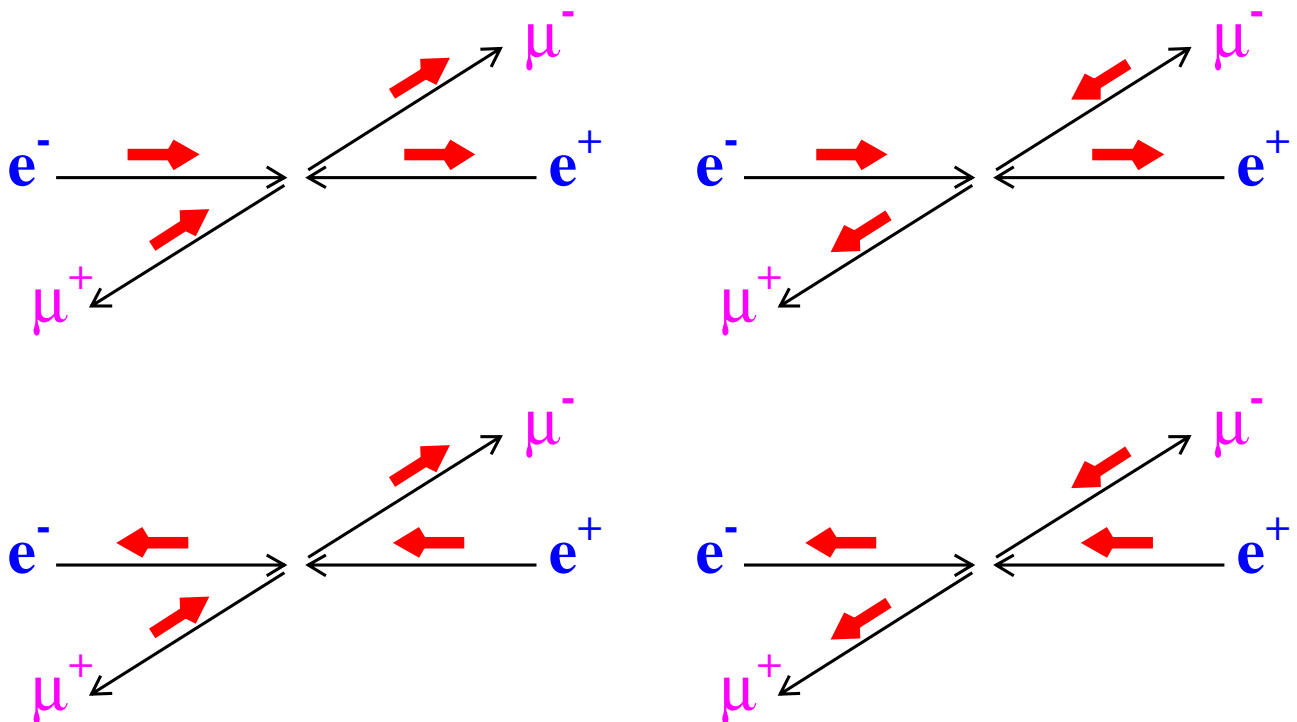


Particle Physics

Dr M.A. Thomson

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$



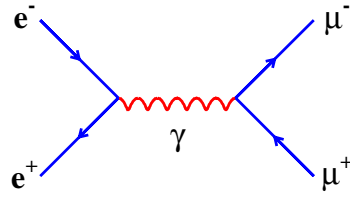
Part II, Lent Term 2004
HANDOUT V

Spin, Helicity and the Dirac Equation

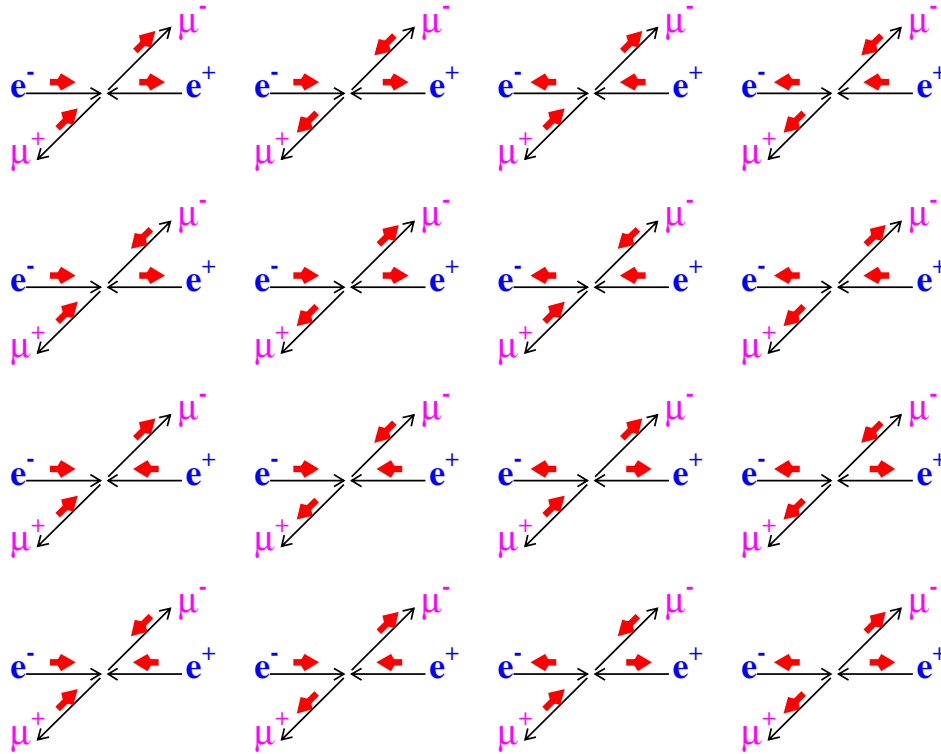
- ★ Upto this point we have taken a hands-off approach to “spin”.
- ★ Scattering cross sections calculated for spin-less particles
- ★ To understand the **WEAK** interaction need to understand **SPIN**
- ★ Need a relativistic theory of quantum mechanics that includes spin
- ★ \Rightarrow The **DIRAC EQUATION**

★ SPIN complicates things....

The process



represents the sum over all possible spin states



$$M \rightarrow \sum_i M_i$$

$$|M|^2 \rightarrow \left| \sum_i M_i \right|^2 = \sum_i |M_i|^2$$

since ORTHOGONAL SPIN states.

$$\sigma = \frac{1}{4} \left[2\pi \sum_i |M_i|^2 \rho(E_f) \right]$$

Cross-section : sum over all spin assignments,
averaged over initial spin states.

The Klein-Gordon Equation Revisited

Schrödinger Equation for a free particle can be written as

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \nabla^2 \psi$$

Derivatives : 1st order in time and 2nd order in space coordinates \Rightarrow **not Lorentz invariant**

From Special Relativity:

$$E^2 = p^2 + m^2$$

from Quantum Mechanics:

$$\hat{E} = i \frac{\partial}{\partial t} \quad , \quad \hat{p} = -i \nabla$$

Combine to give the **Klein-Gordon Equation**:

$$\frac{\partial^2 \psi}{\partial t^2} = (\nabla^2 - m^2) \psi$$

Second order in both space and time derivatives - by construction Lorentz invariant.

★ Negative energy solutions

\Rightarrow anti-particles

★ BUT negative energy solutions also give negative particle densities !?

$$\psi^* \psi < 0$$

Try another approach.....

Weyl Equations

(The massless version of the Dirac Equation)

Klein-Gordon Eqn. for massless particles:

$$\left(\frac{\partial^2 \psi}{\partial t^2} - \nabla^2\right)\psi = 0$$

$$\text{i.e. } (\hat{E}^2 - \hat{p}^2)\psi = 0$$

Try to factorize 2nd Order KG equation →
equation linear in ∇ AND $\frac{\partial}{\partial t}$:

$$\left(\frac{\partial}{\partial t} - \tilde{\sigma} \cdot \nabla\right)\left(\frac{\partial}{\partial t} + \tilde{\sigma} \cdot \nabla\right)\psi = 0$$

with as yet **undetermined** constants $\tilde{\sigma}$

Gives the two decoupled **WEYL** Equations

$$\left(\sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y} + \sigma_z \frac{\partial}{\partial z}\right)\psi = + \frac{\partial \psi}{\partial t}$$

$$\left(\sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y} + \sigma_z \frac{\partial}{\partial z}\right)\psi = - \frac{\partial \psi}{\partial t}$$

both linear in space and time derivatives.

BUT must satisfy the Klein-Gordon Equation

i.e. in operator form

$$(\hat{E} - \tilde{\sigma} \cdot \hat{p})(\hat{E} + \tilde{\sigma} \cdot \hat{p})\psi = 0$$

must satisfy

$$(\hat{E}^2 - \hat{p}^2)\psi = 0$$

Weyl equations give:

$$\begin{aligned}
 (\hat{E}^2 &- \sigma_x \hat{p}_x \sigma_x \hat{p}_x - \sigma_y \hat{p}_y \sigma_y \hat{p}_y - \sigma_z \hat{p}_z \sigma_z \hat{p}_z \\
 &- \sigma_x \hat{p}_x \sigma_y \hat{p}_y - \sigma_y \hat{p}_y \sigma_x \hat{p}_x \\
 &- \sigma_y \hat{p}_y \sigma_z \hat{p}_z - \sigma_z \hat{p}_z \sigma_y \hat{p}_y \\
 &- \sigma_z \hat{p}_z \sigma_x \hat{p}_x - \sigma_x \hat{p}_x \sigma_z \hat{p}_z) \psi = 0
 \end{aligned}$$

Therefore in order to recover the KG equation:

$$(\hat{E}^2 - \hat{p}_x \hat{p}_x - \hat{p}_y \hat{p}_y - \hat{p}_z \hat{p}_z) = 0,$$

require:

$$\begin{aligned}
 \sigma_x^2 &= \sigma_y^2 = \sigma_z^2 = 1 \\
 (\sigma_x \sigma_y + \sigma_y \sigma_x) &= 0 \text{ etc.}
 \end{aligned}$$

$\therefore \sigma_x, \sigma_y, \sigma_z$ **ANTI-COMMUTE**

The simplest choice for σ are the Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence solutions to the Klein-Gordon equation

$$(\hat{E} - \tilde{\sigma} \cdot \hat{p})(\hat{E} + \tilde{\sigma} \cdot \hat{p})\psi = 0$$

are given by the Weyl Equations:

$$\begin{aligned}
 (\hat{E} - \tilde{\sigma} \cdot \hat{p})\phi &= 0 \\
 (\hat{E} + \tilde{\sigma} \cdot \hat{p})\chi &= 0
 \end{aligned}$$

Since σ_i are 2×2 matrices, need 2 component wave-functions - **WEYL SPINORS**.

$$\phi = N \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} e^{-i(\mathbf{E}t - \tilde{\mathbf{p}} \cdot \tilde{\mathbf{r}})}$$

The wave-function is forced to have a new degree of freedom - the spin of the fermion.

Consider the **FIRST** Weyl Equation

$$\begin{aligned}(\hat{E} - \tilde{\sigma} \cdot \hat{\mathbf{p}})\phi &= 0 \\ \left(\frac{\partial}{\partial t} + \sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y} + \sigma_z \frac{\partial}{\partial z} \right) \phi &= 0\end{aligned}$$

For a plane wave solution:

$$\phi = N \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} e^{-i(\mathbf{E}t - \tilde{\mathbf{p}} \cdot \tilde{\mathbf{r}})}$$

the first Weyl Equation gives

$$\begin{aligned}(\mathbf{E} - \sigma_x \mathbf{p}_x - \sigma_y \mathbf{p}_y - \sigma_z \mathbf{p}_z)\phi &= 0 \\ (\mathbf{E} - \tilde{\sigma} \cdot \tilde{\mathbf{p}})\phi &= 0\end{aligned}$$

where

$$\begin{aligned}\tilde{\sigma} \cdot \tilde{\mathbf{p}} &= \sigma_x \mathbf{p}_x + \sigma_y \mathbf{p}_y + \sigma_z \mathbf{p}_z \\ &= \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\end{aligned}$$

Hence for the first WEYL equation, the SPINOR solutions of $(\mathbf{E} - \tilde{\sigma} \cdot \tilde{\mathbf{p}})\phi = 0$ are given the coupled equations:

$$\Rightarrow \left. \begin{aligned} p_z \phi_1 + (p_x - ip_y) \phi_2 &= E \phi_1 \\ (p_x + ip_y) \phi_1 - p_z \phi_2 &= E \phi_2 \end{aligned} \right\}$$

★ Choose $\tilde{\mathbf{p}}$ along z axis, i.e. $p_z = |p|$:

$$(E - |p|)\phi_1 = 0$$

$$(E + |p|)\phi_2 = 0$$

There are two solutions $\phi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\phi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

★ The first solution, $\phi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, requires $E = +|p|$, i.e. a positive energy (particle) solution. Similarly, the second $\phi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, requires $E = -|p|$, i.e. a negative energy (anti-particle) solution.

★ Back to the FIRST WEYL equation

$$(E - \tilde{\sigma} \cdot \hat{\mathbf{p}})\phi = 0$$

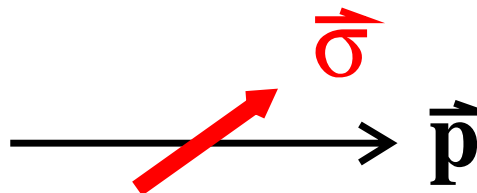
$$\tilde{\sigma} \cdot \hat{\mathbf{p}}\phi = E\phi$$

$$\frac{\tilde{\sigma} \cdot \hat{\mathbf{p}}}{|p|}\phi = \frac{E}{|p|}\phi = \begin{cases} +1 & E > 0 \\ -1 & E < 0 \end{cases}$$

★ The solutions of the WEYL equations are Eigenstates of the HELICITY operator.

$$\hat{H} = \frac{\tilde{\sigma} \cdot \hat{\mathbf{p}}}{|p|}$$

with Eigenvalues $+1$ and -1 respectively.



HELICITY is the projection of a particle's SPIN onto its flight direction.

(Recall WEYL equations applicable for massless particles)

★ Interpret the two solutions of the **FIRST WEYL** equation as a **RIGHT-HANDED** $H = +1$ particle and a **LEFT-HANDED** anti-particle $H = -1$.

$$E > 0, H = +1$$



RH particle

$$E < 0, H = -1$$



LH anti-particle

★ The **SECOND WEYL** equation:

$$(\hat{E} + \tilde{\sigma} \cdot \hat{p})\chi = 0$$

has **LEFT-HANDED** particle and **RIGHT-HANDED** anti-particle solutions.

$$E < 0, H = +1$$



RH anti-particle

$$E > 0, H = -1$$



LH particle

SUMMARY:

- ★ By factorizing the **Klein-Gordon** equation into a form linear in the derivatives \Rightarrow force particles to have a non-commuting degree of freedom, **SPIN** !
- ★ Still obtain anti-particle solutions
- ★ Probability densities always positive
- ★ ‘Natural’ states are the **Helicity Eigenstates**
- ★ Weyl Equations are the ultra-relativistic (massless) limit of the Dirac Equation

Spin in the Fundamental Interactions

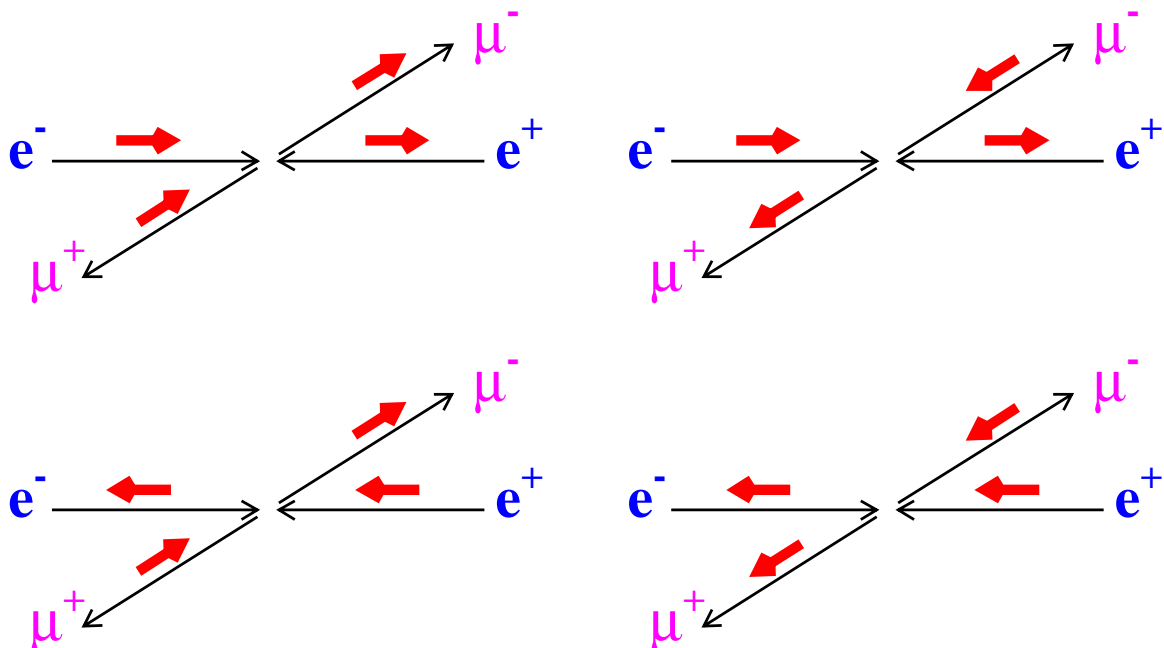
★ The **ELECTROMAGNETIC**, **STRONG**, and **WEAK** interactions are all mediated by **VECTOR** (spin-1) fields. In the massless limit, the fundamental fermion states are eigenstates of the helicity operator. **HANDEDNESS**

★ (**CHIRALITY**) plays a central rôle in the interactions between the field bosons and the fermions; the only allowed couplings are:

LH particle	to	LH particle
RH particle	to	RH particle
LH anti-particle	to	LH anti-particle
RH anti-particle	to	RH anti-particle
LH particle	to	RH anti-particle
RH particle	to	LH anti-particle

EXAMPLE $e^+e^- \rightarrow \mu^+\mu^-$

Of the **16** possibilities **ONLY** the following **SPIN** combinations contribute to the cross-section



All other **SPIN** combinations give zero $|M_i|^2$

Solutions of the Weyl Equations

Consider the general case of a particle with travelling at an angle θ with respect to the z -axis

$$p_z = |p| \cos \theta$$

$$p_x = |p| \sin \theta$$

$$\text{WEYL 1} \quad (\hat{E} - \hat{\sigma} \cdot \hat{p})\phi = 0$$

$$(\hat{\sigma} \cdot \hat{p})\phi = \hat{E}\phi$$

$$\Rightarrow \left. \begin{aligned} p_z \phi_1 + p_x \phi_2 &= E \phi_1 \\ p_x \phi_1 - p_z \phi_2 &= E \phi_2 \end{aligned} \right\}$$

For the positive energy solution $E = +|p|$:

$$\Rightarrow \left. \begin{aligned} \phi_1 \cos \theta + \phi_2 \sin \theta &= \phi_1 \\ \phi_1 \sin \theta - \phi_2 \cos \theta &= \phi_2 \end{aligned} \right\}$$

$$\Rightarrow \phi_1 = \phi_2 \frac{\sin \theta}{(1 - \cos \theta)}$$

Therefore

$$\frac{\phi_1}{\phi_2} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{(1 - \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2})}$$

$$\frac{\phi_1}{\phi_2} = \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}}$$

Normalizing such that $\phi_1^2 + \phi_2^2 = 1$ gives:

$$\phi_1 = \cos \frac{\theta}{2}$$

$$\phi_2 = \sin \frac{\theta}{2}$$

★ So the positive energy solution to the first Weyl equation (RH particle) gives

$$\phi_{\text{RH}} = N \begin{pmatrix} + \cos \frac{\theta}{2} \\ + \sin \frac{\theta}{2} \end{pmatrix} e^{-i(\mathbf{E}t - \tilde{\mathbf{p}} \cdot \tilde{\mathbf{r}})} \quad \text{RH fermion}$$

This is still a Eigenvalue of the helicity operator with ($H = +1$) i.e. a RH particle but now referred to an external axis.

The positive energy solution to the **SECOND** Weyl equation (LH particle) gives

$$\chi_{\text{LH}} = N \begin{pmatrix} - \sin \frac{\theta}{2} \\ + \cos \frac{\theta}{2} \end{pmatrix} e^{-i(\mathbf{E}t - \tilde{\mathbf{p}} \cdot \tilde{\mathbf{r}})} \quad \text{LH fermion}$$

Not much more than the Quantum Mechanical rotation properties of $\text{spin-}\frac{1}{2}$.

★ The spin part of a RH particle/anti-particle wave-function can be written

$$\psi_R(\theta) = \begin{pmatrix} + \cos \frac{\theta}{2} \\ + \sin \frac{\theta}{2} \end{pmatrix} = \cos \frac{\theta}{2} \uparrow + \sin \frac{\theta}{2} \downarrow$$

★ Similarly for a LH particle/anti-particle

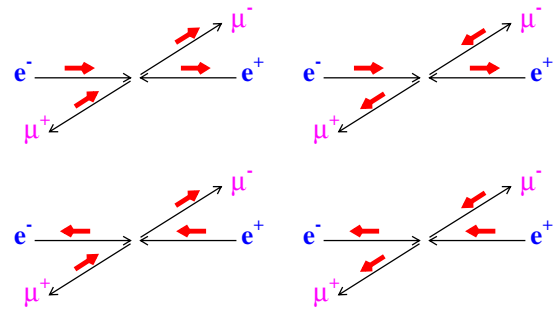
$$\psi_L(\theta) = \begin{pmatrix} - \sin \frac{\theta}{2} \\ + \cos \frac{\theta}{2} \end{pmatrix} = -\sin \frac{\theta}{2} \uparrow + \cos \frac{\theta}{2} \downarrow$$

For particles/anti-particles with momentum $-\tilde{\mathbf{p}}(\theta)$, i.e. an angle $\theta + \pi$ to the z-axis:

$$\begin{aligned} \psi_R(\theta + \pi) &= -\sin \frac{\theta}{2} \uparrow + \cos \frac{\theta}{2} \downarrow \\ \psi_L(\theta + \pi) &= -\cos \frac{\theta}{2} \uparrow - \sin \frac{\theta}{2} \downarrow \end{aligned}$$

Application to $e^+e^- \rightarrow \mu^+\mu^-$

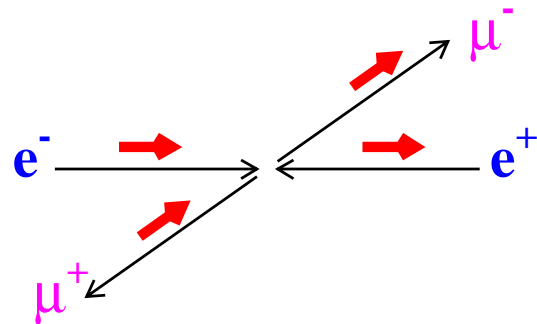
★ Four helicity combinations contribute to the cross-section.



★ Consider the first diagram

$$e_{R}^{-}e_{L}^{+} \rightarrow \mu_{R}^{-}\mu_{L}^{+}$$

With the e^{-} direction defining the z axis:



★ The spin parts of the wave-functions :

$$\psi_{e_{R}^{-}e_{L}^{+}} = \psi_R(0)\psi_L(\pi) = \uparrow\uparrow$$

$$\psi_{\mu_{R}^{-}\mu_{L}^{+}} = \psi_R(\theta)\psi_L(\theta + \pi)$$

$$= -\cos^2 \frac{\theta}{2} \uparrow\uparrow - \cos \frac{\theta}{2} \sin \frac{\theta}{2} (\uparrow\downarrow + \downarrow\uparrow) - \sin^2 \frac{\theta}{2} \downarrow\downarrow$$

★ Giving the contribution the matrix element:

$$\begin{aligned} |M_1|^2 &= \left| \langle \psi_{\mu_{R}^{-}\mu_{L}^{+}} | \frac{e^2}{q^2} | \psi_{e_{R}^{-}e_{L}^{+}} \rangle \right|^2 \\ &= \frac{e^4}{q^4} \left| \cos^2 \frac{\theta}{2} \right|^2 \\ &= \frac{e^4}{q^4} \left(\frac{1}{2} \right)^2 (1 + \cos \theta)^2 \end{aligned}$$

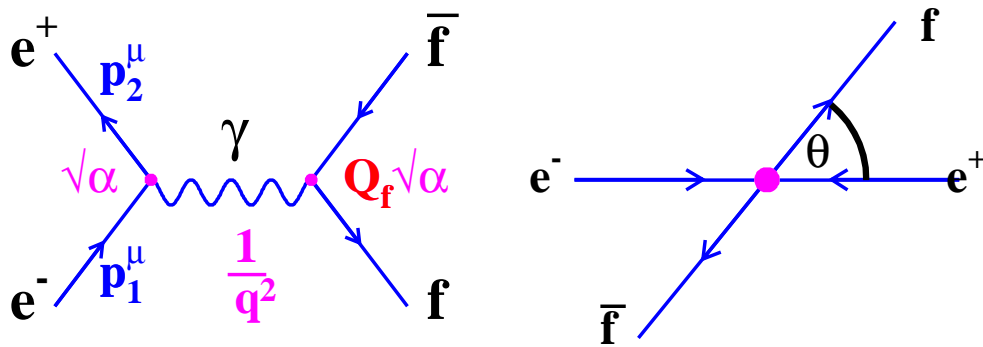
★ Perform same calculation for the four allowed helicity combinations

$$\begin{aligned}
 & \text{Diagram 1: } M_1 \propto \cos^2(\theta/2) \rightarrow \frac{1}{2}(1+\cos\theta) \\
 & \text{Diagram 2: } M_2 \propto \sin^2(\theta/2) \rightarrow \frac{1}{2}(1-\cos\theta) \\
 & \text{Diagram 3: } M_3 \propto \sin^2(\theta/2) \rightarrow \frac{1}{2}(1-\cos\theta) \\
 & \text{Diagram 4: } M_4 \propto \cos^2(\theta/2) \rightarrow \frac{1}{2}(1+\cos\theta)
 \end{aligned}$$

★ For unpolarized electron/positron beams : each of the above process contributes equally. Therefore SUM over all matrix elements and AVERAGE over initial spin states. Giving the total Matrix Element (remember spin-states are orthogonal so sum squared matrix elements) :

$$\begin{aligned}
 |M|^2 &= \frac{1}{4} \{ |M_1|^2 + |M_2|^2 + |M_3|^2 + |M_4|^2 \} \\
 |M|^2 &= \frac{1}{4} \frac{e^4}{q^4} \left\{ \frac{1}{4}(1 + \cos \theta)^2 + \frac{1}{4}(1 - \cos \theta)^2 + \right. \\
 &\quad \left. \frac{1}{4}(1 - \cos \theta)^2 + \frac{1}{4}(1 + \cos \theta)^2 \right\} \\
 |M|^2 &= \frac{e^4}{4q^4} (1 + \cos^2 \theta)
 \end{aligned}$$

★ Nothing more than the QM properties of a SPIN-1 particle decaying to two SPIN- $\frac{1}{2}$ particles



Electron/Positron beams along z -axis ($q^2 = 4E^2 = s$)

Using the spin-averaged matrix element

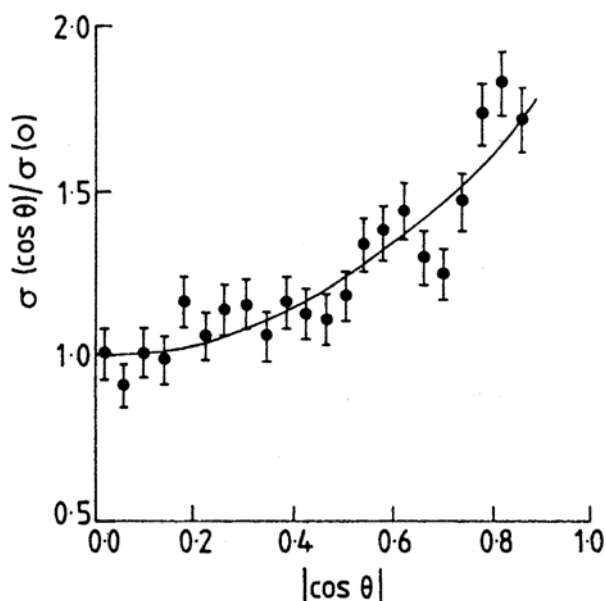
$$|M|^2 = \frac{e^4}{4q^4} (1 + \cos^2 \theta)$$

$$|M|^2 = \frac{(4\pi\alpha)^2}{4s^2} (1 + \cos^2 \theta)$$

$$\frac{d\sigma}{d\Omega} = 2\pi |M|^2 \frac{E^2}{(2\pi)^3}$$

$$= 2\pi \frac{(4\pi\alpha)^2}{4s^2} (1 + \cos^2 \theta) \frac{s}{4} \frac{1}{(2\pi)^3}$$

$$= \frac{\alpha^2 Q_f^2}{4s} (1 + \cos^2 \theta)$$



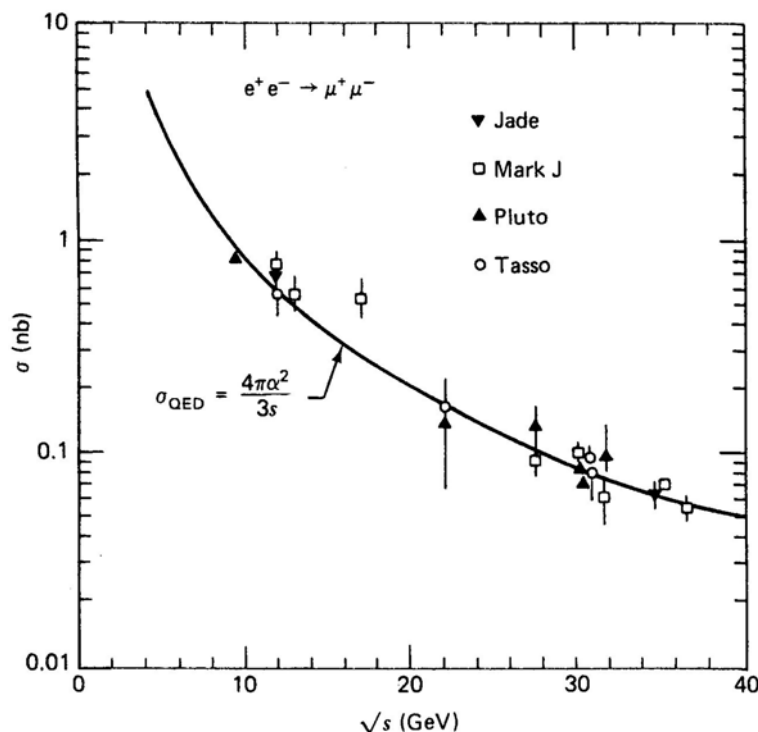
$\frac{d\sigma}{d|\cos \theta|}$ for $e^+e^- \rightarrow q\bar{q}$.
The angle θ is determined from the measured directions of the jets. $|\cos \theta|$ is plotted since it is not possible to uniquely identify which jet corresponds to the quark and which corresponds to the anti-quark. The curve shows the expected $(1 + \cos^2 \theta)$ distribution.
QUARKS are SPIN- $\frac{1}{2}$

(yet again)

Total cross section for $e^+e^- \rightarrow f\bar{f}$

$$\begin{aligned}
 \sigma &= \int \frac{d\sigma}{d\Omega} d\Omega \\
 &= \int_0^{2\pi} \int_0^\pi \frac{\alpha^2 Q_f^2}{4s} (1 + \cos^2 \theta) \sin \theta d\theta d\phi \\
 &= \frac{\pi \alpha^2 Q_f^2}{2s} \int_{-1}^{+1} (1 + y^2) dy \quad (y = \cos \theta) \\
 &= \frac{4\pi \alpha^2 Q_f^2}{3s}
 \end{aligned}$$

$$\sigma(e^+e^- \rightarrow \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}$$



$\sigma(e^+e^- \rightarrow \mu^+\mu^-)$
for e^+e^- collider data
at centre-of-mass ener-
gies 8-36 GeV

★ This is the complete lowest order calculation of the $e^+e^- \rightarrow \mu^+\mu^-$ cross-section (in the limit of massless fermions).

The Dirac Equation

NON-EXAMINABLE

WEYL Equations describe massless SPIN- $\frac{1}{2}$ particles. But all known fermions are MASSIVE. Again start from the KG equation.

$$\frac{\partial^2 \psi}{\partial t^2} = (\nabla^2 - m^2) \psi$$

$$\hat{H}^2 \psi = (\hat{p}^2 + m^2) \psi$$

Write down equation LINEAR in space and time derivatives

$$\hat{H} \psi = (\vec{\alpha} \cdot \hat{p} + \beta m) \psi$$

and require it to be a solution of the KG equation:

$$\begin{aligned} \hat{H} \psi &= (\alpha_x \cdot \hat{p}_x + \alpha_y \cdot \hat{p}_y + \alpha_z \cdot \hat{p}_z + \beta \cdot m) \psi \\ \hat{H}^2 \psi &= \alpha_i^2 \cdot \hat{p}_x^2 + \dots \\ &\quad + (\alpha_x \alpha_y + \alpha_y \alpha_x) \hat{p}_x \hat{p}_y + \dots \\ &\quad + (\alpha_x \beta + \beta \alpha_x) \hat{p}_x m + \dots \\ &\quad + \beta^2 m^2 \end{aligned}$$

For this to satisfy Klein-Gordon equation:

$$\hat{H}^2 \psi = (\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 + m^2) \psi$$

require

$$\begin{aligned} \alpha_i^2 &= \beta^2 &= 1 & \quad (i = x, y, z) \\ \alpha_i \alpha_j + \alpha_j \alpha_i &= 0 & \quad (i \neq j) \\ \alpha_i \beta + \beta \alpha_i &= 0 \end{aligned}$$

Now require 4 anti-commuting matrices.

The Dirac Equation:

$$\hat{H}\psi = (\vec{\alpha} \cdot \hat{\mathbf{p}} + \beta m)\psi$$

Can be written in a slightly different form

$$i \frac{\partial \psi}{\partial t} = (-i\vec{\alpha} \cdot \nabla + \beta m)\psi$$

$$i\beta \frac{\partial \psi}{\partial t} = (-i\beta\vec{\alpha} \cdot \nabla + m)\psi$$

$$(i\beta \frac{\partial}{\partial t} + i\beta\vec{\alpha} \cdot \nabla - \beta^2 m)\psi = 0$$

$$(i\gamma^0 \frac{\partial}{\partial t} + i\gamma^1 \frac{\partial}{\partial x} + i\gamma^2 \frac{\partial}{\partial y} + i\gamma^3 \frac{\partial}{\partial z} - \beta^2 m)\psi = 0$$

$$\text{with } \gamma^\mu = (\beta, \beta\vec{\alpha})$$

Giving

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

with

$$\begin{aligned} (\gamma^0)^2 &= 1 \\ (\gamma^1)^2 &= (\gamma^2)^2 = (\gamma^3)^2 = -1 \\ (\gamma^i \gamma^j - \gamma^j \gamma^i) &= 0 \quad (i \neq j) \end{aligned}$$

Identify the γ^μ as matrices which must satisfy the anti-commutation relations above. The Pauli spin matrices provide only 3 anti-commuting matrices and the lowest dimension matrices satisfying these requirements are 4×4 . The γ -matrices are closely related to the 2×2 Pauli spin matrices.

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}$$

also define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Solutions to the Dirac Equation are written as four-component Dirac **SPINORS**

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

NOTE: this is not the only possible representation of the γ matrices - just the most commonly used

Rest Frame Solutions of the Dirac Equation

Dirac Equation: $(i\gamma^\mu \partial_\mu - m)\psi = 0$

$$(i\gamma^0 \frac{\partial}{\partial t} + i\gamma^1 \frac{\partial}{\partial x} + i\gamma^2 \frac{\partial}{\partial y} + i\gamma^3 \frac{\partial}{\partial z} - m)\psi = 0$$

Consider a particle at **REST**: $p_x = i \frac{\partial}{\partial x} \psi = 0$, etc.

Dirac Equation becomes:

$$(i\gamma^0 \frac{\partial}{\partial t} - m)\psi = 0$$

$$i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \partial\psi_1/\partial t \\ \partial\psi_2/\partial t \\ \partial\psi_3/\partial t \\ \partial\psi_4/\partial t \end{pmatrix} = m \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

$$i \frac{\partial\psi_1}{\partial t} = m\psi_1, \quad i \frac{\partial\psi_2}{\partial t} = m\psi_2$$

Giving two orthogonal $E = +m$ solutions:

$$u_1(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \quad u_2(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}$$

i.e. positive energy **spin-up** and **spin-down PARTICLES**

The two other equations

$$i \frac{\partial\psi_3}{\partial t} = -m\psi_3, \quad i \frac{\partial\psi_4}{\partial t} = -m\psi_4$$

give two orthogonal $E = -m$ solutions:

$$u_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt}, \quad u_4(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt}$$

i.e. **-ve energy spin-up** and **spin-down ANTI-PARTICLES**

The DIRAC equation

- ★ gives PARTICLE/ANTI-PARTICLE solutions
- ★ requires the particles/anti-particles to have an additional degree of freedom (SPIN) !
- ★ in the massless limit, the DIRAC equation reduces to the two uncoupled WEYL equations
- ★ In general the Dirac Equation gives **FOUR** simultaneous equations for the components of the **SPINOR**.

e.g. more general solutions

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{(E+m)} \\ \frac{(p_x + i p_y)}{(E+m)} \end{pmatrix}, \quad u_2 = N \begin{pmatrix} 0 \\ 1 \\ \frac{(p_x - i p_y)}{(E+m)} \\ \frac{-p_z}{(E+m)} \end{pmatrix}$$

For (u_1, u_2) $E = \sqrt{p^2 + m^2}$

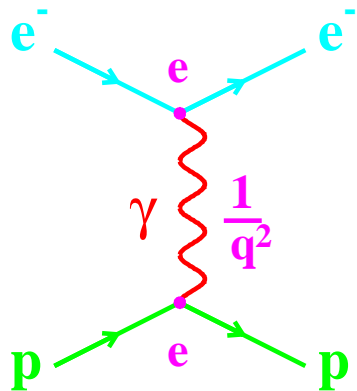
$$u_3 = N \begin{pmatrix} \frac{p_z}{(E-m)} \\ \frac{(p_x + i p_y)}{(E-m)} \\ 1 \\ 0 \end{pmatrix}, \quad u_4 = N \begin{pmatrix} \frac{(p_x - i p_y)}{(E-m)} \\ -\frac{p_z}{(E-m)} \\ 0 \\ 1 \end{pmatrix}$$

For (u_3, u_4) $E = -\sqrt{p^2 + m^2}$

The DIRAC equation lives in the realm of PART III Particle Physics.

Lorentz Structure of Interactions

NON-EXAMINABLE



Electron Current

Propagator

Proton Current

Matrix element M factorises into 3 terms :

$$\begin{aligned}
 -iM &= \langle \bar{u}_e | ie\gamma^\mu | u_e \rangle && \text{Electron Current} \\
 &\times \frac{-ig^{\mu\nu}}{q^2} && \text{Photon Propagator} \\
 &\times \langle \bar{u}_p | ie\gamma^\nu | u_p \rangle && \text{Proton Current}
 \end{aligned}$$

- ★ Fermions are 4-component SPINORS.
- ★ \therefore interaction enters as 4×4 matrices.
- ★ Lorentz invariance allows only five possible forms for the interaction: SCALAR $\bar{u}u$, PSEUDO-SCALAR $\bar{u}\gamma^5 u$, VECTOR $\bar{u}\gamma^\mu u$, AXIAL-VECTOR $\bar{u}\gamma^\mu \gamma^5 u$, TENSOR $\bar{u}\sigma^{\mu\nu} u$
- ★ Electro-magnetic and Strong forces are VECTOR interactions - which determines the HELICITY structure. Treats helicity states symmetrically \Rightarrow PARITY CONSERVATION
- ★ The WEAK interaction has a different form: (V-A) i.e. $\gamma^\mu(1 - \gamma^5)$. Projects out a single helicity combination : \Rightarrow PARITY VIOLATION

The WEAK interaction is the subject of next lecture