

# Particle Physics

Michaelmas Term 2009  
Prof Mark Thomson



## Handout 5 : Electron-Proton Elastic Scattering

Prof. M.A. Thomson

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## Electron-Proton Scattering

- In this handout aiming towards a study of electron-proton scattering as a probe of the structure of the proton

- Two main topics:

- $e^-p \rightarrow e^-p$  elastic scattering (this handout)
- $e^-p \rightarrow e^-X$  deep inelastic scattering (handout 6)

- But first consider scattering from a point-like particle e.g.

$$e^-\mu^- \rightarrow e^-\mu^-$$

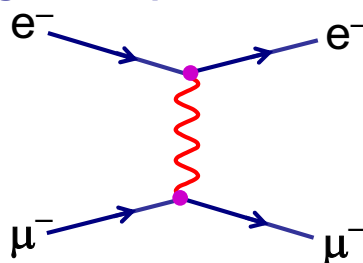
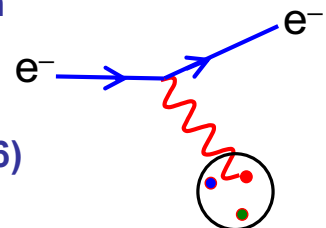
i.e. the QED part of  
( $e^-q \rightarrow e^-q$ )

- Two ways to proceed:

- perform QED calculation from scratch (Q10 on examples sheet)

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} [(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)^2] \quad (1)$$

- take results from  $e^+e^- \rightarrow \mu^+\mu^-$  and use “Crossing Symmetry” to obtain the matrix element for  $e^-\mu^- \rightarrow e^-\mu^-$  (Appendix I)



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$$\rightarrow \langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1 \cdot p_4)^2 + (p_1 \cdot p_2)^2}{(p_1 \cdot p_3)^2} \quad (2)$$

c.f. annihilation:

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2}$$

• **Work in the C.o.M:**

$$p_1 = (E, 0, 0, E) \quad p_2 = (E, 0, 0, -E)$$

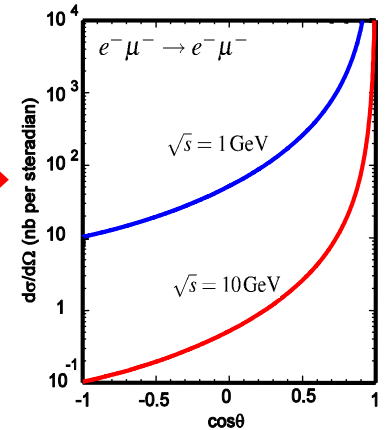
$$p_3 = (E, E \sin \theta, 0, E \cos \theta)$$

$$p_4 = (E, -E \sin \theta, 0, -E \cos \theta)$$

giving  $p_1 \cdot p_2 = 2E^2$ ;  $p_1 \cdot p_3 = E^2(1 - \cos \theta)$ ;  $p_1 \cdot p_4 = E^2(1 + \cos \theta)$

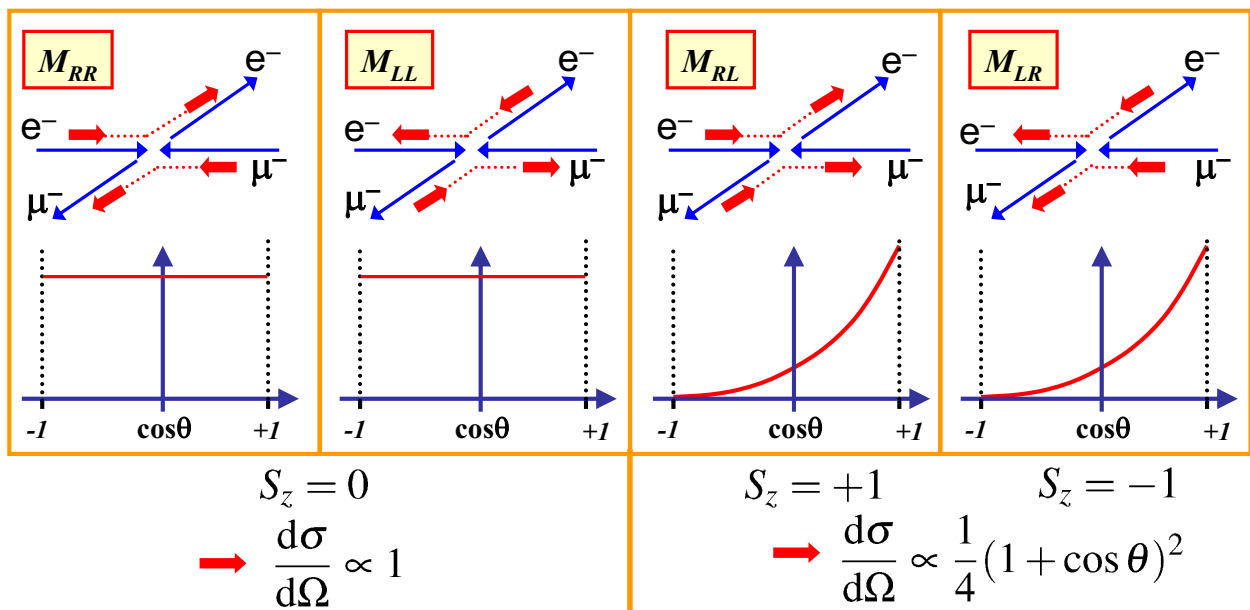
$$\rightarrow \langle |M_{fi}|^2 \rangle = 2e^4 \frac{E^4(1 + \cos \theta)^2 + 4E^4}{E^4(1 - \cos \theta)^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \langle |M_{fi}|^2 \rangle = \frac{e^4}{8\pi^2 s} \frac{[1 + \frac{1}{4}(1 + \cos \theta)^2]}{(1 - \cos \theta)^2}$$



- The **denominator** arises from the propagator  $-ig_{\mu\nu}/q^2$   
 here  $q^2 = (p_1 - p_3)^2 = E^2(1 - \cos \theta)$   
 as  $q^2 \rightarrow 0$  the cross section tends to infinity.

- What about the angular dependence of the **numerator** ?  $\frac{d\sigma}{d\Omega} = \frac{e^4}{8\pi^2 s} \frac{[1 + \frac{1}{4}(1 + \cos \theta)^2]}{(1 - \cos \theta)^2}$
- The factor  $1 + \frac{1}{4}(1 + \cos \theta)^2$  reflects helicity (really chiral) structure of QED
- Of the 16 possible helicity combinations only 4 are non-zero:



i.e. no preferred polar angle

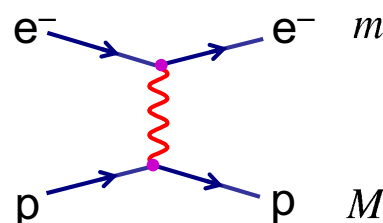
spin 1 rotation again

- The cross section calculated above is appropriate for the scattering of two spin half Dirac (i.e. point-like) particles in the ultra-relativistic limit (where both electron and muon masses can be neglected). In this case

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1 \cdot p_4)^2 + (p_1 \cdot p_2)^2}{(p_1 \cdot p_3)^2}$$

- We will use this again in the discussion of “Deep Inelastic Scattering” of **electrons** from the **quarks** within a proton (**handout 6**).
- Before doing so we will consider the scattering of electrons from the composite proton - i.e. how do we know the proton isn't fundamental “point-like” particle ?

- In this discussion we will not be able to use the relativistic limit and require the general expression for the matrix element (**derived in the optional part of Q10 in the examples sheet**):

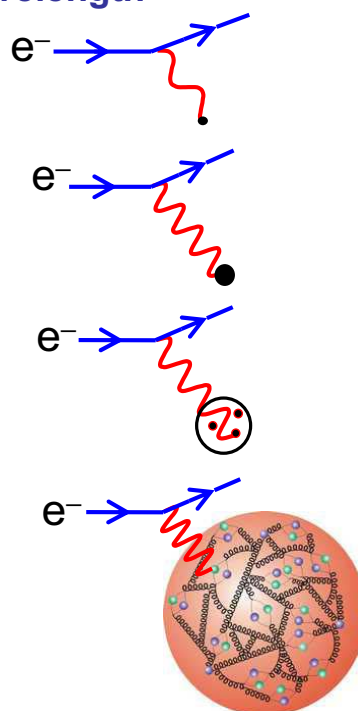


$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} [(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)M^2 - (p_1 \cdot p_4)m^2 + 2m^2M^2] \quad (3)$$

## Probing the Structure of the Proton

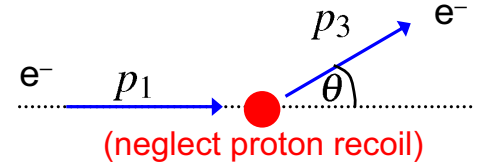
- ★ In  $e^-p \rightarrow e^-p$  scattering the nature of the interaction of the virtual photon with the proton depends strongly on wavelength

- ♦ At **very low** electron energies  $\lambda \gg r_p$  : the scattering is equivalent to that from a “point-like” spin-less object
- ♦ At **low** electron energies  $\lambda \sim r_p$  : the scattering is equivalent to that from a extended charged object
- ♦ At **high** electron energies  $\lambda < r_p$  : the wavelength is sufficiently short to resolve sub-structure. Scattering from constituent quarks
- ♦ At **very high** electron energies  $\lambda \ll r_p$  : the proton appears to be a sea of quarks and gluons.



# Rutherford Scattering Revisited

- ★ Rutherford scattering is the **low energy limit** where the recoil of the proton can be neglected and the **electron is non-relativistic**



- Start from RH and LH Helicity particle spinors

$$u_{\uparrow} = N \begin{pmatrix} c \\ e^{i\phi} s \\ \frac{|\vec{p}|}{E+m} c \\ \frac{|\vec{p}|}{E+m} e^{i\phi} s \end{pmatrix} \quad u_{\downarrow} = N \begin{pmatrix} -s \\ e^{i\phi} c \\ \frac{|\vec{p}|}{E+m} s \\ -\frac{|\vec{p}|}{E+m} e^{i\phi} c \end{pmatrix} \quad \begin{aligned} N &= \sqrt{E+m}; \\ s &= \sin(\theta/2); \quad c = \cos(\theta/2) \end{aligned}$$

- Now write in terms of:

$$\alpha = \frac{|\vec{p}|}{E+m_e} \quad \begin{aligned} \text{Non-relativistic limit: } \alpha &\rightarrow 0 \\ \text{Ultra-relativistic limit: } \alpha &\rightarrow 1 \end{aligned}$$

$$\Rightarrow u_{\uparrow} = N \begin{pmatrix} c \\ e^{i\phi} s \\ \alpha c \\ \alpha e^{i\phi} s \end{pmatrix} \quad u_{\downarrow} = N \begin{pmatrix} -s \\ e^{i\phi} c \\ \alpha s \\ -\alpha e^{i\phi} c \end{pmatrix}$$

and the possible initial and final state electron spinors are:

$$u_{\uparrow}(p_1) = N_e \begin{pmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix} \quad u_{\downarrow}(p_1) = N_e \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\alpha \end{pmatrix} \quad u_{\uparrow}(p_3) = N_e \begin{pmatrix} c \\ s \\ \alpha c \\ \alpha s \end{pmatrix} \quad u_{\downarrow}(p_3) = N_e \begin{pmatrix} -s \\ c \\ \alpha s \\ -\alpha c \end{pmatrix}$$

- Consider all four possible electron currents, i.e. Helicities **R→R, L→L, L→R, R→L**

$$\bar{u}_{\uparrow}(p_3) \gamma^{\mu} u_{\uparrow}(p_1) = (E+m_e) [(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c] \quad (4)$$

$$\bar{u}_{\downarrow}(p_3) \gamma^{\mu} u_{\downarrow}(p_1) = (E+m_e) [(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c] \quad (5)$$

$$\bar{u}_{\uparrow}(p_3) \gamma^{\mu} u_{\downarrow}(p_1) = (E+m_e) [(1-\alpha^2)s, 0, 0, 0] \quad (6)$$

$$\bar{u}_{\downarrow}(p_3) \gamma^{\mu} u_{\uparrow}(p_1) = (E+m_e) [(\alpha^2-1)s, 0, 0, 0] \quad (7)$$

- In the relativistic limit ( $\alpha = 1$ ), i.e.  $E \gg m$

(6) and (7) are identically zero; only **R→R** and **L→L** combinations non-zero

- In the non-relativistic limit,  $|\vec{p}| \ll E$  we have  $\alpha = 0$

$$\bar{u}_{\uparrow}(p_3) \gamma^{\mu} u_{\uparrow}(p_1) = \bar{u}_{\downarrow}(p_3) \gamma^{\mu} u_{\downarrow}(p_1) = (2m_e) [c, 0, 0, 0]$$

$$\bar{u}_{\uparrow}(p_3) \gamma^{\mu} u_{\downarrow}(p_1) = -\bar{u}_{\downarrow}(p_3) \gamma^{\mu} u_{\uparrow}(p_1) = (2m_e) [s, 0, 0, 0]$$

**All four electron helicity combinations have non-zero Matrix Element**

i.e. Helicity eigenstates  $\neq$  Chirality eigenstates

- The initial and final state proton spinors (assuming no recoil) are:

$$u_{\uparrow}(0) = \sqrt{2M_p} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad u_{\downarrow}(0) = \sqrt{2M_p} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Solutions of Dirac equation for a particle at rest

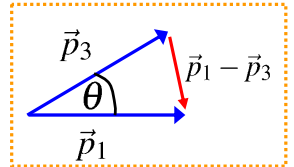
giving the proton currents:

$$j_{p\uparrow\uparrow} = j_{p\downarrow\downarrow} = 2M_p(1, 0, 0, 0)$$

$$j_{p\uparrow\downarrow} = j_{p\downarrow\uparrow} = 0$$

- The spin-averaged ME summing over the 8 allowed helicity states

$$\langle |M_{fi}^2| \rangle = \frac{1}{4} \frac{e^4}{q^4} (16M_p^2 m_e^2) (4c^2 + 4s^2) = \frac{16M_p^2 m_e^2 e^4}{q^4}$$



where  $q^2 = (p_1 - p_3)^2 = (0, \vec{p}_1 - \vec{p}_3)^2 = -4|\vec{p}|^2 \sin^2(\theta/2)$

$$\langle |M_{fi}^2| \rangle = \frac{M_p^2 m_e^2 e^4}{|\vec{p}|^4 \sin^4(\theta/2)}$$

Note: in this limit all angular dependence is in the propagator

- The formula for the differential cross-section in the lab. frame was derived in handout 1:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left( \frac{1}{M + E_1 - E_1 \cos \theta} \right)^2 |M_{fi}|^2 \quad (8)$$

- Here the electron is non-relativistic so  $E \sim m_e \ll M_p$  and we can neglect  $E_1$  in the denominator of equation (8)

$$\rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 M_p^2} |M_{fi}|^2 = \frac{m_e^2 e^4}{64\pi^2 |\vec{p}|^4 \sin^4(\theta/2)}$$

- Writing  $e^2 = 4\pi\alpha$  and the kinetic energy of the electron as  $E_K = p^2/2m_e$

$$\rightarrow \left( \frac{d\sigma}{d\Omega} \right)_{\text{Rutherford}} = \frac{\alpha^2}{16E_K^2 \sin^4 \theta/2} \quad (9)$$

- ★ This is the normal expression for the Rutherford cross section. It could have been derived by considering the scattering of a non-relativistic particle in the static Coulomb potential of the proton  $V(\vec{r})$ , without any consideration of the interaction due to the intrinsic magnetic moments of the electron or proton. From this we can conclude, that in this non-relativistic limit only the interaction between the electric charges of the particles matters.

# The Mott Scattering Cross Section

- For Rutherford scattering we are in the limit where the target recoil is neglected and the scattered particle is non-relativistic  $E_K \ll m_e$
- The limit where the target recoil is neglected and the scattered particle is **relativistic** (i.e. just neglect the electron mass) is called Mott Scattering
- In this limit the electron currents, equations (4) and (6), become:

$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2E [c, s, -is, c] \quad \bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = E [0, 0, 0, 0]$$

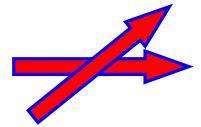
**Relativistic  $\Rightarrow$  Electron “helicity conserved”**

- It is then straightforward to obtain the result:

$$\rightarrow \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} = \underbrace{\frac{\alpha^2}{4E^2 \sin^4 \theta / 2}}_{\text{Rutherford formula with } E_K = E \ (E \gg m_e)} \underbrace{\cos^2 \frac{\theta}{2}}_{\text{Overlap between initial/final state electron wave-functions. Just QM of spin } \frac{1}{2}} \quad (10)$$

**Rutherford formula**  
with  $E_K = E \ (E \gg m_e)$

**Overlap between initial/final state electron wave-functions.**  
Just QM of spin  $\frac{1}{2}$

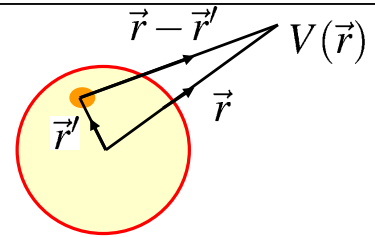


- ★ **NOTE:** we could have derived this expression from scattering of electrons in a static potential from a fixed point in space  $V(\vec{r})$ . The interaction is **ELECTRIC** rather than magnetic (spin-spin) in nature.
- ★ Still haven't taken into account the charge distribution of the proton.....

## Form Factors

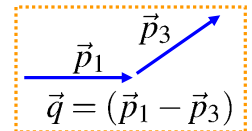
- Consider the scattering of an electron in the static potential due to an extended charge distribution.
- The potential at  $\vec{r}$  from the centre is given by:

$$V(\vec{r}) = \int \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' \quad \text{with} \quad \int \rho(\vec{r}) d^3\vec{r} = 1$$



- In first order perturbation theory the matrix element is given by:

$$\begin{aligned} M_{fi} &= \langle \psi_f | V(\vec{r}) | \psi_i \rangle = \int e^{-i\vec{p}_3 \cdot \vec{r}} V(\vec{r}) e^{i\vec{p}_1 \cdot \vec{r}} d^3\vec{r} \\ &= \int \int e^{i\vec{q} \cdot \vec{r}} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' d^3\vec{r} = \int \int e^{i\vec{q} \cdot (\vec{r}-\vec{r}')} e^{i\vec{q} \cdot \vec{r}'} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' d^3\vec{r} \end{aligned}$$



- Fix  $\vec{r}'$  and integrate over  $d^3\vec{r}$  with substitution  $\vec{R} = \vec{r} - \vec{r}'$

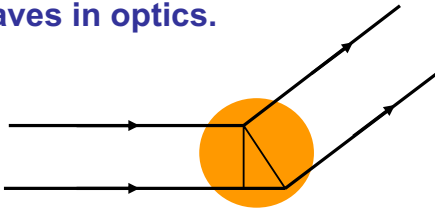
$$M_{fi} = \int e^{i\vec{q} \cdot \vec{R}} \frac{Q}{4\pi|\vec{R}|} d^3\vec{R} \int \rho(\vec{r}') e^{i\vec{q} \cdot \vec{r}'} d^3\vec{r}' = (M_{fi})_{\text{point}} F(\vec{q}^2)$$

- ★ The resulting matrix element is equivalent to the matrix element for scattering from a **point source** multiplied by the **form factor**

$$F(\vec{q}^2) = \int \rho(\vec{r}) e^{i\vec{q} \cdot \vec{r}} d^3\vec{r}$$

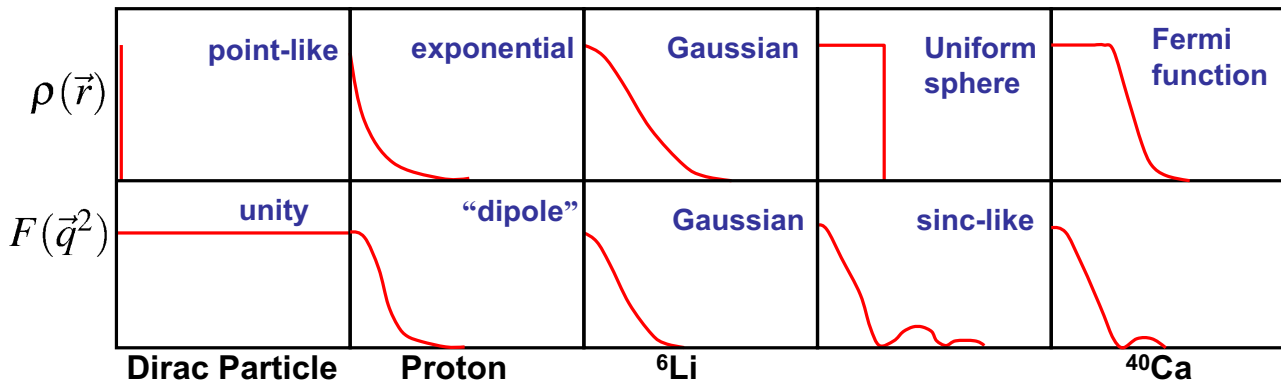
$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \rightarrow \frac{\alpha^2}{4E^2 \sin^4 \theta / 2} \cos^2 \frac{\theta}{2} |F(\vec{q}^2)|^2$$

- There is nothing mysterious about form factors – similar to diffraction of plane waves in optics.



- The finite size of the scattering centre introduces a phase difference between plane waves “scattered from different points in space”. If wavelength is long compared to size all waves in phase and  $F(\vec{q}^2) = 1$

**For example:**

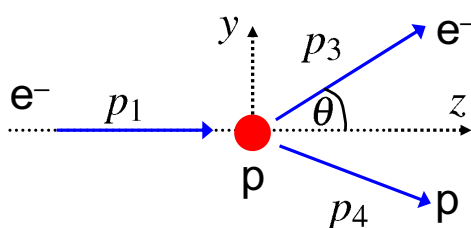


- **NOTE** that for a point charge the form factor is unity.

## Point-like Electron-Proton Elastic Scattering

- So far have only considered the case where the proton does not recoil...

For  $E_1 \gg m_e$  the general case is



$$p_1 = (E_1, 0, 0, E_1)$$

$$p_2 = (M, 0, 0, 0)$$

$$p_3 = (E_3, 0, E_3 \sin \theta, E_3 \cos \theta)$$

$$p_4 = (E_4, \vec{p}_4)$$

- From Eqn. (2) with  $m = m_e = 0$  the matrix element for this process is:

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} [(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)M^2] \quad (11)$$

- Experimentally observe scattered electron so eliminate  $p_4$

- The scalar products not involving  $p_4$  are:

$$p_1 \cdot p_2 = E_1 M \quad p_1 \cdot p_3 = E_1 E_3 (1 - \cos \theta) \quad p_2 \cdot p_3 = E_3 M$$

- From momentum conservation can eliminate  $p_4$ :  $p_4 = p_1 + p_2 - p_3$

$$p_3 \cdot p_4 = p_3 \cdot p_1 + p_3 \cdot p_2 - p_3 \cdot p_3 = E_1 E_3 (1 - \cos \theta) + E_3 M$$

$$p_1 \cdot p_4 = p_1 \cdot p_1 + p_1 \cdot p_2 - p_1 \cdot p_3 = E_1 M - E_1 E_3 (1 - \cos \theta)$$

$$p_1 \cdot p_1 = E_1^2 - |\vec{p}_1|^2 = m_e^2 \approx 0$$

i.e. neglect  $m_e$



- Substituting these scalar products in Eqn. (11) gives

$$\begin{aligned}\langle |M_{fi}|^2 \rangle &= \frac{8e^4}{(p_1 - p_3)^4} ME_1 E_3 [(E_1 - E_3)(1 - \cos \theta) + M(1 + \cos \theta)] \\ &= \frac{8e^4}{(p_1 - p_3)^4} 2ME_1 E_3 [(E_1 - E_3) \sin^2(\theta/2) + M \cos^2(\theta/2)] \quad (12)\end{aligned}$$

- Now obtain expressions for  $q^4 = (p_1 - p_3)^4$  and  $(E_1 - E_3)$

$$q^2 = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = -2E_1 E_3 (1 - \cos \theta) \quad (13)$$

$$= -4E_1 E_3 \sin^2 \theta/2 \quad (14)$$

**NOTE:**  $q^2 < 0$  Space-like

- For  $(E_1 - E_3)$  start from

$$q \cdot p_2 = (p_1 - p_3) \cdot p_2 = M(E_1 - E_3)$$

and use  $(q + p_2)^2 = p_4^2$   $q = (p_1 - p_3) = (p_4 - p_2)$

$$q^2 + p_2^2 + 2q \cdot p_2 = p_4^2$$

$$q^2 + M^2 + 2q \cdot p_2 = M^2$$

→  $q \cdot p_2 = -q^2/2$

- Hence the energy transferred to the proton:

$$E_1 - E_3 = -\frac{q^2}{2M} \quad (15)$$

Because  $q^2$  is always negative  $E_1 - E_3 > 0$  and the scattered electron is always lower in energy than the incoming electron

- Combining equations (11), (13) and (14):

$$\begin{aligned}\langle |M_{fi}|^2 \rangle &= \frac{8e^4}{16E_1^2 E_3^2 \sin^4 \theta/2} 2ME_1 E_3 \left[ M \cos^2 \theta/2 - \frac{q^2}{2M} \sin^2 \theta/2 \right] \\ &= \frac{M^2 e^4}{E_1 E_3 \sin^4 \theta/2} \left[ \cos^2 \theta/2 - \frac{q^2}{2M^2} \sin^2 \theta/2 \right]\end{aligned}$$

- For  $E \gg m_e$  we have (see handout 1)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left( \frac{E_3}{ME_1} \right)^2 |M_{fi}|^2$$

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

→  $\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta/2} \frac{E_3}{E_1} \left( \cos^2 \theta/2 - \frac{q^2}{2M^2} \sin^2 \theta/2 \right)$  (16)



# Interpretation

- So far have derived the differential cross-section for  $e^-p \rightarrow e^-p$  **elastic** scattering assuming point-like Dirac spin  $\frac{1}{2}$  particles. How should we interpret the equation?

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta/2} \frac{E_3}{E_1} \left( \cos^2 \theta/2 - \frac{q^2}{2M^2} \sin^2 \theta/2 \right)$$

- Compare with  $\left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} = \frac{\alpha^2}{4E^2 \sin^4 \theta/2} \cos^2 \frac{\theta}{2}$

the important thing to note about the Mott cross-section is that it is equivalent to scattering of spin  $\frac{1}{2}$  electrons in a fixed **electro-static** potential. Here the term  $E_3/E_1$  is due to the proton recoil.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta/2} \frac{E_3}{E_1} \left( \cos^2 \theta/2 - \underbrace{\frac{q^2}{2M^2} \sin^2 \theta/2}_{\text{new term}} \right)$$

- the new term:  $\propto \sin^2 \frac{\theta}{2}$

**Magnetic interaction : due to the spin-spin interaction**

- The above differential cross-section depends on a single parameter. For an electron scattering angle  $\theta$ , both  $q^2$  and the energy,  $E_3$ , are fixed by kinematics

- Equating (13) and (15)

$$\begin{aligned} -2M(E_1 - E_3) &= -2E_1 E_3 (1 - \cos \theta) \\ \Rightarrow \frac{E_3}{E_1} &= \frac{M}{M + E_1(1 - \cos \theta)} \end{aligned}$$

- Substituting back into (13):

$$\Rightarrow q^2 = -\frac{2ME_1^2(1 - \cos \theta)}{M + E_1(1 - \cos \theta)}$$

- e.g.  $e^-p \rightarrow e^-p$  at  $E_{\text{beam}} = 529.5$  MeV, look at scattered electrons at  $\theta = 75^\circ$

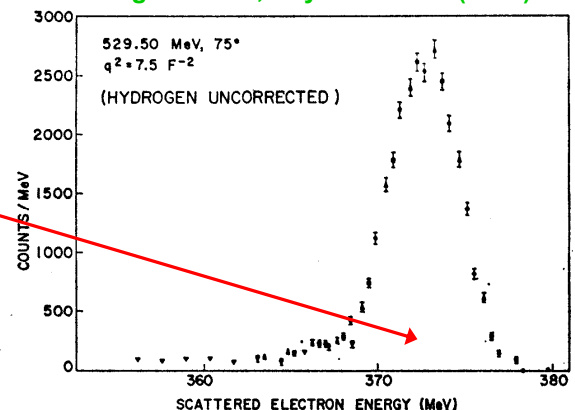
For **elastic** scattering expect:

$$\begin{aligned} E_3 &= \frac{ME_1}{M + E_1(1 - \cos \theta)} \\ E_3 &= \frac{938 \times 529}{938 + 529(1 - \cos 75^\circ)} = 373 \text{ MeV} \end{aligned}$$

The energy identifies the scatter as elastic.  
Also know squared four-momentum transfer

$$|q^2| = \frac{2 \times 938 \times 529^2(1 - \cos 75^\circ)}{938 + 529(1 - \cos 75^\circ)} = 294 \text{ MeV}^2$$

E.B.Hughes et al., Phys. Rev. 139 (1965) B458



# Elastic Scattering from a Finite Size Proton

- ★ In general the finite size of the proton can be accounted for by introducing **two structure functions**. One related to the **charge distribution** in the proton,  $G_E(q^2)$  and the other related to the distribution of the **magnetic moment** of the proton,  $G_M(q^2)$

- It can be shown that equation (16) generalizes to the **ROSENBLUTH FORMULA**.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta/2} \frac{E_3}{E_1} \left( \frac{G_E^2 + \tau G_M^2}{(1 + \tau)} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right)$$

with the Lorentz Invariant quantity:

$$\tau = -\frac{q^2}{4M^2} > 0$$

- Unlike our previous discussion of form factors, here the form factors are a function of  $q^2$  rather than  $\vec{q}^2$  and cannot simply be considered in terms of the FT of the charge and magnetic moment distributions.

But  $q^2 = (E_1 - E_3)^2 - \vec{q}^2$  and from eq (15) obtain

$$\rightarrow -\vec{q}^2 = q^2 \left[ 1 - \left( \frac{q}{2M} \right)^2 \right]$$

So for  $\frac{q^2}{4M^2} \ll 1$  we have  $q^2 \approx -\vec{q}^2$  and  $G(q^2) \approx G(\vec{q}^2)$

- Hence in the limit  $q^2/4M^2 \ll 1$  we can interpret the structure functions in terms of the Fourier transforms of the charge and magnetic moment distributions

$$G_E(q^2) \approx G_E(\vec{q}^2) = \int e^{i\vec{q} \cdot \vec{r}} \rho(\vec{r}) d^3\vec{r}$$

$$G_M(q^2) \approx G_M(\vec{q}^2) = \int e^{i\vec{q} \cdot \vec{r}} \mu(\vec{r}) d^3\vec{r}$$

- Note in deriving the Rosenbluth formula we assumed that the proton was a spin-half Dirac particle, i.e.

$$\vec{\mu} = \frac{e}{M} \vec{S}$$

- However, the experimentally measured value of the proton magnetic moment is larger than expected for a point-like Dirac particle:

$$\vec{\mu} = 2.79 \frac{e}{M} \vec{S}$$

So for the **proton** expect

$$G_E(0) = \int \rho(\vec{r}) d^3\vec{r} = 1 \quad G_M(0) = \int \mu(\vec{r}) d^3\vec{r} = \mu_p = +2.79$$

- Of course the anomalous magnetic moment of the proton is already evidence that it is not point-like !

# Measuring $G_E(q^2)$ and $G_M(q^2)$

- Express the Rosenbluth formula as:

$$\frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_0 \left( \frac{G_E^2 + \tau G_M^2}{(1 + \tau)} + 2\tau G_M^2 \tan^2 \frac{\theta}{2} \right)$$

where  $\left( \frac{d\sigma}{d\Omega} \right)_0 = \frac{\alpha^2}{4E_1^2 \sin^4 \theta/2} \frac{E_3}{E_1} \cos^2 \frac{\theta}{2}$

i.e. the Mott cross-section including the proton recoil. It corresponds to scattering from a spin-0 proton.

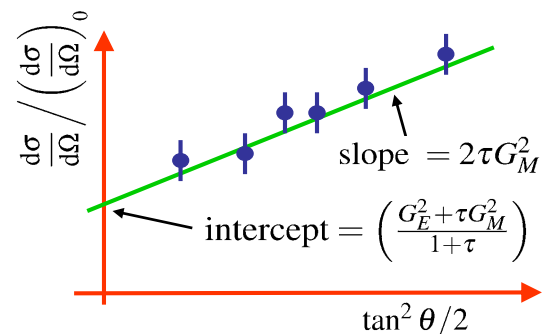
- At very low  $q^2$ :  $\tau = -q^2/4M^2 \approx 0$

$$\frac{d\sigma}{d\Omega} / \left( \frac{d\sigma}{d\Omega} \right)_0 \approx G_E^2(q^2)$$

- At high  $q^2$ :  $\tau \gg 1$

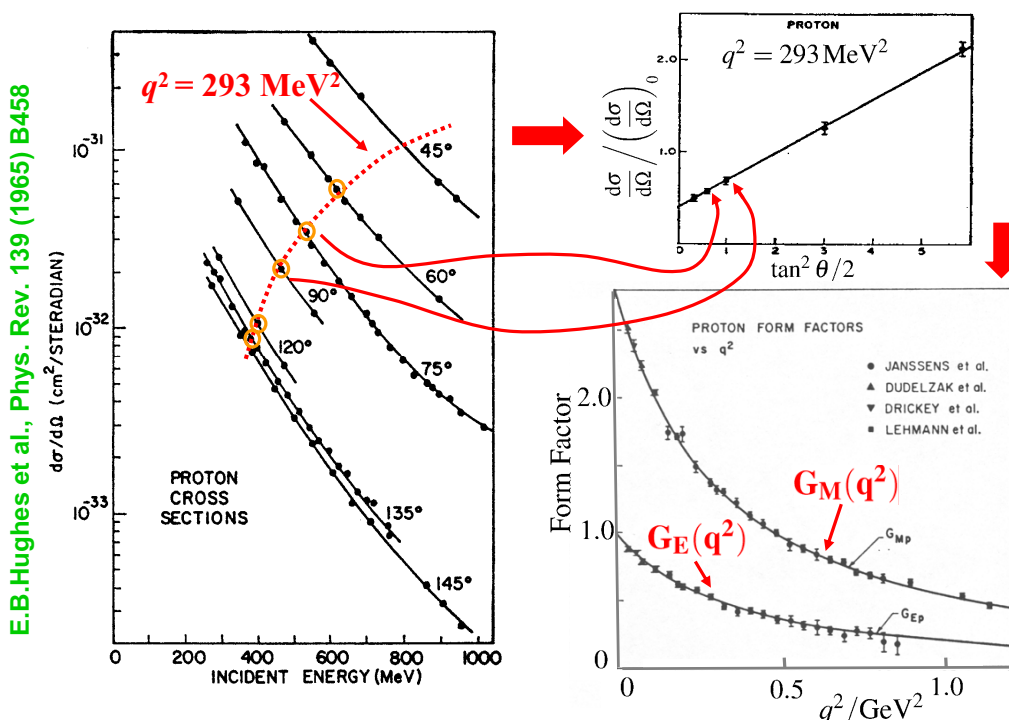
$$\frac{d\sigma}{d\Omega} / \left( \frac{d\sigma}{d\Omega} \right)_0 \approx \tau \left( 1 + 2 \tan^2 \frac{\theta}{2} \right) G_M^2(q^2)$$

- In general we are sensitive to both structure functions! These can be resolved from the angular dependence of the cross section at **FIXED**  $q^2$



## EXAMPLE: $e^-p \rightarrow e^-p$ at $E_{\text{beam}} = 529.5$ MeV

- Electron beam energies chosen to give certain values of  $q^2$
- Cross sections measured to 2-3 %



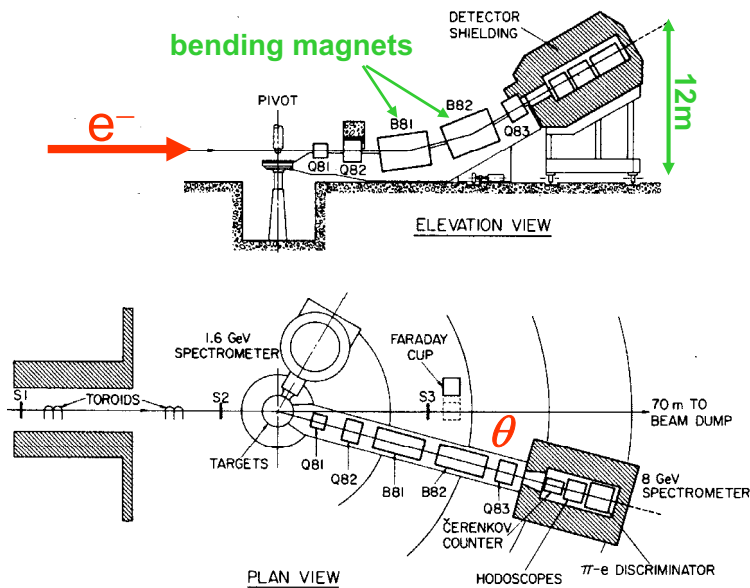
### NOTE

Experimentally find  $G_M(q^2) = 2.79 G_E(q^2)$ , i.e. the electric and magnetic form factors have same distribution

# Higher Energy Electron-Proton Scattering

★ Use electron beam from SLAC LINAC:  $5 < E_{\text{beam}} < 20 \text{ GeV}$

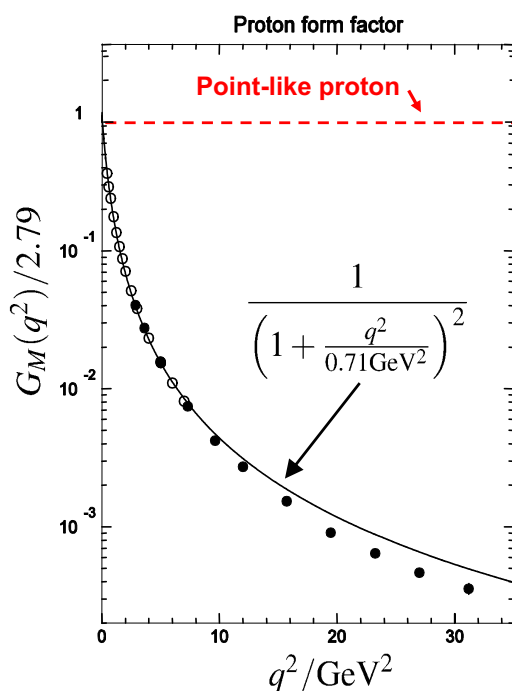
• Detect scattered electrons using the “8 GeV Spectrometer”



High  $q^2 \rightarrow$  Measure  $G_M(q^2)$

P.N.Kirk et al., Phys Rev D8 (1973) 63

## High $q^2$ Results



R.C.Walker et al., Phys. Rev. D49 (1994) 5671  
A.F.Sill et al., Phys. Rev. D48 (1993) 29

★ Form factor falls rapidly with  $q^2$

- Proton is not point-like
- Good fit to the data with “dipole form”:

$$G_E^p(q^2) \approx \frac{G_M^p}{2.79} \approx \frac{1}{(1 + q^2/0.71 \text{ GeV}^2)^2}$$

★ Taking FT find spatial charge and magnetic moment distribution

$$\rho(r) \approx \rho_0 e^{-r/a}$$

with  $a \approx 0.24 \text{ fm}$

• Corresponds to a rms charge radius

$$r_{\text{rms}} \approx 0.8 \text{ fm}$$

★ Although suggestive, does not imply proton is composite !

★ Note: so far have only considered **ELASTIC scattering**; Inelastic scattering is the subject of next handout

( Try Question 11)

# Summary: Elastic Scattering

- ★ For elastic scattering of relativistic electrons from a point-like Dirac proton:

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{\alpha^2}{4E_1^2 \sin^4 \theta/2}}_{\text{Rutherford}} \underbrace{\frac{E_3}{E_1}}_{\text{Proton recoil}} \underbrace{\left( \cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} \sin^2 \frac{\theta}{2} \right)}_{\substack{\text{Electric/} \\ \text{Magnetic} \\ \text{scattering}}} \underbrace{\left( \cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} \sin^2 \frac{\theta}{2} \right)}_{\text{Magnetic term due to spin}}$$

- ★ For elastic scattering of relativistic electrons from an extended proton:

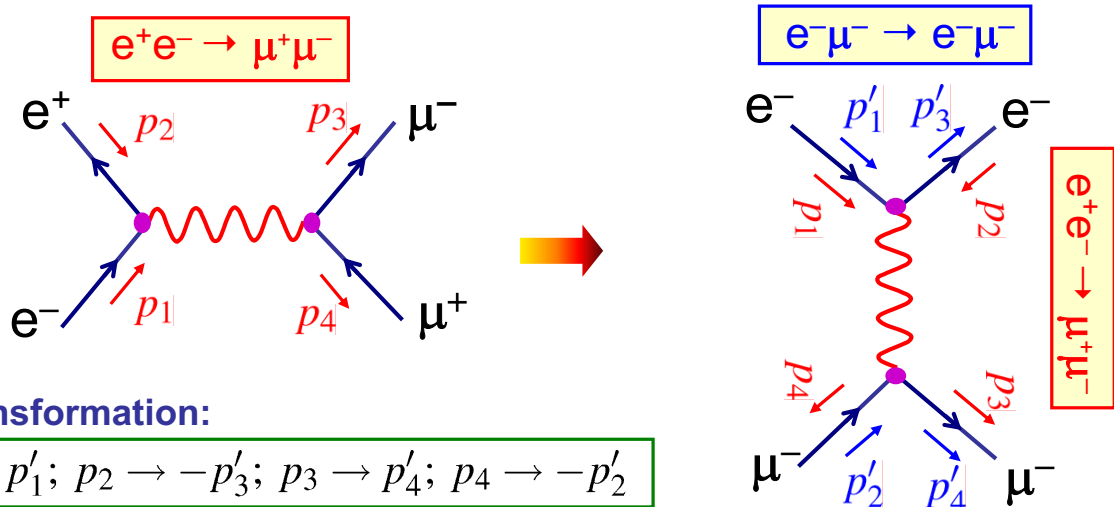
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta/2} \frac{E_3}{E_1} \left( \frac{G_E^2 + \tau G_M^2}{(1 + \tau)} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right)$$

**Rosenbluth Formula**

- ★ Electron elastic scattering from protons demonstrates that the proton is an extended object with rms charge radius of  $\sim 0.8$  fm

## Appendix I : Crossing Symmetry

- ★ Having derived the Lorentz invariant matrix element for  $e^+e^- \rightarrow \mu^+\mu^-$  “rotate” the diagram to correspond to  $e^-\mu^- \rightarrow e^-\mu^-$  and apply the principle of crossing symmetry to write down the matrix element !



- ★ The transformation:

$$p_1 \rightarrow p'_1; p_2 \rightarrow -p'_3; p_3 \rightarrow p'_4; p_4 \rightarrow -p'_2$$

Changes the spin averaged matrix element for

$$\begin{array}{ccc} \boxed{e^- e^+ \rightarrow \mu^- \mu^+} & \longrightarrow & \boxed{e^- \mu^- \rightarrow e^- \mu^-} \\ p_1 \ p_2 & & p'_1 \ p'_2 \quad p'_3 \ p'_4 \end{array}$$

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•Take ME for  $e^+e^- \rightarrow \mu^+\mu^-$  (page 143) and apply crossing symmetry:

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2} \quad \Rightarrow \quad \boxed{\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p'_1 \cdot p'_4)^2 + (p'_1 \cdot p'_2)^2}{(p'_1 \cdot p'_3)^2}} \quad (1)$$