_y &= \frac{y_B - y_A}{r_{AB}} = \frac{3-2}{\sqrt{(3-2)^2 + (3-2)^2 + (3-3)^2}} = \frac{1}{2.2361} = 0.447, \\
u_z &= \frac{z_B - z_A}{r_{AB}} = \frac{3-3}{\sqrt{(3-3)^2 + (3-2)^2 + (3-3)^2}} = \frac{0}{2.2361} = 0,
\end{aligned}$$

where the magnitude of the vector $\mathbf{r}_{AB}$ is

$$r_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} = \sqrt{(3-3)^2 + (3-2)^2 + (3-3)^2}$$
$$= \sqrt{5} = 2.2361.$$

To obtain the numerical values in MATLAB, `x_A`, `y_A`, `z_A` are replaced with `1, 2, 3` and `x_B`, `y_B`, `z_B` are replaced with `3, 3, 3`. For this purpose two lists are created: a list with the symbolical variables {`x_A, y_A, z_A, x_B, y_B, z_B`} and a list with the corresponding numeric values {`1, 2, 3, 3, 3, 3`}:

```
slist = {x_A, y_A, z_A, x_B, y_B, z_B};
nlist = {1, 2, 3, 3, 3, 3};
```

To obtain the numerical value for the symbolic unit vector `u_AB` the following statement is introduced:

```
u_ABn = subs(u_AB, slist, nlist);
```

The statement `subs(expr,lhs,rhs)` replaces `lhs` with `rhs` in the symbolic expression `expr`. The numerical results are printed with the following command:

```
fprintf('u_AB = [%6.3f %6.3f %6.3f] \n', u_ABn)
```

The escape character `\n` specifies new line.

Next the vectors $\mathbf{r}_A$, $\mathbf{r}_B$, and $\mathbf{r}_{AB}$ will be plotted using MATLAB. The numerical values of the vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ are obtained with:

```
rA = eval(subs(r_A, slist, nlist));
rB = eval(subs(r_B, slist, nlist));
```

The command `eval(x)`, where `x` is a string, executes the string as an expression. The command is `axis(([xMIN xMAX yMIN yMAX zMIN zMAX])` put the scaling for the `x`, `y` and `z` axes on the 3-D plot. The statement `axis ij` positions MATLAB into its "matrix" axes mode, the coordinate system origin is at `y=z=0`, the `y`-axis is numbered from top to bottom, the `x`-axis is numbered from left to right, and the `z`-axis is vertical with values increasing from bottom to top. For this example the axes are defined by:

```
a=3.5;
axis ([0 a  0 a  0 a])
axis ij, grid on, hold on
```

The MATLAB command `grid on` adds major grid lines to the current axes and `hold on` locks up the current plot and all axis properties so that following graphing commands add to the existing graph. The vectors `rA` and `rB` are defined in MATLAB as:

```
quiver3(0,0,0, rA(1),rA(2),rA(3),1,...
    'Color','k','LineWidth',1)
quiver3(0,0,0, rB(1),rB(2),rB(3),1,...
    'Color','k','LineWidth',1)
```

where the statement `quiver3(x,y,z,u,v,w)` represent a vector as arrows at the point `(x,y,z)` with the components `(u,v,w)`. The dashed line (−−) between the points `A` and `B` is plotted with the command:

```
line([rA(1) rB(1)],[rA(2) rB(2)],[rA(3) rB(3)],...
    'LineStyle','--')
```

and the unit vector `u` is represented with:

```
quiver3(...
    rA(1),rA(2),rA(3), u_ABn(1),u_ABn(2),u_ABn(3),...
    1,'Color','r','LineWidth',2)
```

The labels for the vectors and the axes are printed in MATLAB with:

```
text(rA(1)/2, rA(2)/2, rA(3)/2+.3,...
    'r_A','fontsize',14,'fontweight','b')
text(rB(1)/2, rB(2)/2, rB(3)/2+.3,...
    'r_B','fontsize',14,'fontweight','b')
text(...
    (rA(1)+rB(1))/2-.4,...
    (rA(2)+rB(2))/2,...
    (rA(3)+ rB(3))/2+.3,...
    'u','fontsize',14,'fontweight','b')
xlabel('x'), ylabel('y'), zlabel('z')
```

The MATLAB representation of the vectors is shown in Fig. E1.2.

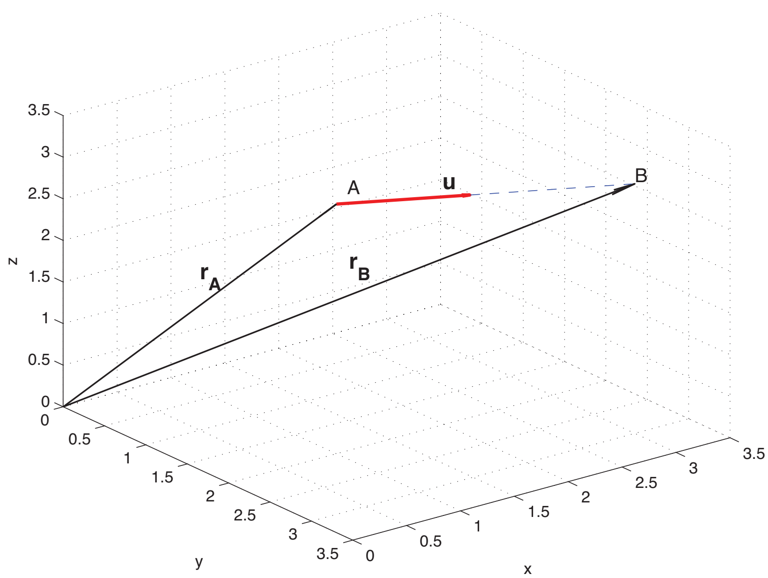


Fig. E1.2 MATLAB figure for Example 1.2

Example 1.3

The vectors $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, and $\mathbf{V}_4$ with the magnitude $|\mathbf{V}_1| = V_1$, $|\mathbf{V}_2| = V_2$, $|\mathbf{V}_3| = V_3$, and $|\mathbf{V}_4| = V_4$ are concurrent at the origin $O(0, 0, 0)$ and are directed through the points of coordinates $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$, $A_3(x_3, y_3, z_3)$, and $A_4(x_4, y_4, z_4)$, respectively. Determine the resultant vector of the system. Numerical application: $V_1 = 10$, $V_2 = 25$, $V_3 = 15$, $V_4 = 40$, $A_1(3, 1, 7)$, $A_2(5, -3, 4)$, $A_3(-4, -3, 1)$, and $A_4(4, 2, -3)$.

Solution

The magnitudes, V_i, of the vectors $\mathbf{V}_i$ and the coordinates, x_i, y_i, z_i, of the points A_i, $i = 1, 2, 3, 4$ are introduced with MATLAB as:

```
V(1)=10; V(2)=25; V(3)=15; V(4)=40; % magnitudes V_i
x(1)= 3; y(1)= 1; z(1)= 7; % A_1
x(2)= 5; y(2)=-3; z(2)= 4; % A_2
x(3)=-4; y(3)=-3; z(3)= 1; % A_3
x(4)= 4; y(4)= 2; z(4)=-3; % A_4
```

The direction cosines of the vectors $\mathbf{V}_i$ are

$$\cos\theta_{ix} = \frac{x_i}{\sqrt{x_i^2 + y_i^2 + z_i^2}},\ \cos\theta_{iy} = \frac{y_i}{\sqrt{x_i^2 + y_i^2 + z_i^2}},\ \cos\theta_{iz} = \frac{z_i}{\sqrt{x_i^2 + y_i^2 + z_i^2}},$$

and the x, y, z components of the vectors $\mathbf{V}_i$ are

$$V_{ix} = V_i \cos\theta_{ix},\ V_{iy} = V_i \cos\theta_{iy},\ V_{iz} = V_i \cos\theta_{iz}.$$

To calculate the direction cosines and components of the vectors for $i = 1, 2, 3, 4$ the MATLAB statement: `for` *var=startval:step:endval, statement* `end` is used. It repeatedly evaluates *statement* in a loop. The counter variable of the loop is *var*. At the start the variable is initialized to value *startval* and is incremented (or decremented when *step* is negative) by the value *step* for each iteration. The *statement* is repeated until *var* has incremented to the value *endval*. For the vectors the following applies `for i=1:4, Program block, end` or:

```
for i = 1:4
% dirction cosines of the vector v(i)
c_x(i) = x(i)/sqrt(x(i)^2+y(i)^2+z(i)^2);
c_y(i) = y(i)/sqrt(x(i)^2+y(i)^2+z(i)^2);
c_z(i) = z(i)/sqrt(x(i)^2+y(i)^2+z(i)^2);
% x, y, z components of the vector v(i)
v_x(i) = V(i)*c_x(i);
v_y(i) = V(i)*c_y(i);
v_z(i) = V(i)*c_z(i);
fprintf('vector %g: \n',i)
fprintf('direction cosines=')
fprintf('[%6.3f,%6.3f,%6.3f]\n',c_x(i),c_y(i),c_z(i))
```

```
fprintf('vector V=')
fprintf('[%6.3f,%6.3f,%6.3f]\n',v_x(i),v_y(i),v_z(i))
fprintf('\n')
end
```

The results in MATLAB are:

```
vector 1:
direction cosines=[ 0.391, 0.130, 0.911]
vector V=[ 3.906, 1.302, 9.113]

vector 2:
direction cosines=[ 0.707,-0.424, 0.566]
vector V=[17.678,-10.607,14.142]

vector 3:
direction cosines=[-0.784,-0.588, 0.196]
vector V=[-11.767,-8.825, 2.942]

vector 4:
direction cosines=[ 0.743, 0.371,-0.557]
vector V=[29.711,14.856,-22.283]
```

or using a table form

i	V_i	A_i	$\cos\theta_{ix}$	$\cos\theta_{iy}$	$\cos\theta_{iz}$	V_{ix}	V_{iy}	V_{iz}
1	10	$(3,1,7)$	0.391	0.130	0.911	3.906	1.302	9.113
2	25	$(5,-3,4)$	0.70	−0.424	0.566	17.678	−10.607	14.142
3	15	$(-4,-3,1)$	−0.784	−0.588	0.196	−11.767	−8.825	2.942
4	40	$(4,2,-3)$	0.743	0.371	−0.557	29.711	14.856	−22.283

The vector $\mathbf{V}_i$ can be written as $\mathbf{V}_i = V_{ix}\mathbf{\imath} + V_{iy}\mathbf{\jmath} + V_{iz}\mathbf{k},\ i = 1,2,3,4$. The resultant of the system is

$$R = \sqrt{(R_x)^2 + (R_y)^2 + (R_z)^2} = \sqrt{\left(\sum V_{ix}\right)^2 + \left(\sum V_{iy}\right)^2 + \left(\sum V_{iz}\right)^2}.$$

The direction cosines of the resultant are

$$\cos\theta_x = \frac{\sum V_{ix}}{R},\ \cos\theta_y = \frac{\sum V_{iy}}{R},\ \cos\theta_z = \frac{\sum V_{iz}}{R}.$$

The resultant and the direction cosines in MATLAB are:

```
Rx = sum(v_x);
Ry = sum(v_y);
Rz = sum(v_z);
R = [Rx Ry Rz];
modR = norm(R);
```

```
fprintf('R=V1+V2+V3+V4=[%6.3f,%6.3f,%6.3f]\n',R)
fprintf('|R|=%6.3g\n',modR)
fprintf('direction cosines=')
fprintf('uR=R/|R|=[%6.3f,%6.3f,%6.3f]\n',R/modR)
```

The MATLAB results are:

```
R=V1+V2+V3+V4=[39.528,-3.274, 3.914]
|R|=  39.9
direction cosines=uR=R/|R|=[ 0.992,-0.082, 0.098]
```

or in table form

R	R_x	R_y	R_z	$\cos\theta_x$	$\cos\theta_y$	$\cos\theta_z$
39.9	39.528	−3.274	3.914	0.992	−0.082	0.098

The negative value of $\cos\theta_y$ signifies that the resultant has a negative component in the y direction.

Next the vectors will be plotted using MATLAB. The axes are defined in MATLAB with:

```
a = 26;
axis([-a a  -a a  -a a])
axis ij, grid on, hold on
xlabel('x'), ylabel('y'), zlabel('z')
text(0,0,0-1.5,'   O','HorizontalAlignment','right')
```

The vectors $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, $\mathbf{V}_4$, and $\mathbf{R}$ are plotted and labeled with the statements:

```
quiver3(0,0,0,v_x(1),v_y(1),v_z(1),1,...
    'Color','k','LineWidth',1.5)
text(v_x(1),v_y(1),v_z(1),' V_1',...
    'fontsize',12,'fontweight','b')

quiver3(0,0,0,v_x(2),v_y(2),v_z(2),1,...
    'Color','k','LineWidth',1.5)
text(v_x(2),v_y(2),v_z(2),' V_2',...
    'fontsize',12,'fontweight','b')

quiver3(0,0,0,v_x(3),v_y(3),v_z(3),1,...
    'Color','k','LineWidth',1.5)
text(v_x(3),v_y(3),v_z(3)+1,'V_3',...
    'fontsize',12,'fontweight','b')

quiver3(0,0,0,v_x(4),v_y(4),v_z(4),1,...
    'Color','k','LineWidth',1.5)
text(v_x(4),v_y(4),v_z(4),' V_4',...
    'fontsize',12,'fontweight','b')
```

```
quiver3(0,0,0,Rx,Ry,Rz,1,...
    'Color','r','LineWidth',2.5)
text(Rx,Ry,Rz,' R','fontsize',14,'fontweight','b')
```

The MATLAB representation of the vectors is shown in Fig. E1.3.

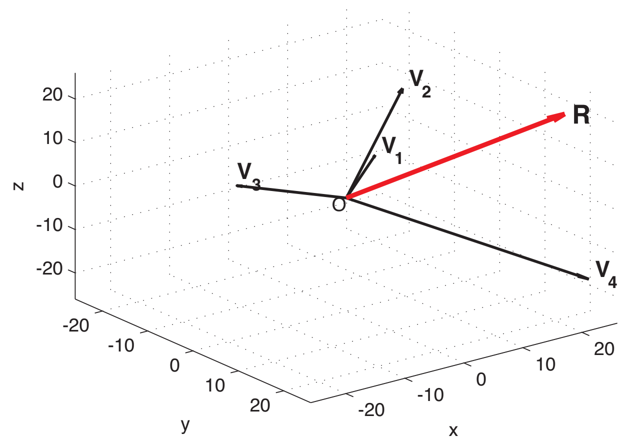


Fig. E1.3 MATLAB figure for Example 1.3

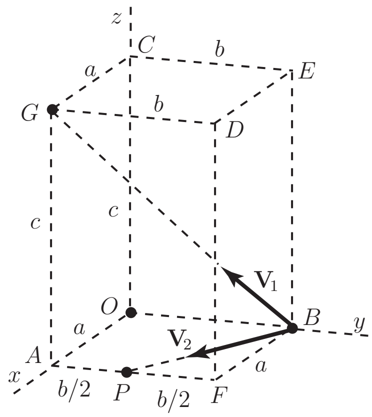
Example 1.4
Two vector system $\mathbf{V}_1$ and $\mathbf{V}_2$, is shown in Fig. E1.4(a). a) Find the resultant of the system. b) Determine the cross product $\mathbf{V}_1 \times \mathbf{V}_1$. c) Find the angle between the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$. Numerical application: $|\mathbf{V}_1| = V_1 = 3$, $|\mathbf{V}_2| = V_2 = 3$, $a = 4$, $b = 5$, and $c = 3$.

Solution
a) The vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are given by

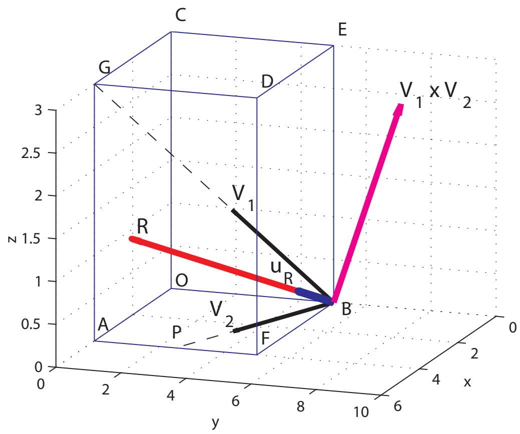
$$\mathbf{V}_1 = V_{1x}\mathbf{ı} + V_{1y}\mathbf{ȷ} + V_{1z}\mathbf{k} = |\mathbf{V}_1|\frac{\mathbf{r}_{BG}}{|\mathbf{r}_{BG}|} = V_1\frac{\mathbf{r}_{BG}}{r_{BG}},$$
$$\mathbf{V}_2 = V_{2x}\mathbf{ı} + V_{2y}\mathbf{ȷ} + V_{2z}\mathbf{k} = |\mathbf{V}_2|\frac{\mathbf{r}_{BP}}{|\mathbf{r}_{BP}|} = V_2\frac{\mathbf{r}_{BP}}{r_{BP}}.$$

Next the vectors $\mathbf{r}_{BG}$ and $\mathbf{r}_{BP}$ will be calculated. From Fig. E1.4 the coordinates of the points B, D, P, and Q are $B = B(x_B, y_B, z_B) = B(0, b, 0) = B(0, 5, 0)$, $G = G(x_G, y_G, z_G) = G(a, 0, c) = G(4, 0, 3)$, and $P = P(x_P, y_P, z_P) = P(a, b/2, 0) = P(4, 5/2, 0)$. The position vectors vectors of the points B, G, and P are

$$\mathbf{r}_B = x_B\,\mathbf{ı} + y_B\,\mathbf{ȷ} + z_B\,\mathbf{k} = b\,\mathbf{ȷ} = 5\,\mathbf{ȷ},$$
$$\mathbf{r}_G = x_G\,\mathbf{ı} + y_G\,\mathbf{ȷ} + z_G\,\mathbf{k} = a\,\mathbf{ı} + c\,\mathbf{k} = 4\,\mathbf{ı} + 3\,\mathbf{k},$$
$$\mathbf{r}_P = x_P\,\mathbf{ı} + y_P\,\mathbf{ȷ} + z_P\,\mathbf{k} = a\,\mathbf{ı} + b/2\,\mathbf{ȷ} = 4\,\mathbf{ı} + 5/2\,\mathbf{ȷ}.$$



(a)



(b)

Fig. E1.4 Example 1.4

The MATLAB commands for input data and for $\mathbf{r}_B$, $\mathbf{r}_G$, and $\mathbf{r}_P$ are:

```
V_1=3; V_2=3;
a=4; b=5; c=3;

x_B=0; y_B=b; z_B=0; r_B=[x_B, y_B, z_B];
x_G=a; y_G=0; z_G=c; r_G=[x_G, y_G, z_G];
x_P=a; y_P=b/2; z_P=0; r_P=[x_P, y_P, z_P];
```

The vectors $\mathbf{r}_{BG}$ and $\mathbf{r}_{BP}$ are

$$\begin{aligned}
\mathbf{r}_{BG} &= \mathbf{r}_G - \mathbf{r}_B = (x_G - x_B)\,\boldsymbol{\imath} + (y_G - y_B)\,\boldsymbol{\jmath} + (z_G - z_B)\,\mathbf{k} \\
&= (a-0)\,\boldsymbol{\imath} + (0-b)\,\boldsymbol{\jmath} + (c-0)\,\mathbf{k} \\
&= a\,\boldsymbol{\imath} - b\,\boldsymbol{\jmath} + c\,\mathbf{k} = 4\,\boldsymbol{\imath} - 5\,\boldsymbol{\jmath} + 3\,\mathbf{k}, \\
\mathbf{r}_{BP} &= \mathbf{r}_P - \mathbf{r}_B = (x_P - x_B)\,\boldsymbol{\imath} + (y_P - y_B)\,\boldsymbol{\jmath} + (z_P - z_B)\,\mathbf{k} \\
&= (a-0)\,\boldsymbol{\imath} + \left(\frac{b}{2} - b\right)\boldsymbol{\jmath} + (0-0)\,\mathbf{k} \\
&= a\,\boldsymbol{\imath} - \frac{b}{2}\,\boldsymbol{\jmath} = 4\,\boldsymbol{\imath} - \frac{5}{2}\,\boldsymbol{\jmath}.
\end{aligned}$$

The magnitudes of the vectors $\mathbf{r}_{BG}$ and $\mathbf{r}_{BP}$ are

$$\begin{aligned}
|\mathbf{r}_{BG}| = r_{BG} &= \sqrt{(x_G - x_B)^2 + (y_G - y_B)^2 + (z_G - z_B)^2} \\
&= \sqrt{(a-0)^2 + (0-b)^2 + (c-0)^2} = \sqrt{a^2 + b^2 + c^2} \\
&= \sqrt{4^2 + 5^2 + 3^2} = 7.071,
\end{aligned}$$

$$|\mathbf{r}_{BP}| = r_{BP} = \sqrt{(x_P - x_B)^2 + (y_P - y_B)^2 + (z_P - z_B)^2}$$
$$= \sqrt{(a-0)^2 + \left(\frac{b}{2} - b\right)^2 + (0-0)^2} = \sqrt{a^2 + \frac{b^2}{4}}$$
$$= \sqrt{4^2 + \frac{5^2}{4}} = 4.717.$$

The vectors $\mathbf{r}_{BG}$ and $\mathbf{r}_{BP}$ and their magnitudes in MATLAB are:

```
r_BG = r_G-r_B;
r_BP = r_P-r_B;
fprintf('r_BG = [%6.3f %6.3f %6.3f]\n', r_BG)
fprintf('r_BP = [%6.3f %6.3f %6.3f]\n', r_BP)

mr_BG = sqrt(dot(r_BG, r_BG));
mr_BP = sqrt(dot(r_BP, r_BP));
fprintf('|r_BG| = %6.3f\n', mr_BG)
fprintf('|r_BP| = %6.3f\n', mr_BP)
```

The vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are

$$\mathbf{V}_1 = V_1 \frac{\mathbf{r}_{BG}}{r_{BG}} = V_1 \frac{a\,\mathbf{ı} - b\,\mathbf{ȷ} + c\,\mathbf{k}}{\sqrt{a^2+b^2+c^2}} = 3\frac{4\,\mathbf{ı} - 5\,\mathbf{ȷ} + 3\,\mathbf{k}}{7.071}$$
$$= 1.697\,\mathbf{ı} - 2.121\,\mathbf{ȷ} + 1.273\,\mathbf{k},$$
$$\mathbf{V}_2 = V_2 \frac{\mathbf{r}_{BP}}{r_{BP}} = V_2 \frac{a\,\mathbf{ı} - (b/2)\,\mathbf{ȷ}}{\sqrt{a^2+b^2/4}} = 3\frac{4\,\mathbf{ı} - (5/2)\,\mathbf{ȷ}}{4.717}$$
$$= 2.544\,\mathbf{ı} - 1.590\,\mathbf{ȷ},$$

or with MATLAB:

```
u_BD = r_BD/mr_BD;
u_PQ = r_PQ/mr_PQ;
V1 = V_1*u_BD
V2 = V_2*u_PQ
V1n = eval(subs(V1, slist, nlist));
V2n = eval(subs(V2, slist, nlist));
fprintf('V1 = [%6.3f %6.3f %6.3f]\n', V1n)
fprintf('V2 = [%6.3f %6.3f %6.3f]\n', V2n)
```

The cartesian components of the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are

$$V_{1x} = 1.697,\ V_{1y} = -2.121,\ V_{1z} = 1.273,\ V_{2x} = 2.544,\ V_{2y} = -1.590,\ V_{2z} = 0.$$

The resultant vector has the components

$$R_x = \sum V_{ix} = V_{1x} + V_{2x} = 1.697 + 2.544 = 4.241,$$
$$R_y = \sum V_{iy} = V_{1y} + V_{2y} = -2.121 - 1.590 = -3.711,$$

$$R_z = \sum V_{iz} = V_{1z} + V_{2z} = 1.273 + 0 = 1.273,$$

and can be written in a vector form as

$$\mathbf{R} = R_x \mathbf{\imath} + R_y \mathbf{\jmath} + R_z \mathbf{k} = 4.241\,\mathbf{\imath} - 3.711\,\mathbf{\jmath} + 1.273\,\mathbf{k}.$$

The magnitude of $\mathbf{R}$ is

$$|\mathbf{R}| = R = \sqrt{R_x^2 + R_y^2 + R_z^2} = \sqrt{(4.241)^2 + (-3.711)^2 + (1.273)^2} = 5.778.$$

The angles of the vector $\mathbf{R}$ with the cartesian axes are calculated from the direction cosines

$$\cos\alpha = \frac{R_x}{|\mathbf{R}|} = \frac{4.241}{5.778} = 0.734,\ \cos\beta = \frac{R_y}{|\mathbf{R}|} = \frac{-3.711}{5.778} = -0.642,\ \text{and}$$
$$\cos\gamma = \frac{R_z}{|\mathbf{R}|} = \frac{1.273}{5.778} = 0.220.$$

The MATLAB commands for the resultant and direction cosines are

```
R_x = V1n(1) + V2n(1);
R_y = V1n(2) + V2n(2);
R_z = V1n(3) + V2n(3);
R = [R_x, R_y, R_z];
nR = norm(R);
u_R = R/nR; % direction cosines
fprintf('R = [%6.3f %6.3f %6.3f]\n', R)
fprintf('|R| = %6.3f\n', nR)
fprintf('u_R = [%6.3f %6.3f %6.3f]\n', u_R)
```

b) The cross product between the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ is

$$\mathbf{V}_1 \times \mathbf{V}_2 = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ V_{1x} & V_{1y} & V_{1z} \\ V_{2x} & V_{2y} & V_{2z} \end{vmatrix} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 1.697 & -2.121 & 1.273 \\ 2.544 & -1.590 & 0 \end{vmatrix} = 2.024\,\mathbf{\imath} + 3.238\,\mathbf{\jmath} + 2.698\,\mathbf{k},$$

or with MATLAB:

```
VC = cross(V1, V2);
fprintf('V1 x V2 = [%6.3f %6.3f %6.3f]\n', VC)
```

c) The angle θ between the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ is calculated with

$$\begin{aligned}\cos\theta &= \frac{\mathbf{V}_1 \cdot \mathbf{V}_2}{V_1\,V_2} = \frac{V_{1x}\,V_{2x} + V_{1y}\,V_{2y} + V_{1z}\,V_{2z}}{V_1\,V_2} \\ &= \frac{2.024(2.544) + (-2.121)(-1.590) + 1.273(0)}{3(3)} = 0.8545.\end{aligned}$$

The angle is $\theta = 31.299°$. The MATLAB commands for calculating the angle between the vectors are:

```
costheta = dot(V1, V2)/(V_1*V_2);
fprintf('theta = %6.3f (deg)\n', acosd(costheta))
```

The MATLAB function `acos(phi)` is the arccosine of the element `phi` and `acosd(phi)` is the inverse cosine , expressed in degrees, of the element of `phi`.

Next the vectors $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{R}$, $\mathbf{u}_R$, and $\mathbf{V}_1 \times \mathbf{V}_2$ will be plotted using MATLAB. The axes are defined in MATLAB with:

```
axis(1.5*[0 a 0 b 0 c])
grid on, hold on
xlabel('x'), ylabel('y'), zlabel('z')
```

For the default "Cartesian" axes mode the coordinate system origin is at `x=y=0`. The `x`-axis is numbered from left to right, the `y`-axis is numbered from bottom to top, and the `z`-axis is vertical with values increasing from bottom to top. The coordinates of the points A, C, D, E, and F are:

```
x_A=a; y_A=0; z_A=0;
x_C=0; y_C=0; z_C=c;
x_D=a; y_D=b; z_D=c;
x_E=0; y_E=b; z_E=c;
x_F=a; y_F=b; z_F=0;
```

The labels of the points O, A, B, C, D, E, F, G, and P are:

```
text(0, 0, 0+.1,' O','fontsize',12)
text(x_A, y_A, z_A+.2,' A','fontsize',12)
text(x_B-.2, y_B, z_B-.1,' B','fontsize',12)
text(x_C, y_C, z_C+.2,' C','fontsize',12)
text(x_D, y_D, z_D+.2,' D','fontsize',12)
text(x_E, y_E, z_E+.2,' E','fontsize',12)
text(x_F, y_F, z_F+.2,' F','fontsize',12)
text(x_G, y_G, z_G+.2,' G','fontsize',12)
text(x_P, y_P, z_P+.2,' P','fontsize',12)
```

The parallelepiped $OABCDEFG$ is plotted using the MATLAB commands:

```
line([0 x_A],[0 y_A],[0 z_A])
line([0 x_B],[0 y_B],[0 z_B])
line([0 x_C],[0 y_C],[0 z_C])
line([x_B x_E],[y_B y_E],[z_B z_E])
line([x_B x_F],[y_B y_F],[z_B z_F])
line([x_A x_F],[y_A y_F],[z_A z_F])
line([x_A x_G],[y_A y_G],[z_A z_G])
line([x_C x_G],[y_C y_G],[z_C z_G])
line([x_C x_E],[y_C y_E],[z_C z_E])
line([x_D x_G],[y_D y_G],[z_D z_G])
```

```
line([x_D x_E],[y_D y_E],[z_D z_E])
line([x_D x_F],[y_D y_F],[z_D z_F])
```

Another way of drawing the parallelepiped *OABCDEFG* is:

```
plot3(...
[x_G x_A x_F x_D x_G x_C x_E x_B 0 x_C],...
[y_G y_A y_F y_D y_G y_C y_E y_B 0 y_C],...
[z_G z_A z_F z_D z_G z_C z_E z_B 0 z_C])
line([0 x_A],[0 y_A],[0 z_A])
line([x_B x_F],[y_B y_F],[z_B z_F])
line([x_D x_E],[y_D y_E],[z_D z_E])
```

where the MATLAB statement `plot3(x,y,z)` plots a line in 3D through the points whose coordinates are the elements of the vectors `x`, `y`, and `z`.

The lines *BG* and *BP* are plotted with:

```
line([x_B x_G],[y_B y_G],[z_B z_G],...
    'Color','k','LineStyle','--')
line([x_B x_P],[y_B y_P],[z_B z_P],...
    'Color','k','LineStyle','--')
```

The vectors $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{R}$, $\mathbf{u}_R$, and $\mathbf{V}_1 \times \mathbf{V}_2$ and their labels are described by the following MATLAB commands:

```
quiver3(x_B,y_B,z_B, V1(1),V1(2),V1(3),1,...
    'Color','k','LineWidth',2)
quiver3(x_B,y_B,z_B, V2(1),V2(2),V2(3),1,...
    'Color','k','LineWidth',2)
quiver3(x_B,y_B,z_B, R(1),R(2),R(3),1,...
    'Color','r','LineWidth',3)
quiver3(x_B,y_B,z_B, u_R(1),u_R(2),u_R(3),1,...
    'Color','b','LineWidth',4)
quiver3(x_B,y_B,z_B, VC(1),VC(2),VC(3),1,...
    'Color','m','LineWidth',3)

text(x_B+V1(1), y_B+V1(2), z_B+V1(3)+.1,...
    'V_1','fontsize',14,'fontweight','b')
text(x_B+V2(1), y_B+V2(2), z_B+V2(3)+.1,...
    'V_2','fontsize',14,'fontweight','b')
text(x_B+R(1), y_B+R(2), z_B+R(3)+.1,...
    'R','fontsize',14,'fontweight','b')
text(x_B+u_R(1), y_B+u_R(2), z_B+u_R(3)+.1,...
    'u_R','fontsize',14,'fontweight','b')
text(x_B+VC(1), y_B+VC(2), z_B+VC(3)+.1,...
    'V_1 x V_2','fontsize',14,'fontweight','b')
```

A rotated MATLAB drawing of the vectors is shown in Fig. E1.4(b).

Example 1.5
The vector $\mathbf{p}$ of magnitude $|\mathbf{p}| = p$ is located in the $x-z$ plane and makes an angle θ with x-axis as shown in Fig. E1.5(a). The vector $\mathbf{q}$ of magnitude $|\mathbf{q}| = q$ is situated along the x-axis Compute the vector (cross) product $\mathbf{v} = \mathbf{p} \times \mathbf{q}$. Numerical application: $|\mathbf{p}| = p = 5$, $|\mathbf{q}| = q = 4$, and $\theta = 30°$.

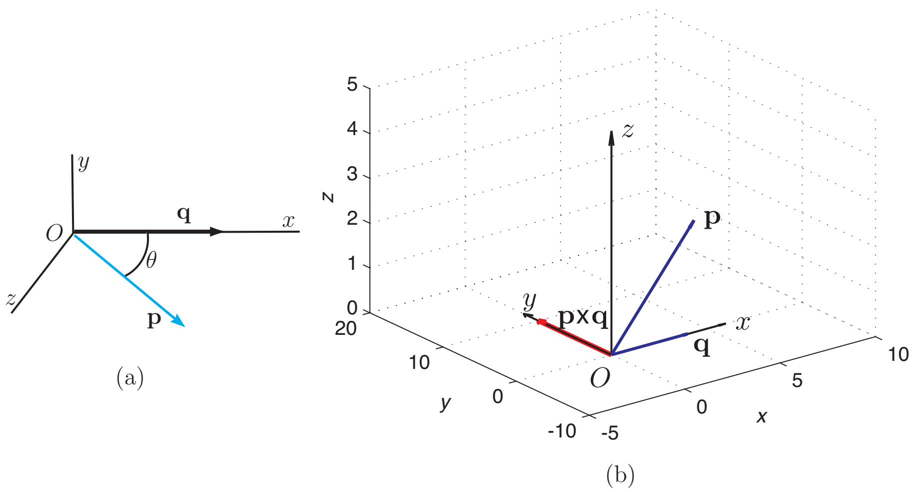


Fig. E1.5 Example 1.5

Solution
The vector product $\mathbf{v}$ is perpendicular to the vectors $\mathbf{p}$ and $\mathbf{q}$ and that is why the vector $\mathbf{v}$ is along the y-axis and with has the magnitude

$$|\mathbf{v}| = |\mathbf{p}|\,|\mathbf{q}|\sin\theta = pq\sin\theta = 5(4)\sin 30° = 10.$$

From Fig. E1.5(a) the direction of the vector $\mathbf{v}$ is upward.

The solution could also be obtained by expressing the vector product $\mathbf{v} = \mathbf{p} \times \mathbf{q}$ of the given vectors $\mathbf{p}$ and $\mathbf{q}$ in terms of the their rectangular components. Resolving $\mathbf{p}$ and $\mathbf{q}$ into components, one can write

$$\begin{aligned}\mathbf{v} = \mathbf{p} \times \mathbf{q} &= (p_x\mathbf{\imath} + p_y\mathbf{\jmath} + p_z\mathbf{k}) \times (q_x\mathbf{\imath} + q_y\mathbf{\jmath} + q_z\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{vmatrix} \\ &= (p_yq_z - p_zq_y)\mathbf{\imath} + (p_zq_x - p_xq_z)\mathbf{\jmath} + (p_xq_y - p_yq_x)\mathbf{k}.\end{aligned}$$

The components p_x, p_y, and p_z of the vector $\mathbf{p}$ are

$p_x = |\mathbf{p}|\cos\theta = p\cos\theta = 5\cos 30° = 5\dfrac{\sqrt{3}}{2} = \dfrac{5\sqrt{3}}{2}$, $p_y = 0$, and

$p_z = |\mathbf{p}|\sin\theta = p\sin\theta = 5\left(\frac{1}{2}\right) = \frac{5}{2}.$
The components q_x, q_y,and q_z of the vector $\mathbf{q}$ are $q_x = q = 4$, $q_y = 0$ and $q_z = 0$.
It results

$$\begin{aligned}
\mathbf{v} &= \mathbf{p}\times\mathbf{q} = (p_y q_z - p_z q_y)\,\imath + (p_z q_x - p_x q_z)\,\jmath + (p_x q_y - p_y q_x)\,\mathbf{k}\\
&= \left(0(0) - \frac{5}{2}(0)\right)\imath + \left(\frac{5}{2}(4) - \frac{5\sqrt{3}}{2}(0)\right)\jmath + \left(\frac{5\sqrt{3}}{2}(0) - 0(4)\right)\mathbf{k}\\
&= \frac{5}{2}(4)\,\jmath = 10\jmath.
\end{aligned}$$

The MATLAB program for the cross product $\mathbf{v} = \mathbf{p}\times\mathbf{q}$ is:

```
syms p q theta real
p_x = p*cos(theta); p_y = 0; p_z = p*sin(theta);
q_x = q; q_y = 0; q_z = 0;
v = cross([p_x p_y p_z],[q_x q_y q_z]);
slist = {p, q, theta}; nlist = {5, 4, pi/6};
vn = subs(v, slist, nlist);
fprintf('p x q = ')
fprintf('[%s %s %s]',char(v(1)),char(v(2)),char(v(3)))
fprintf(' = [%g %g %g] \n', vn)
```

and the output is:

```
p x q = [0 p*sin(theta)*q 0] = [0 10 0]
```

The function `char(x)` converts the array `x` into MATLAB character array.

Next the vectors $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{p}\times\mathbf{q}$ will be plotted using MATLAB. The numerical values of the components of the vectors $\mathbf{p}$ and $\mathbf{q}$ are calculated with:

```
p_xn=double(subs(p_x,slist,nlist));
p_yn=double(subs(p_y,slist,nlist));
p_zn=double(subs(p_z,slist,nlist));

q_xn=double(subs(q_x,slist,nlist));
q_yn=double(subs(q_y,slist,nlist));
q_zn=double(subs(q_z,slist,nlist));
```

The statement `double(x)` converts the symbolic matrix `x` to a matrix of double precision floating point numbers. The Cartesian axes x, y, z are ploted with:

```
axis ([0 6 0 8 0 5])
axis auto,  grid on, hold on
xlabel('\it x'), ylabel('\it y'), zlabel('\it z')

quiver3(0,0,0,6,0,0,1,'Color','k','LineWidth',1)
text('Interpreter','latex','String','  $x$',...
    'Position',[6,0,0],'FontSize',14)
```

```
quiver3(0,0,0,0,12,0,1,'Color','k','LineWidth',1)
text('Interpreter','latex','String','  $y$',...
    'Position',[0,13,0],'FontSize',14)
quiver3(0,0,0,0,0,5,1,'Color','k','LineWidth',1)
text('Interpreter','latex','String','  $z$',...
    'Position',[0,0,5],'FontSize',14)
```

The statement `axis auto` returns the axis scaling to its default automatic mode. The vectors $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{v} = \mathbf{p} \times \mathbf{q}$ are plotted with the MATLAB commands:

```
quiver3(0,0,0,p_xn,p_yn,p_zn,1,...
    'Color','b','LineWidth',1.5)

quiver3(0,0,0,q_xn,q_yn,q_zn,1,...
    'Color','b','LineWidth',1.5)

quiver3(0,0,0,vn(1),vn(2),vn(3),1,...
    'Color','r','LineWidth',2.5)

text('Interpreter','latex',...
 'String','  \bf q',...
 'Position',[q_xn,q_yn,q_zn],...
 'FontSize',14)

text('Interpreter','latex',...
 'String','  \bf p',...
 'Position',[p_xn,p_yn,p_zn],...
 'FontSize',14)

text('Interpreter','latex',...
 'String','  {\bf p}$\times${\bf q}',...
 'Position',[vn(1)+.5,vn(2),vn(3)],...
 'FontSize',14)

text('Interpreter','latex','String','   $O$',...
 'Position',[0,0,0-.5],'FontSize',14,...
 'HorizontalAlignment','right')
```

The MATLAB drawing of the vectors is shown in Fig. E1.5(b).

Example 1.6
Compute $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ and $(\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a}$ where $\mathbf{a} = a_x \boldsymbol{\imath} + a_y \boldsymbol{\jmath} + a_z \mathbf{k}$, $\mathbf{b} = b_x \boldsymbol{\imath} + b_y \boldsymbol{\jmath} + b_z \mathbf{k}$, and $\mathbf{c} = c_x \boldsymbol{\imath} + c_y \boldsymbol{\jmath} + c_z \mathbf{k}$. Numerical application: $a_x = 2$, $a_y = 1$, $a_z = 3$, $b_x = 2$, $b_y = 1$, $b_z = 0$, $c_x = 2$, $c_y = 0$, and $c_z = 0$.

Solution
The scalar $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is

$$
\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= (a_x\mathbf{\imath}+a_y\mathbf{\jmath}+a_z\mathbf{k})\cdot\begin{vmatrix}\mathbf{\imath} & \mathbf{\jmath} & \mathbf{k}\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}\\
&= \begin{vmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}\\
&= a_x\,(b_yc_z-b_zc_y)+a_y\,(b_zc_x-b_xc_z)+a_z\,(b_xc_y-b_yc_x)\\
&= 2\,(1(0)-0(0))+1\,(0(2)-2(0))+3\,(2(0)-1(2))=-6
\end{aligned}
$$

The scalar $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ is

$$
\begin{aligned}
(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= \begin{vmatrix}\mathbf{\imath} & \mathbf{\jmath} & \mathbf{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}\cdot(c_x\mathbf{\imath}+c_y\mathbf{\jmath}+c_z\mathbf{k})\\
&= (c_x\mathbf{\imath}+c_y\mathbf{\jmath}+c_z\mathbf{k})\cdot\begin{vmatrix}\mathbf{\imath} & \mathbf{\jmath} & \mathbf{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}\\
&= \begin{vmatrix}c_x & c_y & c_z\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} = -\begin{vmatrix}a_x & a_y & a_z\\ c_x & c_y & c_z\\ b_x & b_y & b_z\end{vmatrix} = \begin{vmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}\\
&= a_x\,(b_yc_z-b_zc_y)+a_y\,(b_zc_x-b_xc_z)+a_z\,(b_xc_y-b_yc_x)\\
&= 2\,(1(0)-0(0))+1\,(0(2)-2(0))+3\,(2(0)-1(2))=-6
\end{aligned}
$$

The scalar $(\mathbf{c}\times\mathbf{b})\cdot\mathbf{a}$ is

$$
\begin{aligned}
(\mathbf{c}\times\mathbf{b})\cdot\mathbf{a} &= \begin{vmatrix}\mathbf{\imath} & \mathbf{\jmath} & \mathbf{k}\\ c_x & c_y & c_z\\ b_x & b_y & b_z\end{vmatrix}\cdot(a_x\mathbf{\imath}+a_y\mathbf{\jmath}+a_z\mathbf{k})\\
&= (a_x\mathbf{\imath}+a_y\mathbf{\jmath}+a_z\mathbf{k})\cdot\begin{vmatrix}\mathbf{\imath} & \mathbf{\jmath} & \mathbf{k}\\ c_x & c_y & c_z\\ b_x & b_y & b_z\end{vmatrix}\\
&= \begin{vmatrix}a_x & a_y & a_z\\ c_x & c_y & c_z\\ b_x & b_y & b_z\end{vmatrix} = -\begin{vmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}\\
&= -\left[a_x\,(b_yc_z-b_zc_y)+a_y\,(b_zc_x-b_xc_z)+a_z\,(b_xc_y-b_yc_x)\right]\\
&= -\left[2\,(1(0)-0(0))+1\,(0(2)-2(0))+3\,(2(0)-1(2))\right]=6
\end{aligned}
$$

Note that: $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=-(\mathbf{c}\times\mathbf{b})\cdot\mathbf{a}$.

The MATLAB program for the example is

```
syms a_x a_y a_z b_x b_y b_z c_x c_y c_z real
a=[a_x a_y a_z]; b=[b_x b_y b_z]; c=[c_x c_y c_z];
d = dot(a,cross(b,c)); %a.(b x c)
```

```
e = dot(cross(a,b),c); %(a x b).c
f = dot(cross(c,b),a); %(c x b).a
fprintf('a.(b x c)-(a x b).c=%s\n',char(simplify(d-e)))
fprintf('a.(b x c)+(c x b).a=%s\n',char(simplify(d+f)))
slist={a_x,a_y,a_z,b_x,b_y,b_z,c_x,c_y,c_z};
nlist={2,1,3,2,1,0,2,0,0};
fprintf('a.(b x c)=%g\n',subs(d,slist,nlist))
fprintf('(a x b).c=%g\n',subs(e,slist,nlist))
fprintf('(c x b).a=%g\n',subs(f,slist,nlist))
```

Example 1.7
Find the c_z component of the vector $\mathbf{c}$ such as the vectors $\mathbf{a} = a_x\,\mathbf{\imath} + a_y\,\mathbf{\jmath} + a_z\,\mathbf{k}$, $\mathbf{b} = b_x\,\mathbf{\imath} + b_y\,\mathbf{\jmath} + b_z\,\mathbf{k}$, and $\mathbf{c} = c_x\,\mathbf{\imath} + c_y\,\mathbf{\jmath} + c_z\,\mathbf{k}$ are coplanar. Numerical application: $a_x = 2$, $a_y = 3$, $a_z = 0$, $b_x = 3$, $b_y = 2$, $b_z = -2$, $c_x = 2$, and $c_y = 3$.

Solution
The three vectors are coplanar if $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$. The scalar $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is

$$\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= (a_x\mathbf{\imath} + a_y\mathbf{\jmath} + a_z\mathbf{k})\cdot\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \\
&= a_x(b_yc_z - b_zc_y) + a_y(b_zc_x - b_xc_z) + a_z(b_xc_y - b_yc_x) \\
&= a_xb_yc_z - a_xb_zc_y + a_yb_zc_x - a_yb_xc_z + a_zb_xc_y - a_zb_yc_x \\
&= a_xb_yc_z - a_yb_xc_z - a_xb_zc_y + a_yb_zc_x + a_zb_xc_y - a_zb_yc_x \\
&= c_z(a_xb_y - a_yb_x) - a_xb_zc_y + a_yb_zc_x + a_zb_xc_y - a_zb_yc_x.
\end{aligned}$$

The scalar triple product of the three vectors in MATLAB is given by

```
syms a_x a_y a_z b_x b_y b_z c_x c_y c_z real
a=[a_x a_y a_z]; b=[b_x b_y b_z]; c=[c_x c_y c_z];
d=det([a; b; c]); % a.(b x c)
```

The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar if

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0 \Leftrightarrow c_z(a_xb_y - a_yb_x) - a_xb_zc_y + a_yb_zc_x + a_zb_xc_y - a_zb_yc_x = 0,$$

or

$$c_z = \frac{a_xb_zc_y - a_yb_zc_x - a_zb_xc_y + a_zb_yc_x}{a_xb_y - a_yb_x}.$$

Substituting with the numerical values it results

$$c_z = \frac{2(-2)(3) - 3(-2)(2) - 0(3)(3) + 0(2)(2)}{2(2) - 3(3)} = \frac{-12 + 12 - 0 + 0}{4 - 9} = 0.$$

The given numerical vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar if $c_z = 0$.

To solve the equation $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$, a specific MATLAB command will be used. The command `solve('eqn1', 'eqn2', ..., 'eqnN', 'var1', 'var2', ...'varN')` attempts to solve an equation or set of equations `'eqn1'`, `'eqn2', ..., 'eqnN'` for the variables `'eqnN', 'var1', 'var2', ..., 'varN'`. The set of equations are symbolic expressions or strings specifying equations. The MATLAB command to find the solution `c_z` of the equation `det([a; b; c])=0` is

```
x = solve(d, c_z);
```

and the numerical solution for `c_z` is displayed with

```
slist={a_x,a_y,a_z,b_x,b_y,b_z,c_x,c_y};
nlist={2,3,0,3,2,-2,2,3};
fprintf('c_z= %g\n', subs(x, slist, nlist))
```

1.10 Problems

1.1 a) Find the angle θ made by the vector $\mathbf{v} = -10\mathbf{\imath} + 5\mathbf{\jmath}$ with the positive x-axis and determine the unit vector in the direction of $\mathbf{v}$. The angle θ is measured counterclockwise (ccw) and has the values $0 \le \theta \le 2\pi$ or $-\pi \le \theta \le \pi$.
b) Determine the magnitude of the resultant $\mathbf{p} = \mathbf{v}_1 + \mathbf{v}_2$ and the angle that $\mathbf{p}$ makes with the positive x-axis, where the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are shown in Fig. P1.1. The magnitudes of the vectors are $|\mathbf{v}_1| = v_1 = 10$, $|\mathbf{v}_2| = v_2 = 5$, and the angles of the vectors with the positive x-axis are $\theta_1 = 30°$ and $\theta_2 = 60°$.

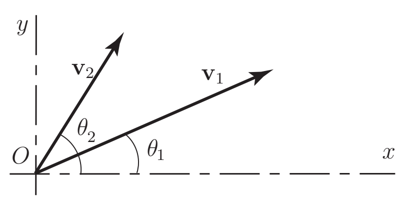


Fig. P1.1 Problem 1.1

1.2 The planar vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are given in xOy plane as shown in Fig. P1.2. The magnitude of the vectors are $a = P$, $b = 2P$, and $c = P\sqrt{2}$. The angles in the figure are $\alpha = 45°$, $\beta = 120°$, and $\gamma = 30°$. Determine the resultant $\mathbf{v} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ and the angle that $\mathbf{v}$ makes with the positive x-axis.

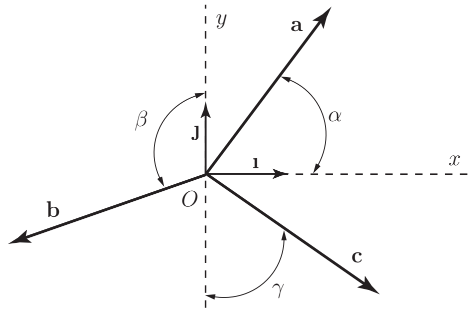


Fig. P1.2 Problem 1.2

1.3 The cube in Fig. P1.3 has the sides equal to $l = 1$. a) Find the direction cosines of the resultant $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4$. b) Determine the angle between the vectors $\mathbf{v}_2$ and $\mathbf{v}_3$. c) Find the projection of the vector $\mathbf{v}_2$ on the vector $\mathbf{v}_4$. d) Calculate $\mathbf{v}_2 \cdot \mathbf{v}_4$, $\mathbf{v}_2 \times \mathbf{v}_4$, $\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$, $(\mathbf{v}_2 \times \mathbf{v}_3) \times \mathbf{v}_4$, and $\mathbf{v}_2 \times (\mathbf{v}_3 \times \mathbf{v}_4)$.

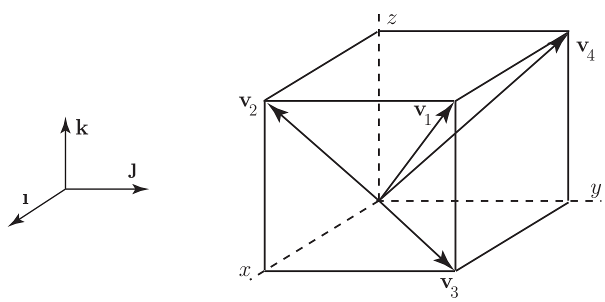


Fig. P1.3 Problem 1.3

1.4 The vectors $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$, shown in Fig. P1.4 , act on the sides of a cube (the side of the cube is $l = 2$). The magnitudes of the vectors are $\mathbf{F}_1 = \mathbf{F}_2 = F = 1$, and $\mathbf{F}_3 = \mathbf{F}_4 = F\sqrt{2}$. a) Find the resultant $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4$. b) Find the direction cosines of the vector $\mathbf{F}_4$. c) Determine the angle between the vectors $\mathbf{F}_1$ and $\mathbf{F}_3$. d) Find the projection of the vector $\mathbf{F}_2$ on the vector $\mathbf{F}_4$. e) Calculate $\mathbf{F}_1 \cdot \mathbf{F}_3$, $\mathbf{F}_2 \times \mathbf{F}_4$, and $\mathbf{F}_1 \cdot (\mathbf{F}_2 \times \mathbf{F}_3)$.

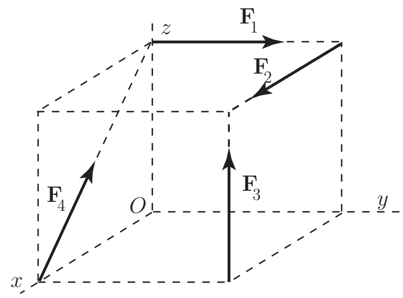


Fig. P1.4 Problem 1.4

1.5 Figure P1.5 represents the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ acting on a cube with the side $l = 2$. The magnitude of the forces are $\mathbf{v}_1 = V = 2$ and $\mathbf{v}_2 = \mathbf{v}_3 = \mathbf{v}_4 = 2V$. a) Find the resultant and the direction cosines of the resultant $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4$. b) Determine the angle between the vectors $\mathbf{v}_1$ and $\mathbf{v}_3$. c) Find the projection of the vector $\mathbf{v}_4$ on the resultant vector $\mathbf{v}$. d) Calculate $\mathbf{v}_2 \cdot \mathbf{v}$, $\mathbf{v}_1 \times \mathbf{v}_2$, and $\mathbf{v}_2 \times \mathbf{v}_4$.

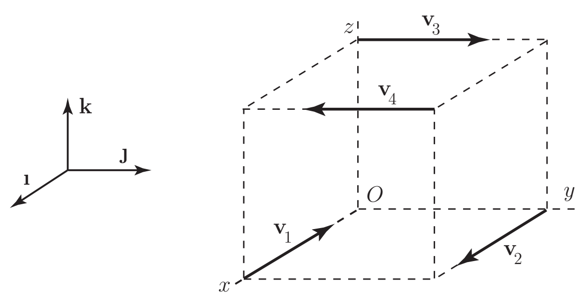


Fig. P1.5 Problem 1.5

1.6 Repeat the previous problem for Fig. P1.6

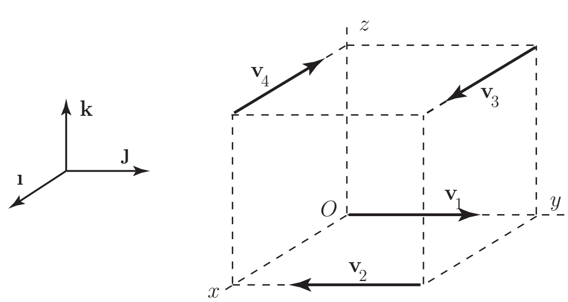


Fig. P1.6 Problem 1.6

1.7 The parallelepiped shown in Fig. P1.7 has the sides $l = 1$ m, $w = 2$ m, and $h = 3$ m. The magnitude of the vectors are $\mathbf{F}_1 = \mathbf{F}_2 = 10$ N, and $\mathbf{F}_3 = \mathbf{F}_4 = 20$ N. a) Find the resultant $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4$. b) Find the unit vectors of the vectors $\mathbf{F}_1$ and $\mathbf{F}_4$. c) Determine the angle between the vectors $\mathbf{F}_1$ and $\mathbf{F}_4$. d) Find the projection of the vector $\mathbf{F}_2$ on the vector $\mathbf{F}_4$. e) Calculate $\mathbf{F}_1 \cdot \mathbf{F}_4$, $\mathbf{F}_2 \times \mathbf{F}_3$, and $\mathbf{F}_1 \cdot (\mathbf{F}_2 \times \mathbf{F}_3)$.

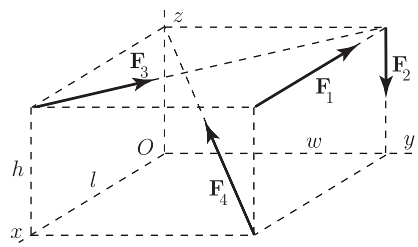


Fig. P1.7 Problem 1.7

1.8 A uniform rectangular plate of length l and width w is held open by a cable (Fig. P1.8). The plate is hinged about an axis parallel to the plate edge of length l. Points A and B are at the extreme ends of this hinged edge. Points D and C are at the ends of the other edge of length l and are respectively adjacent to points A and B. Points D and C move as the plate opens. In the closed position, the plate is in a horizontal plane. When held open by a cable, the plate has rotated through an angle θ relative to the closed position. The supporting cable runs from point D to point E where point E is located a height h directly above the point B on the hinged edge of the plate. The cable tension required to hold the plate open is T. Find the projection of the tension force onto the diagonal axis AC of the plate. Numerical application: $l = 1.0$ m, $w = 0.5$ m, $\theta = 45°$, $h = 1.0$ m, and $T = 100$ N.

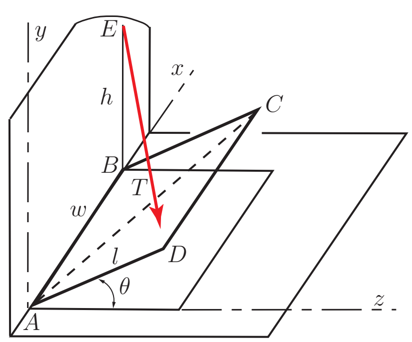


Fig. P1.8 Problem 1.8

1.9 The following spatial vectors are given: $\mathbf{v}_1 = -3\,\imath + 4\,\jmath - 3\,\mathbf{k}$, $\mathbf{v}_2 = 3\,\imath + 3\,\mathbf{k}$, and $\mathbf{v}_3 = 1\,\imath + 2\,\jmath + 3\,\mathbf{k}$. Find the expressions $\mathbf{E}_1 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{E}_2 = \mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$, $\mathbf{E}_3 = (\mathbf{v}_1 \times \mathbf{v}_2) \times \mathbf{v}_3$, and $E_4 = (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3$.

1.10 Find the angle between the vectors $\mathbf{v}_1 = 2\,\imath - 4\,\jmath + 4\,\mathbf{k}$ and $\mathbf{v}_2 = 4\,\imath + 2\,\jmath + 4\,\mathbf{k}$. Find the expressions $\mathbf{v}_1 \times \mathbf{v}_2$ and $\mathbf{v}_1 \cdot \mathbf{v}_2$.

1.11 The following vectors are given $\mathbf{v}_1 = 2\,\imath + 4\,\jmath + 6\,\mathbf{k}$, $\mathbf{v}_2 = 1\,\imath + 3\,\jmath + 5\,\mathbf{k}$, and $\mathbf{v}_3 = -2\,\imath + 2\,\mathbf{k}$. Find the vector triple product of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, and explain the result.

1.12 Solve the vectorial equation $\mathbf{x} \times \mathbf{a} = \mathbf{x} \times \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are two known given vectors.

1.13 Solve the vectorial equation $\mathbf{v} = \mathbf{a} \times \mathbf{x}$, where $\mathbf{v}$ and $\mathbf{a}$ are two known given vectors.

1.14 Solve the vectorial equation $\mathbf{a} \cdot \mathbf{x} = m$, where $\mathbf{a}$ is a known given vector and m is a known given scalar.