Chapter 5 Finite Difference Methods

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References

1. Chapters 5 and 9, Brandimarte
2. Section 17.8, Hull
3. Chapter 7, “Numerical analysis”, Burden and Faires

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Outline

- Finite difference (FD) approximation to the derivatives
- Explicit FD method
- Numerical issues
- Implicit FD method
- Crank-Nicolson method
- Dealing with American options
- Further comments

Chapter 5 Finite Difference Methods

5.1 Finite difference approximations

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Finite-difference mesh

- Aim to approximate the values of the continuous function *f*(t, S) on a set of discrete points in (t, S) plane
- Divide the S-axis into equally spaced nodes at distance ΔS apart, and, the t-axis into equally spaced nodes a distance Δt apart
- (t, S) plane becomes a mesh with mesh points on

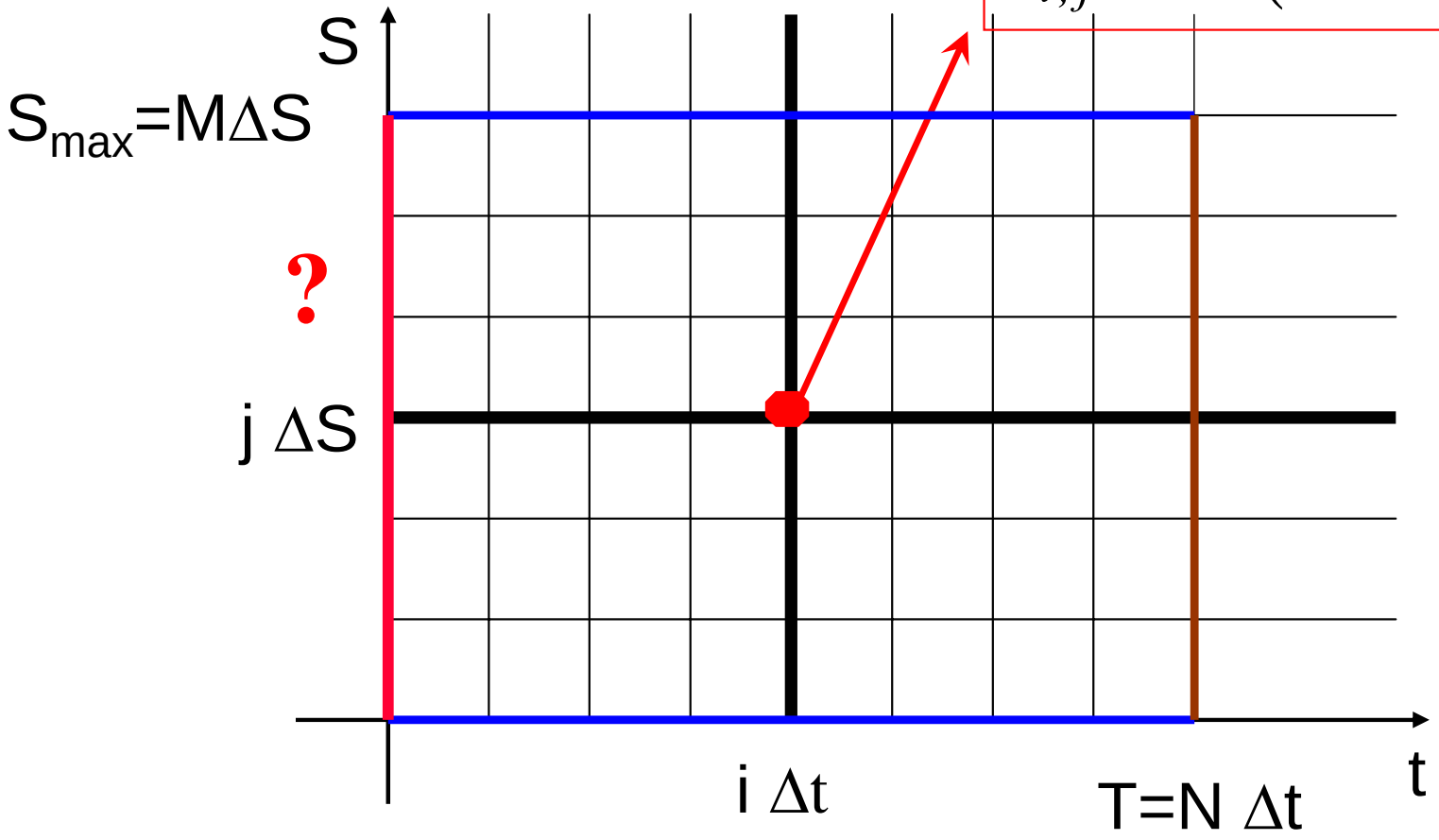
  (i Δt, jΔS)
- We are interested in the values of *f*(t, S) at mesh points

  (i Δt, jΔS), denoted as

$$f_{i,j} = f\left(i\Delta t, j\Delta S\right)$$

The mesh for finite-difference approximation

$$f_{i,j} = f\left(i\Delta t, j\Delta S\right)$$



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Black-Scholes Equation for a European option with value V(S,t)

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial^2 S}+rS\frac{\partial V}{\partial S}-rV=0 \qquad (5.1)$$

where $0 < S < +\infty$ and $0 \le t < T$

with proper final and boundary conditions

Notes:

This is a second-order hyperbolic, elliptic, or parabolic, forward or backward partial differential equation

Its solution is sufficiently well behaved ,i.e. well-posed

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Finite difference approximations

The basic idea of FDM is to replace the partial derivatives by approximations obtained by Taylor expansions near the point of interests

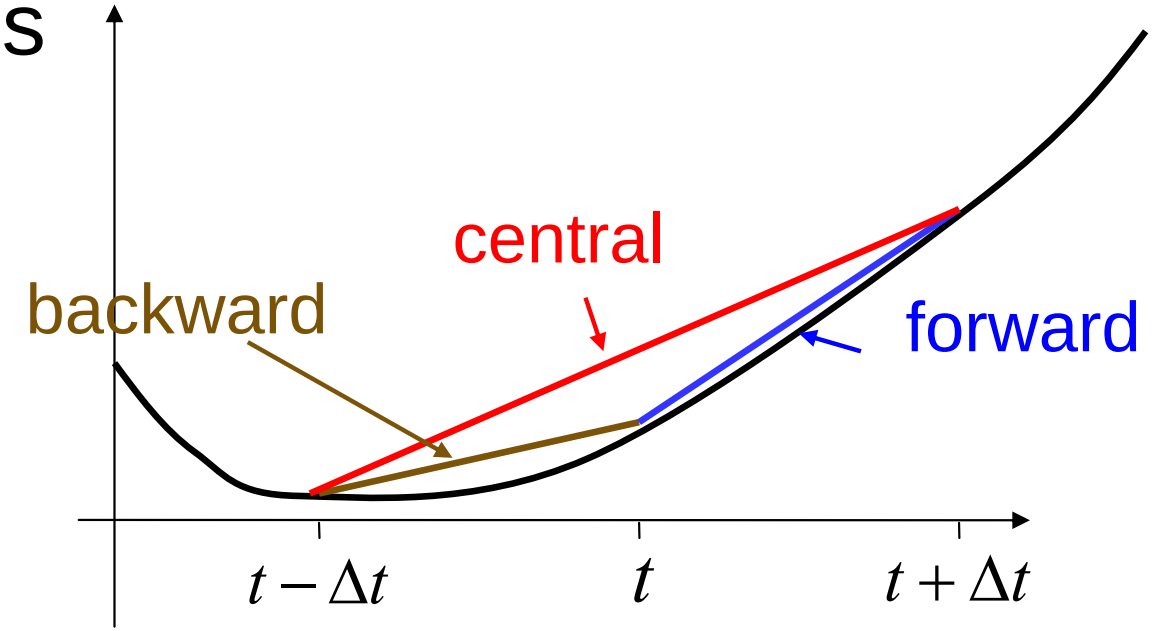
For example,

$$\frac{\partial f(S,t)}{\partial t} = \lim_{\Delta t \to 0} \frac{f(S,t+\Delta t) - f(S,t)}{\Delta t} \approx \frac{f(S,t+\Delta t) - f(S,t)}{\Delta t}$$

for small Δt, using Taylor expansion at point (S,t)

$$f(S,t+\Delta t) = f(S,t) + \frac{\partial f(S,t)}{\partial t}\Delta t + O\left((\Delta t)^2\right)$$

Forward-, Backward-, and Central-difference approximation to 1st order derivatives



Forward: $$\frac{\partial f(t,S)}{\partial t} \approx \frac{f(t+\Delta t,S)-f(t,S)}{\Delta t} + O(\Delta t)$$

Backward: $$\frac{\partial f(t,S)}{\partial t} \approx \frac{f(t,S)-f(t-\Delta t,S)}{\Delta t} + O(\Delta t)$$

Central: $$\frac{\partial f(t,S)}{\partial t} \approx \frac{f(t+\Delta t,S)-f(t-\Delta t,S)}{2\Delta t} + O\left((\Delta t)^2\right)$$

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Symmetric Central-difference approximations to 2nd order derivatives

$$\frac{\partial^2 f(t,S)}{\partial S^2} \approx \frac{f(t,S+\Delta S)-2f(t,S)+f(t,S-\Delta S)}{(\Delta S)^2}+O\left((\Delta S)^2\right)$$

Use Taylor's expansions for $f(t,S+\Delta S)$ and $f(t,S-\Delta S)$ around point (t,S):

$f(t,S+\Delta S)=?$

+

$f(t,S-\Delta S)=?$

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Finite difference approximations

Forward Difference: $\frac{\partial f}{\partial t} \approx \frac{f_{i+1,j} - f_{i,j}}{\Delta t}, \quad \frac{\partial f}{\partial S} \approx \frac{f_{i,j+1} - f_{i,j}}{\Delta S}$

Backward Difference: $\frac{\partial f}{\partial t} \approx \frac{f_{i,j} - f_{i-1,j}}{\Delta t}, \quad \frac{\partial f}{\partial S} \approx \frac{f_{i,j} - f_{i,j-1}}{\Delta S}$

Central Difference: $\frac{\partial f}{\partial t} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta t}, \quad \frac{\partial f}{\partial S} \approx \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S}$

As to the second derivative, we have:

$$\frac{\partial^2 f}{\partial S^2} \approx \left(\frac{f_{i,j+1} - f_{i,j}}{\Delta S} - \frac{f_{i,j} - f_{i,j-1}}{\Delta S} \right) \Big/ \Delta S$$

$$= \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta S)^2}$$

Finite difference approximations

- Depending on which combination of schemes we use in discretizing the equation, we will have explicit, implicit, or Crank-Nicolson methods
- We also need to discretize the boundary and final conditions accordingly. For example, for European Call,

Final Condition:

$$f_{N,j} = max\left(j\Delta S - K, 0\right), \quad \text{for } j = 0,1,\ldots,M$$

Boundary Conditions:

$$\begin{cases} f_{i,0} = 0 \\ f_{i,M} = S_{max} - Ke^{-r(N-i)\Delta t} \end{cases}, \quad \text{for } i = 0,1,\ldots,N$$

where $S_{max} = M\Delta S$.

Chapter 5 Finite Difference Methods

5.2.1 Explicit Finite-Difference Method

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Explicit Finite Difference Methods

In $\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$, at point $(i\Delta t,\ j\Delta S)$, set

backward difference: $\frac{\partial f}{\partial t} \approx \frac{f_{i,j} - f_{i-1,j}}{\Delta t}$

central difference: $\frac{\partial f}{\partial S} \approx \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S}$,

and

$$\frac{\partial^2 f}{\partial S^2} \approx \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2}, \quad rf = rf_{i,j}\ ,\ S = j\Delta S$$

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Explicit Finite Difference Methods

Rewriting the equation, we get an explicit scheme:

$$f_{i-1,j} = a_j^* f_{i,j-1} + b_j^* f_{i,j} + c_j^* f_{i,j+1} \qquad (5.2)$$

where

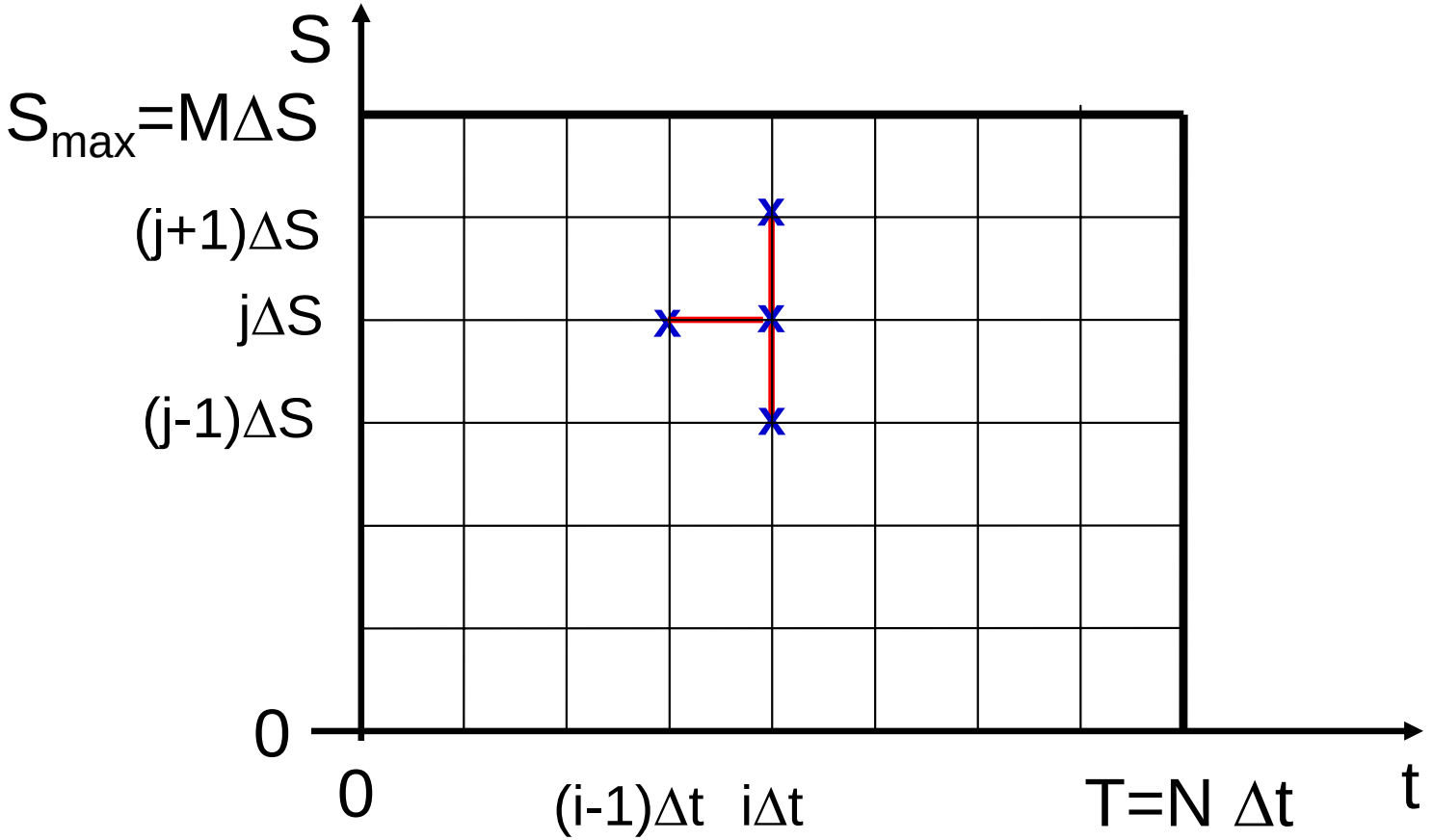
$$a_j^* = \frac{1}{2}\Delta t\left(\sigma^2 j^2 - rj\right)$$

$$b_j^* = 1 - \Delta t\left(\sigma^2 j^2 + r\right)$$

$$c_j^* = \frac{1}{2}\Delta t\left(\sigma^2 j^2 + rj\right)$$

for $i = N-1, N-2, \ldots, 1, 0$ and $j = 1, 2, \ldots, M-1$.

Numerical Computation Dependency



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Implementation

1. Starting with the final values $f_{N,j}$, we apply (5.2) to solve $f_{N-1,j}$ for $1 \le j \le M-1$. We use the boundary condition to determine $f_{N-1,0}$ and $f_{N\text{-}1,M}$.

2. Repeat the process to determine $f_{N-2,j}$ and so on

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Example

We compare explicit finite difference solution for a European put with the exact Black-Scholes formula, where $T = 5/12$ yr, S_0=\$50, K = \$50, σ=30%, r = 10%.

Black-Scholes Price: \$2.8446

EFD Method with S_{max}=\$100, $\Delta S=2$, $\Delta t=5/1200$: \$2.8288

EFD Method with S_{max}=\$100, $\Delta S=1$, $\Delta t=5/4800$: \$2.8406

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Example (Stability)

We compare explicit finite difference solution for a European put with the exact Black-Scholes formula, where T = 5/12 yr, S_0=\$50, K = \$50, σ=30%, r = 10%.

Black-Scholes Price: \$2.8446

EFD Method with S_{max}=\$100, ΔS=2, Δt=5/1200: \$2.8288

EFD Method with S_{max}=\$100, ΔS=1.5, Δt=5/1200: \$3.1414

EFD Method with S_{max}=\$100, ΔS=1, Δt=5/1200: -\$2.8271E22

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5.2.2 Numerical Stability

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Numerical Accuracy

- The problem itself
- The discretization scheme used
- The numerical algorithm used

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Conditioning Issue

Suppose we have mathematically posed problem:

$$y = f(x)$$

where y is to evaluated given an input x.

Let $x^* = x + \delta x$ for small change δx.

If hen $f(x^*)$ is near $f(x)$, then we call the problem is $well\text{-}conditioned$. Otherwise, it is ill-posed/ill-conditioned.

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Conditioning Issue

- Conditioning issue is related to the problem itself, not to the specific numerical algorithm; Stability issue is related to the numerical algorithm
- One can not expect a good numerical algorithm to solve an ill-conditioned problem any more accurately than the data warrant
- But a bad numerical algorithm can produce poor solutions even to well-conditioned problems

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Conditional Issue

The concept "near" can be measured by further information about the particular problem:

$$\frac{\left\| f(x) - f(x^*) \right\|}{\left\| f(x) \right\|} \le C \frac{\left\| \delta x \right\|}{\left\| x \right\|} \qquad \left(f(x) \neq 0 \right)$$

where C is called *condition number* of this problem. If C is large, the problem is ill-conditioned.

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Floating Point Number & Error

Let x be any real number.

Infinite decimal expansion : $x = \pm .x_1 x_2 \cdots x_d \cdots 10^e$

Truncated floating point number : $x \approx fl(x) = \pm .x_1 x_2 \cdots x_d 10^e$

where $x_1 \neq 0, 0 \leq x_i \leq 9,$

d : an integer, precision of the floating point system

e : an bounded integer

Floating point or **roundoff error** : $fl(x) - x$

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Error Propagation

When additional calculations are done, there is an accumulation of these floating point errors.

Example : Let $x = -0.6667$ and $fl(x) = -0.667\ 10^0$ where $d = 3$.

Floating point error : $fl(x) - x = -0.0003$

Error propagation : $fl(x)^2 - x^2 = 0.00040011$

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Numerical Stability or Instability

Stability ensures if the error between the numerical soltuion and the exact solution remains bounded as numerical computation progresses.

That is, $f\left(x^*\right)$(the solution of a slightly perturbed problem) is near $f^*\left(x\right)$(the computed solution).

Stability concerns about the behavior of $\left|f_{i,j} - f\left(i\Delta t, j\Delta S\right)\right|$ as numerical computation progresses for fixed discretization steps Δt and ΔS.

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Convergence issue

Convergence of the numerical algorithm concerns about the behavior of $\left| f_{i,j} - f\left(i\Delta t, j\Delta S\right)\right|$ as $\Delta t, \Delta S \to 0$ for fixed values $\left(i\Delta t, j\Delta S\right)$.

For well-posed linear initial value problem,

Stability $\Leftrightarrow$ Convergence

(Lax's equivalence theorem, Richtmyer and Morton, "Difference Methods for Initial Value Problems" (2nd) 1967)

Numerical Accuracy

- These factors contribute the accuracy of a numerical solution. We can find a “good” estimate if our problem is well-conditioned and the algorithm is stable

$$\left.\begin{array}{l}\text{Stable: } f^*(x) \approx f(x^*) \\ \text{Well-conditioned: } f(x^*) \approx f(x)\end{array}\right\} f^*(x) \approx f(x)$$

Chapter 5 Finite Difference Methods

5.2.3 Financial Interpretation of Numerical Instability

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Financial Interpretation of instability (Hall, page 423-4)

If $\partial f / \partial S$ and $\partial^2 f / \partial S^2$ are assumed to be the same at $(i+1, j)$ as they are at (i, j), we obtain equations of the form:

$$f_{i,j} = \hat{a}_j f_{i+1,j-1} + \hat{b}_j f_{i+1,j} + \hat{c}_j f_{i+1,j+1} \qquad (5.3)$$

where

$$\hat{a}_j = \frac{1}{1+r\Delta t}\left(\frac{1}{2}\Delta t\sigma^2 j^2 - \frac{1}{2}\Delta t\, r\, j\right) = \frac{1}{1+r\Delta t}\pi_d$$

$$\hat{b}_j = \frac{1}{1+r\Delta t}\left(1-\sigma^2 j^2 \Delta t\right) = \frac{1}{1+r\Delta t}\pi_0$$

$$\hat{c}_j = \frac{1}{1+r\Delta t}\left(\frac{1}{2}\Delta t\sigma^2 j^2 + \frac{1}{2}\Delta t\, r\, j\right) = \frac{1}{1+r\Delta t}\pi_u$$

for $i = N-1,\ N-2,\ \ldots,\ 1,\ 0$ and $j = 1,\ 2,\ \ldots,\ M-1$.

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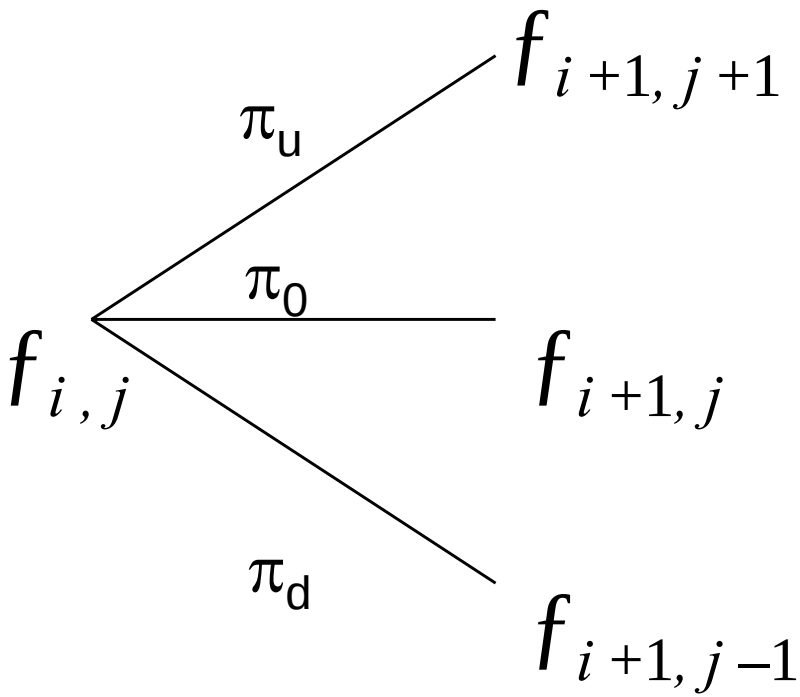
Explicit Finite Difference Methods



These coefficients can be interpreted as probabilities times a discount factor. If one of these probability < 0, instability occurs.

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Explicit Finite Difference Method as Trinomial Tree

Check if the mean and variance of the

Expected value of the increase in asset price during Δt:

$$\mathrm{E}[\Delta] = -\Delta S\,\pi_d + 0\pi_0 + \Delta S\,\pi_u = rj\Delta S\,\Delta t = rS\Delta t$$

Variance of the increment:

$$\mathrm{E}\left[\Delta^2\right] = -(\Delta S)^2\,\pi_d + 0\pi_0 + (\Delta S)^2\,\pi_u = \sigma^2 j^2 (\Delta S)^2 \Delta t = \sigma^2 S^2 \Delta t$$

$$\mathrm{Var}[\Delta] = \mathrm{E}\left[\Delta^2\right] - \mathrm{E}^2[\Delta] = \sigma^2 S^2 \Delta t - r^2 S^2 (\Delta t)^2 \approx \sigma^2 S^2 \Delta t$$

which is coherent with geometric Brownian motion in a risk-neutral world

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Change of Variable

Define $Z = ln\, S$. The B-S equation becomes

$$\frac{\partial f}{\partial t} + \left(r - \frac{\sigma^2}{2} \right) \frac{\partial f}{\partial Z} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial Z^2} = rf$$

The corresponding difference equation is

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + \left(r - \frac{\sigma^2}{2} \right) \frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta Z} + \frac{\sigma^2}{2} \frac{f_{i+1,j+1} - 2f_{i+1,j-1} + f_{i+1,j+1}}{\Delta Z^2} = rf_{i,j}$$

or

$$f_{i,j} = \alpha_j^* f_{i+1,j-1} + \beta_j^* f_{i+1,j} + \gamma_j^* f_{i+1,j+1} \qquad (5.4)$$

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Change of Variable

where

$$\alpha_j^* = \frac{1}{1+r\Delta t}\left[-\left(r-\frac{\sigma^2}{2}\right)\frac{\Delta t}{2\Delta Z}+\frac{\sigma^2}{2}\frac{\Delta t}{\Delta Z^2}\right]$$

$$\beta_j^* = \frac{1}{1+r\Delta t}\left(1-\frac{\Delta t}{\Delta Z^2}\sigma^2\right)$$

$$\gamma_j^* = \frac{1}{1+r\Delta t}\left[\left(r-\frac{\sigma^2}{2}\right)\frac{\Delta t}{2\Delta Z}+\frac{\sigma^2}{2}\frac{\Delta t}{\Delta Z^2}\right]$$

It can be shown that it is numerically most efficient if $\Delta Z = \sigma\sqrt{3\Delta t}$.

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Reduced to Heat Equation

Get rid of the varying coefficients S and S² by using change of variables:

$$S = Ee^{x}, t = T - 2\tau/\sigma^{2}, V(S,t) = Ee^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^{2}\tau}u(x,\tau)$$

$$k = r\Big/\left(\frac{1}{2}\sigma^{2}\right)$$

Equation (5.1) becomes heat equation (5.5):

$$\frac{\partial u}{\partial \tau} = \frac{\partial^{2} u}{\partial x^{2}} \qquad (5.5)$$

for $-\infty < x < +\infty$ and $\tau > 0$

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Explicit Finite Difference Method

With $u_n^m = u(n\delta x, m\delta\tau)$, this involves solving a system of finite difference equations of the form :

$$\frac{u_n^{m+1} - u_n^m}{\delta\tau} + O(\delta\tau) = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2} + O\left((\delta x)^2\right)$$

Ignoring terms of $O(\delta\tau)$ and $O\left((\delta x)^2\right)$, we can approximat e this by

$$u_n^{m+1} = \alpha u_{n+1}^m + (1-2\alpha)u_n^m + \alpha u_{n-1}^m$$

where $\alpha = \dfrac{\delta\tau}{(\delta x)^2}$, for $-N^- \le n \le N^+$ and $m = 0,1,\ldots M$ $\left(= \dfrac{\frac{1}{2}\sigma^2 T}{\delta\tau}\right)$

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Stability and Convergence

(P. Wilmott, et al, Option Pricing)

Stability:

The solution of Eqn (5.5) is

i) Stable if $0 < \alpha = \frac{\delta\tau}{(\delta x)^2} \le \frac{1}{2}$; ii) Unstable if $\alpha > \frac{1}{2}$

Convergence:

If $0 < \alpha \le \frac{1}{2}$, then the explicit finite-difference approximation converges to the exact solution as $\delta\tau, \delta x \to 0$

(in the sense that $u_n^m \to u(n\delta x, m\delta\tau)$ as $\delta\tau, \delta x \to 0$)

Rate of Convergence is $O(\delta\tau)$

Chapter 5 Finite Difference Methods

5.3.1 Implicit Finite-Difference Method

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Implicit Finite Difference Methods

In $\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r f$, we use

forward difference: $\frac{\partial f}{\partial t} \approx \frac{f_{i+1,\mathrm{j}} - f_{i,j}}{\Delta t}$

central difference: $\frac{\partial f}{\partial S} \approx \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S}$,

and

$$\frac{\partial^2 f}{\partial S^2} \approx \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2}, \quad rf = rf_{i,j}$$

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Implicit Finite Difference Methods

Rewriting the equation, we get an implicit scheme:

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j} \qquad (5.6)$$

where

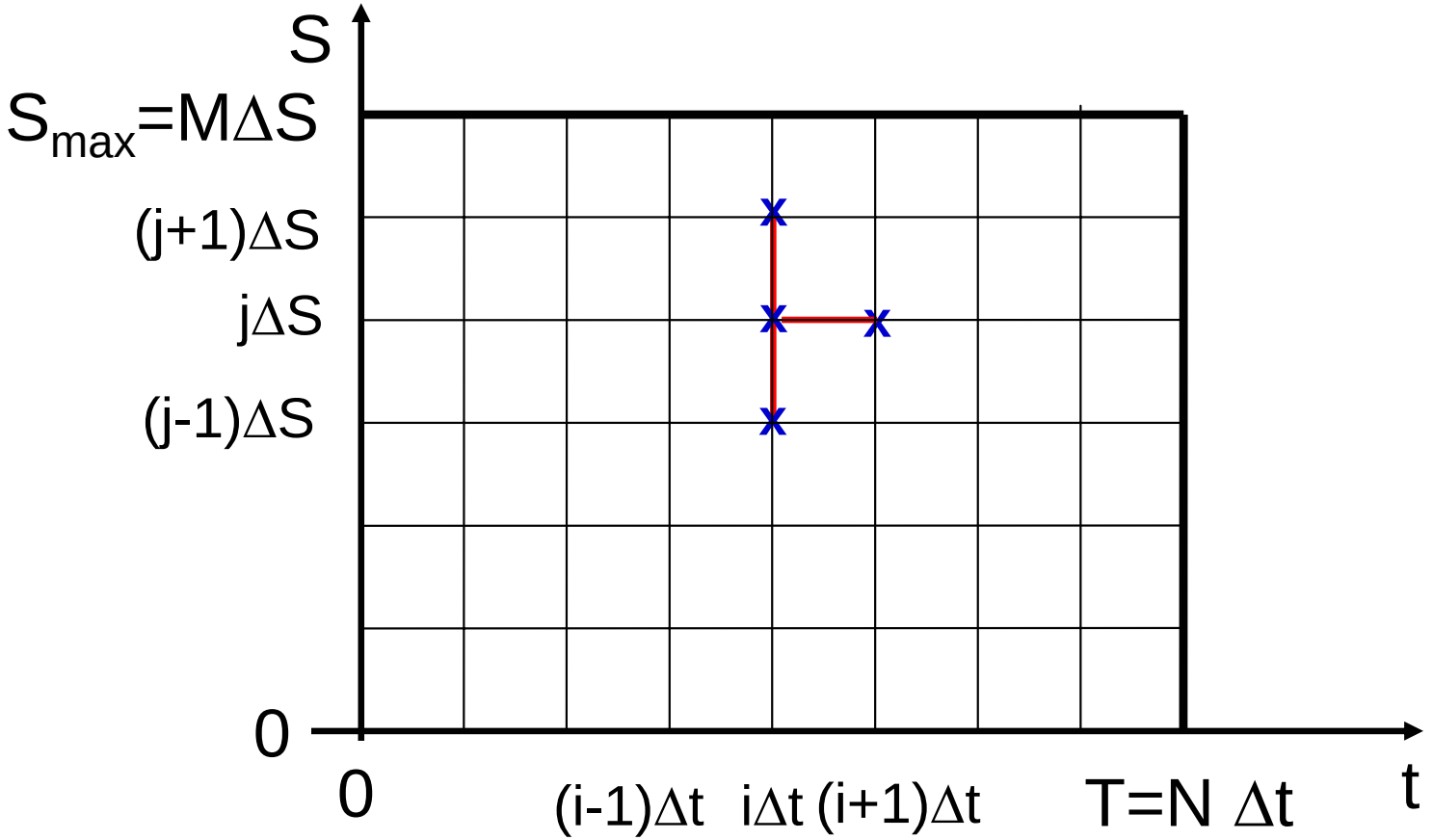
$$a_j = \frac{1}{2}\Delta t\left(-\sigma^2 j^2 + rj\right)$$

$$b_j = 1 + \Delta t\left(\sigma^2 j^2 + r\right)$$

$$c_j = -\frac{1}{2}\Delta t\left(\sigma^2 j^2 + rj\right)$$

for $i = N-1,\ N-2,\ \ldots,\ 1,\ 0$ and $j = 1,\ 2,\ \ldots,\ M-1$.

Numerical Computation Dependency



Implementation

Equation (5.6) can be rewritten in matrix form:

$$\mathbf{C}\mathbf{f}_i = \mathbf{f}_{i+1} + \mathbf{b}_i \qquad (5.7)$$

where $\mathbf{f}_i$ and $\mathbf{b}_i$ are $(M-1)$ dimensional vectors

$$\mathbf{f}_i = \left[f_{i,1}, f_{i,2}, f_{i,3} \cdots, f_{i,M-1} \right]^T, \mathbf{b}_i = \left[-a_1 f_{i,0}, 0, 0, \cdots, 0, -c_{M-1} f_{i,M} \right]^T$$

and $\mathbf{C}$ is $(M-1)\times(M-1)$ symmetric matrices

$$\mathbf{C} = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & \ddots & 0 \\ 0 & a_3 & b_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & c_{M-2} \\ 0 & \cdots & 0 & a_{M-1} & b_{M-1} \end{bmatrix}$$

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Implementation

1. Starting with the final values $f_{N,j}$, we need to solve a linear system (5.7) to obtain $f_{N-1,j}$ for $1 \le j \le M-1$ using LU factorization or iterative methods. We use the boundary condition to determine $f_{N-1,0}$ and $f_{N\text{-}1,M}$.

2. Repeat the process to determine $f_{N-2,j}$ and so on

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Example

We compare implicit finite difference solution for a European put with the exact Black-Scholes formula, where T = 5/12 yr, S_0=\$50, K = \$50, σ=30%, r = 10%.

Black-Scholes Price: \$2.8446

IFD Method with S_{max}=\$100, ΔS=2, Δt=5/1200: \$2.8194

IFD Method with S_{max}=\$100, ΔS=1, Δt=5/4800: \$2.8383

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Example (Stability)

We compare implicit finite difference solution for a European put with the exact Black-Scholes formula, where T = 5/12 yr, S_0=\$50, K = \$50, σ=30%, r = 10%.

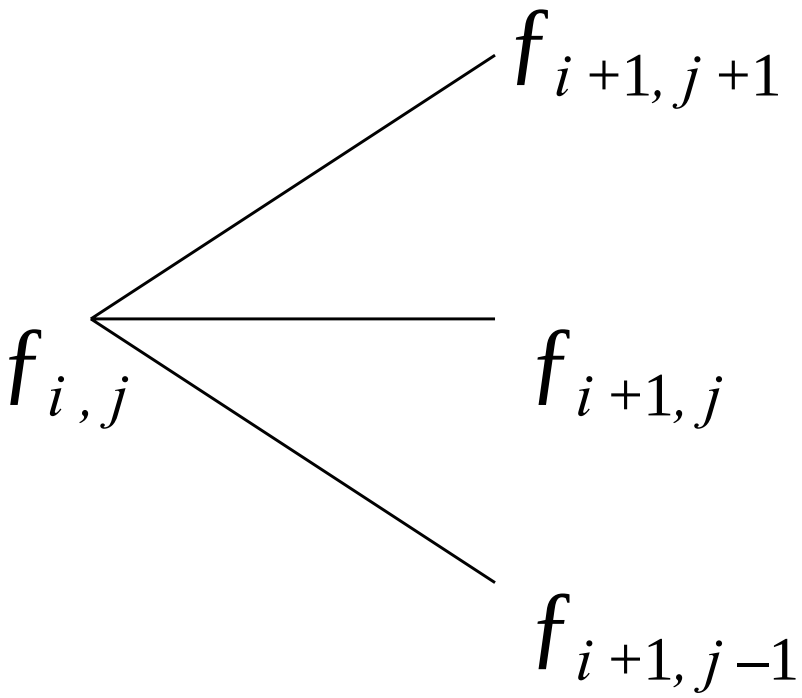
Black-Scholes Price: \$2.8846

IFD Method with S_{max}=\$100, ΔS=2, Δt=5/1200: \$2.8288

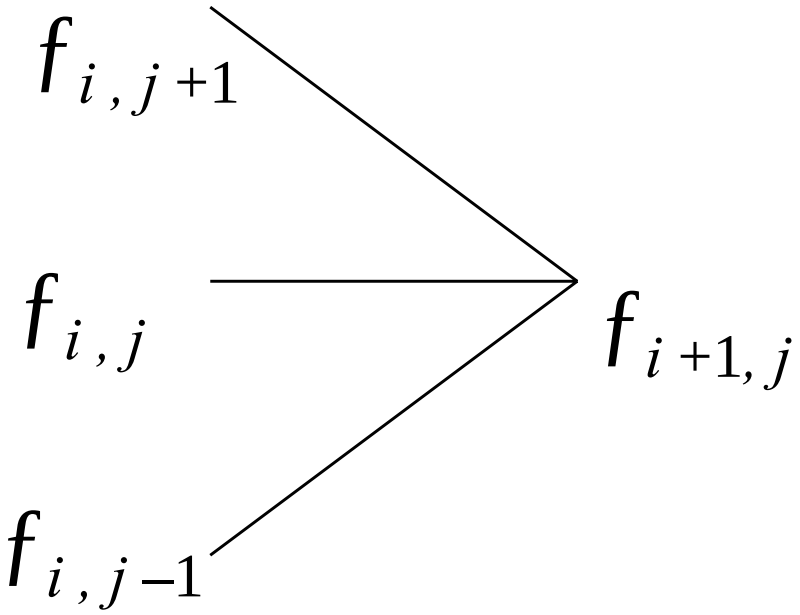
IFD Method with S_{max}=\$100, ΔS=1.5, Δt=5/1200: \$3.1325

IFD Method with S_{max}=\$100, ΔS=1, Δt=5/1200: \$2.8348

Implicit vs Explicit Finite Difference Methods



Explicit Method



Implicit Method

(always stable)

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Implicit vs Explicit Finite Difference Method

- The explicit finite difference method is equivalent to the trinomial tree approach:
  - Truncation error: $O(\Delta t)$
  - Stability: not always
- The implicit finite difference method is equivalent to a multinomial tree approach:
  - Truncation error: $O(\Delta t)$
  - Stability: always

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Other Points on Finite Difference Methods

- It is better to have ln S rather than S as the underlying variable in general
- Improvements over the basic implicit and explicit methods:
  - Crank-Nicolson method, average of explicit and implicit FD methods, trying to achieve
    - Truncation error: $O((\Delta t)^2)$
    - Stability: always

Chapter 5 Finite Difference Methods

5.3.2 Solving a linear system using direct methods

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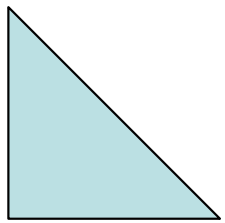
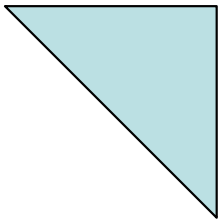
Solve Ax=b

$$A\,x=b$$

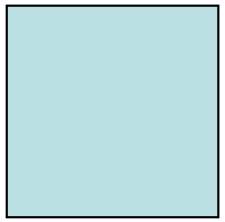
Various shapes of matrix A



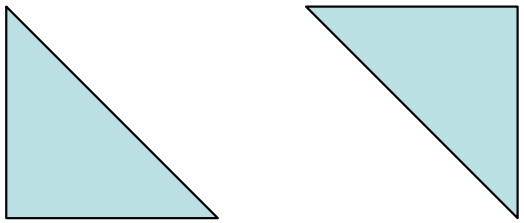
Lower triangular



Upper triangular



General



5.3.2.A Triangular Solvers

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Example: 3 x 3 upper triangular system

$$\begin{bmatrix} 4 & 6 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 10 \\ 20 \end{bmatrix}$$

$$\Rightarrow x_3 = 20/4 = 5$$

$$\Rightarrow x_2 = 10 - x_3 = 5$$

$$\Rightarrow 4x_1 = 100 - x_3 - 6 * x_2 = 65$$

$$\therefore x_1 = 65/4$$

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Solve an upper triangular system Ax=b

$$\begin{pmatrix} 0 & \cdots & 0 & a_{ii} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = a_{ii}x_i + \sum_{j>i} a_{ij}x_j = b_i, i = 1, \ldots, n$$

$$\Rightarrow x_n = b_n / a_{nn}$$

$$x_i = \left(b_i - \sum_{j>i} a_{ij}x_j \right) \Big/ a_{ii}, i = 1, \cdots n-1$$

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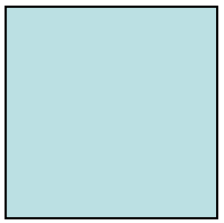
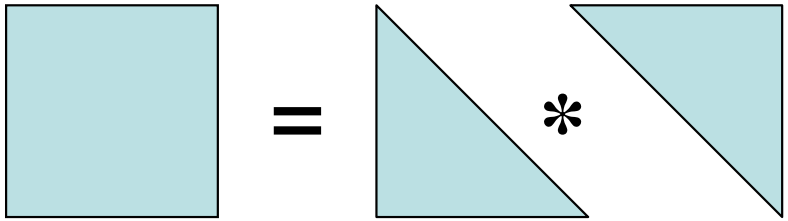
Implementation

```
Function x = UpperTriSolve(A, b)
        n = length(b);
        x(n) = b(n)/A(n,n);
        for i = n-1:-1:1
                sum = b(i);
                for j = i+1:n
                        sum = sum - A(i,j)*x(j);
                end
                x(i) = sum/A(i,i);
        end
```



5.3.2.B Gauss Elimination

To solve Ax=b for general form A



To solve Ax = b :

Suppose A = LU. Then

$$Ax = L\underline{Ux} = b$$

$$\Downarrow$$

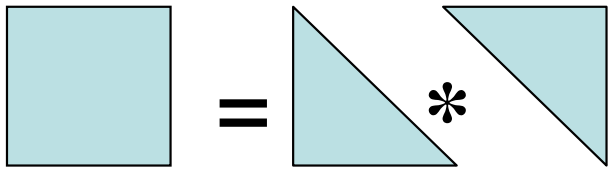
$$z$$

Solve two triangular systems :

1) solve z from Lz = b

2) solve x from Ux = z

Gauss Elimination



Goal: Make A an upper triangular matrix through using fundamental row operations

For example, to zero the elements in the lower triangular part of A

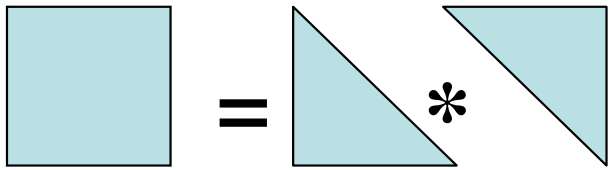
1) Zero a_{21}

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - \frac{a_{21}a_{12}}{a_{11}} & a_{23} - \frac{a_{21}a_{13}}{a_{11}} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Downarrow$$

$$\mathbf{E}_{21}\mathbf{A}$$

Gauss Elimination



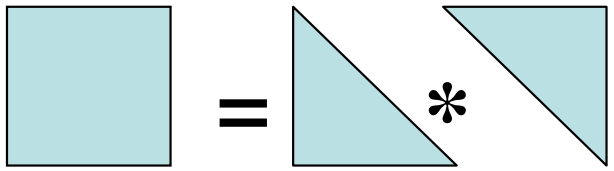
2) Zero a_{31}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}-\frac{a_{21}a_{12}}{a_{11}} & a_{23}-\frac{a_{21}a_{13}}{a_{11}} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}-\frac{a_{21}a_{12}}{a_{11}} & a_{23}-\frac{a_{21}a_{13}}{a_{11}} \\ 0 & a_{32}-\frac{a_{31}a_{12}}{a_{11}} & a_{33}-\frac{a_{31}a_{13}}{a_{11}} \end{bmatrix}$$

$$\Downarrow$$

$$\mathbf{E}_{31}\mathbf{E}_{21}\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} \\ 0 & \tilde{a}_{32} & \tilde{a}_{33} \end{bmatrix}$$

Gauss Elimination



3) Zero $\tilde{a}_{32}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\dfrac{\tilde{a}_{32}}{\tilde{a}_{22}} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} \\ 0 & \tilde{a}_{32} & \tilde{a}_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} \\ 0 & 0 & \tilde{a}_{33} - \dfrac{\tilde{a}_{32}\tilde{a}_{23}}{\tilde{a}_{22}} \end{bmatrix} \equiv \mathbf{U}$$

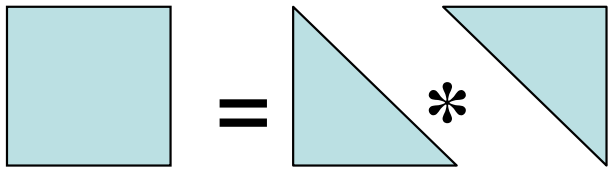
$$\Downarrow$$

$$\mathbf{E}_{32}\mathbf{E}_{31}\mathbf{E}_{21}\mathbf{A} = \mathbf{U}$$

$$\Downarrow$$

lower triangular

Gauss Elimination



Claim 1:

$$\mathbf{E}_{32}\mathbf{E}_{31}\mathbf{E}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & -\frac{\tilde{a}_{32}}{\tilde{a}_{22}} & 1 \end{bmatrix}$$

Claim 2:

$$\left(\mathbf{E}_{32}\mathbf{E}_{31}\mathbf{E}_{21}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & \frac{\tilde{a}_{32}}{\tilde{a}_{22}} & 1 \end{bmatrix}$$

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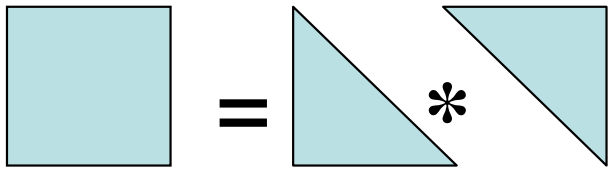
LU Factorization



Therefore, through Gauss Elimination, we have

$$\mathbf{E}_{32}\mathbf{E}_{31}\mathbf{E}_{21}\mathbf{A} = \mathbf{U}$$

$$\mathbf{A} = \left(\mathbf{E}_{32}\mathbf{E}_{31}\mathbf{E}_{21}\right)^{-1}\mathbf{U}$$

$$\mathbf{A} = \mathbf{L}\mathbf{U}$$

It is called LU factorization. When A is symmetric,

$\mathbf{A} = \mathbf{L}\mathbf{L}^T$ which is called *Cholesky Decomposition*

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To solve Ax=b for general form A

To solve $Ax = b$ becomes

1) Use Gauss elimination to make A upper triangular, i.e.

$$L^{-1}Ax = L^{-1}b \Rightarrow Ux = z$$

2) Solve x from $Ux = z$

This suggests that when doing Gauss elimination, we can do it to the augmented matrix $[A\ b]$ associated with the linear system.

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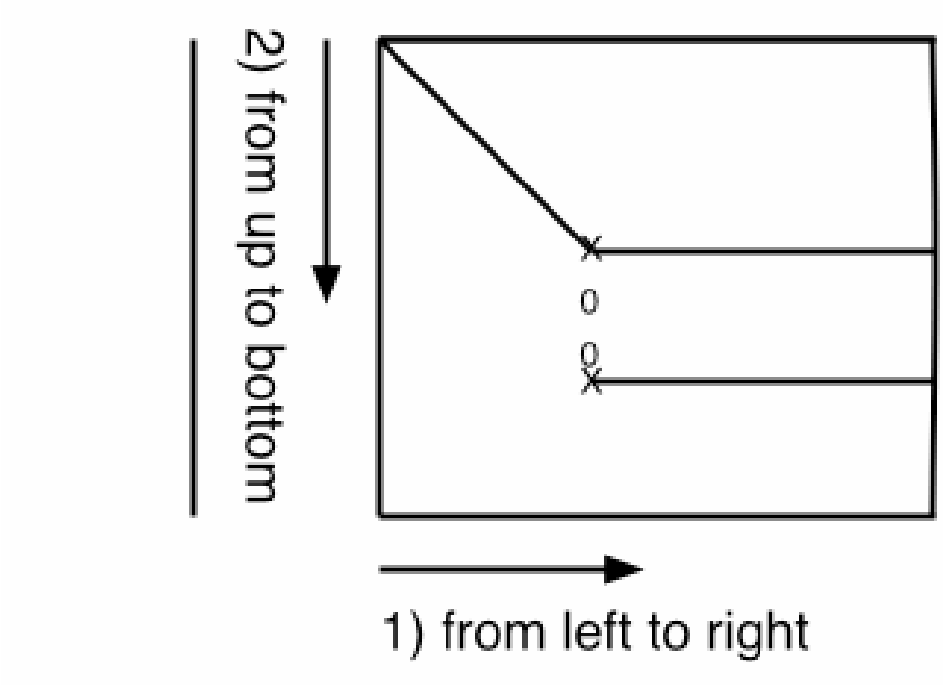
An exercise

$$\begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

```
% build the matrix A
A = [2, -1, 0; -1, 2, -1; 0, -1, 2]

% build the vector b
x_true = [1:3]';
b = A * x_true;

% lu decomposition of A
[l, u] = lu(A)

% solve z from lz = b where z = ux
z = l\b;

% solve x from ux = z
x = u\z
```

General Gauss Elimination



row i

row j - $\frac{a_{ji}}{a_{ii}}$ * **row i**

What if we have zero or very small diagonal elements?

Somtimes, we need to permute rows of A so that Gauss eliminatio n can be computed or computed stably, i.e., PA = LU. This is called partial pivoting.

For example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\mathbf{PA} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0.0001 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 10{,}000 & 1 \end{bmatrix}\begin{bmatrix} 0.0001 & 1 \\ 0 & -9999 \end{bmatrix} = \mathbf{LU}$$

$$\mathbf{PA} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0.0001 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.0001 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & .9999 \end{bmatrix} = \mathbf{LU}$$

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Can we always have A = LU?

No!

If $det\left(\mathbf{A}\left(1{:}k,1{:}k\right)\right) \neq 0$ for k=1,…,n-1,

then A $\in \mathbb{R}^{nxn}$ has an LU factorization.

Proof:

See Golub and Loan, Matrix Computation, 3rd edition

Chapter 5 Finite Difference Methods

5.4.1 Crank-Nicolson Method

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redefine THE POSSIBLE.

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Explicit Finite Difference Methods

With $\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf,$

we can obtain an explicite form:

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + O(\Delta t) =$$

$$-rj\Delta S \frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta S} - \frac{1}{2}\sigma^2 j^2 (\Delta S)^2 \frac{f_{i+1,j+1} + f_{i+1,j-1} - 2f_{i+1,j}}{\Delta S^2}$$

$$+rf_{i+1,j} \quad + O\left((\Delta S)^2\right)$$

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Implicit Finite Difference Methods

With $$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r f,$$

we can obtain an implicit form :

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + O(\Delta t) =$$

$$-rj\Delta S \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} - \frac{1}{2}\sigma^2 j^2 (\Delta S)^2 \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2}$$

$$+rf_{i,j} + O\left((\Delta S)^2\right)$$

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Crank-Nicolson Methods: average of explicit and implicit methods (Brandmarte, p485)

$$\frac{f_{i+1,j}-f_{i,j}}{\Delta t}+O\left((\Delta t)^2\right)=$$

$$-\frac{rj\Delta S}{2}\left(\frac{f_{i+1,j+1}-f_{i+1,j-1}}{2\Delta S}+\frac{f_{i,j+1}-f_{i,j-1}}{2\Delta S}\right)$$

$$-\frac{1}{4}\sigma^2 j^2(\Delta S)^2\left(\frac{f_{i+1,j+1}+f_{i+1,j-1}-2f_{i+1,j}}{\Delta S^2}+\frac{f_{i,j+1}+f_{i,j-1}-2f_{i,j}}{\Delta S^2}\right)$$

$$+\frac{r}{2}\left(f_{i+1,j}+f_{i,j}\right)+O\left((\Delta S)^2\right)$$

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Crank-Nicolson Methods

Rewriting the equation, we get an Crank-Nicolson scheme:

$$-\alpha_j f_{i,j-1} + (1-\beta_j) f_{i,j} - \gamma_j f_{i,j+1} =$$

$$\alpha_j f_{i+1,j-1} + (1+\beta_j) f_{i+1,j} + \gamma_j f_{i+1,j+1} \qquad (5.5)$$

where

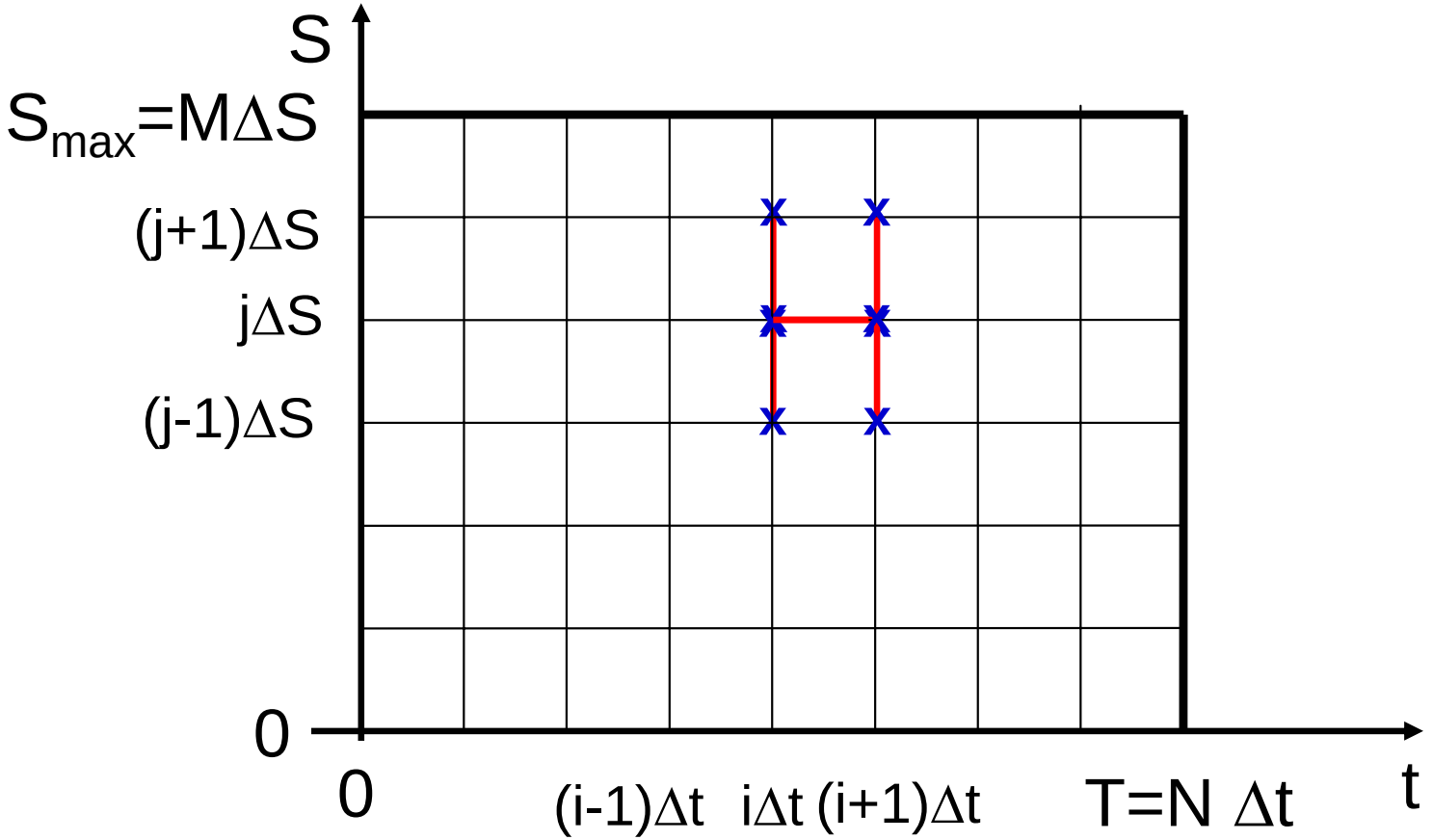
$$\alpha_j = \frac{\Delta t}{4}\left(\sigma^2 j^2 - rj\right)$$

$$\beta_j = -\frac{\Delta t}{2}\left(\sigma^2 j^2 + r\right)$$

$$\gamma_j = \frac{\Delta t}{4}\left(\sigma^2 j^2 + rj\right)$$

for $i = N-1, N-2, \ldots, 1, 0$ and $j = 1, 2, \ldots, M-1$.

Numerical Computation Dependency



Implementation

Equation (5.5) can be rewritten in matrix form:

$$\mathbf{M}_1\mathbf{f}_i = \mathbf{M}_2\mathbf{f}_{i+1} + \mathbf{b}$$

where $\mathbf{f}_i$ and $\mathbf{b}_i$ are $(M-1)$ dimensional vectors

$$\mathbf{f}_i = \left[f_{i,1}, f_{i,2}, f_{i,3} \cdots, f_{i,M-1} \right]^T,$$

$$\mathbf{b} = \left[\alpha_1 \left(f_{i,0} + f_{i+1,0} \right), 0 \cdots, \gamma_{M-1} \left(f_{i,M} + f_{i+1,M} \right) \right]^T$$

and $\mathbf{M}_1$ and $\mathbf{M}_2$ are $(M-1)\times(M-1)$ symmetric matrices

$$\mathbf{M}_1 = \begin{bmatrix} 1-\beta_1 & -\gamma_1 & 0 & \cdots & 0 \\ -\alpha_2 & 1-\beta_2 & \gamma_2 & \ddots & 0 \\ 0 & \alpha_3 & & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\gamma_{M-2} \\ 0 & \cdots & 0 & -\alpha_{M-1} & 1-\beta_{M-1} \end{bmatrix}$$

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Implementation

and

$$\mathbf{M}_2 = \begin{bmatrix} 1+\beta_1 & \gamma_1 & 0 & \cdots & 0 \\ \alpha_2 & 1+\beta_2 & \gamma_2 & \ddots & 0 \\ 0 & \alpha_3 & & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \gamma_{M-2} \\ 0 & \cdots & 0 & \alpha_{M-1} & 1+\beta_{M-1} \end{bmatrix}$$

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Example

We compare Crank-Nicolson Methods for a European put with the exact Black-Scholes formula, where T = 5/12 yr, S_0=\$50, K = \$50, σ=30%, r = 10%.

Black-Scholes Price: \$2.8446

CN Method with S_{max}=\$100, ΔS=2, Δt=5/1200: \$2.8241

CN Method with S_{max}=\$100, ΔS=1, Δt=5/4800: \$2.8395

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Example (Stability)

We compare Crank-Nicolson Method for a European put with the exact Black-Scholes formula, where T = 5/12 yr, S_0=\$50, K = \$50, σ=30%, r = 10%.

Black-Scholes Price: \$2.8446

CN Method with S_{max}=\$100, ΔS=1.5, Δt=5/1200: \$3.1370

CN Method with S_{max}=\$100, ΔS=1, Δt=5/1200: \$2.8395

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Example: Barrier Options

Barrier options are options where the payoff depends on whether the underlying asset's price reaches a certain level during a certain period of time.

Types:

knock-out: option ceases to exist when the asset price reaches a barrier

Knock-in: option comes into existence when the asset price reaches a barrier

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Example: Barrier Option

We compare Crank-Nicolson Method for a European down-and-out put, where T = 5/12 yr, S_0=\$50, K = \$50, S_b=\$50 σ=40%, r = 10%.

What are the boundary conditions for S?

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Example: Barrier Option

We compare Crank-Nicolson Method for a European down-and-out put, where T = 5/12 yr, S_0=\$50, K = \$50, S_b=\$50 σ=40%, r = 10%.

Boundary conditions are $f\left(t, S_{max}\right) = 0$ and $f\left(t, S_b\right) = 0$

Exact Price (Hall, p533-535): \$0.5424

CN Method with S_{max}=\$100, ΔS=0.5, Δt=1/1200: \$0.5414

Appendix A.

Matrix Norms

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redefine THE POSSIBLE.

Vector Norms

- Norms serve as a way to measure the length of a vector or a matrix
- A vector norm is a function mapping $\mathbf{x} \in \Re^n$ to a real number $\|\mathbf{x}\|$ s.t.
  - $\|\mathbf{x}\| > 0$ for any $\mathbf{x} \neq \mathbf{0}$; $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$
  - $\|c\,\mathbf{x}\| = |c|\|\mathbf{x}\|$ for any $c \in \Re$
  - $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for any $\mathbf{x}, \mathbf{y} \in \Re^n$
- There are various ways to define a norm

$$\|\mathbf{x}\|_p \equiv \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \quad (\text{p} = 2 \text{ is the Euclidean norm})$$

$$\|\mathbf{x}\|_\infty \equiv \max_{1 \leq i \leq n} |x_i|$$

– For example, $\mathbf{v} = [2 \;\; 4 \;\; -1 \;\; 3]$. $\|\mathbf{v}\|_1 = ?$, $\|\mathbf{v}\|_\infty = ?$, $\|\mathbf{v}\|_2 = ?$

Matrix Norms

- Similarly, a matrix norm is a function mapping $\mathbf{A} \in \Re^{m \times n}$ to a real number $\|\mathbf{A}\|$ s.t.
  - $\|\mathbf{A}\| > 0$ for any $\mathbf{A} \neq \mathbf{0}$; $\|\mathbf{A}\| = \mathbf{0}$ iff $\mathbf{A} = \mathbf{0}$
  - $\|c\ \mathbf{A}\| = |c| \|\mathbf{A}\|$ for any $c \in \Re$
  - $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$ for any $\mathbf{A}$, $\mathbf{B}\ \in \Re^{m \times n}$
- Various commonly used matrix norms

$$\|\mathbf{A}\|_p \equiv \sup_{\mathbf{x} \neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p} \qquad \|\mathbf{A}\|_F \equiv \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

$$\|\mathbf{A}\|_1 \equiv \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}| \qquad \|\mathbf{A}\|_\infty \equiv \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$$

$\|\mathbf{A}\|_2 \equiv \sqrt{\rho(\mathbf{A}^T \mathbf{A})}$, the spectral norm, where

$\rho(\mathbf{B}) \equiv \max\{|\lambda_k| : \lambda_k$ is an eigenvalue of $\mathbf{B}\}$

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An Example

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -1 \\ 3 & 1 & 5 \\ -2 & 3 & -1 \end{bmatrix}$$

$$\|\mathbf{A}\|_\infty = ? \qquad \|\mathbf{A}\|_2 = ?$$

$$\|\mathbf{A}\|_1 = ? \qquad \|\mathbf{A}\|_F = ?$$

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Basic Properties of Norms

Let $\mathbf{A}, \mathbf{B} \in \Re^{n\times n}$ and $\mathbf{x}, \mathbf{y} \in \Re^{n}$. Then

1. $\|\mathbf{x}\| \geq 0$; and $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$
2. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$
3. $\|\alpha\, \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$ where α is a real number
4. $\|\mathbf{A}\mathbf{x}\| \leq \|\mathbf{A}\| \|\mathbf{x}\|$
5. $\|\mathbf{A}\mathbf{B}\| \leq \|\mathbf{A}\| \|\mathbf{B}\|$

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Condition number of a square matrix

All norms in $\Re^n$ $\left(\Re^{m\times n}\right)$ are equivalent. That is, if $\|\bullet\|_\alpha$ and $\|\bullet\|_\beta$ are norms on $\Re^n$, then $\exists\, c_1, c_2 > 0$ such that for all $\mathbf{x} \in \Re^n$, we have

$$c_1 \|\mathbf{x}\|_\alpha \le \|\mathbf{x}\|_\beta \le c_2 \|\mathbf{x}\|_\alpha$$

Condition Number of A Matrix: $C \equiv \|\mathbf{A}\|\|\mathbf{A}^{-1}\|$, where $A \in \Re^{n\times n}$.

The condition number gives a measure of how close a matrix is close to singular. The bigger the C, the harder it is to solve **Ax = b**.

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Convergence

- **vectors $\mathbf{x}^k$ converges to $\mathbf{x}$** $\Leftrightarrow \|\mathbf{x}^k - \mathbf{x}\|$ **converges to 0**

- **matrix $\mathbf{A}^k \to \mathbf{0}$** $\Leftrightarrow \|\mathbf{A}^k - \mathbf{0}\| \to 0$

Appendix B.

Basic Row Operations

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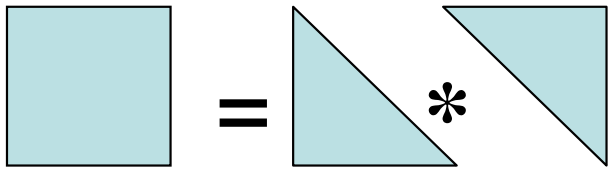
redefine THE POSSIBLE.

Basic row operations



Three kinds of basic row operations:

1) Interchange the order of two rows or (equations)

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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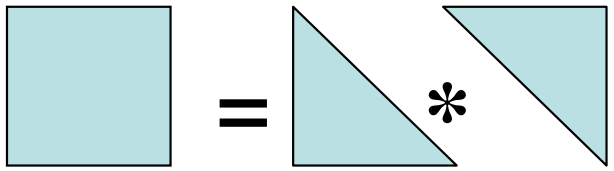
Basic row operations



2) Multiply a row by a nonzero constant

$$\begin{bmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} c\,a_{11} & c\,a_{12} & c\,a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3) Add or subtract rows

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}-a_{11} & a_{22}-a_{12} & a_{23}-a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$