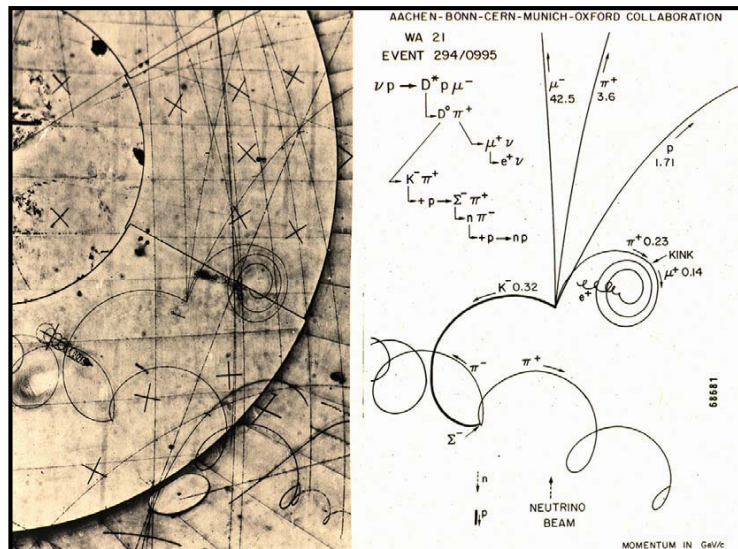


Particle Physics

Michaelmas Term 2009

Prof Mark Thomson



Handout 7 : Symmetries and the Quark Model

Introduction/Aims

- ★ Symmetries play a central role in particle physics; one aim of particle physics is to discover the fundamental symmetries of our universe
- ★ In this handout will apply the idea of symmetry to the quark model with the aim of :
 - ♦ Deriving hadron wave-functions
 - ♦ Providing an introduction to the more abstract ideas of colour and QCD (handout 8)
 - ♦ Ultimately explaining why hadrons only exist as $\bar{q}q$ (mesons) qqq (baryons) or $\bar{q}\bar{q}\bar{q}$ (anti-baryons)
- + introduce the ideas of the SU(2) and SU(3) symmetry groups which play a major role in particle physics (see handout 13)

Symmetries and Conservation Laws

- ★ Suppose physics is invariant under the transformation

$$\psi \rightarrow \psi' = \hat{U} \psi \quad \text{e.g. rotation of the coordinate axes}$$

- To conserve probability normalisation require

$$\langle \psi | \psi \rangle = \langle \psi' | \psi' \rangle = \langle \hat{U} \psi | \hat{U} \psi \rangle = \langle \psi | \hat{U}^\dagger \hat{U} | \psi \rangle$$

$$\rightarrow \boxed{\hat{U}^\dagger \hat{U} = 1} \quad \text{i.e. } \hat{U} \text{ has to be unitary}$$

- For physical predictions to be unchanged by the symmetry transformation, also require all QM matrix elements unchanged

$$\langle \psi | \hat{H} | \psi \rangle = \langle \psi' | \hat{H} | \psi' \rangle = \langle \psi | \hat{U}^\dagger \hat{H} \hat{U} | \psi \rangle$$

i.e. require

$$\hat{U}^\dagger \hat{H} \hat{U} = \hat{H}$$

$\times \hat{U}$

$$\hat{U} \hat{U}^\dagger \hat{H} \hat{U} = \hat{U} \hat{H} \rightarrow \hat{H} \hat{U} = \hat{U} \hat{H}$$

therefore

$$\boxed{[\hat{H}, \hat{U}] = 0}$$

$$\boxed{\hat{U} \text{ commutes with the Hamiltonian}}$$

- ★ Now consider the infinitesimal transformation (ϵ small)

$$\hat{U} = 1 + i\epsilon \hat{G}$$

(\hat{G} is called the **generator** of the transformation)

- For \hat{U} to be unitary

$$\hat{U} \hat{U}^\dagger = (1 + i\epsilon \hat{G})(1 - i\epsilon \hat{G}^\dagger) = 1 + i\epsilon(\hat{G} - \hat{G}^\dagger) + O(\epsilon^2)$$

$$\text{neglecting terms in } \epsilon^2 \quad \hat{U} \hat{U}^\dagger = 1 \rightarrow \boxed{\hat{G} = \hat{G}^\dagger}$$

i.e. \hat{G} is Hermitian and therefore corresponds to an observable quantity G !

- Furthermore, $[\hat{H}, \hat{U}] = 0 \Rightarrow [\hat{H}, 1 + i\epsilon \hat{G}] = 0 \Rightarrow [\hat{H}, \hat{G}] = 0$

$$\text{But from QM} \quad \frac{d}{dt} \langle \hat{G} \rangle = i \langle [\hat{H}, \hat{G}] \rangle = 0$$

i.e. G is a **conserved** quantity.

$$\boxed{\text{Symmetry} \longleftrightarrow \text{Conservation Law}}$$

- ★ For each symmetry of nature have an observable **conserved** quantity

Example: Infinitesimal spatial translation $x \rightarrow x + \epsilon$

i.e. expect physics to be invariant under $\psi(x) \rightarrow \psi' = \psi(x + \epsilon)$

$$\psi'(x) = \psi(x + \epsilon) = \psi(x) + \frac{\partial \psi}{\partial x} \epsilon = \left(1 + \epsilon \frac{\partial}{\partial x} \right) \psi(x)$$

$$\text{but } \hat{p}_x = -i \frac{\partial}{\partial x} \rightarrow \psi'(x) = (1 + i\epsilon \hat{p}_x) \psi(x)$$

The generator of the symmetry transformation is $\hat{p}_x \rightarrow p_x$ is conserved

- Translational invariance of physics implies momentum conservation !

- In general the symmetry operation may depend on more than one parameter

$$\hat{U} = 1 + i\vec{\epsilon} \cdot \vec{G}$$

For example for an infinitesimal 3D linear translation : $\vec{r} \rightarrow \vec{r} + \vec{\epsilon}$

$$\rightarrow \hat{U} = 1 + i\vec{\epsilon} \cdot \vec{p} \quad \vec{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$$

- So far have only considered an infinitesimal transformation, however a finite transformation can be expressed as a series of infinitesimal transformations

$$\hat{U}(\vec{\alpha}) = \lim_{n \rightarrow \infty} \left(1 + i \frac{\vec{\alpha}}{n} \cdot \vec{G} \right)^n = e^{i\vec{\alpha} \cdot \vec{G}}$$

Example: Finite spatial translation in 1D: $x \rightarrow x + x_0$ with $\hat{U}(x_0) = e^{ix_0 \hat{p}_x}$

$$\begin{aligned} \psi'(x) = \psi(x + x_0) &= \hat{U} \psi(x) = \exp \left(x_0 \frac{d}{dx} \right) \psi(x) & \left(p_x = -i \frac{\partial}{\partial x} \right) \\ &= \left(1 + x_0 \frac{d}{dx} + \frac{x_0^2}{2!} \frac{d^2}{dx^2} + \dots \right) \psi(x) \\ &= \psi(x) + x_0 \frac{d\psi}{dx} + \frac{x_0^2}{2} \frac{d^2\psi}{dx^2} + \dots \end{aligned}$$

i.e. obtain the expected Taylor expansion

Symmetries in Particle Physics : Isospin

- The proton and neutron have very similar masses and the nuclear force is found to be approximately charge-independent, i.e.

$$V_{pp} \approx V_{np} \approx V_{nn}$$

- To reflect this symmetry, Heisenberg (1932) proposed that if you could “switch off” the electric charge of the proton

There would be no way to distinguish
between a proton and neutron

- Proposed that the neutron and proton should be considered as two states of a single entity; the nucleon

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- ★ Analogous to the spin-up/spin-down states of a spin- $\frac{1}{2}$ particle

ISOSPIN

- ★ Expect physics to be invariant under rotations in this space

- The neutron and proton form an isospin doublet with total isospin $I = \frac{1}{2}$ and third component $I_3 = \pm \frac{1}{2}$

Flavour Symmetry of the Strong Interaction

We can extend this idea to the quarks:

★ Assume the strong interaction treats all quark flavours equally (it does)

• **Because** $m_u \approx m_d$:

The strong interaction possesses an **approximate** flavour symmetry
i.e. from the point of view of the strong interaction nothing changes
if all up quarks are replaced by down quarks and *vice versa*.

• Choose the basis

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

• Express the invariance of the strong interaction under $u \leftrightarrow d$ as invariance under “rotations” in an abstract isospin space

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

The 2x2 **unitary** matrix depends on 4 complex numbers, i.e. 8 real parameters
But there are four constraints from $\hat{U}^\dagger \hat{U} = 1$

➡ **8 – 4 = 4 independent matrices**

• In the language of group theory the four matrices form the **U(2)** group

• One of the matrices corresponds to multiplying by a phase factor

$$\hat{U}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e^{i\phi}$$

not a flavour transformation and of no relevance here.

• The remaining three matrices form an **SU(2)** group (**special unitary**) with $\det U = 1$

• For an infinitesimal transformation, in terms of the **Hermitian** generators \hat{G}

$$\hat{U} = 1 + i\varepsilon \hat{G}$$

• $\det U = 1 \Rightarrow \text{Tr}(\hat{G}) = 0$

• A linearly independent choice for \hat{G} are the **Pauli spin matrices**

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

• The proposed flavour symmetry of the strong interaction has the same transformation properties as **SPIN** !

• Define **ISOSPIN**: $\vec{T} = \frac{1}{2} \vec{\sigma} \quad \hat{U} = e^{i\vec{\alpha} \cdot \vec{T}}$

• Check this works, for an infinitesimal transformation

$$\hat{U} = 1 + \frac{1}{2} i\vec{\varepsilon} \cdot \vec{\sigma} = 1 + \frac{i}{2} (\varepsilon_1 \sigma_1 + \varepsilon_2 \sigma_2 + \varepsilon_3 \sigma_3) = \begin{pmatrix} 1 + \frac{1}{2} i\varepsilon_3 & \frac{1}{2} i(\varepsilon_1 - i\varepsilon_2) \\ \frac{1}{2} i(\varepsilon_1 + i\varepsilon_2) & 1 - \frac{1}{2} i\varepsilon_3 \end{pmatrix}$$

Which is, as required, unitary and has unit determinant

$$U^\dagger U = I + O(\varepsilon^2) \quad \det U = 1 + O(\varepsilon^2)$$

Properties of Isospin

- Isospin has the exactly the same properties as spin

$$[T_1, T_2] = iT_3 \quad [T_2, T_3] = iT_1 \quad [T_3, T_1] = iT_2$$

$$[T^2, T_3] = 0 \quad T^2 = T_1^2 + T_2^2 + T_3^2$$

As in the case of spin, have three non-commuting operators, T_1, T_2, T_3 and even though all three correspond to observables, can't know them simultaneously. So label states in terms of **total isospin** I and the third component of isospin I_3

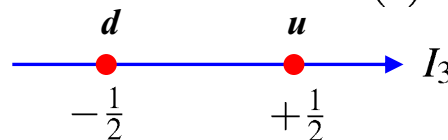
NOTE: isospin has nothing to do with spin – just the same mathematics

- The eigenstates are exact analogues of the eigenstates of ordinary angular momentum $|s, m\rangle \rightarrow |I, I_3\rangle$

with $T^2|I, I_3\rangle = I(I+1)|I, I_3\rangle \quad T_3|I, I_3\rangle = I_3|I, I_3\rangle$

- In terms of isospin:

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left| \frac{1}{2}, +\frac{1}{2} \right\rangle \quad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$



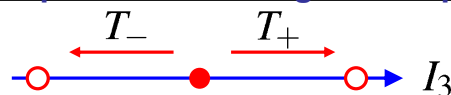
$$I = \frac{1}{2}, \quad I_3 = \pm \frac{1}{2}$$

- In general $I_3 = \frac{1}{2}(N_u - N_d)$

- Can define isospin ladder operators – analogous to spin ladder operators

$$T_- \equiv T_1 - iT_2$$

$$u \rightarrow d$$



$$T_+ \equiv T_1 + iT_2$$

$$d \rightarrow u$$

$$T_+|I, I_3\rangle = \sqrt{I(I+1) - I_3(I_3+1)}|I, I_3+1\rangle$$

$$T_-|I, I_3\rangle = \sqrt{I(I+1) - I_3(I_3-1)}|I, I_3-1\rangle$$

Step up/down in I_3 until reach end of **multiplet** $T_+|I, +I\rangle = 0 \quad T_-|I, -I\rangle = 0$

$$T_+u = 0 \quad T_+d = u \quad T_-u = d \quad T_-d = 0$$

- Ladder operators turn $u \rightarrow d$ and $d \rightarrow u$

- ★ Combination of isospin: **e.g.** what is the isospin of a system of two **d** quarks, is exactly analogous to combination of spin (i.e. angular momentum)

$$|I^{(1)}, I_3^{(1)}\rangle |I^{(2)}, I_3^{(2)}\rangle \rightarrow |I, I_3\rangle$$

- I_3 additive: $I_3 = I_3^{(1)} + I_3^{(2)}$

- I in integer steps from $|I^{(1)} - I^{(2)}|$ to $|I^{(1)} + I^{(2)}|$

- ★ Assumed **symmetry** of Strong Interaction under isospin transformations implies the existence of **conserved quantities**

- In strong interactions I_3 and I are conserved, analogous to conservation of J_z and J for angular momentum

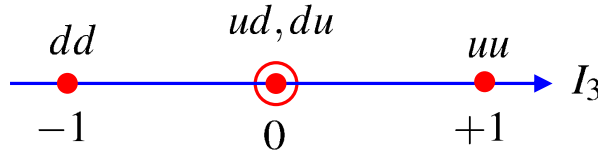
Combining Quarks

Goal: derive proton wave-function

- First combine two quarks, then combine the third
- Use requirement that fermion wave-functions are anti-symmetric

Isospin starts to become useful in defining states of more than one quark.

e.g. two quarks, here we have four possible combinations:



Note: ⊙ represents two states with the same value of I_3

- We can immediately identify the extremes (I_3 additive)

$$uu \equiv \left| \frac{1}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle = |1, +1\rangle$$

$$dd \equiv \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = |1, -1\rangle$$

To obtain the $|1, 0\rangle$ state use ladder operators

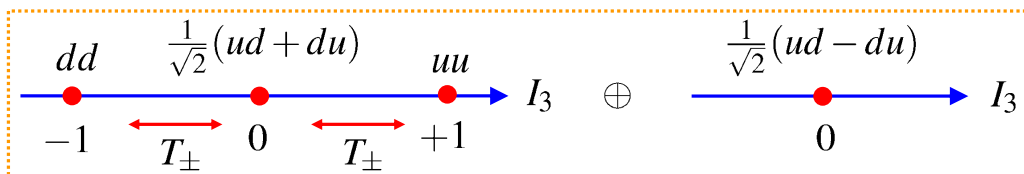
$$T_- |1, +1\rangle = \sqrt{2} |1, 0\rangle = T_-(uu) = ud + du$$

$$\rightarrow |1, 0\rangle = \frac{1}{\sqrt{2}}(ud + du)$$

The final state, $|0, 0\rangle$, can be found from orthogonality with $|1, 0\rangle$

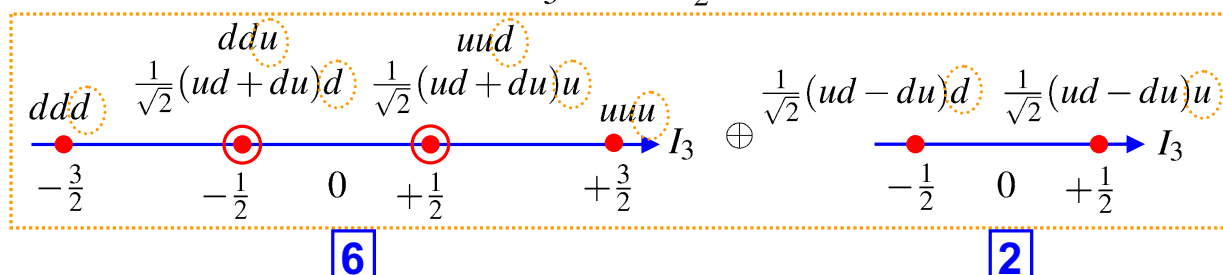
$$\rightarrow |0, 0\rangle = \frac{1}{\sqrt{2}}(ud - du)$$

- From four possible combinations of isospin doublets obtain a triplet of isospin 1 states and a singlet isospin 0 state $2 \otimes 2 = 3 \oplus 1$



- Can move around within multiplets using ladder operators
- note, as anticipated $I_3 = \frac{1}{2}(N_u - N_d)$
- States with different total isospin are physically different – the isospin 1 triplet is **symmetric** under interchange of quarks 1 and 2 whereas singlet is **anti-symmetric**

- ★ Now add an additional up or down quark. From each of the above 4 states get two new isospin states with $I'_3 = I_3 \pm \frac{1}{2}$



- Use ladder operators and orthogonality to group the 6 states into isospin multiplets, e.g. to obtain the $I = \frac{3}{2}$ states, step up from ddd

★ Derive the $I = \frac{3}{2}$ states from $ddd \equiv |\frac{3}{2}, -\frac{3}{2}\rangle$

$$\begin{aligned}
 T_+|\frac{3}{2}, -\frac{3}{2}\rangle &= T_+(ddd) = (T_+d)dd + d(T_+d)d + dd(T_+)d \\
 \sqrt{3}|\frac{3}{2}, -\frac{1}{2}\rangle &= udd + dud + ddu \\
 |\frac{3}{2}, -\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}}(udd + dud + ddu) \\
 T_+|\frac{3}{2}, -\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}}T_+(udd + dud + ddu) \\
 2|\frac{3}{2}, +\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}}(uud + udu + uud + duu + udu + duu) \\
 |\frac{3}{2}, +\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}}(uud + udu + duu) \\
 T_+|\frac{3}{2}, +\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}}T_+(uud + udu + duu) \\
 \sqrt{3}|\frac{3}{2}, +\frac{3}{2}\rangle &= \frac{1}{\sqrt{3}}(uuu + uuu + uuu) \\
 |\frac{3}{2}, +\frac{3}{2}\rangle &= uuu
 \end{aligned}$$

★ From the [6] states on previous page, use orthogonality to find $|\frac{1}{2}, \pm\frac{1}{2}\rangle$ states

★ The [2] states on the previous page give another $|\frac{1}{2}, \pm\frac{1}{2}\rangle$ doublet

★ The eight states $uuu, uud, udu, udd, duu, dud, ddu, ddd$ are grouped into an **isospin quadruplet** and two **isospin doublets**

$$2 \otimes 2 \otimes 2 = 2 \otimes (3 \oplus 1) = (2 \otimes 3) \oplus (2 \otimes 1) = 4 \oplus 2 \oplus 2$$

• Different multiplets have different symmetry properties

$$\left. \begin{aligned}
 |\frac{3}{2}, +\frac{3}{2}\rangle &= uuu \\
 |\frac{3}{2}, +\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}}(uud + udu + duu) \\
 |\frac{3}{2}, -\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}}(ddu + dud + udd) \\
 |\frac{3}{2}, -\frac{3}{2}\rangle &= ddd
 \end{aligned} \right\} \text{S} \quad \text{A quadruplet of states which are symmetric under the interchange of any two quarks}$$

$$\left. \begin{aligned}
 |\frac{1}{2}, -\frac{1}{2}\rangle &= -\frac{1}{\sqrt{6}}(2ddu - udd - dud) \\
 |\frac{1}{2}, +\frac{1}{2}\rangle &= \frac{1}{\sqrt{6}}(2uud - udu - duu)
 \end{aligned} \right\} \text{M}_S \quad \text{Mixed symmetry. Symmetric for } 1 \leftrightarrow 2$$

$$\left. \begin{aligned}
 |\frac{1}{2}, -\frac{1}{2}\rangle &= \frac{1}{\sqrt{2}}(udd - dud) \\
 |\frac{1}{2}, +\frac{1}{2}\rangle &= \frac{1}{\sqrt{2}}(udu - duu)
 \end{aligned} \right\} \text{M}_A \quad \text{Mixed symmetry. Anti-symmetric for } 1 \leftrightarrow 2$$

• Mixed symmetry states have no definite symmetry under interchange of quarks $1 \leftrightarrow 3$ etc.

Combining Spin

- Can apply exactly the same mathematics to determine the possible spin wave-functions for a combination of 3 spin-half particles

$$|\frac{3}{2}, +\frac{3}{2}\rangle = \uparrow\uparrow\uparrow$$

$$|\frac{3}{2}, +\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$$

$$|\frac{3}{2}, -\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow)$$

$$|\frac{3}{2}, -\frac{3}{2}\rangle = \downarrow\downarrow\downarrow$$

S

A quadruplet of states which are symmetric under the interchange of any two quarks

$$|\frac{1}{2}, -\frac{1}{2}\rangle = -\frac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow - \uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$

$$|\frac{1}{2}, +\frac{1}{2}\rangle = \frac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

M_S

Mixed symmetry.
Symmetric for 1 ↔ 2

$$|\frac{1}{2}, -\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$

$$|\frac{1}{2}, +\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

M_A

Mixed symmetry.
Anti-symmetric for 1 ↔ 2

- Can now form total wave-functions for combination of three quarks

Baryon Wave-functions (ud)

- ★ Quarks are fermions so require that the total wave-function is anti-symmetric under the interchange of any two quarks

- ★ the total wave-function can be expressed in terms of:

$$\psi = \phi_{\text{flavour}} \chi_{\text{spin}} \xi_{\text{colour}} \eta_{\text{space}}$$

- ★ The colour wave-function for all bound qqq states is anti-symmetric (see handout 8)

- Here we will only consider the lowest mass, ground state, baryons where there is no internal orbital angular momentum.

- For **L=0** the spatial wave-function is symmetric $(-1)^L$.

$$\rightarrow \xi_{\text{colour}} \eta_{\text{space}}$$

anti-symmetric

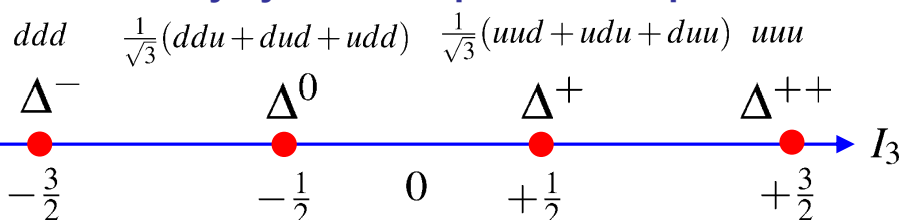
$$\rightarrow \phi_{\text{flavour}} \chi_{\text{spin}}$$

symmetric

Overall anti-symmetric

- ★ Two ways to form a totally symmetric wave-function from spin and isospin states:

- ① combine totally symmetric spin and isospin wave-functions $\phi(S)\chi(S)$



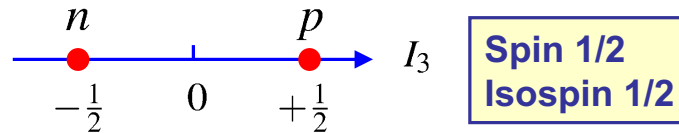
**Spin 3/2
Isospin 3/2**

② combine mixed symmetry spin and mixed symmetry isospin states

- Both $\phi(M_S)\chi(M_S)$ and $\phi(M_A)\chi(M_A)$ are sym. under inter-change of quarks $1 \leftrightarrow 2$
- Not sufficient, these combinations have no definite symmetry under $1 \leftrightarrow 3, \dots$
- However, it is not difficult to show that the (normalised) linear combination:

$$\frac{1}{\sqrt{2}}\phi(M_S)\chi(M_S) + \frac{3}{\sqrt{2}}\phi(M_A)\chi(M_A)$$

is **totally symmetric** (i.e. symmetric under $1 \leftrightarrow 2$; $1 \leftrightarrow 3$; $2 \leftrightarrow 3$)



- The spin-up proton wave-function is therefore:

$$|p \uparrow\rangle = \frac{1}{6\sqrt{2}}(2uud - udu - duu)(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + \frac{3}{2\sqrt{2}}(udu - duu)(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

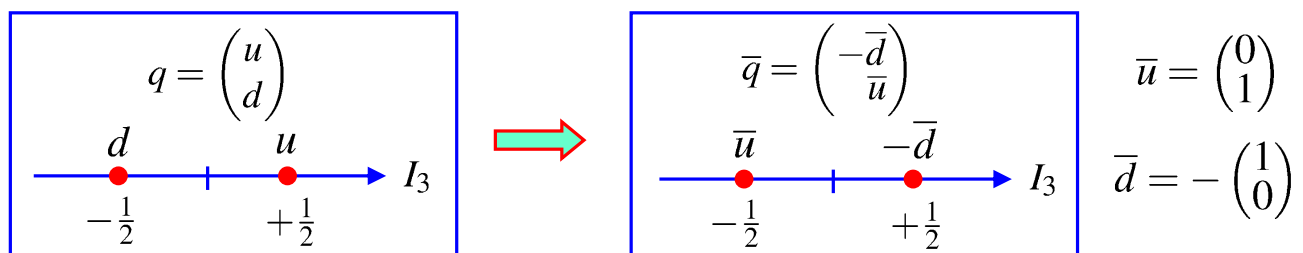


$$|p \uparrow\rangle = \frac{1}{\sqrt{18}}(2u \uparrow u \uparrow d \downarrow - u \uparrow u \downarrow d \uparrow - u \downarrow u \uparrow d \uparrow + \\ 2u \uparrow d \downarrow u \uparrow - u \uparrow d \uparrow u \downarrow - u \downarrow d \uparrow u \uparrow + \\ 2d \downarrow u \uparrow u \uparrow - d \uparrow u \downarrow u \uparrow - d \uparrow u \uparrow u \uparrow)$$

NOTE: not always necessary to use the fully symmetrised proton wave-function, e.g. the first 3 terms are sufficient for calculating the proton magnetic moment

Anti-quarks and Mesons (u and d)

- ★ The u, d quarks and \bar{u} , \bar{d} anti-quarks are represented as isospin doublets



- **Subtle point:** The ordering and the minus sign in the anti-quark doublet ensures that anti-quarks and quarks transform in the same way (see Appendix I). This is necessary if we want physical predictions to be invariant under $u \leftrightarrow d$; $\bar{u} \leftrightarrow \bar{d}$

- Consider the effect of ladder operators on the anti-quark isospin states

e.g. $T_+ \bar{u} = T_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\bar{d}$

- The effect of the ladder operators on anti-particle isospin states are:

$$T_+ \bar{u} = -\bar{d} \quad T_+ \bar{d} = 0 \quad T_- \bar{u} = 0 \quad T_- \bar{d} = -\bar{u}$$

Compare with

$$T_+ u = 0 \quad T_+ d = u \quad T_- u = d \quad T_- d = 0$$

Light ud Mesons

- ★ Can now construct meson states from combinations of up/down quarks



- Consider the $q\bar{q}$ combinations in terms of isospin

$$|1, +1\rangle = |\frac{1}{2}, +\frac{1}{2}\rangle |\frac{1}{2}, +\frac{1}{2}\rangle = -u\bar{d}$$

$$|1, -1\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle |\frac{1}{2}, -\frac{1}{2}\rangle = d\bar{u}$$

The bar indicates this is the isospin representation of an anti-quark

To obtain the $I_3 = 0$ states use ladder operators and orthogonality

$$T_- |1, +1\rangle = T_- [-u\bar{d}]$$

$$\sqrt{2} |1, 0\rangle = -T_- [u] \bar{d} - u T_- [\bar{d}]$$

$$= -d\bar{d} + u\bar{u}$$

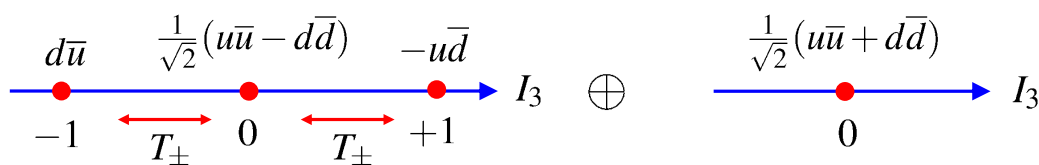
$$\Rightarrow |1, 0\rangle = \frac{1}{\sqrt{2}} (u\bar{u} - d\bar{d})$$

• Orthogonality gives: $|0, 0\rangle = \frac{1}{\sqrt{2}} (u\bar{u} + d\bar{d})$

- ★ To summarise:



⇒ Triplet of $I = 1$ states and a singlet $I = 0$ state



- You will see this written as $2 \otimes \bar{2} = 3 \oplus 1$

Quark doublet

Anti-quark doublet

- To show the state obtained from orthogonality with $|1, 0\rangle$ is a singlet use ladder operators

$$T_+ |0, 0\rangle = T_+ \frac{1}{\sqrt{2}} (u\bar{u} + d\bar{d}) = \frac{1}{\sqrt{2}} (-u\bar{d} + u\bar{d}) = 0$$

similarly $T_- |0, 0\rangle = 0$

★ A singlet state is a “dead-end” from the point of view of ladder operators

SU(3) Flavour

- ★ Extend these ideas to include the strange quark. Since $m_s > m_u/m_d$ don't have an **exact symmetry**. But m_s not so very different from m_u/m_d and can treat the strong interaction (and resulting hadron states) as if it were symmetric under $u \leftrightarrow d \leftrightarrow s$

- **NOTE:** any results obtained from this assumption are only **approximate** as the symmetry is not exact.

- The assumed uds flavour symmetry can be expressed as

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- The 3x3 **unitary** matrix depends on **9** complex numbers, i.e. **18** real parameters
There are **9** constraints from $\hat{U}^\dagger \hat{U} = 1$

➡ Can form **18 - 9 = 9** linearly independent matrices

These 9 matrices form a U(3) group.

- As before, one matrix is simply the identity multiplied by a complex phase and is of no interest in the context of flavour symmetry
- The remaining **8** matrices have $\det U = 1$ and form an **SU(3)** group
- The **eight** matrices (the Hermitian generators) are: $\vec{T} = \frac{1}{2}\vec{\lambda}$ $\hat{U} = e^{i\vec{\alpha} \cdot \vec{T}}$

- ★ In SU(3) flavour, the three quark states are represented by:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- ★ In SU(3) uds flavour symmetry contains SU(2) ud flavour symmetry which allows us to write the first three matrices:

$$\lambda_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}$$

i.e. u ↔ d $\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- The third component of isospin is now written $I_3 = \frac{1}{2}\lambda_3$

with $I_3 u = +\frac{1}{2}u \quad I_3 d = -\frac{1}{2}d \quad I_3 s = 0$

- I_3 “counts the number of up quarks – number of down quarks in a state

- As before, ladder operators $T_{\pm} = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$ $d \bullet \xleftarrow{T_{\pm}} \bullet u$

- Now consider the matrices corresponding to the $u \leftrightarrow s$ and $d \leftrightarrow s$

$u \leftrightarrow s$	$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$d \leftrightarrow s$	$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

- Hence in addition to $\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ have two other traceless diagonal matrices

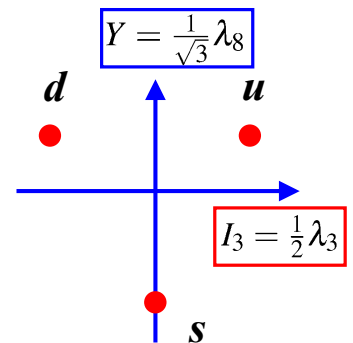
- However the three diagonal matrices are not be independent.

- Define the eighth matrix, λ_8 , as the linear combination:

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

which specifies the “vertical position” in the 2D plane

“Only need two axes (quantum numbers) to specify a state in the 2D plane”: (I_3, Y)



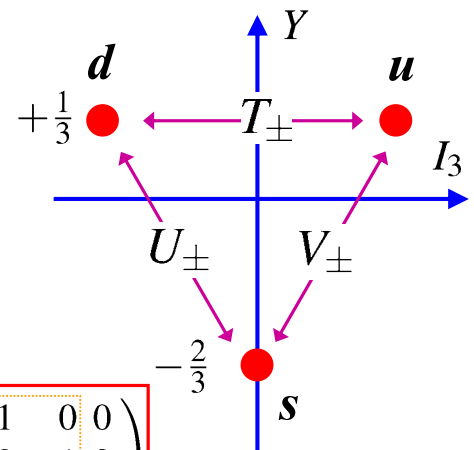
- The other six matrices form six ladder operators which step between the states

$$\begin{aligned} T_{\pm} &= \frac{1}{2}(\lambda_1 \pm i\lambda_2) \\ V_{\pm} &= \frac{1}{2}(\lambda_4 \pm i\lambda_5) \\ U_{\pm} &= \frac{1}{2}(\lambda_6 \pm i\lambda_7) \end{aligned}$$

with

$$I_3 = \frac{1}{2}\lambda_3 \quad Y = \frac{1}{\sqrt{3}}\lambda_8$$

and the eight Gell-Mann matrices



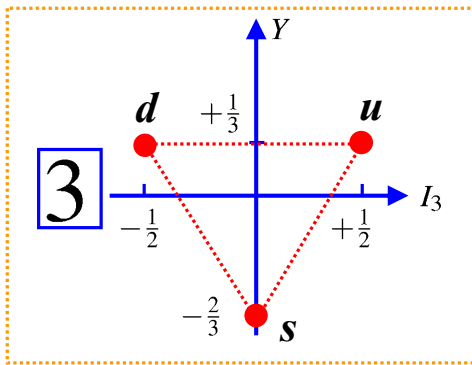
$u \leftrightarrow d$	$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
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$u \leftrightarrow s$	$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$
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$d \leftrightarrow s$	$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$
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$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Quarks and anti-quarks in SU(3) Flavour

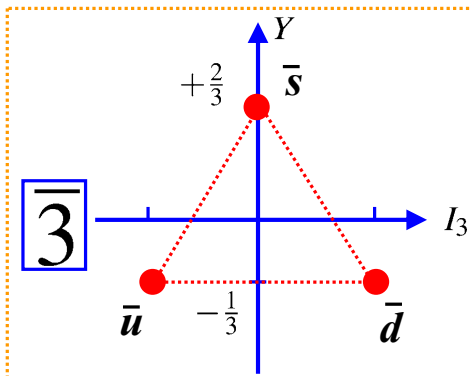


Quarks

$$I_3 u = +\frac{1}{2}u; \quad I_3 d = -\frac{1}{2}d; \quad I_3 s = 0$$

$$Y u = +\frac{1}{3}u; \quad Y d = +\frac{1}{3}d; \quad Y s = -\frac{2}{3}s$$

- The anti-quarks have opposite SU(3) flavour quantum numbers



Anti-Quarks

$$I_3 \bar{u} = -\frac{1}{2}\bar{u}; \quad I_3 \bar{d} = +\frac{1}{2}\bar{d}; \quad I_3 \bar{s} = 0$$

$$Y \bar{u} = -\frac{1}{3}\bar{u}; \quad Y \bar{d} = -\frac{1}{3}\bar{d}; \quad Y \bar{s} = +\frac{2}{3}\bar{s}$$

SU(3) Ladder Operators

- SU(3) uds flavour symmetry contains ud , us and ds SU(2) symmetries
- Consider the $u \leftrightarrow s$ symmetry "V-spin" which has the associated $s \rightarrow u$ ladder operator

$$V_+ = \frac{1}{2}(\lambda_4 + i\lambda_5) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with

$$V_+ s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = +u$$

- The effects of the six ladder operators are:

$T_+ d = u; \quad T_- u = d;$	$T_+ \bar{u} = -\bar{d}; \quad T_- \bar{d} = -\bar{u}$
$V_+ s = u; \quad V_- u = s;$	$V_+ \bar{u} = -\bar{s}; \quad V_- \bar{s} = -\bar{u}$
$U_+ s = d; \quad U_- d = s;$	$U_+ \bar{d} = -\bar{s}; \quad U_- \bar{s} = -\bar{d}$

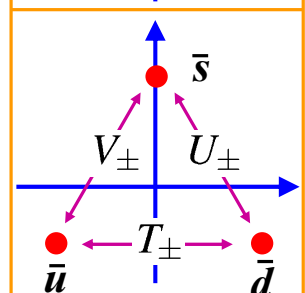
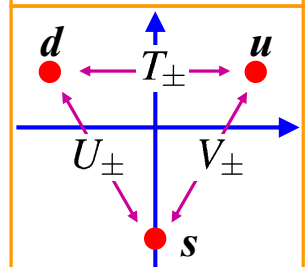
all other combinations give zero

SU(3) LADDER OPERATORS

$$T_{\pm} = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$$

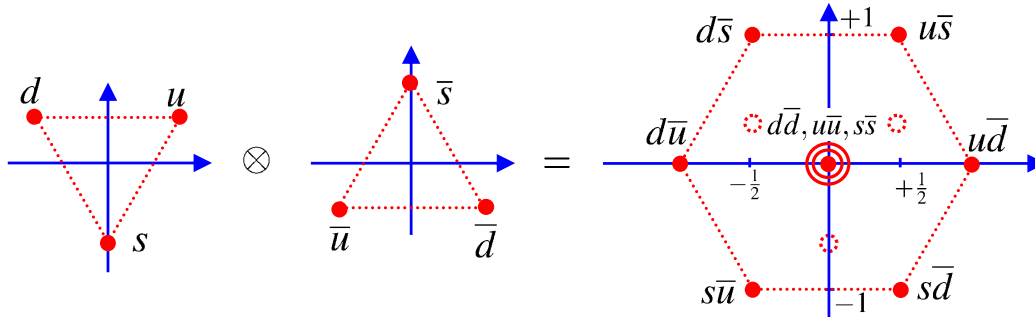
$$V_{\pm} = \frac{1}{2}(\lambda_4 \pm i\lambda_5)$$

$$U_{\pm} = \frac{1}{2}(\lambda_6 \pm i\lambda_7)$$

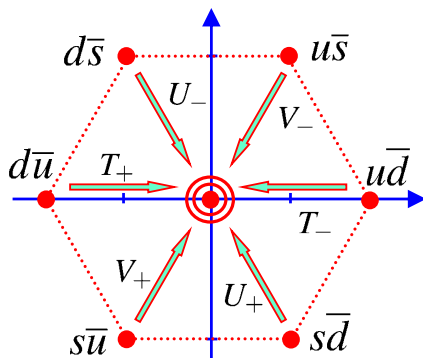


Light (uds) Mesons

- Use ladder operators to construct **uds** mesons from the nine possible $q\bar{q}$ states



- The three central states, all of which have $Y = 0$; $I_3 = 0$ can be obtained using the ladder operators and orthogonality. Starting from the outer states can reach the centre in six ways



$$\begin{aligned} T_+|d\bar{u}\rangle &= |u\bar{u}\rangle - |d\bar{d}\rangle & T_-|u\bar{d}\rangle &= |d\bar{d}\rangle - |u\bar{u}\rangle \\ V_+|s\bar{u}\rangle &= |u\bar{u}\rangle - |s\bar{s}\rangle & V_-|u\bar{s}\rangle &= |s\bar{s}\rangle - |u\bar{u}\rangle \\ U_+|s\bar{d}\rangle &= |d\bar{d}\rangle - |s\bar{s}\rangle & U_-|d\bar{s}\rangle &= |s\bar{s}\rangle - |d\bar{d}\rangle \end{aligned}$$

- Only **two** of these six states are linearly independent.
- But there are **three** states with $Y = 0$; $I_3 = 0$
- Therefore one state is not part of the same multiplet, i.e. cannot be reached with ladder ops.

- First form two linearly independent orthogonal states from:

$$|u\bar{u}\rangle - |d\bar{d}\rangle \quad |u\bar{u}\rangle - |s\bar{s}\rangle \quad |d\bar{d}\rangle - |s\bar{s}\rangle$$

- ★ If the SU(3) flavour symmetry were exact, the choice of states wouldn't matter. However, $m_s > m_{u,d}$ and the symmetry is only approximate.

- Experimentally observe three light mesons with $m \sim 140$ MeV: π^+ , π^0 , π^-
- Identify **one state** (the π^0) with the isospin triplet (derived previously)

$$\psi_1 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$

- The second state can be obtained by taking the linear combination of the other two states which is orthogonal to the π^0

$$\psi_2 = \alpha(|u\bar{u}\rangle - |s\bar{s}\rangle) + \beta(|d\bar{d}\rangle - |s\bar{s}\rangle)$$

with orthonormality: $\langle \psi_1 | \psi_2 \rangle = 0$; $\langle \psi_2 | \psi_2 \rangle = 1$

$$\rightarrow \psi_2 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

- The final state (which is not part of the same multiplet) can be obtained by requiring it to be orthogonal to ψ_1 and ψ_2

$$\rightarrow \psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

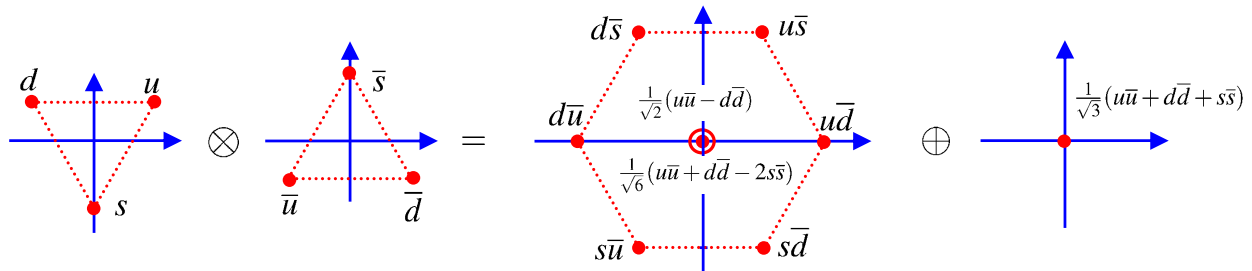
SINGLET

- ★ It is easy to check that ψ_3 is a singlet state using ladder operators

$$T_+ \psi_3 = T_- \psi_3 = U_+ \psi_3 = U_- \psi_3 = V_+ \psi_3 = V_- \psi_3 = 0$$

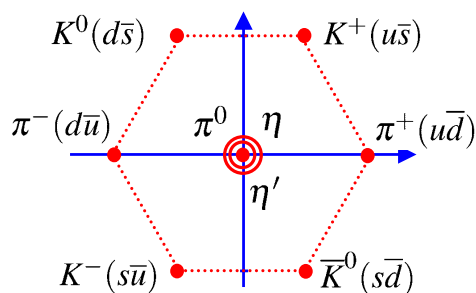
which confirms that $\psi_3 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$ is a “flavourless” singlet

- Therefore the combination of a quark and anti-quark yields nine states which breakdown into an **OCTET** and a **SINGLET**



- In the language of group theory: $3 \otimes \bar{3} = 8 \oplus 1$
- ★ Compare with combination of two spin-half particles $2 \otimes 2 = 3 \oplus 1$
TRIPLET of spin-1 states: $|1, -1\rangle, |1, 0\rangle, |1, +1\rangle$
spin-0 **SINGLET**: $|0, 0\rangle$
- These spin triplet states are connected by ladder operators just as the meson uds octet states are connected by **SU(3)** flavour ladder operators
- The singlet state carries no angular momentum – in this sense the **SU(3)** flavour singlet is “flavourless”

PSEUDOSCALAR MESONS ($L=0, S=0, J=0, P=-1$)



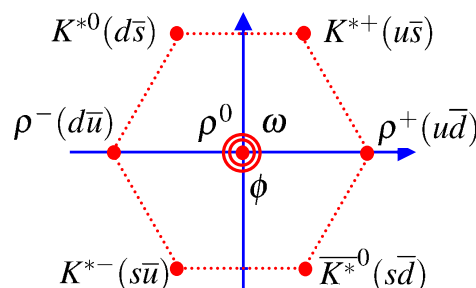
- Because **SU(3)** flavour is only approximate the physical states with $I_3 = 0, Y = 0$ can be mixtures of the octet and singlet states.

Empirically find:

$$\begin{aligned} \pi^0 &= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \\ \eta &\approx \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \\ \eta' &\approx \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \end{aligned}$$

singlet

VECTOR MESONS ($L=0, S=1, J=1, P=-1$)



- For the vector mesons the physical states are found to be approximately “ideally mixed”:

$$\begin{aligned} \rho^0 &= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \\ \omega &\approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \\ \phi &\approx s\bar{s} \end{aligned}$$

MASSSES

$$\begin{aligned} \pi^\pm &: 140 \text{ MeV} & \pi^0 &: 135 \text{ MeV} \\ K^\pm &: 494 \text{ MeV} & K^0/\bar{K}^0 &: 498 \text{ MeV} \\ \eta &: 549 \text{ MeV} & \eta' &: 958 \text{ MeV} \end{aligned}$$

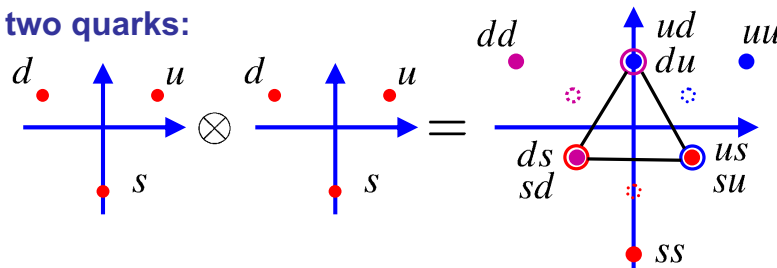
$$\begin{aligned} \rho^\pm &: 770 \text{ MeV} & \rho^0 &: 770 \text{ MeV} \\ K^{*\pm} &: 892 \text{ MeV} & K^{*0}/\bar{K}^{*0} &: 896 \text{ MeV} \\ \omega &: 782 \text{ MeV} & \phi &: 1020 \text{ MeV} \end{aligned}$$

Combining uds Quarks to form Baryons

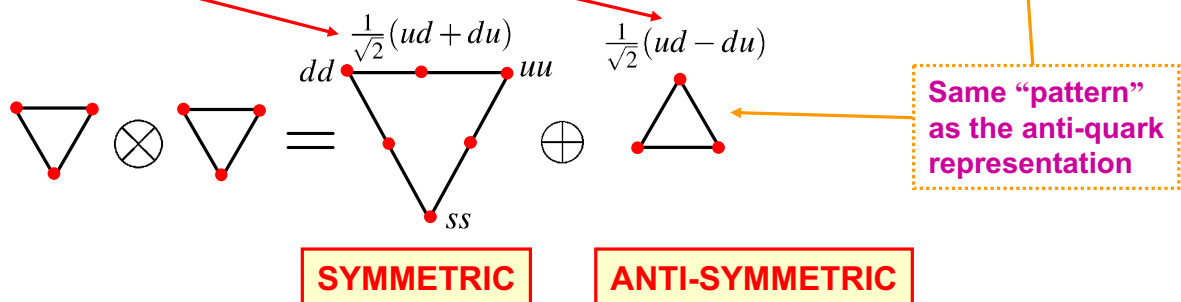
- ★ Have already seen that constructing Baryon states is a fairly tedious process when we derived the proton wave-function. Concentrate on multiplet structure rather than deriving all the wave-functions.

- ★ **Everything we do here is relevant to the treatment of colour**

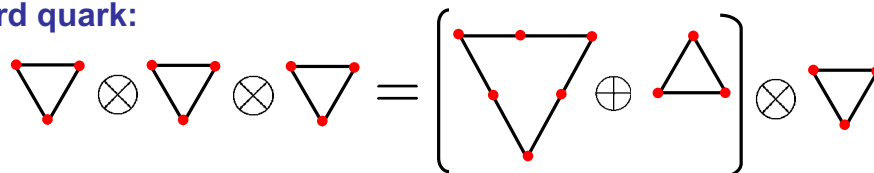
- First combine two quarks:



- ★ Yields a **symmetric sextet** and **anti-symmetric triplet**: $3 \otimes 3 = 6 \oplus \bar{3}$

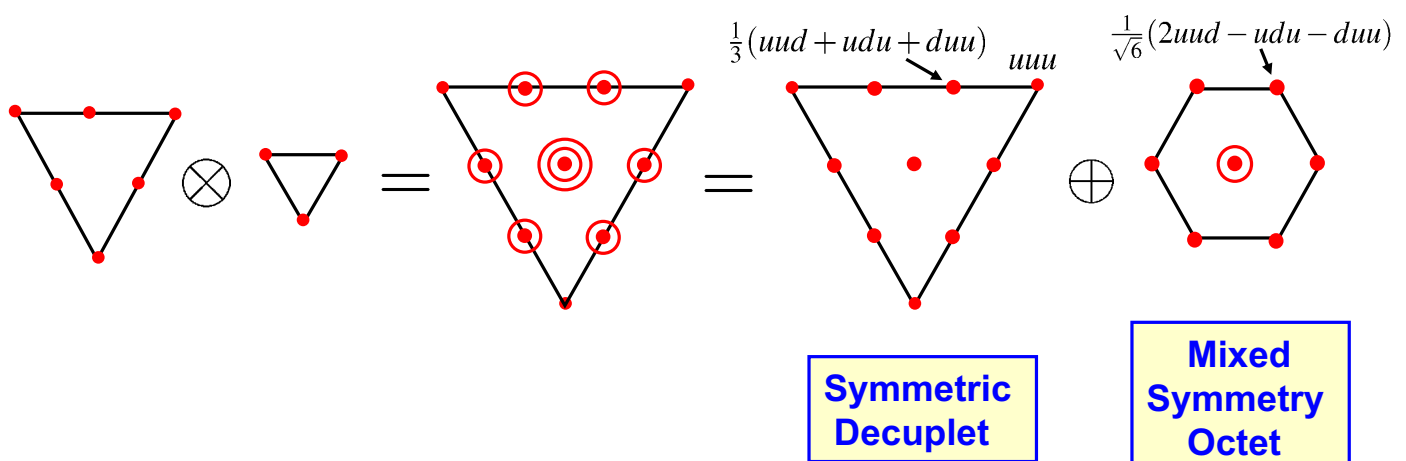


- Now add the third quark:



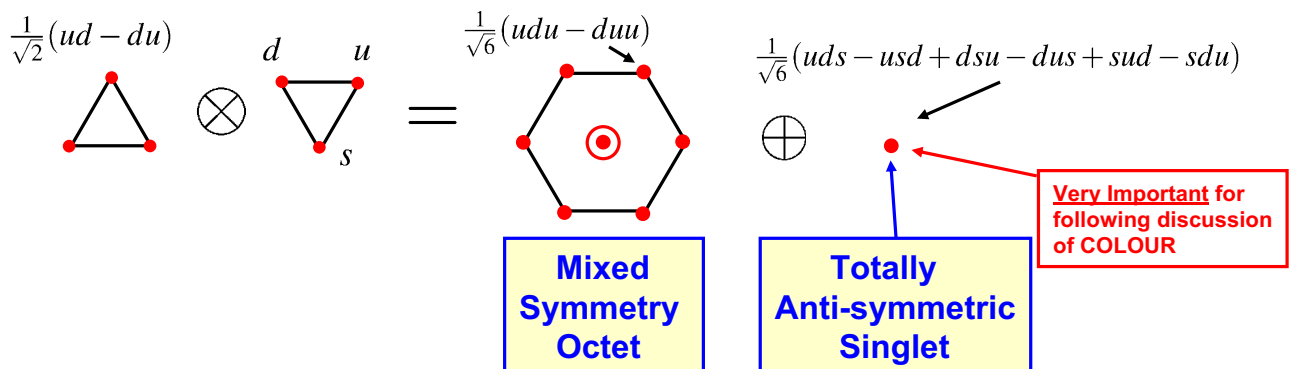
- Best considered in two parts, building on the **sextet** and **triplet**. Again concentrate on the multiplet structure (for the wave-functions refer to the discussion of proton wave-function).

- ① Building on the sextet: $3 \otimes 6 = 10 \oplus 8$



② Building on the triplet:

- Just as in the case of uds mesons we are combining $\bar{3} \times 3$ and again obtain an octet and a singlet



- Can verify the wave-function $\psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uds - usd + dsu - dus + sud - sdu)$ is a singlet by using ladder operators, e.g.

$$T_+ \psi_{\text{singlet}} = \frac{1}{\sqrt{6}}(uus - usu + usu - uus + suu - suu) = 0$$

- ★ In summary, the combination of three uds quarks decomposes into

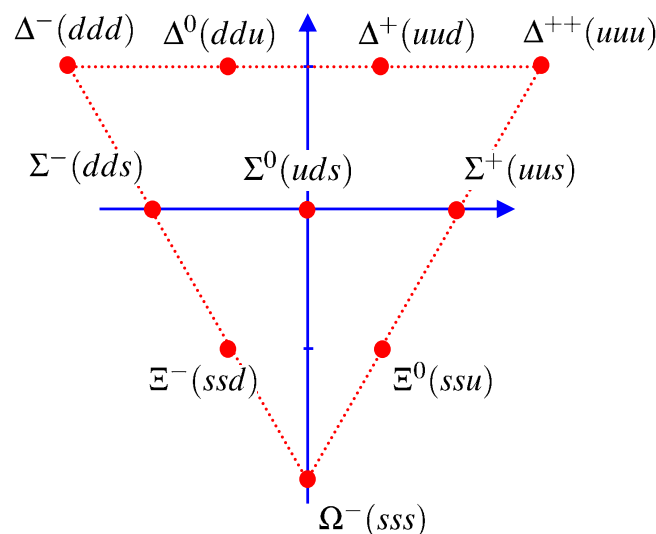
$$3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \bar{3}) = 10 \oplus 8 \oplus 8 \oplus 1$$

Baryon Decuplet

- ★ The baryon states ($L=0$) are:

- the **spin 3/2 decuplet** of symmetric flavour and symmetric spin wave-functions $\phi(S)\chi(S)$

BARYON DECUPLET ($L=0$, $S=3/2$, $J=3/2$, $P=+1$)



Mass in MeV

$\Delta(1232)$

$\Sigma(1318)$

$\Xi(1384)$

$\Omega(1672)$

- ★ If $SU(3)$ flavour were an exact symmetry all masses would be the same (broken symmetry)

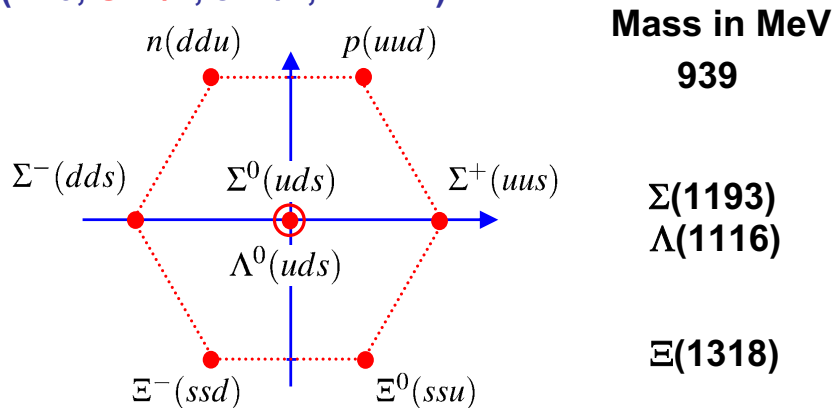
Baryon Octet

- ★ The **spin 1/2 octet** is formed from mixed symmetry flavour and mixed symmetry spin wave-functions

$$\alpha \phi(M_S) \chi(M_S) + \beta \phi(M_A) \chi(M_A)$$

See previous discussion proton for how to obtain wave-functions

BARYON OCTET ($L=0$, $S=1/2$, $J=1/2$, $P=+1$)



- ★ **NOTE:** Cannot form a totally symmetric wave-function based on the anti-symmetric flavour singlet as there no totally anti-symmetric spin wave-function for 3 quarks

Summary

- ★ Considered $SU(2)$ **ud** and $SU(3)$ **uds** flavour symmetries
- ★ Although these flavour symmetries are only approximate can still be used to explain observed multiplet structure for mesons/baryons
- ★ In case of $SU(3)$ flavour symmetry results, e.g. predicted wave-functions should be treated with a pinch of salt as $m_s \neq m_{u/d}$
- ★ Introduced idea of singlet states being “spinless” or “flavourless”
- ★ In the next handout apply these ideas to colour and QCD...

Appendix: the SU(2) anti-quark representation

Non-examinable

- Define anti-quark doublet $\bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix} = \begin{pmatrix} -d^* \\ u^* \end{pmatrix}$

- The quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$ transforms as $q' = Uq$

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = U \begin{pmatrix} u \\ d \end{pmatrix} \xrightarrow[\text{conjugate}]{\text{Complex}} \begin{pmatrix} u'^* \\ d'^* \end{pmatrix} = U^* \begin{pmatrix} u^* \\ d^* \end{pmatrix}$$

- Express in terms of anti-quark doublet

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{q}' = U \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{q}$$

- Hence \bar{q} transforms as

$$\bar{q}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{q}$$

- In general a 2x2 unitary matrix can be written as

$$U = \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix}$$

- Giving

$$\begin{aligned} \bar{q}' &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_{11}^* & c_{12}^* \\ -c_{12} & c_{11} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{q} \\ &= \begin{pmatrix} c_{11} & c_{12} \\ -c_{12}^* & c_{11}^* \end{pmatrix} \\ &= U \bar{q} \end{aligned}$$

- Therefore the anti-quark doublet $\bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$

transforms in the same way as the quark doublet $q = \begin{pmatrix} u \\ d \end{pmatrix}$

★ **NOTE:** this is a special property of SU(2) and for SU(3) there is no analogous representation of the anti-quarks