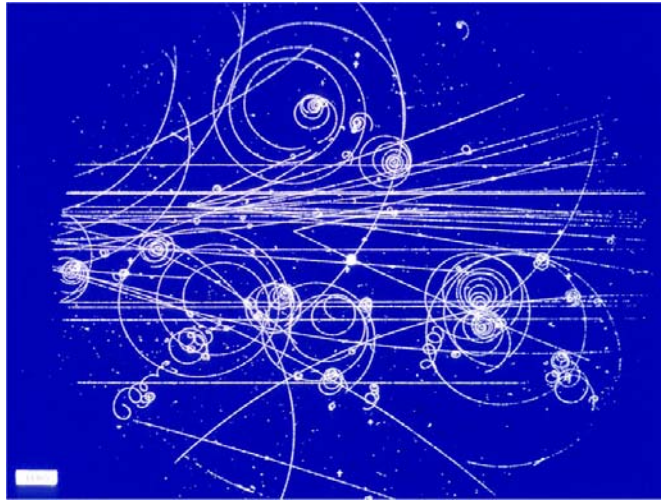


# Particle Physics

Michaelmas Term 2009  
Prof Mark Thomson



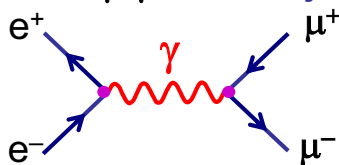
## Handout 4 : Electron-Positron Annihilation

## QED Calculations

- How to calculate a cross section using QED (e.g.  $e^+e^- \rightarrow \mu^+\mu^-$ ):

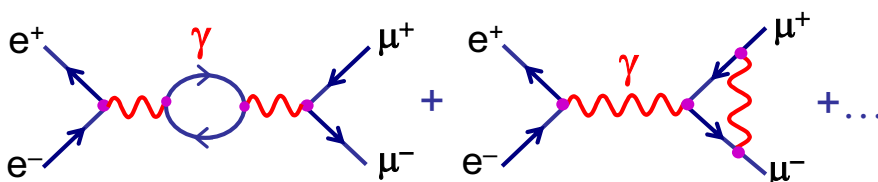
- ① Draw all possible Feynman Diagrams

- For  $e^+e^- \rightarrow \mu^+\mu^-$  there is just one lowest order diagram



$$M \propto e^2 \propto \alpha_{em}$$

+ many **second order** diagrams + ...



$$M \propto e^4 \propto \alpha_{em}^2$$

- ② For each diagram calculate the matrix element using Feynman rules derived in handout 4.
- ③ Sum the individual matrix elements (i.e. sum the amplitudes)

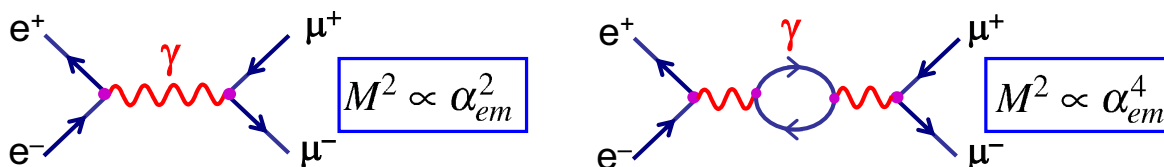
$$M_{fi} = M_1 + M_2 + M_3 + \dots$$

- **Note:** summing amplitudes therefore different diagrams for the same final state can interfere either positively or negatively!

and then square  $|M_{fi}|^2 = (M_1 + M_2 + M_3 + \dots)(M_1^* + M_2^* + M_3^* + \dots)$

➡ this gives the full perturbation expansion in  $\alpha_{em}$

- For QED  $\alpha_{em} \sim 1/137$  the lowest order diagram dominates and for most purposes it is sufficient to **neglect** higher order diagrams.



- ④ Calculate decay rate/cross section using formulae from handout 1.

- e.g. for a decay

$$\Gamma = \frac{p^*}{32\pi^2 m_a^2} \int |M_{fi}|^2 d\Omega$$

- For scattering in the centre-of-mass frame

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} |M_{fi}|^2 \quad (1)$$

- For scattering in lab. frame (neglecting mass of scattered particle)

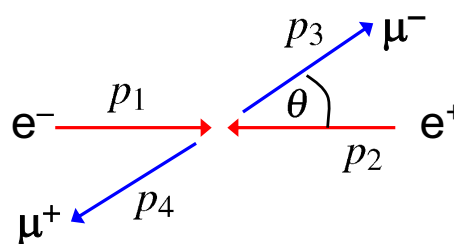
$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left( \frac{E_3}{ME_1} \right)^2 |M_{fi}|^2$$

## Electron Positron Annihilation

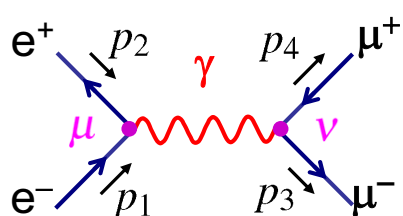
- ★ Consider the process:  $e^+e^- \rightarrow \mu^+\mu^-$

- Work in C.o.M. frame (this is appropriate for most  $e^+e^-$  colliders).

$$\begin{aligned} p_1 &= (E, 0, 0, p) & p_2 &= (E, 0, 0, -p) \\ p_3 &= (E, \vec{p}_f) & p_4 &= (E, -\vec{p}_f) \end{aligned}$$



- Only consider the lowest order Feynman diagram:



- Feynman rules give:

$$-iM = [\bar{v}(p_2)ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}(p_3)ie\gamma^\nu v(p_4)]$$

NOTE:

- Incoming anti-particle  $\bar{v}$
- Incoming particle  $u$
- Adjoint spinor written first

- In the C.o.M. frame have

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f|}{|\vec{p}_i|} |M_{fi}|^2 \quad \text{with} \quad s = (p_1 + p_2)^2 = (E + E)^2 = 4E^2$$

# Electron and Muon Currents

- Here  $q^2 = (p_1 + p_2)^2 = s$  and matrix element

$$-iM = [\bar{v}(p_2)ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}(p_3)ie\gamma^\nu v(p_4)]$$

$$\Rightarrow M = -\frac{e^2}{s} g_{\mu\nu} [\bar{v}(p_2)\gamma^\mu u(p_1)] [\bar{u}(p_3)\gamma^\nu v(p_4)]$$

- In handout 2 introduced the **four-vector current**

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

which has same form as the two terms in [ ] in the matrix element

- The matrix element can be written in terms of the electron and muon currents

$$(j_e)^\mu = \bar{v}(p_2)\gamma^\mu u(p_1) \quad \text{and} \quad (j_\mu)^\nu = \bar{u}(p_3)\gamma^\nu v(p_4)$$

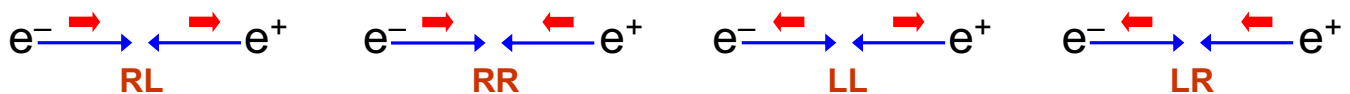
$$\Rightarrow M = -\frac{e^2}{s} g_{\mu\nu} (j_e)^\mu (j_\mu)^\nu$$

$$M = -\frac{e^2}{s} j_e \cdot j_\mu$$

- Matrix element is a four-vector scalar product – confirming it is **Lorentz Invariant**

# Spin in $e^+e^-$ Annihilation

- In general the electron and positron will not be polarized, i.e. there will be equal numbers of positive and negative helicity states
- There are four possible combinations of spins in the **initial state** !



- Similarly there are four possible helicity combinations in the final state
- In total there are **16** combinations e.g. **RL**→**RR**, **RL**→**RL**, ....
- To account for these states we need to **sum** over **all 16 possible helicity combinations** and then **average** over the number of **initial helicity states**:

$$\langle |M|^2 \rangle = \frac{1}{4} \sum_{\text{spins}} |M_i|^2 = \frac{1}{4} (|M_{LL \rightarrow LL}|^2 + |M_{LL \rightarrow LR}|^2 + \dots)$$

- ★ i.e. need to evaluate:

$$M = -\frac{e^2}{s} j_e \cdot j_\mu$$

for all 16 helicity combinations !

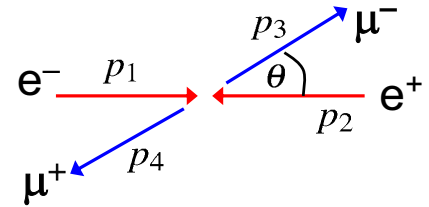
- ★ Fortunately, in the limit  $E \gg m_\mu$  only 4 helicity combinations give non-zero matrix elements – we will see that this is an important feature of QED/QCD

- In the C.o.M. frame in the limit  $E \gg m$

$$p_1 = (E, 0, 0, E); \quad p_2 = (E, 0, 0, -E)$$

$$p_3 = (E, E \sin \theta, 0, E \cos \theta);$$

$$p_4 = (E, -\sin \theta, 0, -E \cos \theta)$$



- Left- and right-handed helicity spinors (handout 3) for particles/anti-particles are:

$$u_{\uparrow} = N \begin{pmatrix} c \\ e^{i\phi} s \\ \frac{|\vec{p}|}{E+m} c \\ \frac{|\vec{p}|}{E+m} e^{i\phi} s \end{pmatrix} \quad u_{\downarrow} = N \begin{pmatrix} -s \\ e^{i\phi} c \\ \frac{|\vec{p}|}{E+m} s \\ -\frac{|\vec{p}|}{E+m} e^{i\phi} c \end{pmatrix} \quad v_{\uparrow} = N \begin{pmatrix} \frac{|\vec{p}|}{E+m} s \\ -\frac{|\vec{p}|}{E+m} e^{i\phi} c \\ \frac{s}{E+m} \\ e^{i\phi} c \end{pmatrix} \quad v_{\downarrow} = N \begin{pmatrix} \frac{|\vec{p}|}{E+m} c \\ \frac{|\vec{p}|}{E+m} e^{i\phi} s \\ \frac{c}{E+m} \\ e^{i\phi} s \end{pmatrix}$$

where  $s = \sin \frac{\theta}{2}$ ;  $c = \cos \frac{\theta}{2}$  and  $N = \sqrt{E+m}$

- In the limit  $E \gg m$  these become:

$$u_{\uparrow} = \sqrt{E} \begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}; \quad u_{\downarrow} = \sqrt{E} \begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}; \quad v_{\uparrow} = \sqrt{E} \begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}; \quad v_{\downarrow} = \sqrt{E} \begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

- The initial-state electron can either be in a left- or right-handed helicity state

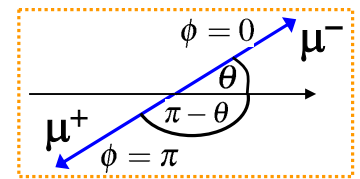
$$u_{\uparrow}(p_1) = \sqrt{E} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad u_{\downarrow}(p_1) = \sqrt{E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix};$$

- For the initial state positron ( $\theta = \pi$ ) can have either:

$$v_{\uparrow}(p_2) = \sqrt{E} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}; \quad v_{\downarrow}(p_2) = \sqrt{E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

- Similarly for the final state  $\mu^-$  which has polar angle  $\theta$  and choosing  $\phi = 0$

$$u_{\uparrow}(p_3) = \sqrt{E} \begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}; \quad u_{\downarrow}(p_3) = \sqrt{E} \begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix};$$

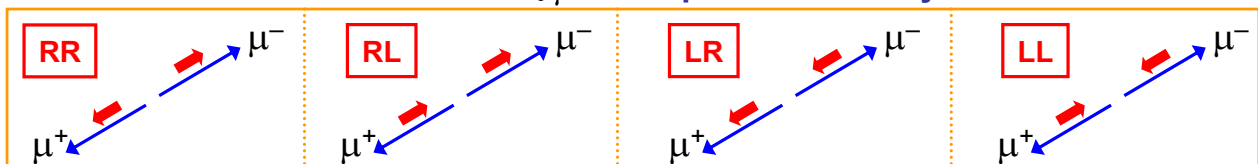


- And for the final state  $\mu^+$  replacing  $\theta \rightarrow \pi - \theta$ ;  $\phi \rightarrow \pi$  obtain

$$v_{\uparrow}(p_4) = \sqrt{E} \begin{pmatrix} c \\ s \\ -c \\ -s \end{pmatrix}; \quad v_{\downarrow}(p_4) = \sqrt{E} \begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}; \quad \left. \begin{array}{l} \text{using} \\ \sin(\frac{\pi-\theta}{2}) = \cos \frac{\theta}{2} \\ \cos(\frac{\pi-\theta}{2}) = \sin \frac{\theta}{2} \\ e^{i\pi} = -1 \end{array} \right\}$$

- Wish to calculate the matrix element  $M = -\frac{e^2}{s} j_e \cdot j_{\mu}$

- ★ first consider the muon current  $j_{\mu}$  for 4 possible helicity combinations



# The Muon Current

- Want to evaluate  $(j_\mu)^\nu = \bar{u}(p_3)\gamma^\nu v(p_4)$  for all four helicity combinations
- For arbitrary spinors  $\psi, \phi$  with it is straightforward to show that the components of  $\bar{\psi}\gamma^\mu\phi$  are

$$\bar{\psi}\gamma^0\phi = \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \quad (3)$$

$$\bar{\psi}\gamma^1\phi = \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \quad (4)$$

$$\bar{\psi}\gamma^2\phi = \psi^\dagger\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \quad (5)$$

$$\bar{\psi}\gamma^3\phi = \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2 \quad (6)$$

- Consider the  $\mu_R^-\mu_L^+$  combination using  $\psi = u_\uparrow$   $\phi = v_\downarrow$

with  $v_\downarrow = \sqrt{E} \begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}; u_\uparrow = \sqrt{E} \begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix};$

$$\bar{u}_\uparrow(p_3)\gamma^0v_\downarrow(p_4) = E(cs - sc + cs - sc) = 0$$

$$\bar{u}_\uparrow(p_3)\gamma^1v_\downarrow(p_4) = E(-c^2 + s^2 - c^2 + s^2) = 2E(s^2 - c^2) = -2E \cos \theta$$

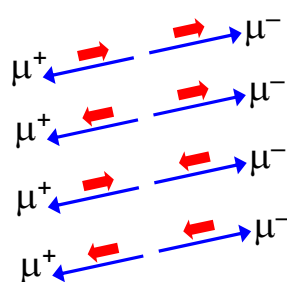
$$\bar{u}_\uparrow(p_3)\gamma^2v_\downarrow(p_4) = -iE(-c^2 - s^2 - c^2 - s^2) = 2iE$$

$$\bar{u}_\uparrow(p_3)\gamma^3v_\downarrow(p_4) = E(cs + sc + cs + sc) = 4Esc = 2E \sin \theta$$

- Hence the four-vector muon current for the **RL** combination is

$$\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos \theta, i, \sin \theta)$$

- The results for the 4 helicity combinations (obtained in the same manner) are:



$$\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos \theta, i, \sin \theta) \quad \text{RL}$$

$$\bar{u}_\uparrow(p_3)\gamma^\nu v_\uparrow(p_4) = (0, 0, 0, 0) \quad \text{RR}$$

$$\bar{u}_\downarrow(p_3)\gamma^\nu v_\downarrow(p_4) = (0, 0, 0, 0) \quad \text{LL}$$

$$\bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) = 2E(0, -\cos \theta, -i, \sin \theta) \quad \text{LR}$$

★ IN THE LIMIT  $E \gg m$  only two helicity combinations are non-zero !

- This is an important feature of QED. It applies equally to QCD.
- In the Weak interaction only one helicity combination contributes.
- Before continuing with the cross section calculation, this feature of QED is discussed in more detail

# CHIRALITY

- The helicity eigenstates for a particle/anti-particle for  $E \gg m$  are:

$$u_{\uparrow} = \sqrt{E} \begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}; \quad u_{\downarrow} = \sqrt{E} \begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}; \quad v_{\uparrow} = \sqrt{E} \begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}; \quad v_{\downarrow} = \sqrt{E} \begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

where  $s = \sin \frac{\theta}{2}$ ;  $c = \cos \frac{\theta}{2}$

- Define the matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

- In the limit  $E \gg m$  the helicity states are also eigenstates of  $\gamma^5$

$$\gamma^5 u_{\uparrow} = +u_{\uparrow}; \quad \gamma^5 u_{\downarrow} = -u_{\downarrow}; \quad \gamma^5 v_{\uparrow} = -v_{\uparrow}; \quad \gamma^5 v_{\downarrow} = +v_{\downarrow}$$

- ★ In general, define the eigenstates of  $\gamma^5$  as LEFT and RIGHT HANDED CHIRAL states

$$u_R; \quad u_L; \quad v_R; \quad v_L$$

$$\text{i.e. } \gamma^5 u_R = +u_R; \quad \gamma^5 u_L = -u_L; \quad \gamma^5 v_R = -v_R; \quad \gamma^5 v_L = +v_L$$

- In the LIMIT  $E \gg m$  (and ONLY IN THIS LIMIT):

$$u_R \equiv u_{\uparrow}; \quad u_L \equiv u_{\downarrow}; \quad v_R \equiv v_{\uparrow}; \quad v_L \equiv v_{\downarrow}$$

- ★ This is a subtle but important point: in general the **HELICITY** and **CHIRAL** eigenstates are not the same. It is **only** in the **ultra-relativistic limit** that the chiral eigenstates correspond to the helicity eigenstates.

- ★ Chirality is an important concept in the structure of QED, and any interaction of the form  $\bar{u}\gamma^\nu u$

- In general, the eigenstates of the chirality operator are:

$$\gamma^5 u_R = +u_R; \quad \gamma^5 u_L = -u_L; \quad \gamma^5 v_R = -v_R; \quad \gamma^5 v_L = +v_L$$

- Define the **projection operators**:

$$P_R = \frac{1}{2}(1 + \gamma^5); \quad P_L = \frac{1}{2}(1 - \gamma^5)$$

- The projection operators, project out the chiral eigenstates

$$\begin{aligned} P_R u_R &= u_R; & P_R u_L &= 0; & P_L u_R &= 0; & P_L u_L &= u_L \\ P_R v_R &= 0; & P_R v_L &= v_L; & P_L v_R &= v_R; & P_L v_L &= 0 \end{aligned}$$

- Note  $P_R$  projects out **right-handed particle states** and **left-handed anti-particle states**
- We can then write any spinor in terms of its left and right-handed chiral components:

$$\psi = \psi_R + \psi_L = \frac{1}{2}(1 + \gamma^5)\psi + \frac{1}{2}(1 - \gamma^5)\psi$$

# Chirality in QED

- In QED the basic interaction between a fermion and photon is:

$$ie\bar{\psi}\gamma^\mu\phi$$

- Can decompose the spinors in terms of **Left** and **Right**-handed chiral components:

$$\begin{aligned} ie\bar{\psi}\gamma^\mu\phi &= ie(\bar{\psi}_L + \bar{\psi}_R)\gamma^\mu(\phi_R + \phi_L) \\ &= ie(\bar{\psi}_R\gamma^\mu\phi_R + \bar{\psi}_R\gamma^\mu\phi_L + \bar{\psi}_L\gamma^\mu\phi_R + \bar{\psi}_L\gamma^\mu\phi_L) \end{aligned}$$

- Using the properties of  $\gamma^5$  (Q8 on examples sheet)

$$(\gamma^5)^2 = 1; \quad \gamma^{5\dagger} = \gamma^5; \quad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$$

it is straightforward to show

(Q9 on examples sheet)

$$\bar{\psi}_R\gamma^\mu\phi_L = 0; \quad \bar{\psi}_L\gamma^\mu\phi_R = 0$$

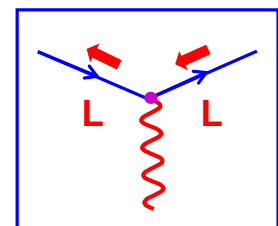
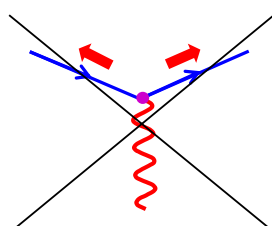
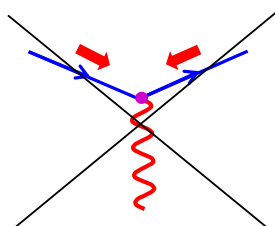
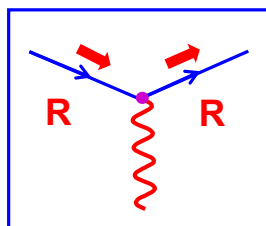
- ★ Hence only certain combinations of **chiral** eigenstates contribute to the interaction. This statement is **ALWAYS** true.
- For  $E \gg m$ , the chiral and helicity eigenstates are equivalent. This implies that for  $E \gg m$  only certain helicity combinations contribute to the QED vertex ! This is why previously we found that for two of the four helicity combinations for the muon current were zero

## Allowed QED Helicity Combinations

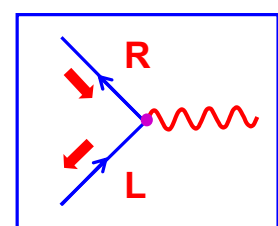
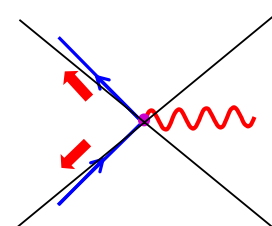
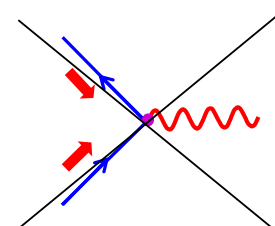
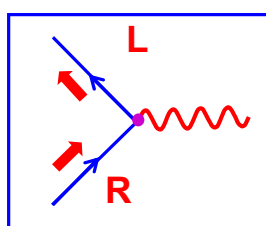
- In the ultra-relativistic limit the helicity eigenstates  $\equiv$  chiral eigenstates
- In this limit, the only non-zero **helicity** combinations in QED are:

### Scattering:

“Helicity conservation”

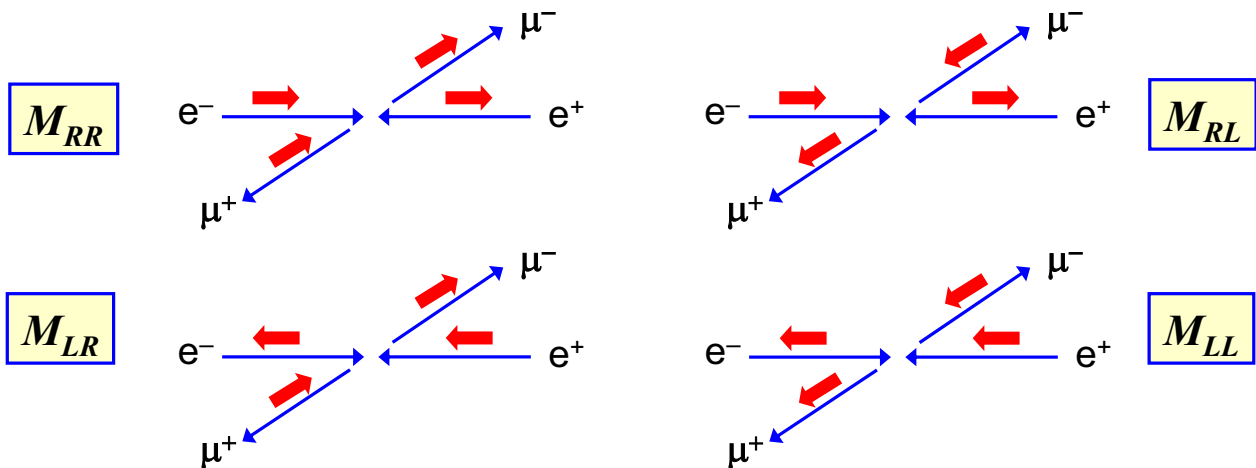


### Annihilation:

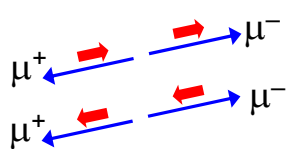


# Electron Positron Annihilation cont.

★ For  $e^+e^- \rightarrow \mu^+\mu^-$  now only have to consider the 4 matrix elements:



•Previously we derived the muon currents for the allowed helicities:



$$\begin{aligned} \mu_R^- \mu_L^+ : \quad & \bar{u}_\uparrow(p_3) \gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos\theta, i, \sin\theta) \\ \mu_L^- \mu_R^+ : \quad & \bar{u}_\downarrow(p_3) \gamma^\nu v_\uparrow(p_4) = 2E(0, -\cos\theta, -i, \sin\theta) \end{aligned}$$

•Now need to consider the electron current

## The Electron Current

•The incoming electron and positron spinors (L and R helicities) are:

$$u_\uparrow = \sqrt{E} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad u_\downarrow = \sqrt{E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}; \quad v_\uparrow = \sqrt{E} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}; \quad v_\downarrow = \sqrt{E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

•The electron current can either be obtained from equations (3)-(6) as before or it can be obtained directly from the expressions for the muon current.

$$(j_e)^\mu = \bar{v}(p_2) \gamma^\mu u(p_1) \qquad (j_\mu)^\mu = \bar{u}(p_3) \gamma^\mu v(p_4)$$

•Taking the Hermitian conjugate of the muon current gives

$$\begin{aligned} [\bar{u}(p_3) \gamma^\mu v(p_4)]^\dagger &= [u(p_3)^\dagger \gamma^0 \gamma^\mu v(p_4)]^\dagger \\ &= v(p_4)^\dagger \gamma^{\mu\dagger} \gamma^{0\dagger} u(p_3) & (AB)^\dagger &= B^\dagger A^\dagger \\ &= v(p_4)^\dagger \gamma^{\mu\dagger} \gamma^0 u(p_3) & \gamma^{0\dagger} &= \gamma^0 \\ &= v(p_4)^\dagger \gamma^0 \gamma^\mu u(p_3) & \gamma^{\mu\dagger} \gamma^0 &= \gamma^0 \gamma^\mu \\ &= \bar{v}(p_4) \gamma^\mu u(p_3) \end{aligned}$$



- Taking the complex conjugate of the muon currents for the two non-zero helicity configurations:

$$\bar{\nu}_{\downarrow}(p_4)\gamma^{\mu}u_{\uparrow}(p_3) = [\bar{u}_{\uparrow}(p_3)\gamma^{\nu}\nu_{\downarrow}(p_4)]^* = 2E(0, -\cos\theta, -i, \sin\theta)$$

$$\bar{\nu}_{\uparrow}(p_4)\gamma^{\mu}u_{\downarrow}(p_3) = [\bar{u}_{\downarrow}(p_3)\gamma^{\nu}\nu_{\uparrow}(p_4)]^* = 2E(0, -\cos\theta, i, \sin\theta)$$

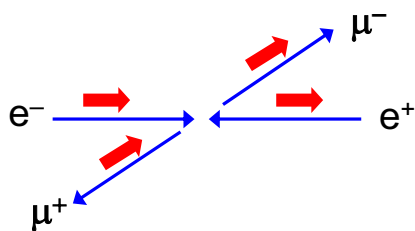
To obtain the electron currents we simply need to set  $\theta = 0$

$e^- \xrightarrow{\text{red}} \xrightarrow{\text{red}} e^+$	$e_R^- e_L^+ :$	$\bar{\nu}_{\downarrow}(p_2)\gamma^{\nu}u_{\uparrow}(p_1) = 2E(0, -1, -i, 0)$
$e^- \xleftarrow{\text{red}} \xleftarrow{\text{red}} e^+$	$e_L^- e_R^+ :$	$\bar{\nu}_{\uparrow}(p_2)\gamma^{\nu}u_{\downarrow}(p_1) = 2E(0, -1, i, 0)$

## Matrix Element Calculation

- We can now calculate  $M = -\frac{e^2}{s} j_e \cdot j_{\mu}$  for the four possible helicity combinations.

e.g. the matrix element for  $e_R^- e_L^+ \rightarrow \mu_R^- \mu_L^+$  which will denote  $M_{RR}$



Here the first subscript refers to the helicity of the  $e^-$  and the second to the helicity of the  $\mu^-$ . Don't need to specify other helicities due to "helicity conservation", only certain chiral combinations are non-zero.

★ Using:

$$e_R^- e_L^+ : (j_e)^{\mu} = \bar{\nu}_{\downarrow}(p_2)\gamma^{\mu}u_{\uparrow}(p_1) = 2E(0, -1, -i, 0)$$

$$\mu_R^- \mu_L^+ : (j_{\mu})^{\nu} = \bar{u}_{\uparrow}(p_3)\gamma^{\nu}\nu_{\downarrow}(p_4) = 2E(0, -\cos\theta, i, \sin\theta)$$

gives

$$M_{RR} = -\frac{e^2}{s} [2E(0, -1, -i, 0)] \cdot [2E(0, -\cos\theta, i, \sin\theta)]$$

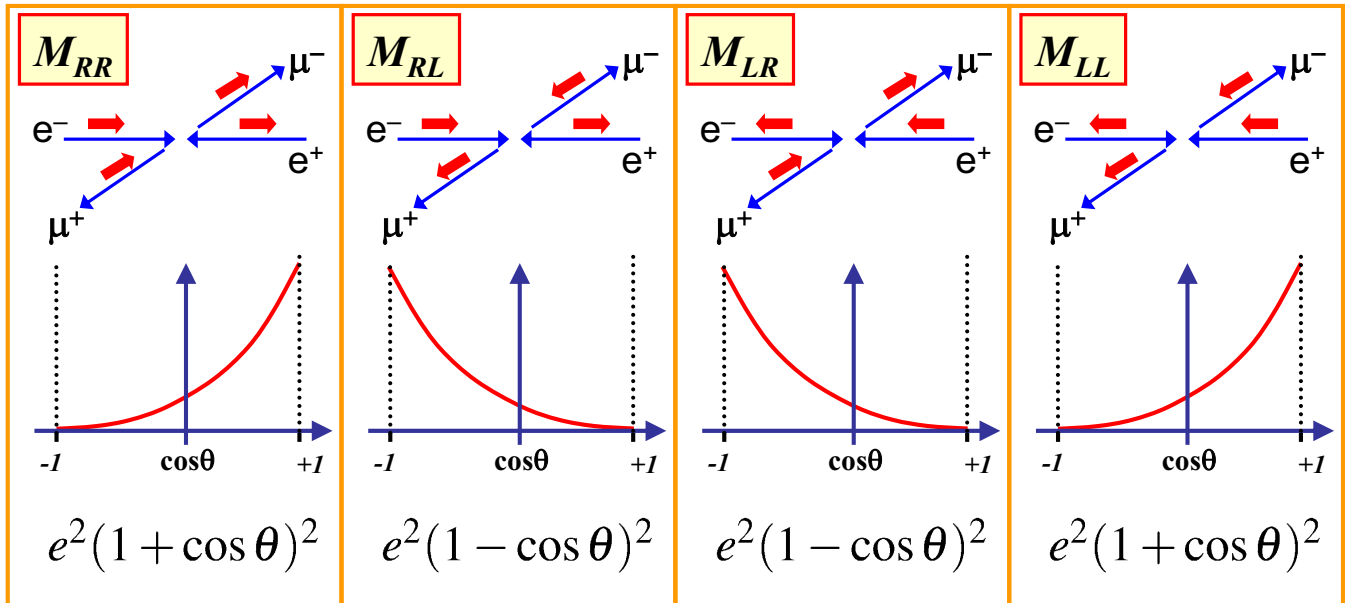
$$= -e^2(1 + \cos\theta)$$

$$= -4\pi\alpha(1 + \cos\theta) \quad \text{where} \quad \alpha = e^2/4\pi \approx 1/137$$

Similarly

$$|M_{RR}|^2 = |M_{LL}|^2 = (4\pi\alpha)^2(1 + \cos\theta)^2$$

$$|M_{RL}|^2 = |M_{LR}|^2 = (4\pi\alpha)^2(1 - \cos\theta)^2$$



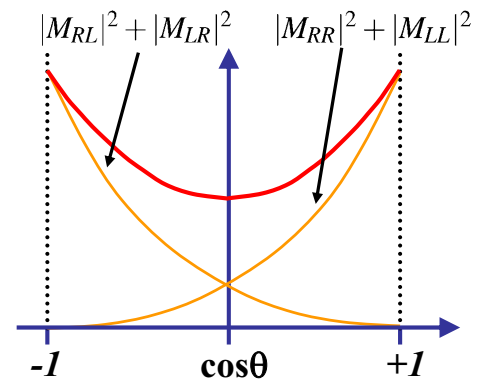
- Assuming that the incoming electrons and positrons are **unpolarized**, all 4 possible initial helicity states are equally likely.

## Differential Cross Section

- The cross section is obtained by averaging over the initial spin states and summing over the final spin states:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{4} \times \frac{1}{64\pi^2 s} (|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + |M_{LL}|^2) \\ &= \frac{(4\pi\alpha)^2}{256\pi^2 s} (2(1 + \cos\theta)^2 + 2(1 - \cos\theta)^2) \end{aligned}$$

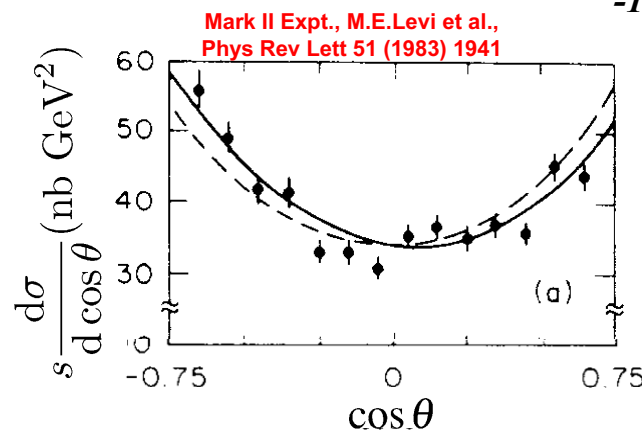
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} (1 + \cos^2\theta)$$



**Example:**

$$e^+e^- \rightarrow \mu^+\mu^-$$

$$\sqrt{s} = 29 \text{ GeV}$$



--- pure QED,  $O(\alpha^3)$   
 — QED plus Z contribution

Angular distribution becomes slightly asymmetric in higher order QED or when Z contribution is included

- The total cross section is obtained by integrating over  $\theta, \phi$  using

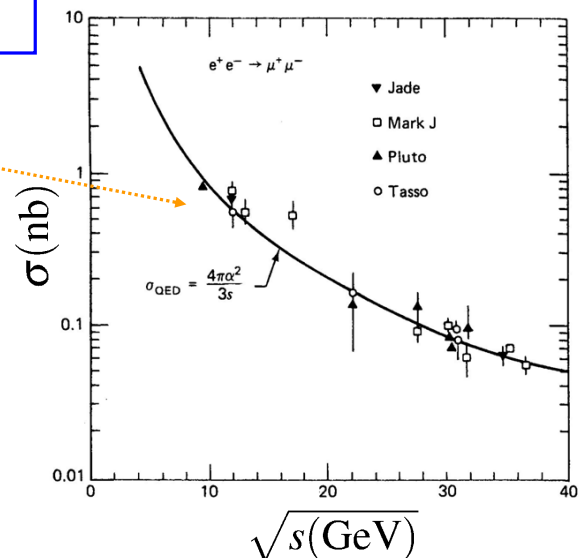
$$\int (1 + \cos^2 \theta) d\Omega = 2\pi \int_{-1}^{+1} (1 + \cos^2 \theta) d\cos \theta = \frac{16\pi}{3}$$

giving the **QED** total cross-section for the process  $e^+e^- \rightarrow \mu^+\mu^-$

$$\sigma = \frac{4\pi\alpha^2}{3s}$$

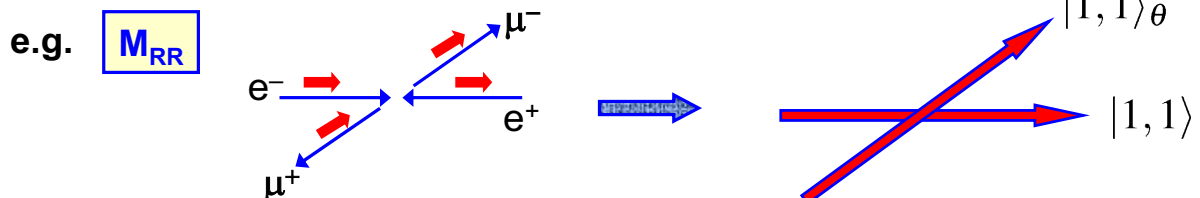
- ★ Lowest order cross section calculation provides a good description of the data !

This is an impressive result. From first principles we have arrived at an expression for the electron-positron annihilation cross section which is good to **1%**



## Spin Considerations ( $E \gg m$ )

- ★ The angular dependence of the QED electron-positron matrix elements can be understood in terms of angular momentum
- Because of the allowed helicity states, the electron and positron interact in a spin state with  $S_z = \pm 1$ , i.e. in a total spin 1 state aligned along the z axis:  $|1, +1\rangle$  or  $|1, -1\rangle$
- Similarly the muon and anti-muon are produced in a total spin 1 state aligned along an axis with polar angle  $\theta$



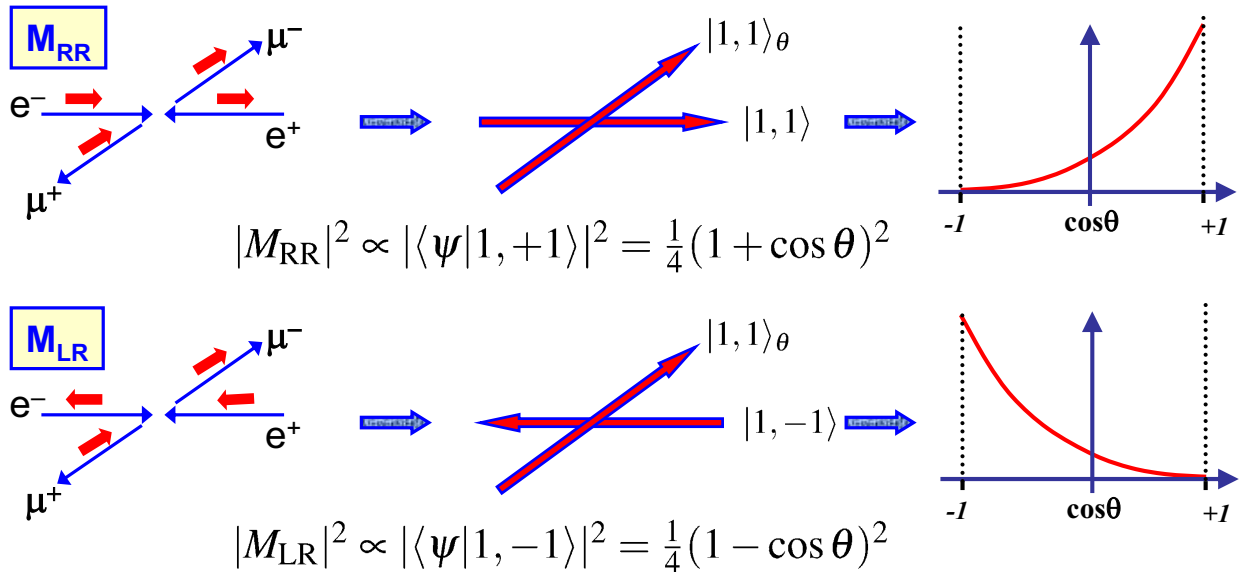
- Hence  $M_{RR} \propto \langle \psi | 1, 1 \rangle$  where  $\psi$  corresponds to the spin state,  $|1, 1\rangle_\theta$ , of the muon pair.
- To evaluate this need to express  $|1, 1\rangle_\theta$  in terms of eigenstates of  $S_z$
- In the appendix (and also in IB QM) it is shown that:

$$|1, 1\rangle_\theta = \frac{1}{2}(1 - \cos \theta)|1, -1\rangle + \frac{1}{\sqrt{2}} \sin \theta |1, 0\rangle + \frac{1}{2}(1 + \cos \theta)|1, +1\rangle$$

- Using the wave-function for a spin 1 state along an axis at angle  $\theta$

$$\psi = |1, 1\rangle_\theta = \frac{1}{2}(1 - \cos \theta)|1, -1\rangle + \frac{1}{\sqrt{2}} \sin \theta |1, 0\rangle + \frac{1}{2}(1 + \cos \theta)|1, +1\rangle$$

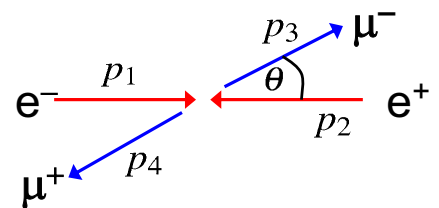
can immediately understand the angular dependence



## Lorentz Invariant form of Matrix Element

- Before concluding, note that the spin-averaged Matrix Element derived above is written in terms of the muon angle in the C.o.M. frame.

$$\begin{aligned} \langle |M_{fi}|^2 \rangle &= \frac{1}{4} \times (|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + |M_{LL}|^2) \\ &= \frac{1}{4} e^4 (2(1 + \cos \theta)^2 + 2(1 - \cos \theta)^2) \\ &= e^4 (1 + \cos^2 \theta) \end{aligned}$$



- The matrix element is **Lorentz Invariant** (scalar product of 4-vector currents) and it is desirable to write it in a frame-independent form, i.e. express in terms of Lorentz Invariant 4-vector scalar products

- In the C.o.M.  $p_1 = (E, 0, 0, E)$   $p_2 = (E, 0, 0, -E)$   
 $p_3 = (E, E \sin \theta, 0, E \cos \theta)$   $p_4 = (E, -E \sin \theta, 0, -E \cos \theta)$   
 giving:  $p_1 \cdot p_2 = 2E^2$ ;  $p_1 \cdot p_3 = E^2(1 - \cos \theta)$ ;  $p_1 \cdot p_4 = E^2(1 + \cos \theta)$

- Hence we can write

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2}$$

★ Valid in any frame !

# Summary

- ★ In the centre-of-mass frame the  $e^+e^- \rightarrow \mu^+\mu^-$  differential cross-section is

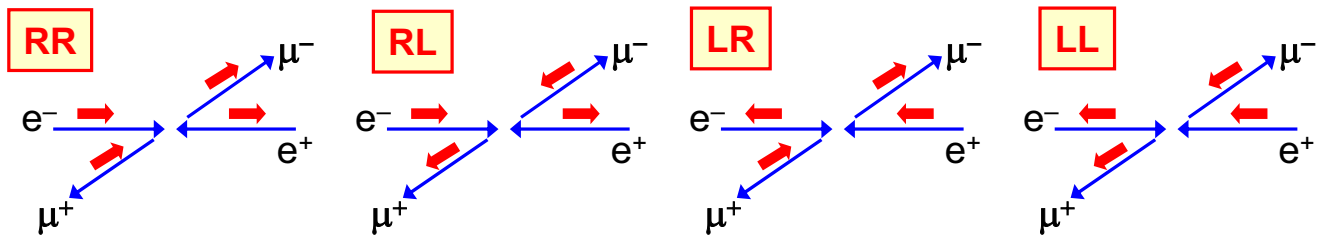
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} (1 + \cos^2 \theta)$$

**NOTE:** neglected masses of the muons, i.e. assumed  $E \gg m_\mu$

- ★ In QED only certain combinations of **LEFT-** and **RIGHT-HANDED CHIRAL** states give non-zero matrix elements
- ★ **CHIRAL** states defined by chiral projection operators

$$P_R = \frac{1}{2}(1 + \gamma^5); \quad P_L = \frac{1}{2}(1 - \gamma^5)$$

- ★ In limit  $E \gg m$  the chiral eigenstates correspond to the **HELICITY** eigenstates and only certain **HELICITY** combinations give non-zero matrix elements



## Appendix : Spin 1 Rotation Matrices

- Consider the spin-1 state with spin +1 along the axis defined by unit vector

$$\vec{n} = (\sin \theta, 0, \cos \theta)$$

- Spin state is an eigenstate of  $\vec{n} \cdot \vec{S}$  with eigenvalue +1

$$(\vec{n} \cdot \vec{S})|\psi\rangle = +1|\psi\rangle$$

(A1)

- Express in terms of linear combination of spin 1 states which are eigenstates of  $S_z$

$$|\psi\rangle = \alpha|1, 1\rangle + \beta|1, 0\rangle + \gamma|1, -1\rangle$$

with

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

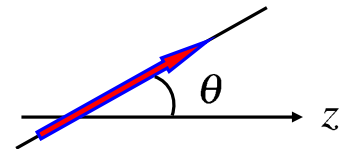
- (A1) becomes

$$(\sin \theta S_x + \cos \theta S_z)(\alpha|1, 1\rangle + \beta|1, 0\rangle + \gamma|1, -1\rangle) = \alpha|1, 1\rangle + \beta|1, 0\rangle + \gamma|1, -1\rangle \quad (\text{A2})$$

- Write  $S_x$  in terms of ladder operators  $S_x = \frac{1}{2}(S_+ + S_-)$

where  $S_+|1, 1\rangle = 0$   $S_+|1, 0\rangle = \sqrt{2}|1, 1\rangle$   $S_+|1, -1\rangle = \sqrt{2}|1, 0\rangle$

$$S_-|1, 1\rangle = \sqrt{2}|1, 0\rangle \quad S_-|1, 0\rangle = \sqrt{2}|1, -1\rangle \quad S_-|1, -1\rangle = 0$$



- **from which we find**

$$S_x|1,1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle$$

$$S_x|1,0\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle)$$

$$S_x|1,-1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle$$

- **(A2) becomes**

$$\sin \theta \left[ \frac{\alpha}{\sqrt{2}}|1,0\rangle + \frac{\beta}{\sqrt{2}}|1,-1\rangle + \frac{\beta}{\sqrt{2}}|1,1\rangle + \frac{\gamma}{\sqrt{2}}|1,0\rangle \right] +$$

$$\alpha \cos \theta |1,1\rangle - \gamma \cos \theta |1,-1\rangle = \alpha |1,1\rangle + \beta |1,0\rangle \gamma |1,-1\rangle$$

- **which gives**

$$\left. \begin{aligned} \beta \frac{\sin \theta}{\sqrt{2}} + \alpha \cos \theta &= \alpha \\ (\alpha + \gamma) \frac{\sin \theta}{\sqrt{2}} &= \beta \\ \beta \frac{\sin \theta}{\sqrt{2}} - \gamma \cos \theta &= \gamma \end{aligned} \right\}$$

- **using**  $\alpha^2 + \beta^2 + \gamma^2 = 1$  **the above equations yield**

$$\alpha = \frac{1}{\sqrt{2}}(1 + \cos \theta) \quad \beta = \frac{1}{\sqrt{2}} \sin \theta \quad \gamma = \frac{1}{\sqrt{2}}(1 - \cos \theta)$$

- **hence**

$$\psi = \frac{1}{2}(1 - \cos \theta)|1,-1\rangle + \frac{1}{\sqrt{2}} \sin \theta |1,0\rangle + \frac{1}{2}(1 + \cos \theta)|1,+1\rangle$$

• **The coefficients**  $\alpha, \beta, \gamma$  **are examples of what are known as quantum mechanical **rotation matrices**. The express how angular momentum eigenstate in a particular direction is expressed in terms of the eigenstates defined in a different direction**

$$d_{m',m}^j(\theta)$$

- **For spin-1** ( $j = 1$ ) **we have just shown that**

$$d_{1,1}^1(\theta) = \frac{1}{2}(1 + \cos \theta) \quad d_{0,1}^1(\theta) = \frac{1}{\sqrt{2}} \sin \theta \quad d_{-1,1}^1(\theta) = \frac{1}{2}(1 - \cos \theta)$$

- **For spin-1/2 it is straightforward to show**

$$d_{\frac{1}{2},\frac{1}{2}}^{\frac{1}{2}}(\theta) = \cos \frac{\theta}{2} \quad d_{-\frac{1}{2},\frac{1}{2}}^{\frac{1}{2}}(\theta) = \sin \frac{\theta}{2}$$