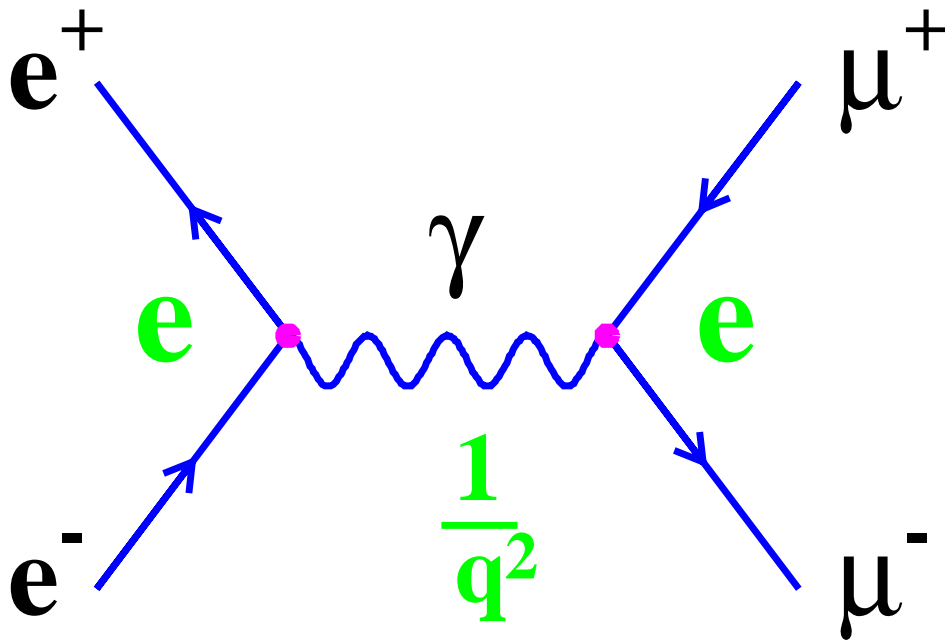


Particle Physics

Dr M.A. Thomson



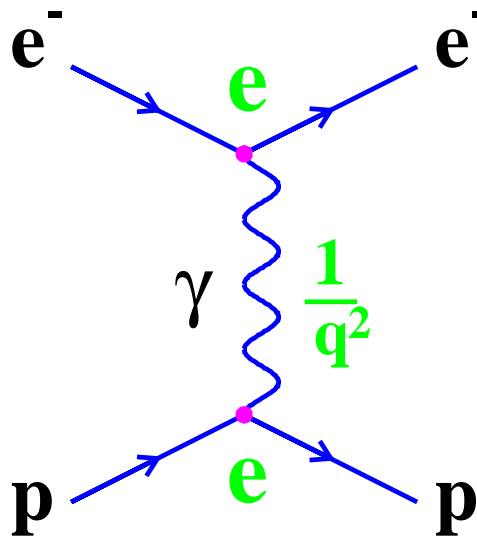
Part II, Lent Term 2004
HANDOUT II

Quantum Electrodynamics

QUANTUM ELECTRODYNAMICS: is the quantum theory of the electromagnetic interaction.

CLASSICAL PICTURE: Action at a distance : forces arise from \vec{E} and \vec{B} fields. Particles act as sources of the fields $\rightarrow V(\vec{r})$.

Q.E.D. PICTURE: Forces arise from the exchange of **virtual field quanta**.



Although a complete derivation of the theory of Q.E.D. and Feynman diagrams is beyond the scope of this course, the main features will be derived.

Interaction via Particle Exchange

NON-EXAMINABLE

FERMI'S GOLDEN RULE for Transition rate, Γ_{fi} :

$$\Gamma_{fi} = \frac{2\pi}{\hbar} |M_{fi}|^2 \rho(E_f)$$

$\rho(E_f)$ = density of final states.

★ From **1st** order perturbation theory, matrix element M_{fi} :

$$M_{fi} = \langle \psi_f | \hat{H}' | \psi_i \rangle$$

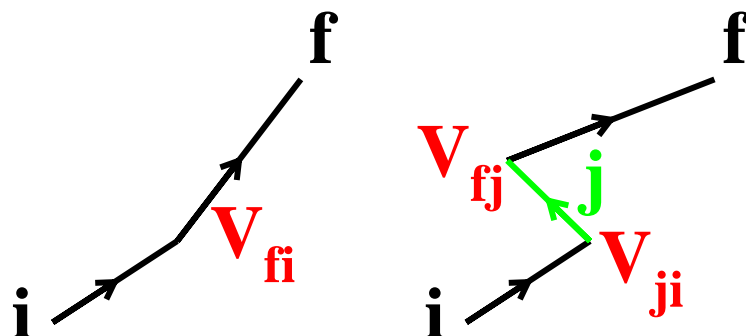
where \hat{H}' is the operator corresponding to the perturbation to the Hamiltonian.

★ This is only the **1st** order term in the perturbation expansion. In **2nd** order perturbation theory:

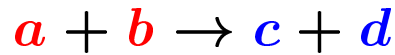
$$M_{fi} \rightarrow M_{fi} + \sum_{j \neq i} |M_{fj}| \frac{1}{E_i - E_j} |M_{ji}|$$

where the sum is over all **intermediate states j** , and E_i and E_j are the energies of the initial and intermediate state

★ For scattering, the **1st** and **2nd** order terms can be viewed as:



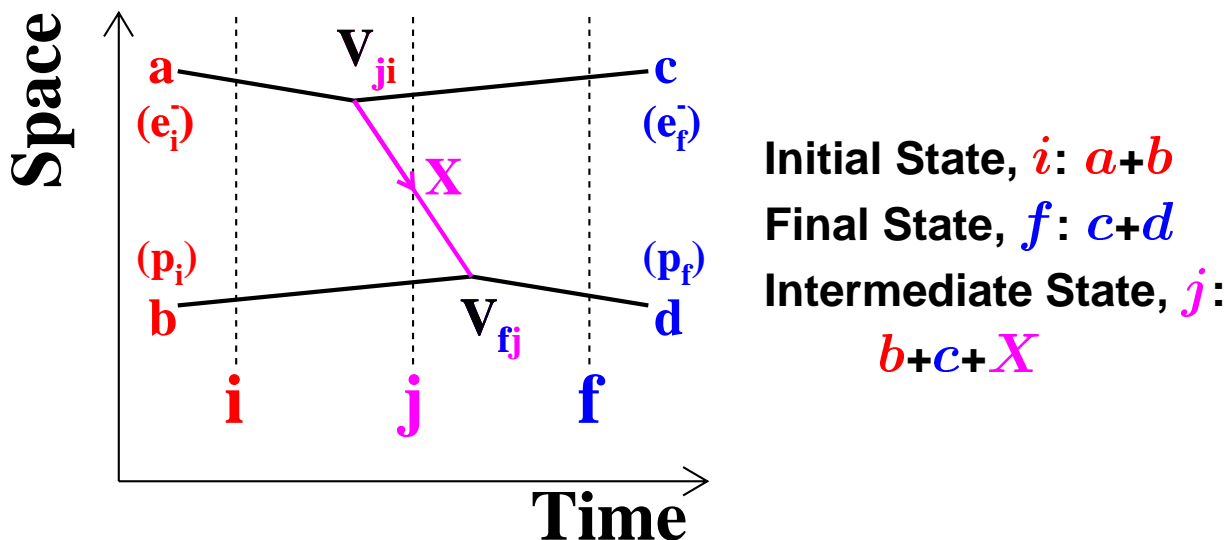
Consider the particle interaction



which involves the exchange a particle X . This could be the elastic scattering of electrons and protons, e.g.

$e^- p \rightarrow e^- p$ where X is an exchanged photon.

★ One possible space-time picture for this process is



★ The Time Ordered interaction consists of $a \rightarrow c + X$ followed by $b + X \rightarrow d$. For example $e_i^- p_i \rightarrow e_f^- p_f$ has the electron emitting a photon ($e_i^- \rightarrow e_f^- \gamma$) followed by the photon being absorbed by the proton ($p_i \gamma \rightarrow p_f$).

★ The corresponding term in 2^{nd} order PT:

$$\begin{aligned}
 M_{fi}^{ab} &= \frac{\langle \psi_f | \hat{H}' | \psi_j \rangle \langle \psi_j | \hat{H}' | \psi_i \rangle}{E_i - E_j} \\
 &= \frac{\langle \psi_d | \hat{H}' | \psi_X \psi_b \rangle \langle \psi_c \psi_X | \hat{H}' | \psi_a \rangle}{(E_a + E_b) - (E_c + E_X + E_b)} \\
 &= \frac{\langle \psi_d | \hat{H}' | \psi_X \psi_b \rangle \langle \psi_c \psi_X | \hat{H}' | \psi_a \rangle}{(E_a - E_c - E_X)}
 \end{aligned}$$

Before we go any further some comments:

- ★ The superscript ab on M_{fi}^{ab} indicates the time ordering where a interacts with X before b

consequently the results are not Lorentz Invariant
i.e. depend on rest frame.

- ★ Momentum is conserved in $a \rightarrow c + X$ and $b + X \rightarrow d$.

- ★ The exchanged particle X is ON MASS SHELL:

$$E_X^2 - p_X^2 = m_X^2$$

- ★ The matrix elements $\langle \psi_d | \hat{H}' | \psi_X \psi_b \rangle$ and $\langle \psi_c \psi_X | \hat{H}' | \psi_a \rangle$ depend on the “strength” of the interaction. e.g. the strength of the γe^- and γp interaction which determines the probability that an electron(proton) will emit(absorb) a photon.

- ★ For the electromagnetic interaction:

$$\langle \psi_k | \hat{H}' | \psi_j \rangle = e\epsilon_0 \langle \psi_k | z | \psi_j \rangle$$

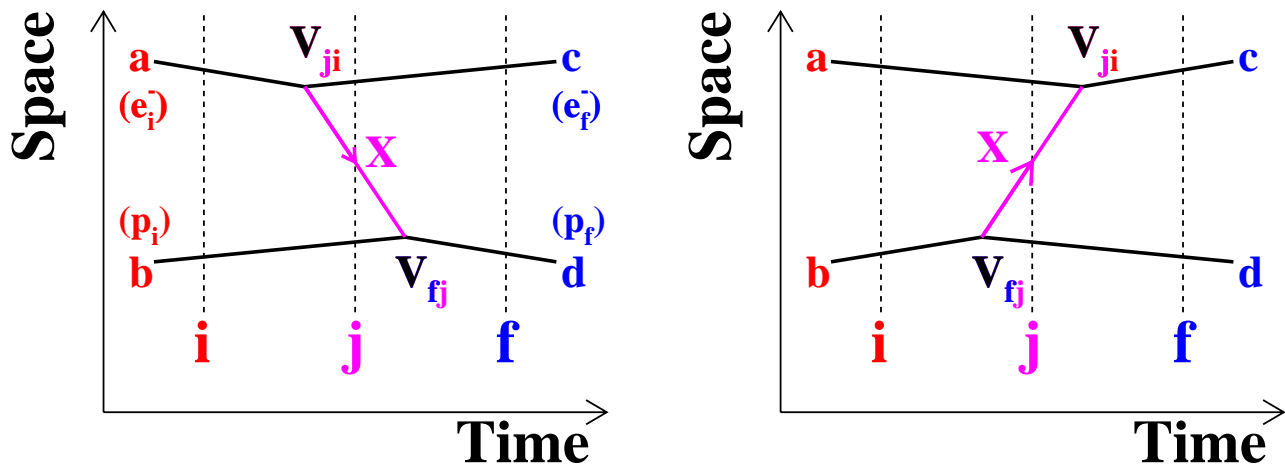
for a photon with polarization in the z-direction. (see Dr Ritchie's QM II lecture 10)

- ★ Neglecting spin (i.e. for assuming all particles are spin-0 i.e. scalars) the ME becomes:

$$\langle \psi_d | \hat{H}' | \psi_X \psi_b \rangle = e$$

- ★ More generally, $\langle \psi_d | \hat{H}' | \psi_X \psi_b \rangle = g$, where g is the interaction strength.

Now consider the other time ordering $b \rightarrow d + X$
followed by $a + X \rightarrow c$



The corresponding term in 2^{nd} order PT:

$$\begin{aligned}
 M_{fi}^{ba} &= \frac{\langle \psi_c | \hat{H}' | \psi_X \psi_a \rangle \langle \psi_d \psi_X | \hat{H}' | \psi_b \rangle}{(E_a + E_b) - (E_d + E_X + E_a)} \\
 &= \frac{\langle \psi_c | \hat{H}' | \psi_X \psi_a \rangle \langle \psi_d \psi_X | \hat{H}' | \psi_b \rangle}{(E_b - E_d - E_X)} \\
 &= \frac{\langle \psi_c | \hat{H}' | \psi_X \psi_a \rangle \langle \psi_d \psi_X | \hat{H}' | \psi_b \rangle}{(E_b - E_d - E_X)}
 \end{aligned}$$

Assume a common interaction strength, g , at both vertices,

$$\text{i.e. } \langle \psi_c | \hat{H}' | \psi_X \psi_a \rangle = \langle \psi_d \psi_X | \hat{H}' | \psi_b \rangle = g$$

$$\Rightarrow M_{fi}^{ba} = \frac{g^2}{(E_b - E_d - E_X)} \times \frac{1}{2E_X}$$

WARNING : I have introduced an (unjustified) factor of $\frac{1}{2E_X}$. This arises from the relativistic normalization of the wave-function for particle X (see appendix). For initial/final state particles the normalisation is cancelled by corresponding terms in the flux/phase-space. For the “intermediate” particle X no such cancelation occurs.

Now sum over two time ordered transition rates

$$M_{fi} = M_{fi}^{ab} + M_{fi}^{ba}$$

$$= g^2 \left(\frac{1}{E_a - E_c - E_X} + \frac{1}{E_b - E_d - E_X} \right) \times \frac{1}{2E_X}$$

$$\text{since } E_a + E_b = E_c + E_d$$

$$\Rightarrow E_b - E_d = E_c - E_a$$

giving:

$$M_{fi} = g^2 \left(\frac{1}{E_a - E_c - E_X} + \frac{1}{E_c - E_a - E_X} \right) \times \frac{1}{2E_X}$$

$$= g^2 \left(\frac{1}{E_a - E_c - E_X} - \frac{1}{E_a - E_c + E_X} \right) \times \frac{1}{2E_X}$$

$$= g^2 \frac{2E_X}{(E_a - E_c)^2 - E_X^2} \times \frac{1}{2E_X}$$

From the first time ordering:

$$E_X^2 = (\tilde{p}_a - \tilde{p}_c)^2 + m_X^2$$

therefore

$$M_{fi} = \frac{g^2}{(E_a - E_c)^2 - (\tilde{p}_a - \tilde{p}_c)^2 - m_X^2}$$

$$M_{fi} = \frac{g^2}{q^2 - m_X^2}$$

$$\text{with } q^2 = q^\mu q_\mu = E^2 - |\tilde{p}|^2$$

where $(E, |\tilde{p}|)$ are energy/momentum carried by the virtual particle. The **SUM** of time-ordered processes depends on q^2 and is therefore Lorentz invariant ! The 'invariant mass' of the exchanged particle, X , $m_{inv}^2 = E^2 - |\tilde{p}|^2$, is **NOT** the **REST MASS**, m_X .

The term

$$\frac{1}{q^2 - m^2}$$

is called the **PROPAGATOR**

It corresponds to the term in the matrix element arising from the exchange of a massive particle which mediates the force. For massless particles e.g. photons :

$$\frac{1}{q^2}$$

NOTE: q^2 is the 4-momentum of the exchanged particle ($q^2 = q^\mu q_\mu = E^2 - |\vec{p}|^2$)

Previously (page 35 of HANDOUT 1) we obtained the matrix element for elastic scattering in the **YUKAWA** potential:

$$M_{fi}^{YUK} = - \frac{g^2}{(m^2 + |\vec{p}|^2)}$$

For elastic scattering $E_X = 0$, and $q^2 = -|\vec{p}|^2$

$$M_{fi}^{YUK} \rightarrow \frac{g^2}{q^2 - m^2}$$

Which is exactly the expression obtained on the previous page. Hence, elastic scattering via particle exchange in 2nd order P.T. is equivalent to scattering in a Yukawa potential using 1st order P.T.

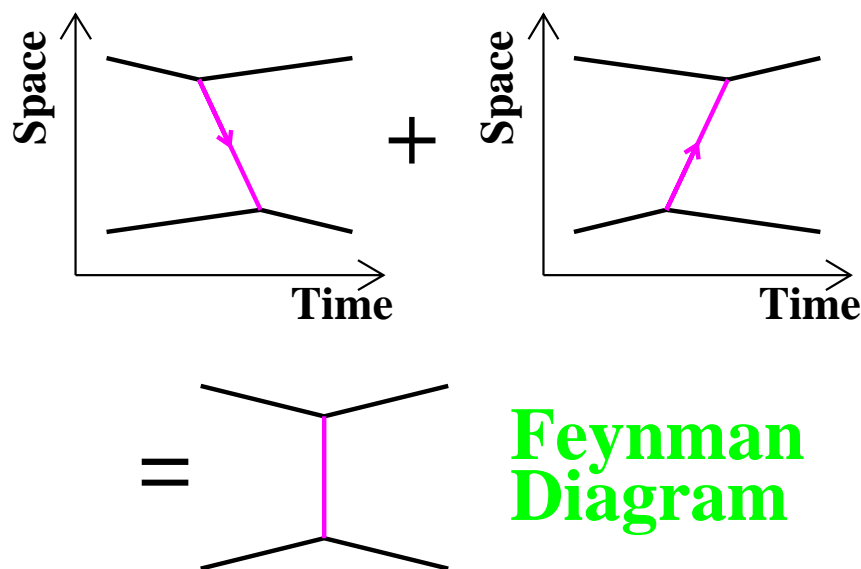
Action at a Distance

NEWTON : “...that one body can **act upon another at a distance**, through a vacuum, without the mediation of anything else,..., is to me a **great absurdity**”

- ★ In Classical Mechanics and non-relativistic Quantum Mechanics forces arise from potentials $V(\tilde{\mathbf{r}})$ which act instantaneously over all space.
- ★ In Quantum Field theory, forces are mediated by the exchange of virtual field quanta - and there is no mysterious action at a distance.
- ★ Matter and Force described by ‘particles’

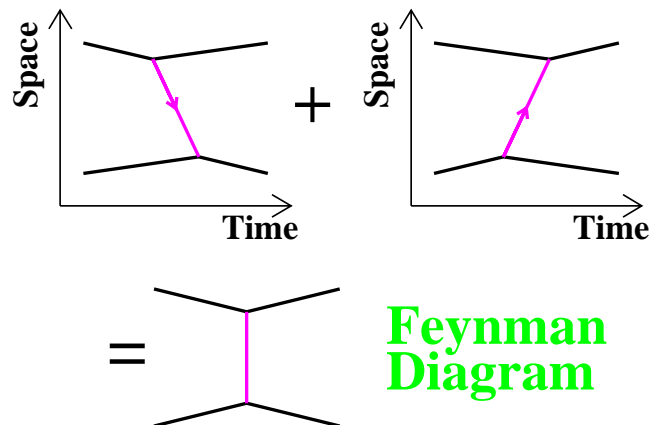
Feynman Diagrams

- ★ The results of calculations based on a single process in Time-Ordered Perturbation Theory (sometimes called old-fashioned, OFPT) depend on the reference frame.
- ★ However, the sum of all time orderings is not frame dependent and provides the basis for our relativistic theory of Quantum Mechanics.
- ★ The sum of time orderings are represented by **FEYNMAN DIAGRAMS**



- ★ Energy and Momentum are conserved at the interaction **vertices**
- ★ But the exchanged particle no longer has $m_X^2 = E_X^2 - p_X^2$, it is **VIRTUAL**

Virtual Particles



Virtual Particles:

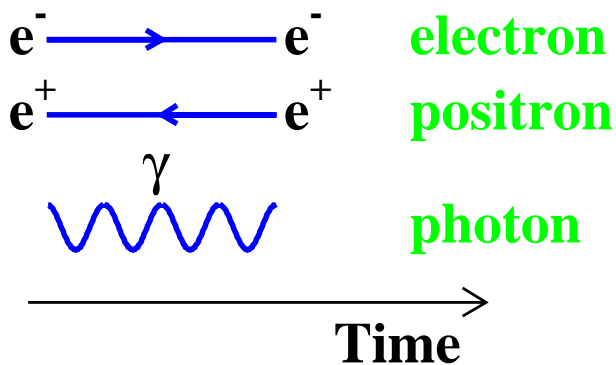
- ★ Forces due to exchanged particle X which is termed **VIRTUAL**.
- ★ The exchanged particle is **off mass-shell**, *i.e.* for the **unobservable** exchanged **VIRTUAL** particle $E^2 \neq p^2 + m_X^2$.
- ★ *i.e.* $m^2 = E_X^2 - p_X^2$ does not give the physical mass, m_X . The mass of the virtual particle $m^2 = E_X^2 - p_X^2$ can be +ve or -ve.

Qualitatively: the propagator is inversely proportional to how far the particle is off-shell. The further off-shell, the smaller the probability of producing such a virtual state.

Understanding Feynman Diagrams

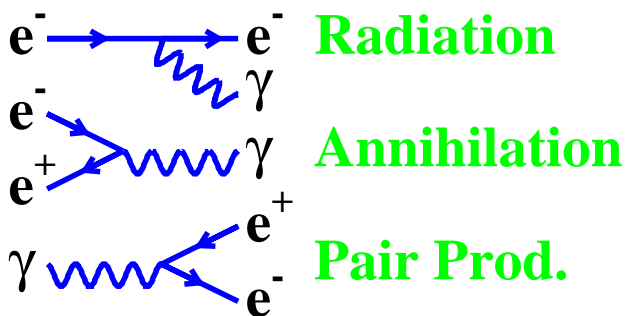
★ Feynman diagrams are the language of modern particle physics. They will be used extensively throughout this course.

The Basic Building Blocks



Note : the positron (e^+) line is drawn as a negative energy electron traveling backwards in time

The e^\pm — photon interactions

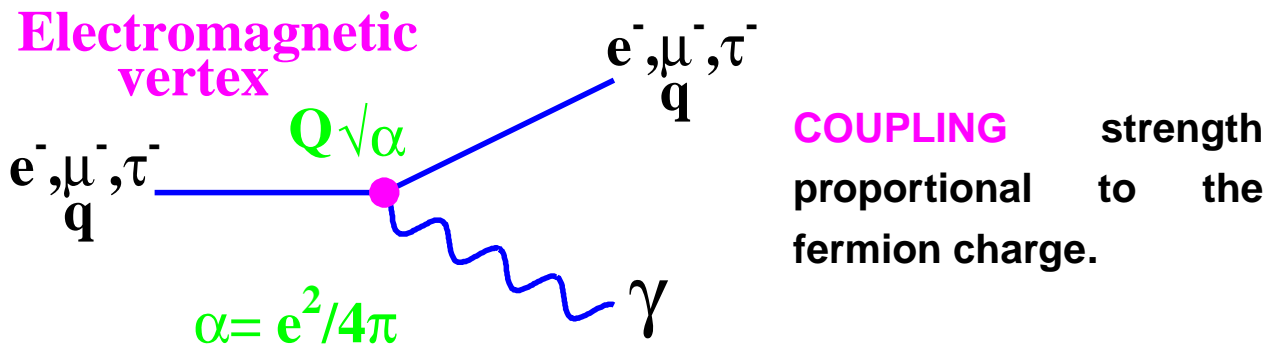


Note: none of these processes are allowed in isolation : Forbidden by (E, \vec{p}) conservation.

★ The strength of the interaction between the virtual photon and fermions is called the coupling strength. For the electromagnetic interaction this is proportional to electric charge e .

The Electromagnetic Vertex

★ The electromagnetic interaction is described by the **photon propagator** and the **vertex**:



★ All electromagnetic interactions can be described in terms of the above diagram

★ Always conserve energy and momentum + (angular momentum, charge)

★ QED Vertex **NEVER** changes flavour i.e. $e^- \rightarrow e^- \gamma$ but **not** $e^- \rightarrow \mu^- \gamma$

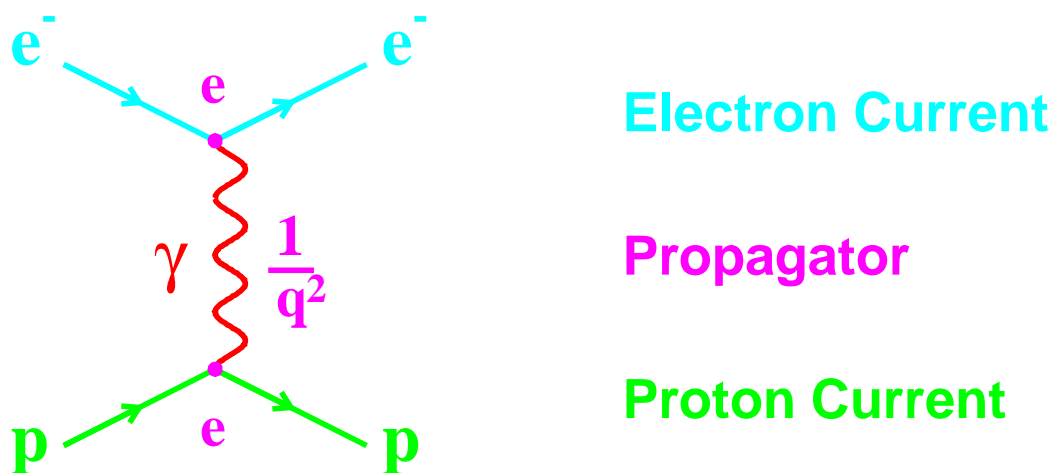
★ QED Vertex also conserves **PARITY**

★ Qualitatively : $Q\sqrt{\alpha}$ can be thought of the probability of a charged particle emitting a photon, the probability is proportional to $1/q^2$ of the photon.

Physics with Feynman Diagrams

Scattering cross sections calculated from:

- ★ Fermion wave functions
- ★ Vertex Factors : coupling strength
- ★ Propagator
- ★ Phase Space



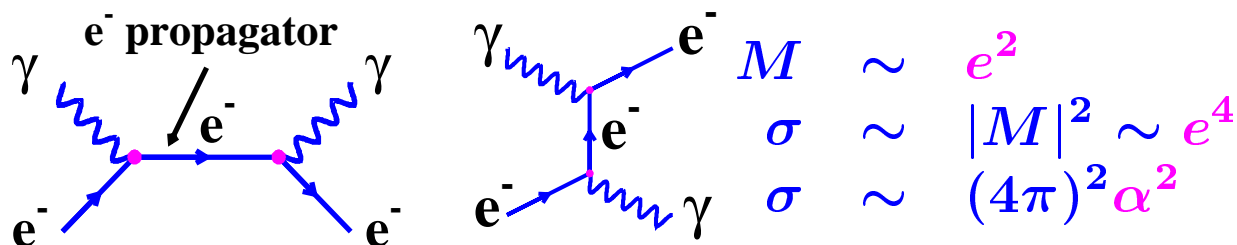
Matrix element M factorises into 3 terms :

$$\begin{aligned}
 -iM &= \langle \bar{u}_e | ie\gamma^\mu | u_e \rangle && \text{Electron Current} \\
 &\times \frac{-ig^{\mu\nu}}{q^2} && \text{Photon Propagator} \\
 &\times \langle \bar{u}_p | ie\gamma^\nu | u_p \rangle && \text{Proton Current}
 \end{aligned}$$

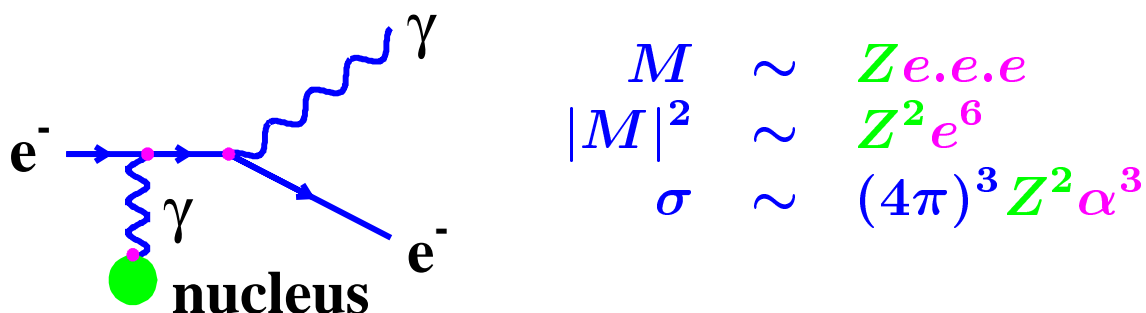
The factors γ^μ and $g^{\mu\nu}$ are 4×4 matrices which account for the spin-structure of the interaction (described in the lecture on the Dirac Equation).

Pure QED Processes

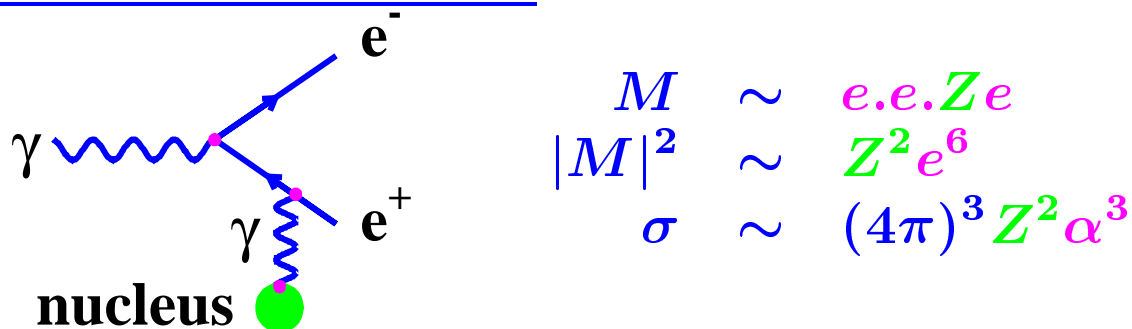
Compton Scattering



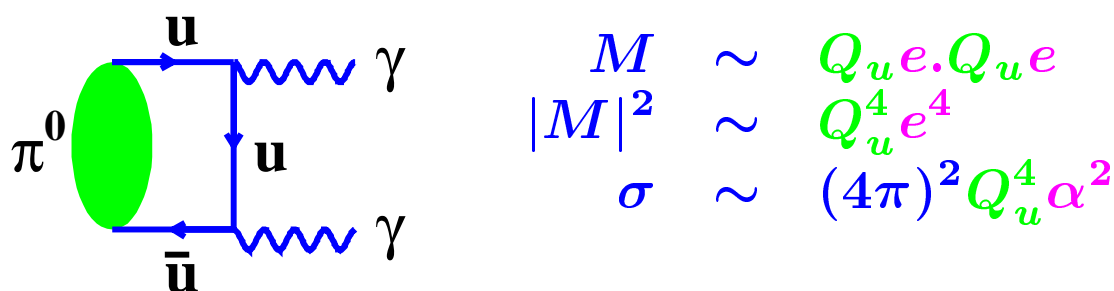
Bremsstrahlung



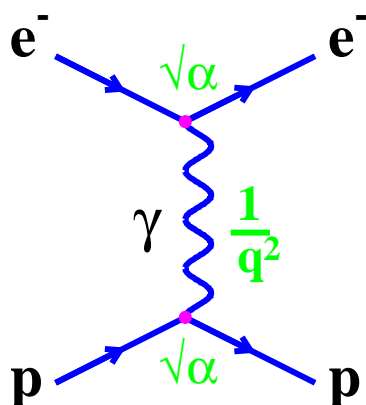
e^+e^- Pair Production



π^0 Decay

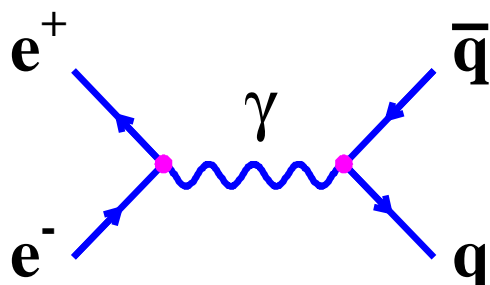


Electron-Proton Scattering



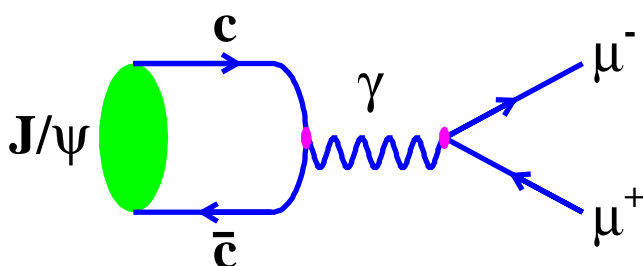
$$\begin{aligned} M &\sim e \cdot e \\ |M|^2 &\sim e^4 \\ \sigma &\sim (4\pi)^2 \alpha^2 \end{aligned}$$

e^+e^- Annihilation



$$\begin{aligned} M &\sim e \cdot Q_u e \\ |M|^2 &\sim Q_u^2 e^4 \\ \sigma &\sim (4\pi)^2 Q_u^2 \alpha^2 \end{aligned}$$

$J/\psi \rightarrow \mu^+\mu^-$



$$\begin{aligned} M &\sim Q_c e \cdot e \\ |M|^2 &\sim Q_c^2 e^4 \\ \sigma &\sim (4\pi)^2 Q_c^2 \alpha^2 \end{aligned}$$

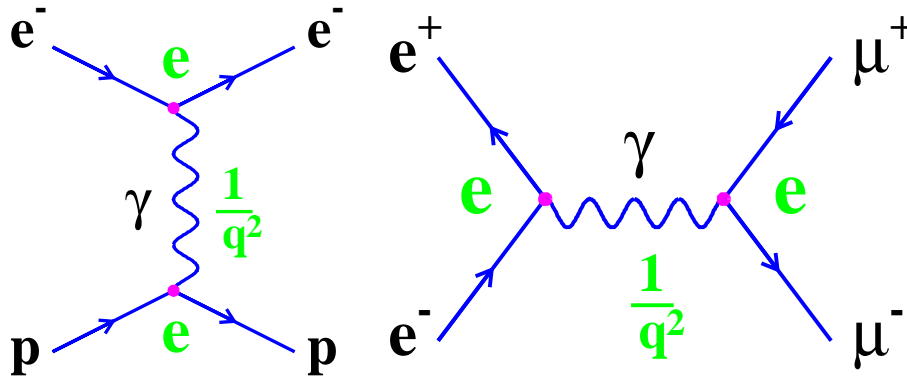
Coupling strength determines 'order of magnitude' of matrix element. For particles interacting/decaying via electromagnetic interaction: typical values for cross sections/lifetimes

$$\begin{aligned} \sigma_{em} &\sim 10^{-2} \text{ mb} \\ \tau_{em} &\sim 10^{-20} \text{ s} \end{aligned}$$

Scattering in QED

EXAMPLE Calculate the “spin-less” cross sections for the two processes:

- ★ electron-proton scattering
- ★ electron-positron annihilation



Here we will consider the case where all particles are spin-0, (see lecture on Dirac Equation for complete treatment)

Fermi's Golden rule and Born Approximation:

$$\frac{d\sigma}{d\Omega} = 2\pi |M|^2 d\rho(E_f)/d\Omega$$

For both processes write the SAME matrix element

$$M = \frac{e^2}{q^2} = \frac{4\pi\alpha}{q^2}$$

However, the four-momentum transfer ($q^2 = E^2 - \tilde{q}^2$) is very different (\tilde{q} is the 3-momentum of the virtual photon)

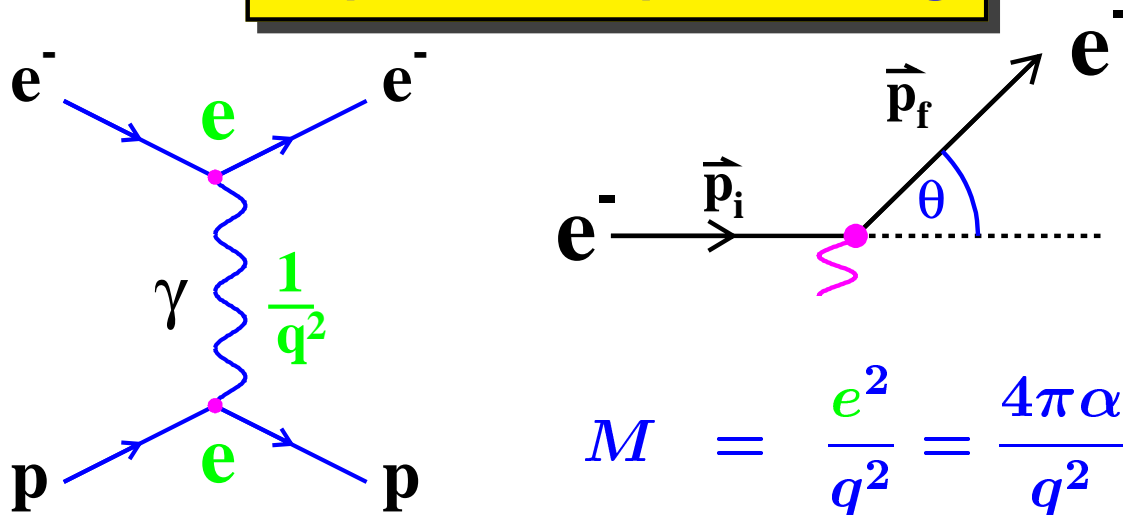
- ★ Elastic e^- -proton scattering : $q = (0, \tilde{q})$

$$q^2 = -|\tilde{q}|^2$$

- ★ e^+e^- annihilation : $q = (2E, 0)$

$$q^2 = +4E^2$$

“Spin-less” e-p Scattering



$$M = \frac{e^2}{q^2} = \frac{4\pi\alpha}{q^2}$$

From Handout 1, pages 31-34:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= 2\pi |M|^2 \frac{E^2}{(2\pi)^3} \\ &= 2\pi \frac{(4\pi\alpha)^2}{q^4} \frac{E^2}{(2\pi)^3} = \frac{4\alpha^2 E^2}{q^4} \end{aligned}$$

q^2 is the four-momentum transfer:

$$\begin{aligned} q^2 &= q^\mu q_\mu = (E_f - E_i)^2 - (\tilde{p}_f - \tilde{p}_i)^2 \\ &= E_f^2 + E_i^2 - 2E_f E_i - \tilde{p}_f^2 - \tilde{p}_i^2 + 2\tilde{p}_f \cdot \tilde{p}_i \\ &= 2m_e^2 - 2E_f E_i + 2|\tilde{p}_f||\tilde{p}_i| \cos \theta \end{aligned}$$

neglecting electron mass: *i.e.* $m_e^2 = 0$ and $|\tilde{p}_f| = E_f$

$$q^2 = -2E_i E_f (1 - \cos \theta)$$

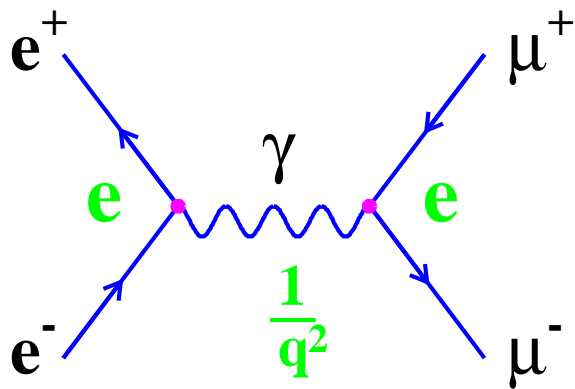
$$q^2 = -4E_i E_f \sin^2 \frac{\theta}{2}$$

Therefore for ELASTIC scattering $E_i = E_f$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}}$$

i.e. the Rutherford scattering formula (Handout 1 p.36)

“Spin-less” e^+e^- Annihilation



$$M = \frac{4\pi\alpha}{q^2}$$

$$\frac{d\sigma}{d\Omega} = 2\pi \frac{(4\pi\alpha)^2}{q^4} \frac{E^2}{(2\pi)^3} = \frac{4\alpha^2 E^2}{q^4}$$

same formula, but different four-momentum transfer:

$$q^2 = q^\mu q_\mu = (E_{e^+} + E_{e^-})^2 - (\tilde{p}_{e^+} + \tilde{p}_{e^-})^2$$

Assuming we are in the centre-of-mass system

$$\begin{aligned} E_{e^+} &= E_{e^-} = E \\ \tilde{p}_{e^-} &= -\tilde{p}_{e^+} \\ \rightarrow q^2 &= (2E)^2 = s \end{aligned}$$

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{4\alpha^2 E^2}{q^4} = \frac{4\alpha^2 E^2}{16E^4} \\ &= \frac{\alpha^2}{s} \end{aligned}$$

Integrating gives total cross section:

$$\sigma = 4\pi \frac{\alpha^2}{s}$$

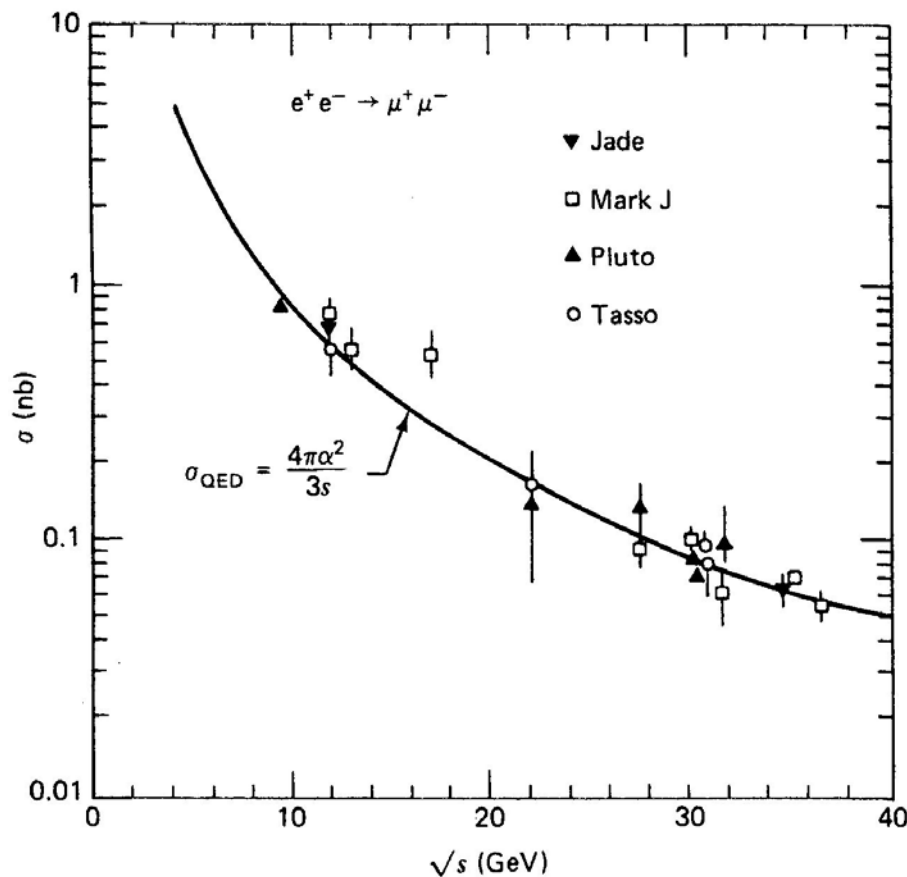
This is not quite correct - because we have neglected spin.
The actual cross section (see lecture on Dirac Equation) is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1 + \cos^2 \theta)$$

$$\sigma(e^+e^- \rightarrow \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}$$

Natural Units Example cross section at $\sqrt{s} = 22 \text{ GeV}$
i.e. 11 GeV electrons colliding with 11 GeV positrons.

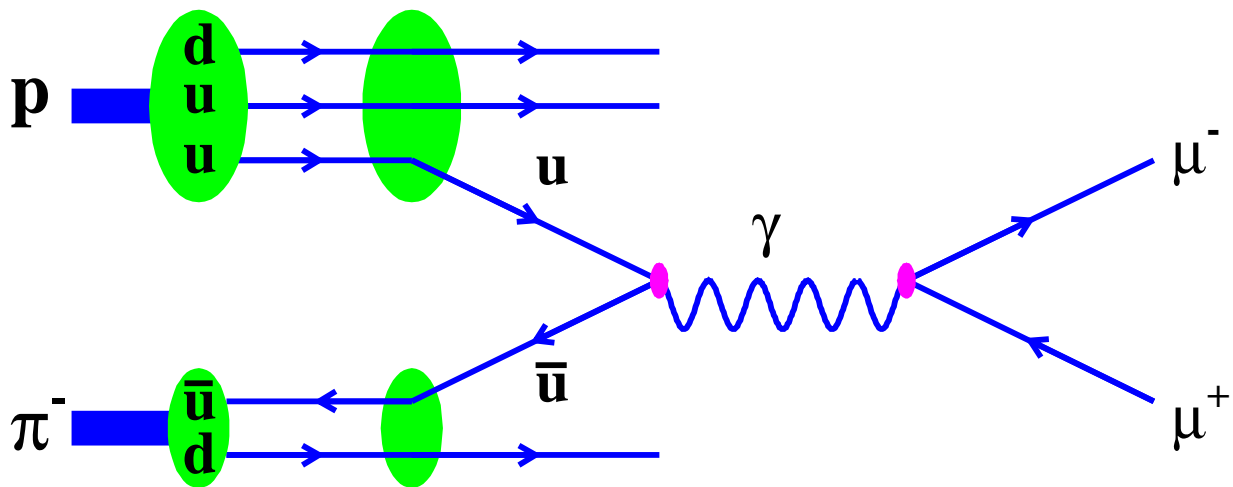
$$\begin{aligned}\sigma_{e^+e^- \rightarrow \mu^+\mu^-} &= \frac{4\pi\alpha^2}{3s} = \frac{4\pi}{137^2} \frac{1}{3 \times 22^2} \\ &= 4.6 \times 10^{-7} \text{ GeV}^{-2} \\ &= 4.6 \times 10^{-7} (\hbar c)^2 / (1.6 \times 10^{-10})^2 \text{ m}^2 \\ &= 1.8 \times 10^{-38} \text{ m}^2 = 0.18 \text{ nb}\end{aligned}$$



The Drell-Yan Process

★ Can also annihilate $q\bar{q}$ as in the Drell-Yan process

e.g. $\pi^- p \rightarrow \mu^+ \mu^- + \text{hadrons}$



$$\sigma(\pi^- p \rightarrow \mu^+ \mu^- + \text{hadrons}) \propto Q_u^2 \alpha^2$$

(see Question 3 on the problem sheet)

Experimental Tests of QED

- ★ QED is an incredibly successful theory

Example

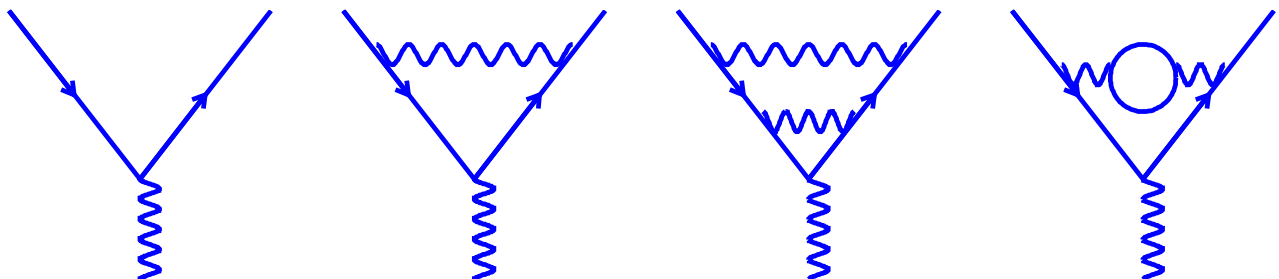
- ★ Magnetic moments of e^\pm, μ^\pm

$$\tilde{\mu} = g \frac{e}{2m} \tilde{s}$$

- ★ For a **point-like** spin 1/2 particle :

$$g = 2$$

However higher order terms induce an **anomalous magnetic moment** i.e. g not quite 2.



$$\frac{(g_e - 2)}{2} = 11596521.869 \pm 0.041 \times 10^{-10} \text{ EXPT}$$

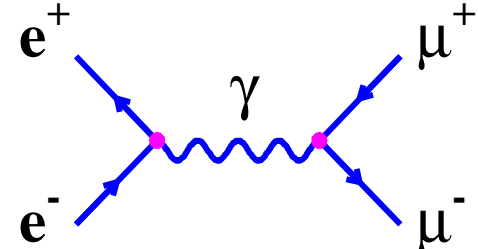
$$\frac{(g_e - 2)}{2} = 11596521.3 \pm 0.3 \times 10^{-10} \text{ THEORY}$$

- ★ Agreement at the level of 1 in 10^8
- ★ Q.E.D. provides a remarkably precise description of the electromagnetic interaction !

Higher Orders

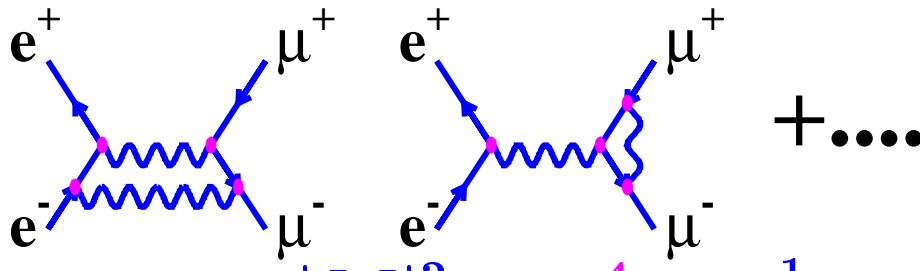
So far only considered **lowest order** term in the perturbation series. Higher order terms also contribute

Lowest Order:



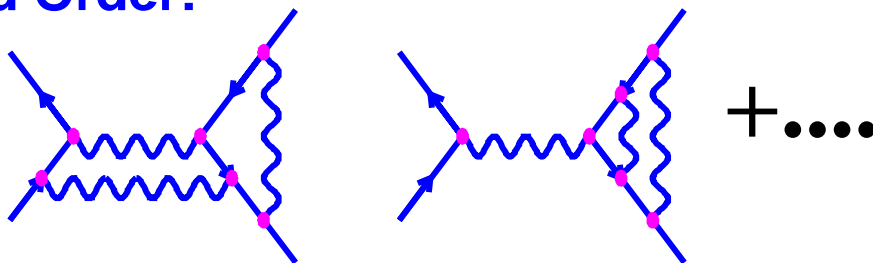
$$|M|^2 \propto \alpha^2 \sim \frac{1}{137^2}$$

Second Order:



$$|M|^2 \propto \alpha^4 \sim \frac{1}{137^4}$$

Third Order:



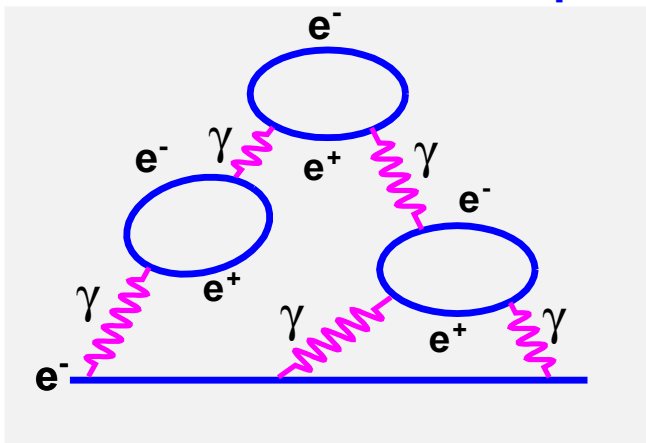
$$|M|^2 \propto \alpha^6 \sim \frac{1}{137^6}$$

Second order suppressed by α^2 relative to first order. Provided α is small, i.e. perturbation is small, lowest order dominates.

Running of α

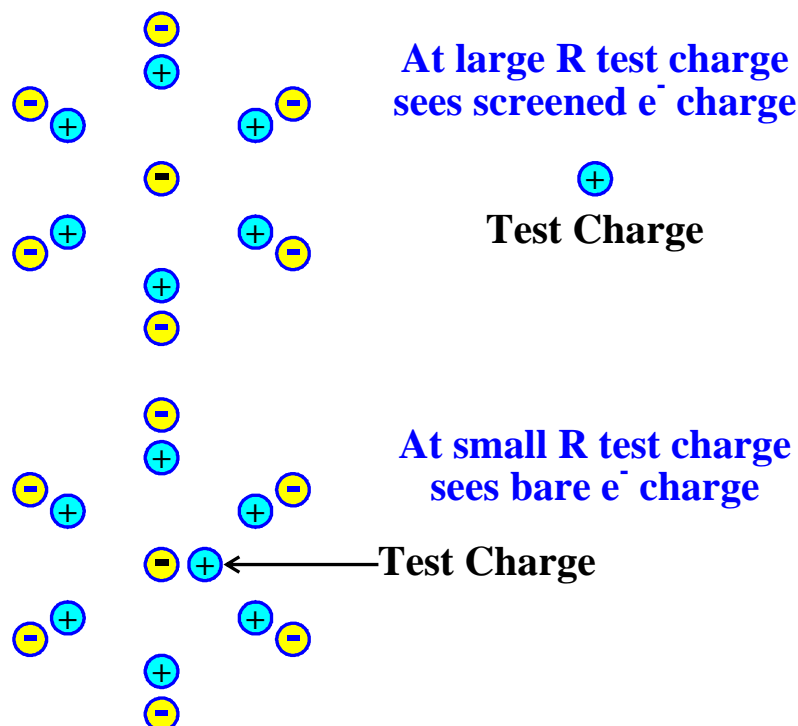
- ★ $\alpha = \frac{e^2}{4\pi}$ specifies the strength of the interaction between an electron and photon.
- ★ BUT α isn't a constant

Consider a free electron: Quantum fluctuations lead to a 'cloud' of virtual electron/positron pairs



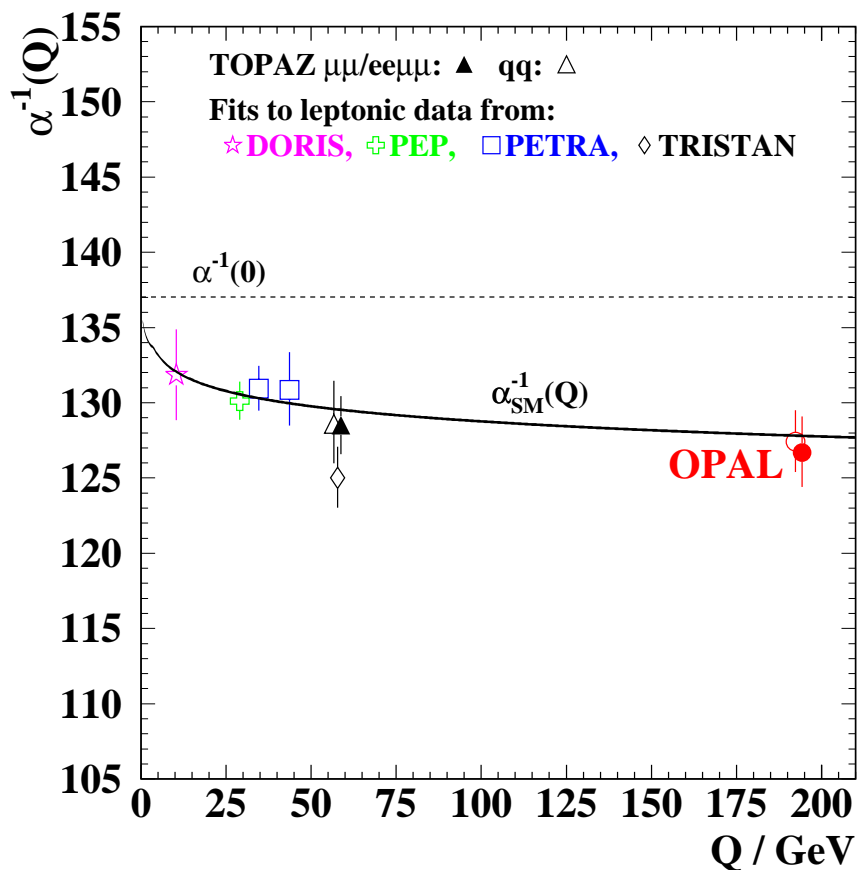
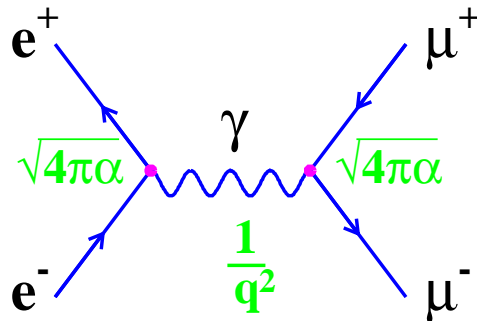
this is just one of many (an infinite set) such diagrams.

- ★ The vacuum acts like a dielectric medium
- ★ The virtual e^+e^- pairs are polarized
- ★ At large distances the **bare** electron charge is screened.



Running of α

Measure $\alpha(q^2)$ from $e^+e^- \rightarrow \mu^+\mu^-$ etc.



★ α increases with the increasing q^2 (i.e. closer to the bare charge).

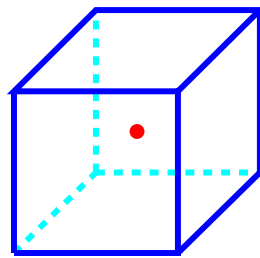
★ At $q^2 = 0$: $\alpha = 1/137$

★ At $q^2 = (100 \text{ GeV})^2$: $\alpha = 1/128$

Appendix: Relativistic Phase Space

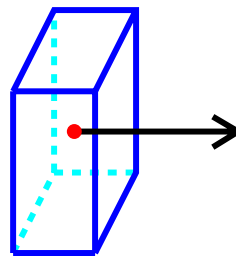
NON-EXAMINABLE

- ★ Previously normalized wave-functions to 1 particle in a box of side L (see Handout 1, pages 33-34).



Rest Frame

1 particle/ V



Lab. Frame

1 particle/ (V/γ)

- ★ In relativity, box will be Lorentz Contracted by a factor of

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{E}{m}$$

$$i.e. \quad V' = V \left(\frac{m}{E} \right)$$

i.e. E/m particles per volume V

NEED to adjust normalization volume with energy

Conventional choice:

$$N = \frac{1}{\sqrt{2E}}$$

- ★ In most scattering process the factors of $\sqrt{2E}$ in the wave-function normalization cancel with corresponding factors in the expressions for flux and density of states, just as the factors of L^3 were canceled previously (Handout 1, pages 35)