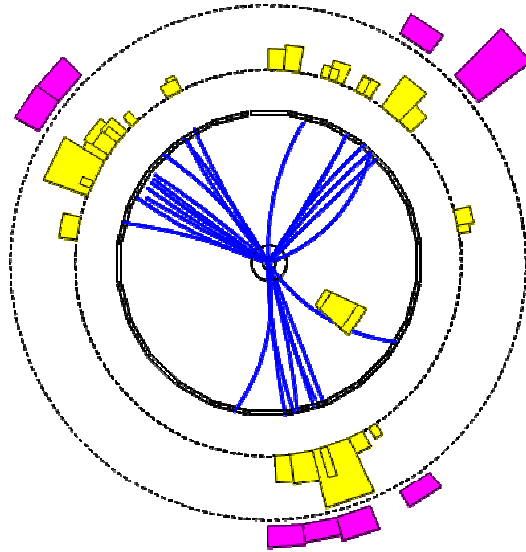


Particle Physics

Michaelmas Term 2009
Prof Mark Thomson



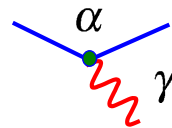
Handout 8 : Quantum Chromodynamics

Colour in QCD

- ★ The theory of the strong interaction, Quantum Chromodynamics (QCD), is very similar to QED but with 3 conserved “colour” charges

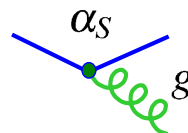
In QED:

- the electron carries one unit of charge $-e$
- the anti-electron carries one unit of anti-charge $+e$
- the force is mediated by a massless “gauge boson” – the photon



In QCD:

- quarks carry colour charge: r, g, b
- anti-quarks carry anti-charge: $\bar{r}, \bar{g}, \bar{b}$
- The force is mediated by massless gluons



- ★ In QCD, the strong interaction is invariant under rotations in colour space
 $r \leftrightarrow b; r \leftrightarrow g; b \leftrightarrow g$
i.e. the same for all three colours



SU(3) colour symmetry

- This is an **exact** symmetry, unlike the approximate uds flavour symmetry discussed previously.

- ★ Represent r, g, b **SU(3) colour states by:**

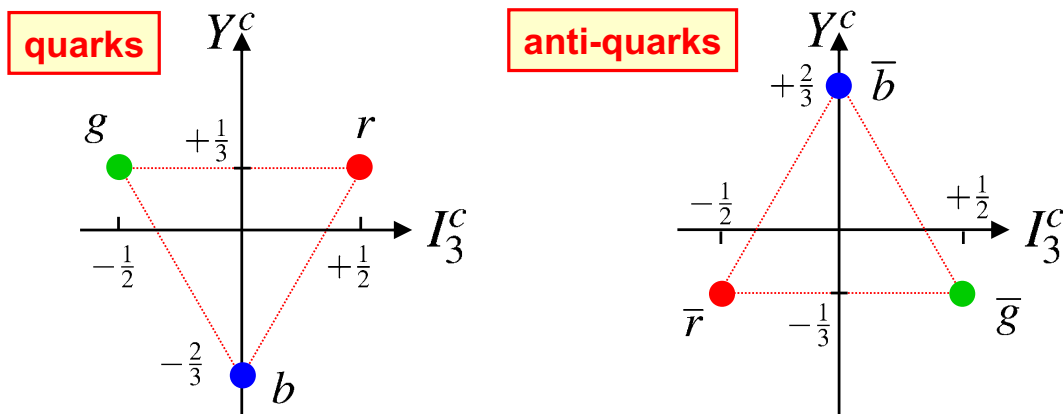
$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- ★ Colour states can be labelled by two quantum numbers:

- ♦ I_3^c colour isospin
- ♦ Y^c colour hypercharge

Exactly analogous to labelling u,d,s flavour states by I_3 and Y

- ★ Each quark (anti-quark) can have the following colour quantum numbers:



Colour Confinement

- ★ It is believed (although not yet proven) that all observed free particles are “colourless”
- i.e. never observe a free quark (which would carry colour charge)
 - consequently quarks are always found in bound states colourless hadrons

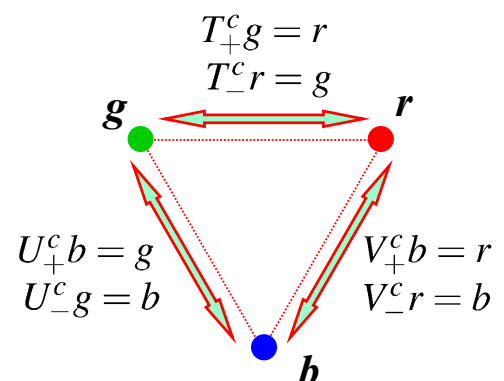
- ★ Colour Confinement Hypothesis:

only colour singlet states can exist as free particles

- ★ All hadrons must be “colourless” i.e. colour **singlets**
- ★ To construct colour wave-functions for hadrons can apply results for **SU(3) flavour** symmetry to **SU(3) colour** with replacement

$$\begin{array}{l} u \rightarrow r \\ d \rightarrow g \\ s \rightarrow b \end{array}$$

- ★ just as for uds flavour symmetry can define colour ladder operators



Colour Singlets

★ It is important to understand what is meant by a **singlet state**

★ Consider spin states obtained from two spin 1/2 particles.

• Four spin combinations: $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$

• Gives four eigenstates of \hat{S}^2, \hat{S}_z $(2 \otimes 2 = 3 \oplus 1)$

$$|1, +1\rangle = \uparrow\uparrow$$

$$|1, 0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$$

$$|1, -1\rangle = \downarrow\downarrow$$

**spin-1
triplet**

$$\oplus |0, 0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

**spin-0
singlet**

★ The singlet state is “spinless”: it has zero angular momentum, is invariant under SU(2) spin transformations and spin ladder operators yield zero

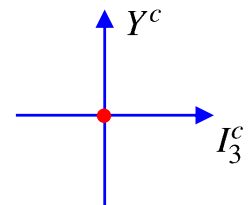
$$S_{\pm}|0, 0\rangle = 0$$

★ In the same way **COLOUR SINGLETS** are “colourless” combinations:

♦ they have zero colour quantum numbers $I_3^c = 0, Y^c = 0$

♦ invariant under SU(3) colour transformations

♦ ladder operators $T_{\pm}, U_{\pm}, V_{\pm}$ all yield zero

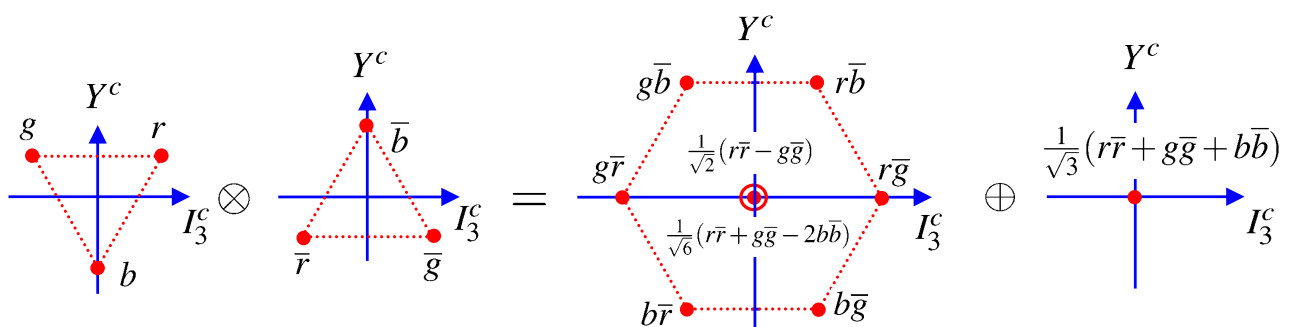


★ NOT sufficient to have $I_3^c = 0, Y^c = 0$: does not mean that state is a singlet

Meson Colour Wave-function

★ Consider colour wave-functions for $q\bar{q}$

★ The combination of colour with anti-colour is mathematically identical to construction of meson wave-functions with uds flavour symmetry



Coloured octet and a colourless singlet

• Colour confinement implies that hadrons only exist in colour singlet states so the colour wave-function for mesons is:

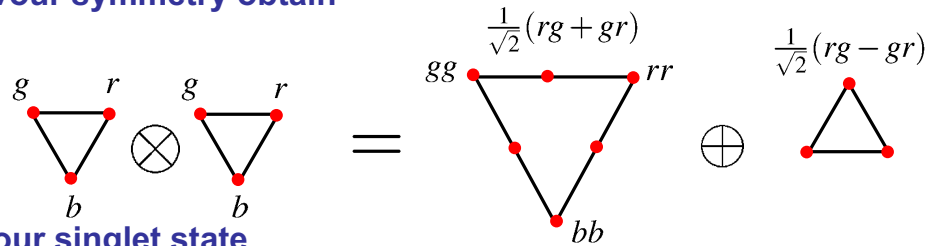
$$\psi_c^{q\bar{q}} = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

★ Can we have a $qq\bar{q}$ state ? i.e. by adding a quark to the above octet can we form a state with $Y^c = 0; I_3^c = 0$. The answer is clear no.

→ $qq\bar{q}$ bound states do not exist in nature.

Baryon Colour Wave-function

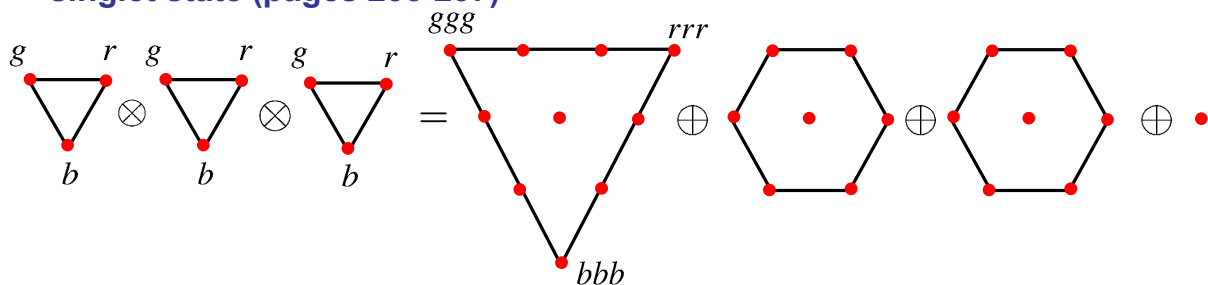
- ★ Do **qq** bound states exist ? This is equivalent to asking whether it possible to form a colour singlet from two colour triplets ?
- Following the discussion of construction of baryon wave-functions in SU(3) flavour symmetry obtain



- No **qq** colour singlet state
- Colour confinement → bound states of **qq** **do not exist**



BUT combination of three quarks (three colour triplets) gives a colour singlet state (pages 235-237)



- ★ The singlet colour wave-function is:

$$\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr)$$

Check this is a colour singlet...

- It has $I_3^c = 0$, $Y^c = 0$: a necessary but not sufficient condition
- Apply ladder operators, e.g. T_+ (recall $T_+g = r$)

$$T_+ \psi_c^{qqq} = \frac{1}{\sqrt{6}}(rrb - rbr + rbr - rrb + brr - brr) = 0$$

- Similarly $T_- \psi_c^{qqq} = 0$; $V_\pm \psi_c^{qqq} = 0$; $U_\pm \psi_c^{qqq} = 0$;



Colourless singlet - therefore qq̄ bound states exist !



Anti-symmetric colour wave-function

Allowed Hadrons i.e. the possible colour singlet states

• $q\bar{q}$, qqq

Mesons and Baryons

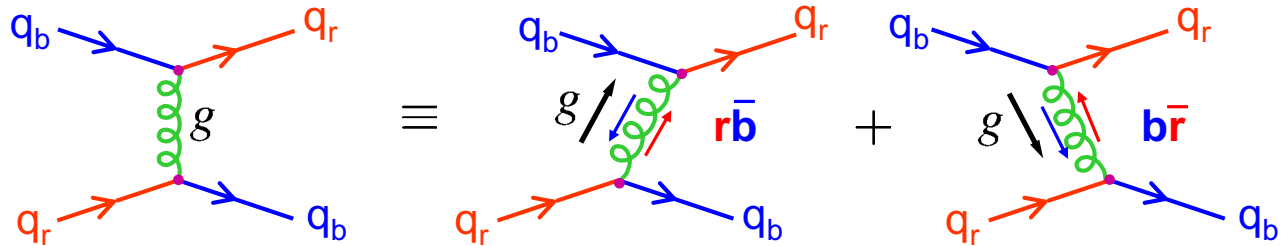
• $q\bar{q}q\bar{q}$, $qqqq\bar{q}$

Exotic states, e.g. pentaquarks

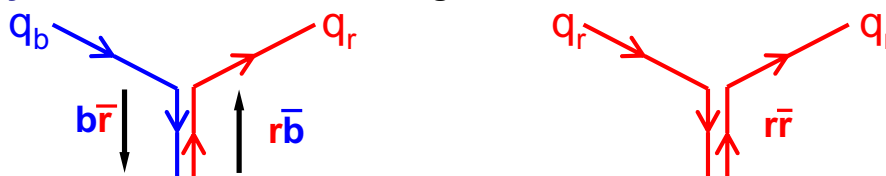
To date all confirmed hadrons are either mesons or baryons. However, some recent (but not entirely convincing) “evidence” for pentaquark states

Gluons

- ★ In QCD quarks interact by exchanging virtual massless gluons, e.g.

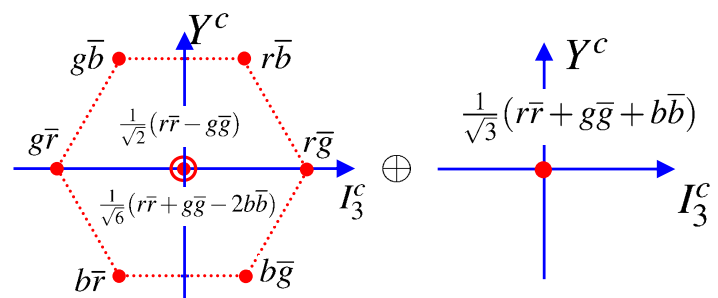


- ★ Gluons carry **colour** and **anti-colour**, e.g.



- ★ Gluon colour wave-functions (colour + anti-colour) are the same as those obtained for mesons (also colour + anti-colour)

⇒ **OCTET + "COLOURLESS" SINGLET**



- ★ So we might expect 9 physical gluons:

OCTET: $r\bar{g}, r\bar{b}, g\bar{r}, g\bar{b}, b\bar{r}, b\bar{g}, \frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g}), \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$
SINGLET: $\frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$

- ★ **BUT**, colour confinement hypothesis:

only colour singlet states
can exist as free particles



Colour singlet gluon would be unconfined.
It would behave like a strongly interacting
photon → infinite range Strong force.

- ★ Empirically, the strong force is short range and therefore know that the physical gluons are confined. The colour singlet state does not exist in nature !

NOTE: this is not entirely ad hoc. In the context of gauge field theory (see minor option) the strong interaction arises from a fundamental **SU(3)** symmetry. The gluons arise from the generators of the symmetry group (the Gell-Mann λ matrices). There are 8 such matrices → 8 gluons. Had nature “chosen” a **U(3)** symmetry, would have 9 gluons, the additional gluon would be the colour singlet state and QCD would be an unconfined long-range force.

NOTE: the “gauge symmetry” determines the exact nature of the interaction
 → **FEYNMAN RULES**

Gluon-Gluon Interactions

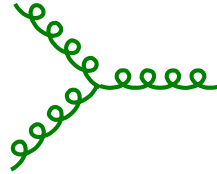
- ★ In QED the **photon** does not carry the charge of the EM interaction (photons are electrically neutral)
- ★ In contrast, in QCD the **gluons** do carry **colour charge**



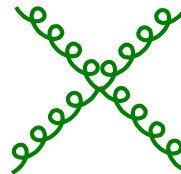
Gluon Self-Interactions

- ★ Two new vertices (no QED analogues)

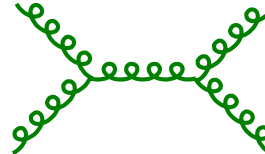
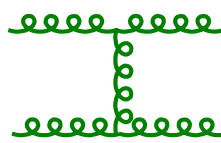
triple-gluon vertex



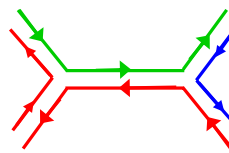
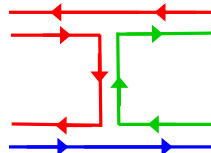
quartic-gluon vertex



- ★ In addition to quark-quark scattering, therefore can have gluon-gluon scattering

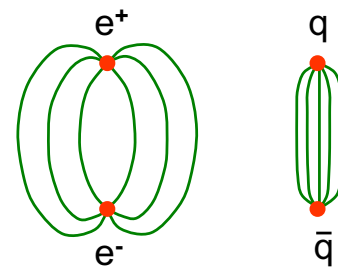


e.g. possible way of arranging the colour flow

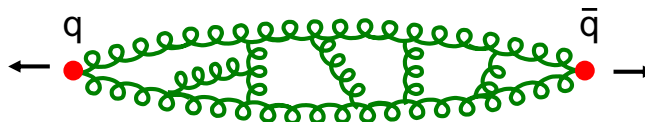


Gluon self-Interactions and Confinement

- ★ Gluon self-interactions are believed to give rise to colour confinement
- ★ Qualitative picture:
 - Compare QED with QCD
 - In QCD “gluon self-interactions squeeze lines of force into a flux tube”



- ★ What happens when try to separate two coloured objects e.g. $q\bar{q}$



- Form a flux tube of interacting gluons of approximately constant energy density $\sim 1 \text{ GeV/fm}$

→ $V(r) \sim \lambda r$

- Require infinite energy to separate coloured objects to infinity
- Coloured quarks and gluons are always **confined** within colourless states
- In this way QCD provides a plausible explanation of confinement – but **not yet proven** (although there has been recent progress with Lattice QCD)

Hadronisation and Jets

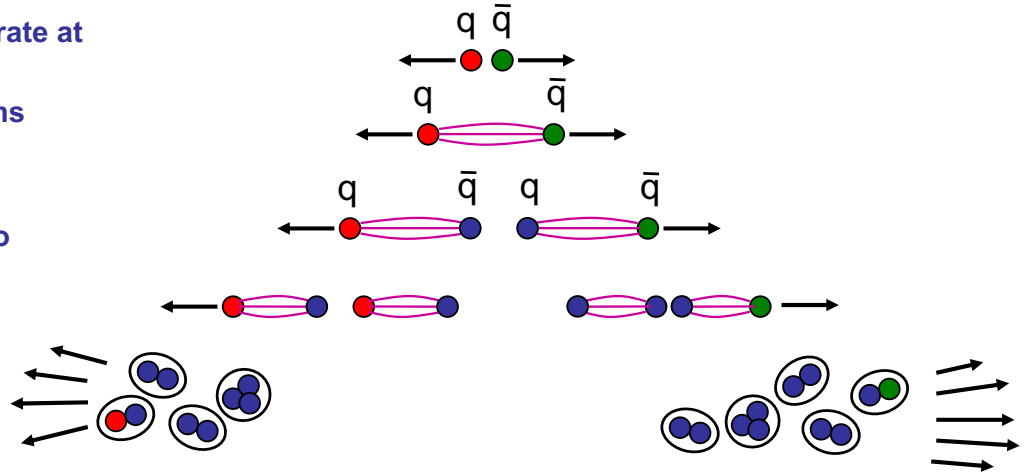
★ Consider a quark and anti-quark produced in electron positron annihilation

i) Initially Quarks separate at high velocity

ii) Colour flux tube forms between quarks

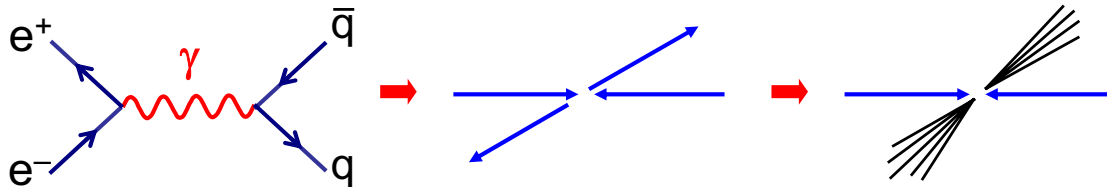
iii) Energy stored in the flux tube sufficient to produce $q\bar{q}$ pairs

iv) Process continues until quarks pair up into jets of colourless hadrons



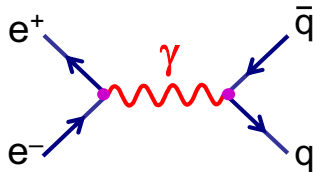
★ This process is called **hadronisation**. It is not (yet) calculable.

★ The main consequence is that at collider experiments quarks and gluons observed as jets of particles



QCD and Colour in e^+e^- Collisions

★ e^+e^- colliders are an excellent place to study QCD



★ Well defined production of quarks

- QED process well-understood
- no need to know parton structure functions
- + experimentally very clean – no proton remnants

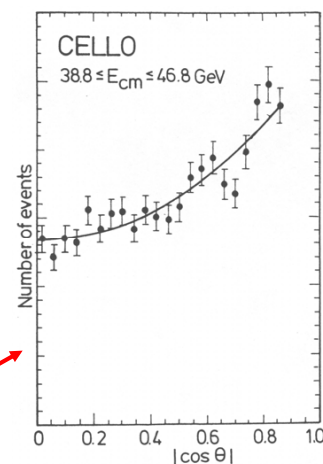
★ In handout 5 obtained expressions for the $e^+e^- \rightarrow \mu^+\mu^-$ cross-section

$$\sigma = \frac{4\pi\alpha^2}{3s} \quad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} (1 + \cos^2 \theta)$$

- In e^+e^- collisions produce all quark flavours for which $\sqrt{s} > 2m_q$
- In general, i.e. unless producing a $q\bar{q}$ bound state, produce jets of hadrons
- Usually can't tell which jet came from the quark and came from anti-quark

★ Angular distribution of jets $\propto (1 + \cos^2 \theta)$

→ Quarks are spin $\frac{1}{2}$



H.J. Behrend et al., Phys Lett 183B (1987) 400

- ★ Colour is conserved and quarks are produced as $r\bar{r}$, $g\bar{g}$, $b\bar{b}$
- ★ For a **single quark flavour and single colour**

$$\sigma(e^+e^- \rightarrow q_i\bar{q}_i) = \frac{4\pi\alpha^2}{3s} Q_q^2$$

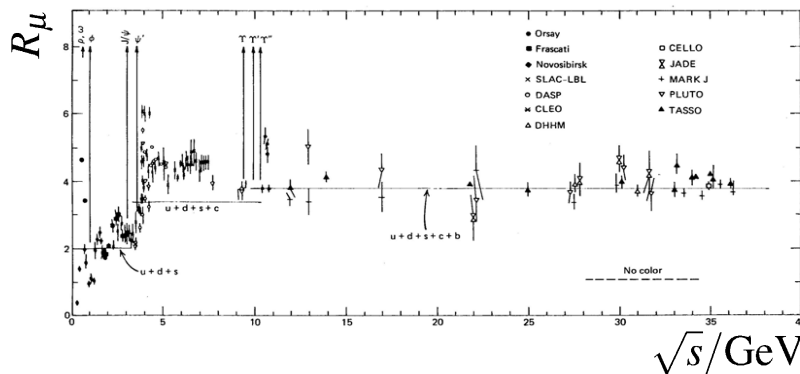
- Experimentally observe jets of hadrons:

$$\sigma(e^+e^- \rightarrow \text{hadrons}) = 3 \sum_{u,d,s,\dots} \frac{4\pi\alpha^2}{3s} Q_q^2$$

Factor 3 comes from colours

- Usual to express as ratio compared to $\sigma(e^+e^- \rightarrow \mu^+\mu^-)$

$$R_\mu = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = 3 \sum_{u,d,s,\dots} Q_q^2$$



u,d,s: $R_\mu = 3 \times \left(\frac{1}{9} + \frac{4}{9} + \frac{1}{9}\right) = 2$

u,d,s,c: $R_\mu = \frac{10}{3}$

u,d,s,c,b: $R_\mu = \frac{11}{3}$

- ★ Data consistent with expectation with factor 3 from colour

Jet production in e^+e^- Collisions

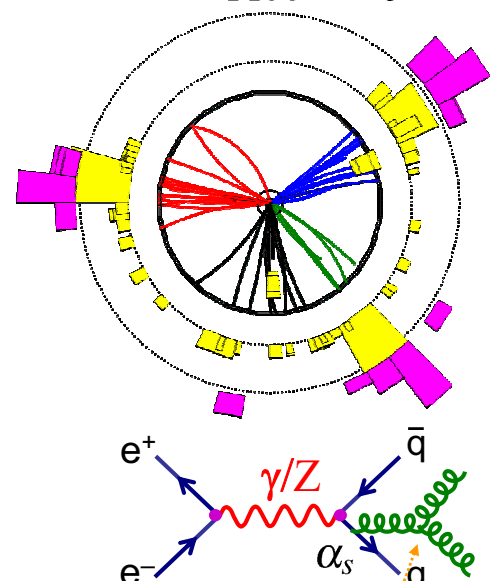
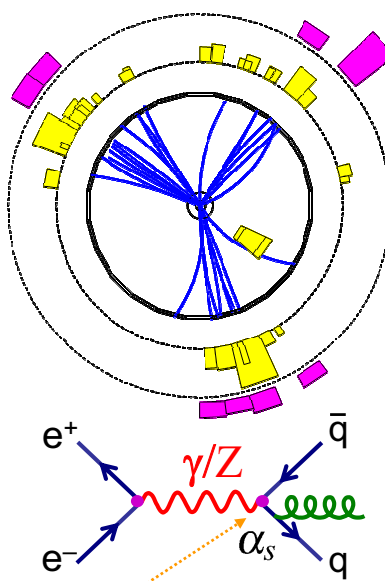
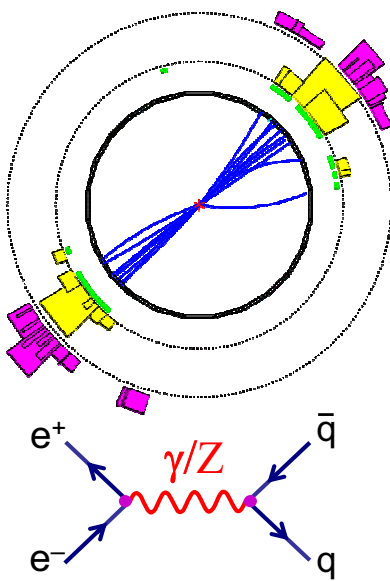
- ★ e^+e^- colliders are also a good place to study gluons

$$e^+e^- \rightarrow q\bar{q} \rightarrow 2\text{jets}$$

$$e^+e^- \rightarrow q\bar{q}g \rightarrow 3\text{jets}$$

$$e^+e^- \rightarrow q\bar{q}gg \rightarrow 4\text{jets}$$

OPAL at LEP (1989-2000)



Experimentally:

- Three jet rate \rightarrow measurement of α_s
- Angular distributions \rightarrow gluons are spin-1
- Four-jet rate and distributions \rightarrow QCD has an underlying SU(3) symmetry

The Quark – Gluon Interaction

- Representing the colour part of the fermion wave-functions by:

$$r = c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad g = c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad b = c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Particle wave-functions $u(p) \longrightarrow c_i u(p)$

- The QCD qqg vertex is written:

$$\bar{u}(p_3) c_j^\dagger \left\{ -\frac{1}{2} i g_s \lambda^a \gamma^\mu \right\} c_i u(p_1)$$

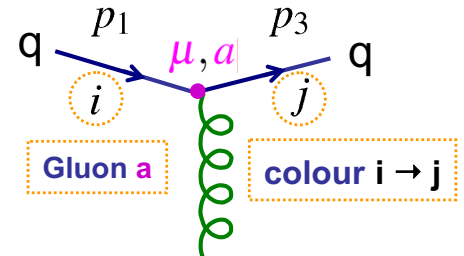
- Only difference w.r.t. QED is the insertion of the 3x3 SU(3) Gell-Mann matrices (justified in handout 13).

- Isolating the colour part:

$$c_j^\dagger \lambda^a c_i = c_j^\dagger \begin{pmatrix} \lambda_{1i}^a \\ \lambda_{2i}^a \\ \lambda_{3i}^a \end{pmatrix} = \lambda_{ji}^a$$

- Hence the fundamental quark - gluon QCD interaction can be written

$$\bar{u}(p_3) c_j^\dagger \left\{ -\frac{1}{2} i g_s \lambda^a \gamma^\mu \right\} c_i u(p_1) \equiv \bar{u}(p_3) \left\{ -\frac{1}{2} i g_s \lambda_{ji}^a \gamma^\mu \right\} u(p_1)$$



Feynman Rules for QCD

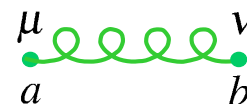
External Lines

spin 1/2	incoming quark	$u(p)$	
	outgoing quark	$\bar{u}(p)$	
	incoming anti-quark	$\bar{v}(p)$	
	outgoing anti-quark	$v(p)$	
spin 1	incoming gluon	$\epsilon^\mu(p)$	
	outgoing gluon	$\epsilon^\mu(p)^*$	

Internal Lines (propagators)

spin 1 gluon

$$\frac{-i g_{\mu\nu}}{q^2} \delta^{ab}$$

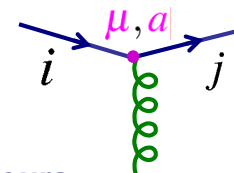


$a, b = 1, 2, \dots, 8$ are gluon colour indices

Vertex Factors

spin 1/2 quark

$$-i g_s \frac{1}{2} \lambda_{ji}^a \gamma^\mu$$



$i, j = 1, 2, 3$ are quark colours,

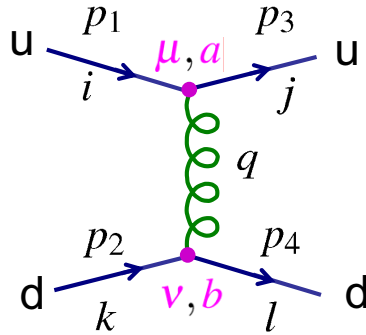
λ^a $a = 1, 2, \dots, 8$ are the Gell-Mann SU(3) matrices

- + 3 gluon and 4 gluon interaction vertices

- Matrix Element $-iM =$ product of all factors

Matrix Element for quark-quark scattering

★ Consider QCD scattering of an up and a down quark



- The incoming and out-going quark colours are labelled by $i, j, k, l = \{1, 2, 3\}$ (or $\{r, g, b\}$)
- In terms of colour this scattering is $ik \rightarrow jl$
- The 8 different gluons are accounted for by the colour indices $a, b = 1, 2, \dots, 8$
- NOTE: the δ -function in the propagator ensures $a = b$, i.e. the gluon “emitted” at a is the same as that “absorbed” at b

★ Applying the Feynman rules:

$$-iM = [\bar{u}_u(p_3) \{ -\frac{1}{2} i g_s \lambda_{ji}^a \gamma^\mu \} u_u(p_1)] \frac{-i g_{\mu\nu}}{q^2} \delta^{ab} [\bar{u}_d(p_4) \{ -\frac{1}{2} i g_s \lambda_{lk}^b \gamma^\nu \} u_d(p_2)]$$

where summation over a and b (and μ and ν) is implied.

★ Summing over a and b using the δ -function gives:

$$M = -\frac{g_s^2}{4} \lambda_{ji}^a \lambda_{lk}^a \frac{1}{q^2} g_{\mu\nu} [\bar{u}_u(p_3) \gamma^\mu u_u(p_1)] [\bar{u}_d(p_4) \gamma^\nu u_d(p_2)]$$

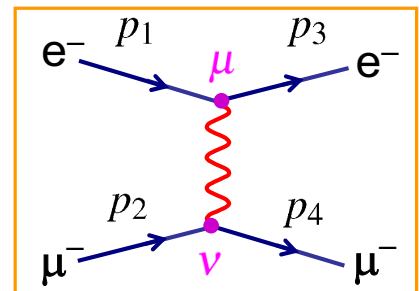
Sum over all 8 gluons (repeated indices)

QCD vs QED

QED

$$-iM = [\bar{u}(p_3) i e \gamma^\mu u(p_1)] \frac{-i g_{\mu\nu}}{q^2} [\bar{u}(p_4) i e \gamma^\nu u(p_2)]$$

$$M = -e^2 \frac{1}{q^2} g_{\mu\nu} [\bar{u}(p_3) \gamma^\mu u(p_1)] [\bar{u}(p_4) \gamma^\nu u(p_2)]$$

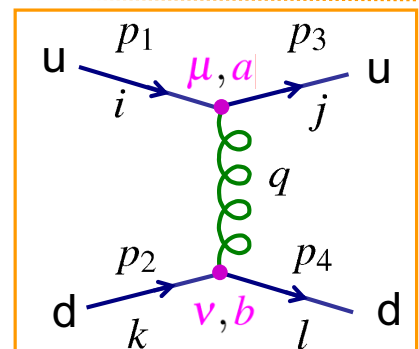


QCD

$$M = -\frac{g_s^2}{4} \lambda_{ji}^a \lambda_{lk}^a \frac{1}{q^2} g_{\mu\nu} [\bar{u}_u(p_3) \gamma^\mu u_u(p_1)] [\bar{u}_d(p_4) \gamma^\nu u_d(p_2)]$$

★ QCD Matrix Element = QED Matrix Element with:

$$e^2 \rightarrow g_s^2 \quad \text{or equivalently} \quad \alpha = \frac{e^2}{4\pi} \rightarrow \alpha_s = \frac{g_s^2}{4\pi}$$



+ QCD Matrix Element includes an additional “colour factor”

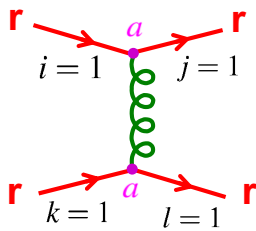
$$C(ik \rightarrow jl) \equiv \frac{1}{4} \sum_{a=1}^8 \lambda_{ji}^a \lambda_{lk}^a$$

Evaluation of QCD Colour Factors

- QCD colour factors reflect the gluon states that are involved

$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
Gluons: $r\bar{g}, g\bar{r}$		$r\bar{b}, b\bar{r}$	$\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g}) \quad \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$

① Configurations involving a single colour



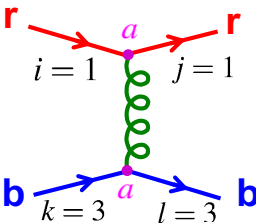
- Only matrices with non-zero entries in 11 position are involved

$$\begin{aligned}
 C(rr \rightarrow rr) &= \frac{1}{4} \sum_{a=1}^8 \lambda_{11}^a \lambda_{11}^a = \frac{1}{4} (\lambda_{11}^3 \lambda_{11}^3 + \lambda_{11}^8 \lambda_{11}^8) \\
 &= \frac{1}{4} \left(1 + \frac{1}{3} \right) = \frac{1}{3}
 \end{aligned}$$

Similarly find

$$C(rr \rightarrow rr) = C(gg \rightarrow gg) = C(bb \rightarrow bb) = \frac{1}{3}$$

② Other configurations where quarks don't change colour e.g. $rb \rightarrow rb$

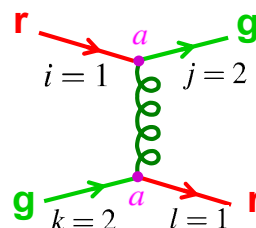


- Only matrices with non-zero entries in 11 and 33 position are involved

$$\begin{aligned}
 C(rb \rightarrow rb) &= \frac{1}{4} \sum_{a=1}^8 \lambda_{11}^a \lambda_{33}^a = \frac{1}{4} (\lambda_{11}^8 \lambda_{33}^8) \\
 &= \frac{1}{4} \left(\frac{1}{\sqrt{3}} \cdot \frac{-2}{\sqrt{3}} \right) = -\frac{1}{6}
 \end{aligned}$$

Similarly $C(rb \rightarrow rb) = C(rg \rightarrow rg) = C(gr \rightarrow gr) = C(gb \rightarrow gb) = C(br \rightarrow br) = C(bg \rightarrow bg) = -\frac{1}{6}$

③ Configurations where quarks swap colours e.g. $rg \rightarrow gr$



- Only matrices with non-zero entries in 12 and 21 position are involved

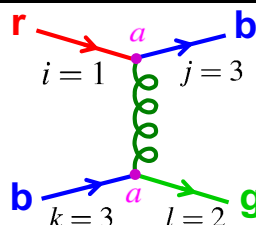
Gluons $r\bar{g}, g\bar{r}$

$$\begin{aligned}
 C(rg \rightarrow gr) &= \frac{1}{4} \sum_{a=1}^8 \lambda_{21}^a \lambda_{12}^a = \frac{1}{4} (\lambda_{21}^1 \lambda_{12}^1 + \lambda_{21}^2 \lambda_{12}^2) \\
 &= \frac{1}{4} (i(-i) + 1) = \frac{1}{2}
 \end{aligned}$$

$$\hat{T}_+^{(ij)} \hat{T}_-^{(kl)}$$

$$C(rb \rightarrow br) = C(rg \rightarrow gr) = C(gr \rightarrow rg) = C(gb \rightarrow bg) = C(br \rightarrow rb) = C(bg \rightarrow gb) = \frac{1}{2}$$

④ Configurations involving 3 colours e.g. $rb \rightarrow bg$



- Only matrices with non-zero entries in the 13 and 32 position
- But none of the λ matrices have non-zero entries in the 13 and 32 positions. Hence the colour factor is zero

★ colour is conserved

Colour Factors : Quarks vs Anti-Quarks

- Recall the colour part of wave-function:

$$r = c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad g = c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad b = c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

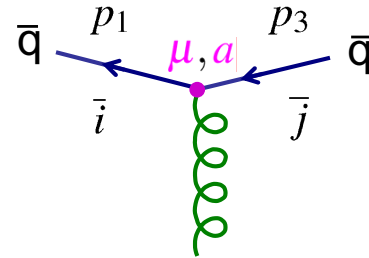
- The QCD qqg vertex was written:

$$\bar{u}(p_3)c_j^\dagger \left\{ -\frac{1}{2}ig_s\lambda^a\gamma^\mu \right\} c_i u(p_1)$$

- ★ Now consider the anti-quark vertex

- The QCD $\bar{q}\bar{q}g$ vertex is:

$$\bar{v}(p_1)c_i^\dagger \left\{ -\frac{1}{2}ig_s\lambda^a\gamma^\mu \right\} c_j v(p_3)$$



Note that the **incoming** anti-particle now enters on the LHS of the expression

- For which the colour part is

$$c_i^\dagger \lambda^a c_j = c_i^\dagger \begin{pmatrix} \lambda_{1j}^a \\ \lambda_{2j}^a \\ \lambda_{3j}^a \end{pmatrix} = \lambda_{ij}^a$$

i.e indices ***ij*** are swapped with respect to the quark case

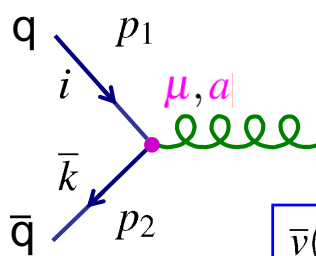
- Hence

$$\bar{v}(p_1)c_i^\dagger \left\{ -\frac{1}{2}ig_s\lambda^a\gamma^\mu \right\} c_j v(p_3) \equiv \bar{v}(p_1) \left\{ -\frac{1}{2}ig_s\lambda_{ij}^a\gamma^\mu \right\} v(p_3)$$

- c.f. the quark - gluon QCD interaction

$$\bar{u}(p_3)c_j^\dagger \left\{ -\frac{1}{2}ig_s\lambda^a\gamma^\mu \right\} c_i u(p_1) \equiv \bar{u}(p_3) \left\{ -\frac{1}{2}ig_s\lambda_{ji}^a\gamma^\mu \right\} u(p_1)$$

- ★ Finally we can consider the quark – anti-quark annihilation



QCD vertex:

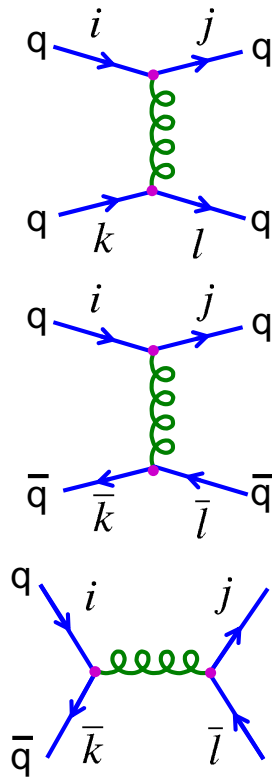
$$\bar{v}(p_2)c_k^\dagger \left\{ -\frac{1}{2}ig_s\lambda^a\gamma^\mu \right\} c_i u(p_1)$$

with

$$c_k^\dagger \lambda^a c_i = \lambda_{ki}^a$$

$$\bar{v}(p_2)c_k^\dagger \left\{ -\frac{1}{2}ig_s\lambda^a\gamma^\mu \right\} c_i u(p_1) \equiv \bar{v}(p_2) \left\{ -\frac{1}{2}ig_s\lambda_{ki}^a\gamma^\mu \right\} u(p_1)$$

- Consequently the colour factors for the different diagrams are:



$$C(ik \rightarrow jl) \equiv \frac{1}{4} \sum_{a=1}^8 \lambda_{ji}^a \lambda_{lk}^a$$

$$C(i\bar{k} \rightarrow j\bar{l}) \equiv \frac{1}{4} \sum_{a=1}^8 \lambda_{ji}^a \lambda_{kl}^a$$

$$C(i\bar{k} \rightarrow j\bar{l}) \equiv \frac{1}{4} \sum_{a=1}^8 \lambda_{ki}^a \lambda_{jl}^a$$

e.g.

$$\begin{aligned} C(rr \rightarrow rr) &= \frac{1}{3} \\ C(rg \rightarrow rg) &= -\frac{1}{6} \\ C(rg \rightarrow gr) &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} C(r\bar{r} \rightarrow r\bar{r}) &= \frac{1}{3} \\ C(r\bar{g} \rightarrow r\bar{g}) &= -\frac{1}{6} \\ C(r\bar{r} \rightarrow g\bar{g}) &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} C(r\bar{r} \rightarrow r\bar{r}) &= \frac{1}{3} \\ C(r\bar{g} \rightarrow r\bar{g}) &= \frac{1}{2} \\ C(r\bar{r} \rightarrow g\bar{g}) &= -\frac{1}{6} \end{aligned}$$

Colour index of adjoint spinor comes first

Quark-Quark Scattering

- Consider the process $u + d \rightarrow u + d$ which can occur in the high energy proton-proton scattering
- There are nine possible colour configurations of the colliding quarks which are all equally likely.
- Need to determine the average matrix element which is the sum over all possible colours divided by the number of possible initial colour states

$$\langle |M_{fi}|^2 \rangle = \frac{1}{3} \cdot \frac{1}{3} \sum_{i,j,k,l=1}^3 |M_{fi}(ij \rightarrow kl)|^2$$

- The colour average matrix element contains the average colour factor

$$\langle |C|^2 \rangle = \frac{1}{9} \sum_{i,j,k,l=1}^3 |C(ij \rightarrow kl)|^2$$

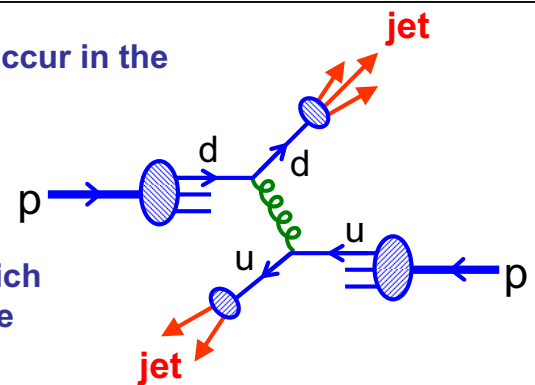
- For $qq \rightarrow qq$

$rr \rightarrow rr, \dots$

$rb \rightarrow rb, \dots$

$rb \rightarrow br, \dots$

$$\langle |C|^2 \rangle = \frac{1}{9} \left[3 \times \left(\frac{1}{3} \right)^2 + 6 \times \left(-\frac{1}{6} \right)^2 + 6 \times \left(\frac{1}{2} \right)^2 \right] = \frac{2}{9}$$



- Previously derived the Lorentz Invariant cross section for $e^- \mu^- \rightarrow e^- \mu^-$ elastic scattering in the ultra-relativistic limit (handout 6).

QED

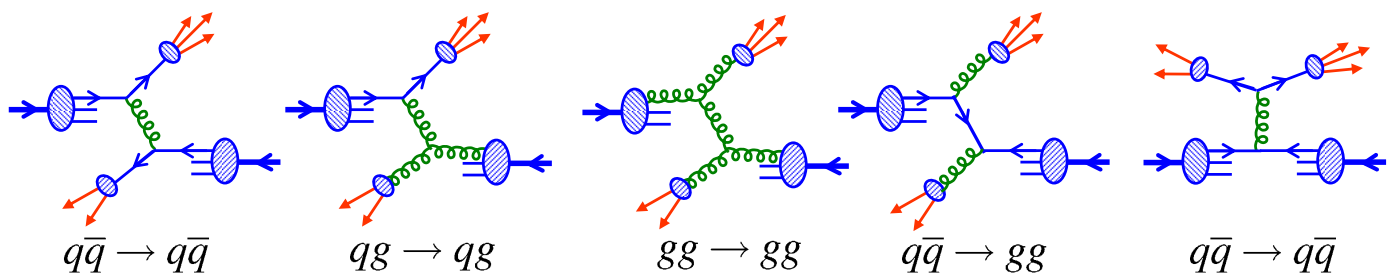
$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2}{q^4} \left[1 + \left(1 + \frac{q^2}{s} \right)^2 \right]$$

- For $ud \rightarrow ud$ in QCD replace $\alpha \rightarrow \alpha_s$ and multiply by $\langle |C|^2 \rangle$

QCD

$$\frac{d\sigma}{dq^2} = \frac{2}{9} \frac{2\pi\alpha_s^2}{q^4} \left[1 + \left(1 + \frac{q^2}{\hat{s}} \right)^2 \right]$$

- Here \hat{s} is the centre-of-mass energy of the quark-quark collision
- The calculation of hadron-hadron scattering is very involved, need to include parton structure functions and include all possible interactions
e.g. two jet production in **proton-antiproton** collisions



Prof. M.A. Thomson

Michaelmas 2009

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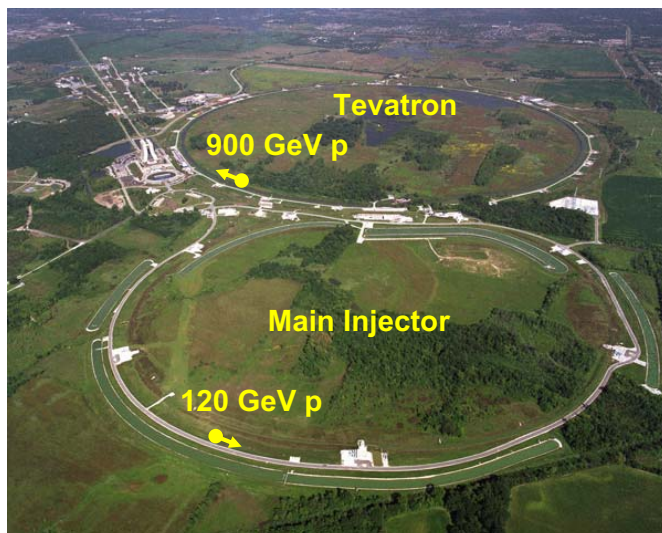
e.g. $p\bar{p}$ collisions at the Tevatron

★ Tevatron collider at Fermi National Laboratory (FNAL)

- located ~40 miles from Chicago, US
- started operation in 1987 (will run until 2009/2010)

★ $p\bar{p}$ collisions at $\sqrt{s} = 1.8$ TeV

c.f. 14 TeV at the LHC



Two main accelerators:

★ Main Injector

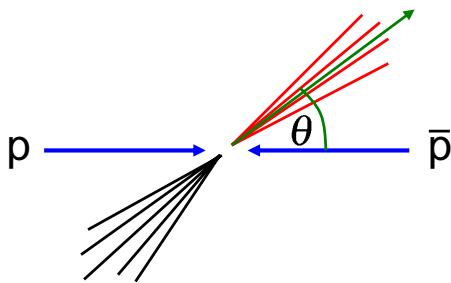
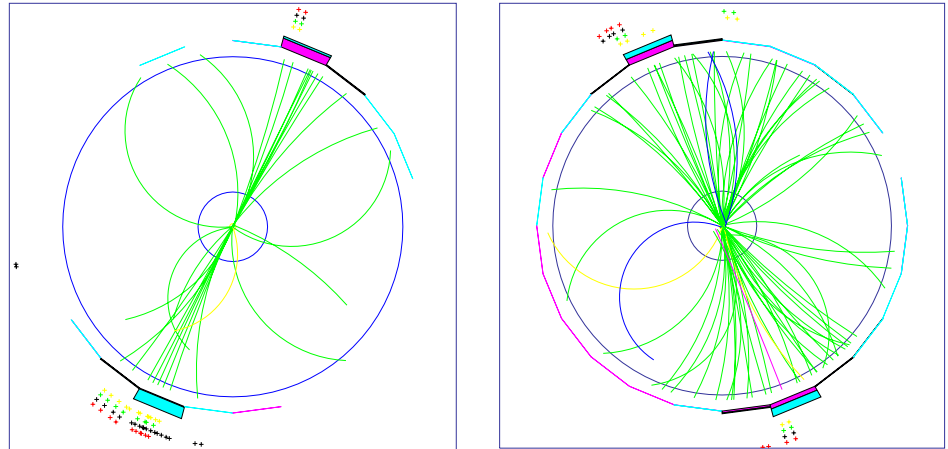
- Accelerates 8 GeV p to 120 GeV
- also \bar{p} to 120 GeV
- Protons sent to **Tevatron & MINOS**
- \bar{p} all go to **Tevatron**

★ Tevatron

- 4 mile circumference
- accelerates p/\bar{p} from 120 GeV to 900 GeV

★ Test QCD predictions by looking at production of pairs of high energy jets

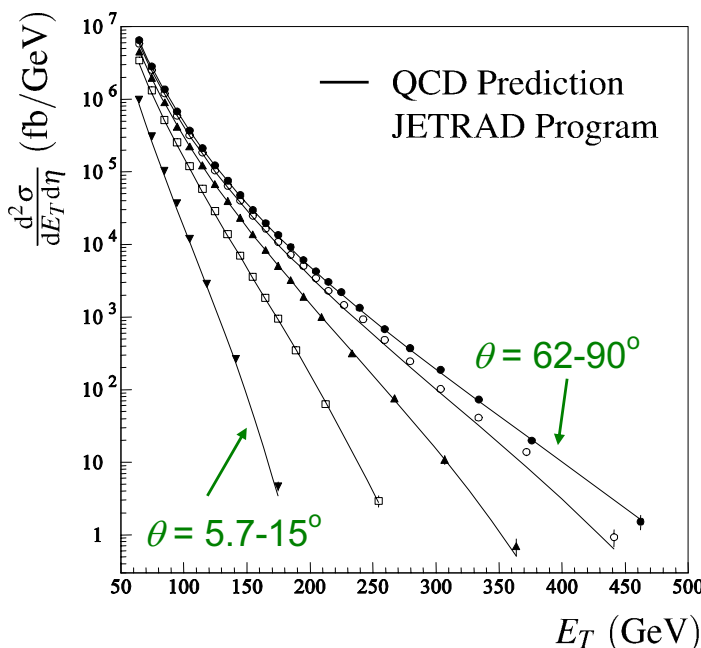
$p\bar{p} \rightarrow \text{jet jet} + X$



★ Measure cross-section in terms of

- “transverse energy” $E_T = E_{\text{jet}} \sin \theta$
- “pseudorapidity” $\eta = \ln \left[\cot \left(\frac{\theta}{2} \right) \right]$

...don't worry too much about the details here, what matters is that...



D0 Collaboration, Phys. Rev. Lett. 86 (2001)

★ QCD predictions provide an excellent description of the data

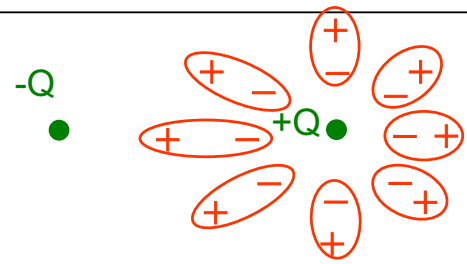
★ NOTE:

- at low E_T cross-section is dominated by low x partons i.e. gluon-gluon scattering
- at high E_T cross-section is dominated by high x partons i.e. quark-antiquark scattering

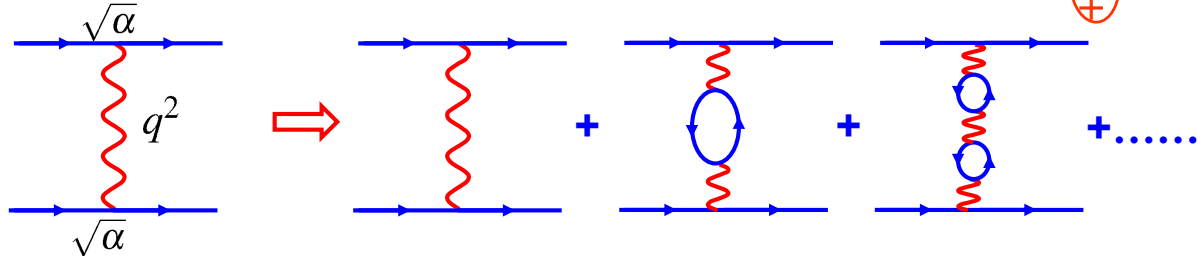
Running Coupling Constants

QED

- “bare” charge of electron screened by virtual e^+e^- pairs
- behaves like a polarizable dielectric



- ★ In terms of Feynman diagrams:



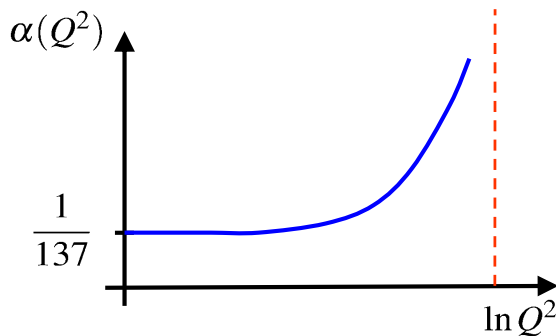
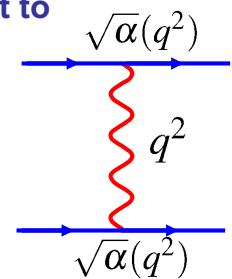
- ★ Some final state so add matrix element **amplitudes**: $M = M_1 + M_2 + M_3 + \dots$

- ★ Giving an infinite series which can be summed and is equivalent to a single diagram with “running” coupling constant

$$\alpha(Q^2) = \alpha(Q_0^2) / \left[1 - \frac{\alpha(Q_0^2)}{3\pi} \ln \left(\frac{Q^2}{Q_0^2} \right) \right]$$

$Q^2 \gg Q_0^2$

Note sign



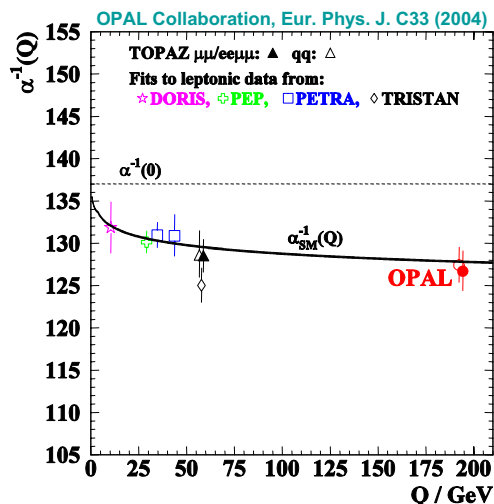
- ★ Might worry that coupling becomes infinite at

$$\ln \left(\frac{Q^2}{Q_0^2} \right) = \frac{3\pi}{1/137}$$

i.e. at

$$Q \sim 10^{26} \text{ GeV}$$

- But quantum gravity effects would come in way below this energy and it is highly unlikely that QED “as is” would be valid in this regime



- ★ In QED, running coupling **increases** very slowly

- Atomic physics: $Q^2 \sim 0$

$$1/\alpha = 137.03599976(50)$$

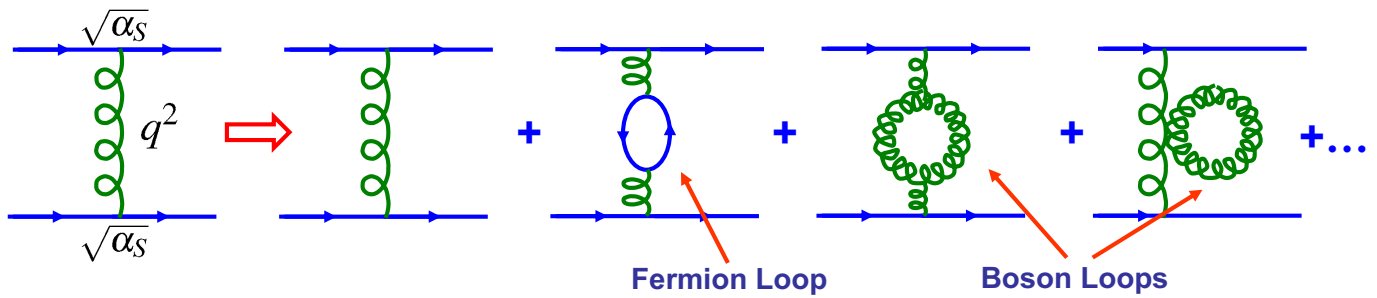
- High energy physics:

$$1/\alpha(193 \text{ GeV}) = 127.4 \pm 2.1$$

Running of α_s

QCD

Similar to QED but also have gluon loops



★ Remembering adding amplitudes, so can get negative interference and the sum can be smaller than the original diagram alone

★ Bosonic loops “interfere negatively”

$$\alpha_s(Q^2) = \alpha_s(Q_0^2) \left/ \left[1 + B\alpha_s(Q_0^2) \ln \left(\frac{Q^2}{Q_0^2} \right) \right] \right.$$

with $B = \frac{11N_c - 2N_f}{12\pi}$ $\left\{ \begin{array}{l} N_c = \text{no. of colours} \\ N_f = \text{no. of quark flavours} \end{array} \right.$

$N_c = 3; N_f = 6 \rightarrow B > 0$

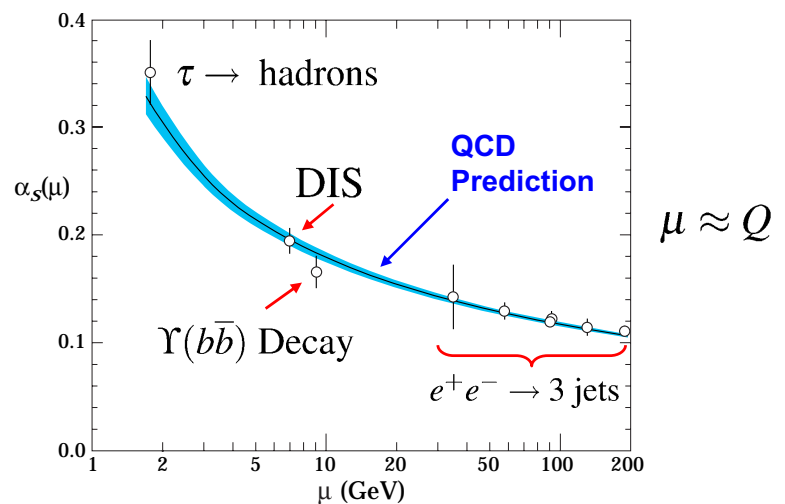
$\rightarrow \alpha_s$ decreases with Q^2

Nobel Prize for Physics, 2004
(Gross, Politzer, Wilczek)

★ Measure α_s in many ways:

- jet rates
- DIS
- tau decays
- bottomonium decays
- +...

★ As predicted by QCD,
 α_s decreases with Q^2



★ At low Q^2 : α_s is large, e.g. at $Q^2 = 1 \text{ GeV}^2$ find $\alpha_s \sim 1$

• Can't use perturbation theory ! This is the reason why QCD calculations at low energies are so difficult, e.g. properties hadrons, hadronisation of quarks to jets,...

★ At high Q^2 : α_s is rather small, e.g. at $Q^2 = M_Z^2$ find $\alpha_s \sim 0.12$

\rightarrow Asymptotic Freedom

• Can use perturbation theory and this is the reason that in DIS at high Q^2 quarks behave as if they are quasi-free (i.e. only weakly bound within hadrons)

Summary

- ★ Superficially QCD very similar to QED
- ★ But gluon self-interactions are believed to result in colour confinement
- ★ All hadrons are colour singlets which explains why only observe

Mesons

Baryons

- ★ A low energies $\alpha_S \sim 1$

→ Can't use perturbation theory !

Non-Perturbative regime

- ★ Coupling constant runs, smaller coupling at higher energy scales

$$\alpha_S(100\text{GeV}) \sim 0.1$$

→ Can use perturbation theory

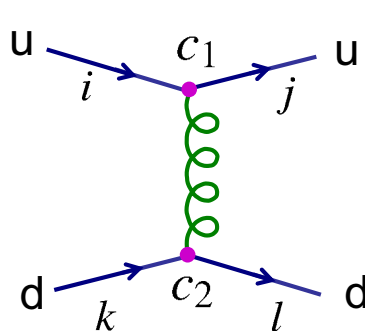
Asymptotic Freedom

- ★ Where calculations can be performed, QCD provides a good description of relevant experimental data

Appendix I: Alternative evaluation of colour factors

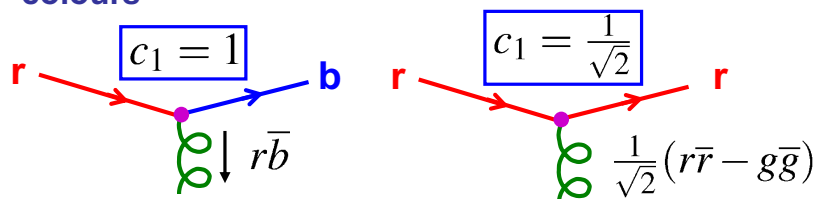
“Non-examinable”
but can be used
as to derive colour
factors.

- ★ The colour factors can be obtained (more intuitively) as follows :



• Write $C(ik \rightarrow jl) = \frac{1}{2}c_1c_2$

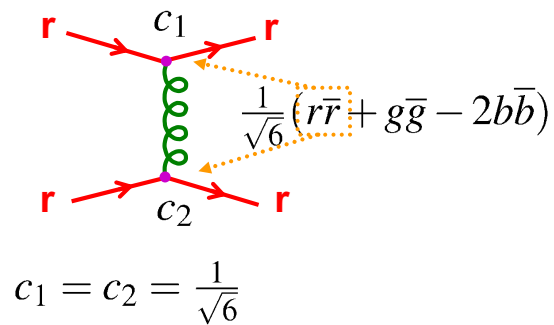
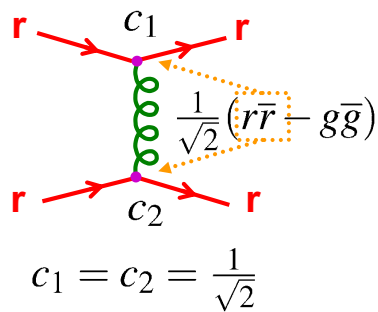
- Where the colour coefficients at the two vertices depend on the quark and gluon colours



- Sum over all possible exchanged gluons conserving colour at both vertices

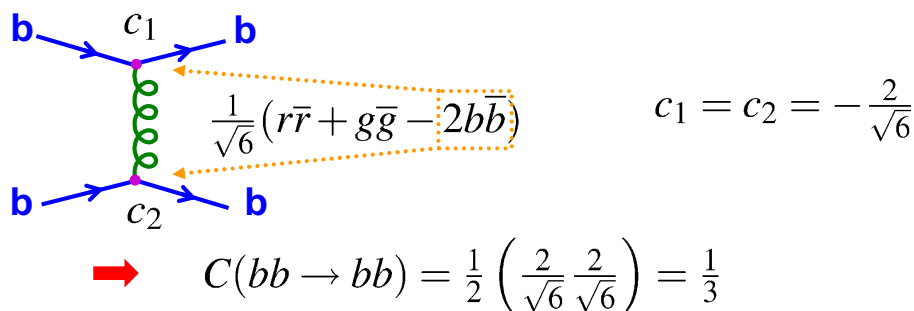
① Configurations involving a single colour

e.g. $rr \rightarrow rr$: two possible exchanged gluons

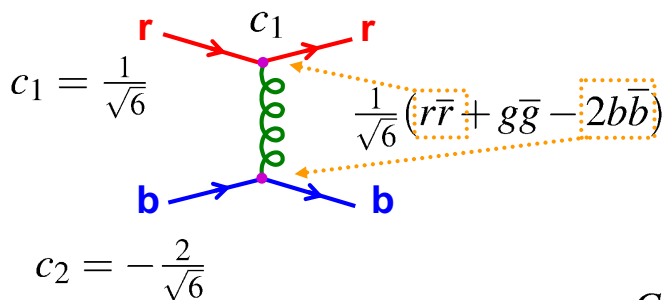


$$C(rr \rightarrow rr) = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{6} \right) = \frac{1}{3}$$

e.g. $bb \rightarrow bb$: only one possible exchanged gluon

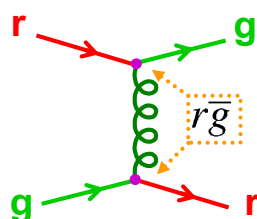


② Other configurations where quarks don't change colour



$$C(rb \rightarrow rb) = \frac{1}{2} \left(-\frac{1}{\sqrt{6}} \frac{2}{\sqrt{6}} \right) = -\frac{1}{6}$$

③ Configurations where quarks swap colours



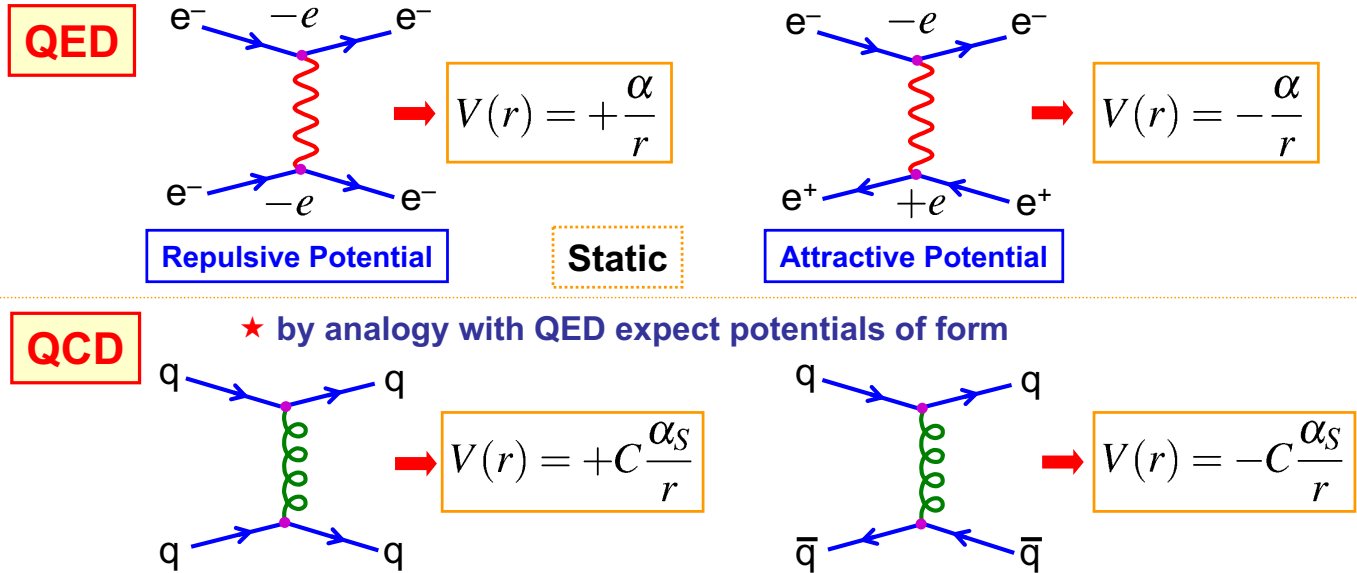
$$c_1 = c_2 = 1$$

$$C(rg \rightarrow gr) = \frac{1}{2}$$

Appendix II: Colour Potentials

Non-examinable

- Previously argued that gluon self-interactions lead to a $+\lambda r$ long-range potential and that this is likely to explain colour confinement
- Have yet to consider the short range potential – i.e. for quarks in mesons and baryons does QCD lead to an attractive potential?
- Analogy with QED: (NOTE this is very far from a formal proof)



★ Whether it is a attractive or repulsive potential depends on **sign of colour factor**

★ Consider the colour factor for a $q\bar{q}$ system in the colour singlet state:

$$\psi = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

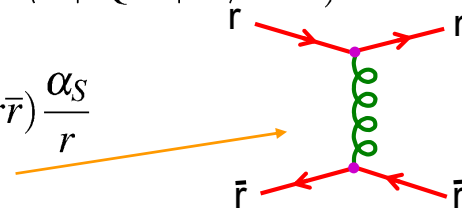
with colour potential $\langle V_{q\bar{q}} \rangle = \langle \psi | V_{\text{QCD}} | \psi \rangle$

$$\Rightarrow \langle V_{q\bar{q}} \rangle = \frac{1}{3} (\langle r\bar{r} | V_{\text{QCD}} | r\bar{r} \rangle + \dots + \langle r\bar{r} | V_{\text{QCD}} | b\bar{b} \rangle + \dots)$$

• Following the QED analogy:

$$\langle r\bar{r} | V_{\text{QCD}} | r\bar{r} \rangle = -C(r\bar{r} \rightarrow r\bar{r}) \frac{\alpha_S}{r}$$

which is the term arising from $r\bar{r} \rightarrow r\bar{r}$



• Have 3 terms like $r\bar{r} \rightarrow r\bar{r}, b\bar{b} \rightarrow b\bar{b}, \dots$ and 6 like $r\bar{r} \rightarrow g\bar{g}, r\bar{r} \rightarrow b\bar{b}, \dots$

$$\langle V_{q\bar{q}} \rangle = -\frac{1}{3} \frac{\alpha_S}{r} [3 \times C(r\bar{r} \rightarrow r\bar{r}) + 6 \times C(r\bar{r} \rightarrow g\bar{g})] = -\frac{1}{3} \frac{\alpha_S}{r} \left[3 \times \frac{1}{3} + 6 \times \frac{1}{2} \right]$$

$$\Rightarrow \langle V_{q\bar{q}} \rangle = -\frac{4}{3} \frac{\alpha_S}{r}$$

NEGATIVE \Rightarrow ATTRACTIVE

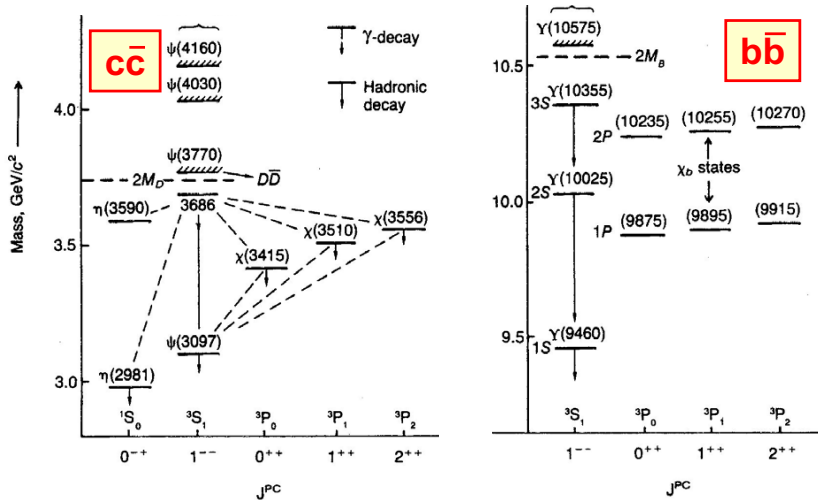
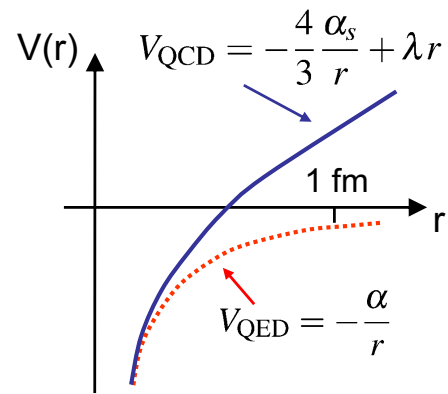
• The same calculation for a $q\bar{q}$ colour octet state, e.g. $r\bar{g}$ gives a positive repulsive potential: $C(r\bar{g} \rightarrow r\bar{g}) = -\frac{1}{6}$

★ Whilst not a formal proof, it is comforting to see that in the colour singlet $q\bar{q}$ state the QCD potential is indeed attractive. (question 15)

- ★ Combining the short-range QCD potential with the linear long-range term discussed previously:

$$V_{\text{QCD}} = -\frac{4}{3} \frac{\alpha_s}{r} + \lambda r$$

- ★ This potential is found to give a good description of the observed charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) bound states.



NOTE:

- c, b are heavy quarks
- approx. non-relativistic
- orbit close together
- probe $1/r$ part of V_{QCD}

Agreement of data with prediction provides strong evidence that V_{QCD} has the Expected form