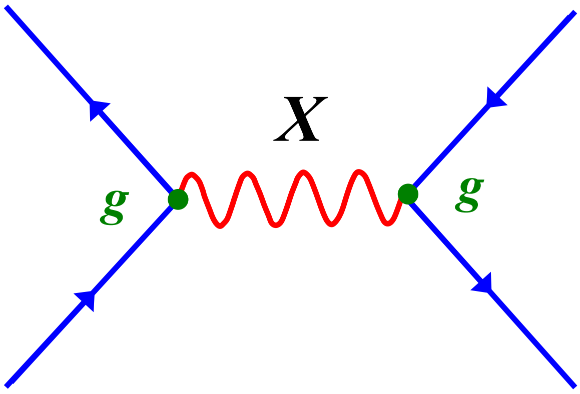
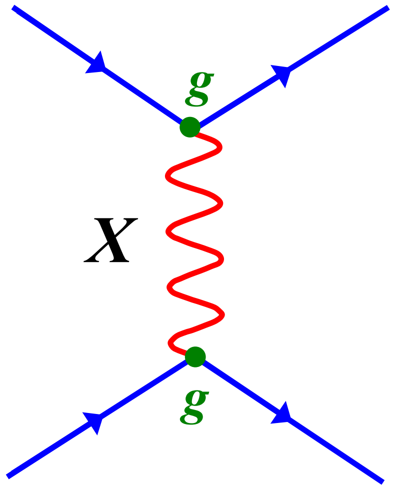
Particle Physics

Michaelmas Term 2009
Prof Mark Thomson





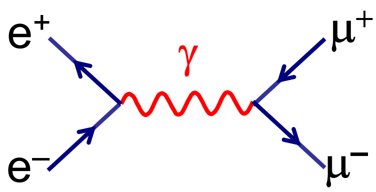
Handout 3 : Interaction by Particle Exchange and QED

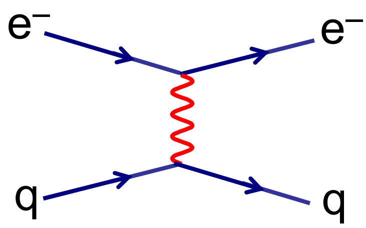
Recap

★ **Working towards a proper calculation of decay and scattering processes**

Initially concentrate on:

- $e^+e^- \to \mu^+\mu^-$
- $e^- q \to e^- q$





In Handout 1 covered the <u>relativistic</u> calculation of particle decay rates and cross sections

$$\sigma \propto \frac{|M|^2}{\text{flux}} \times (\text{phase space})$$

In Handout 2 covered <u>relativistic</u> treatment of spin-half particles

Dirac Equation

This handout concentrate on the Lorentz Invariant Matrix Element

- **Interaction by particle exchange**
- **Introduction to Feynman diagrams**
- **The Feynman rules for QED**

Interaction by Particle Exchange

- Calculate transition rates from **Fermi's Golden Rule**

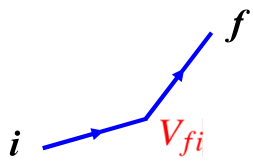
$$\boxed{\Gamma_{fi} = 2\pi|T_{fi}|^2\rho(E_f)}$$

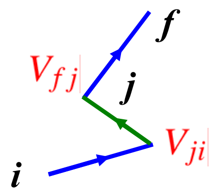
where T_{fi} is perturbation expansion for the Transition Matrix Element

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i}\frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} + \ldots$$

- For particle scattering, the first two terms in the perturbation series can be viewed as:

"scattering in a potential"



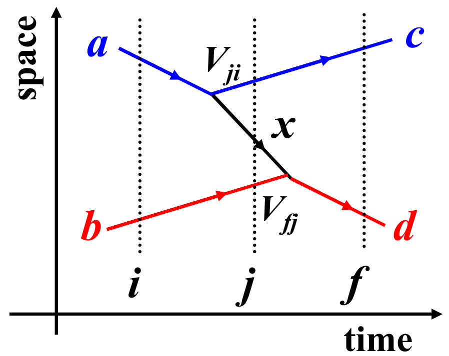


"scattering via an intermediate state"

- **"Classical picture"** – particles act as sources for fields which give rise a potential in which other particles scatter – **"action at a distance"**
- **"Quantum Field Theory picture"** – forces arise due to the exchange of virtual particles. No action at a distance + **forces** between particles now **due to particles**

(start of non-examinable section)

- Consider the particle interaction $a + b \rightarrow c + d$ which occurs via an intermediate state corresponding to the exchange of particle x
- One possible space-time picture of this process is:



Initial state i : $a + b$
Final state f : $c + d$
Intermediate state j : $c + b + x$

- This **time-ordered** diagram corresponds to ***a*** "emitting" ***x*** and then ***b*** absorbing ***x***

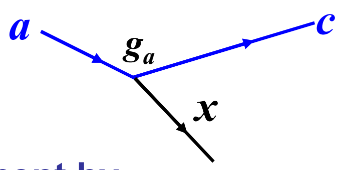
- The corresponding term in the perturbation expansion is:

$$T_{fi} = \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j}$$

$$T_{fi}^{ab} = \frac{\langle d|V|x+b\rangle\langle c+x|V|a\rangle}{(E_a+E_b)-(E_c+E_x+E_b)}$$

- T_{fi}^{ab} refers to the time-ordering where ***a*** emits ***x*** before ***b*** absorbs it

- Need an expression for $\langle c+x|V|a\rangle$ in non-invariant matrix element T_{fi}
- Ultimately aiming to obtain Lorentz Invariant ME
- Recall T_{fi} is related to the invariant matrix element by



$$T_{fi} = \prod_k (2E_k)^{-1/2} M_{fi}$$

where k runs over all particles in the matrix element

- Here we have

$$\langle c+x|V|a\rangle = \frac{M_{(a\to c+x)}}{(2E_a 2E_c 2E_x)^{1/2}}$$

$M_{(a\to c+x)}$ is the "Lorentz Invariant" matrix element for $a \to c + x$

★ The simplest Lorentz Invariant quantity is a scalar, in this case

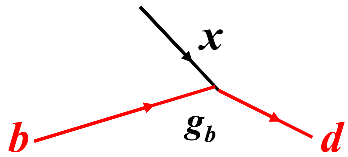
$$\langle c+x|V|a\rangle = \frac{g_a}{(2E_a 2E_c 2E_x)^{1/2}}$$

g_a is a measure of the strength of the interaction $a \to c + x$

Note : the matrix element is only LI in the sense that it is defined in terms of LI wave-function normalisations and that the form of the coupling is LI

Note : in this "illustrative" example g is not dimensionless.

Similarly $$\langle d|V|x+b\rangle = \frac{g_b}{(2E_b 2E_d 2E_x)^{1/2}}$$



Giving
$$\begin{aligned} T_{fi}^{ab} &= \frac{\langle d|V|x+b\rangle\langle c+x|V|a\rangle}{(E_a+E_b)-(E_c+E_x+E_b)} \\ &= \frac{1}{2E_x}\cdot\frac{1}{(2E_a 2E_b 2E_c 2E_d)^{1/2}}\cdot\frac{g_a g_b}{(E_a-E_c-E_x)} \end{aligned}$$

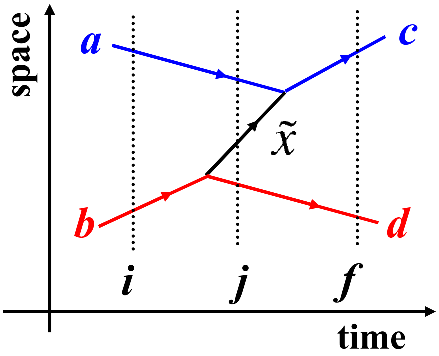
★ The "Lorentz Invariant" matrix element for the entire process is

$$\begin{aligned} M_{fi}^{ab} &= (2E_a 2E_b 2E_c 2E_d)^{1/2} T_{fi}^{ab} \\ &= \frac{1}{2E_x}\cdot\frac{g_a g_b}{(E_a-E_c-E_x)} \end{aligned}$$

Note:

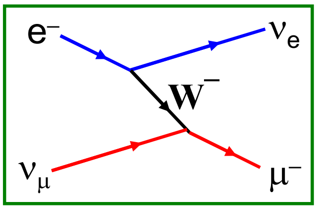
- M_{fi}^{ab} refers to the time-ordering where a emits x before b absorbs it
  It is not Lorentz invariant, order of events in time depends on frame
- Momentum is conserved at each interaction vertex but not energy
  $E_j \neq E_i$
- Particle x is "on-mass shell" i.e. $E_x^2 = \vec{p}_x^{\,2} + m^2$

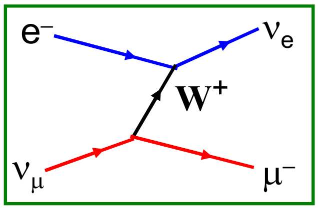
★But need to consider also the other time ordering for the process



•This **time-ordered** diagram corresponds to *b* "emitting" $\tilde{x}$ and then *a* absorbing $\tilde{x}$

• $\tilde{x}$ is the anti-particle of x e.g.





•The **Lorentz invariant matrix element** for this time ordering is:

$$M_{fi}^{ba} = \frac{1}{2E_x}\cdot\frac{g_a g_b}{(E_b - E_d - E_x)}$$

★In QM need to sum over matrix elements corresponding to same final state:

$$\begin{aligned} M_{fi} &= M_{fi}^{ab} + M_{fi}^{ba} \\ &= \frac{g_a g_b}{2E_x}\cdot\left(\frac{1}{E_a - E_c - E_x} + \frac{1}{E_b - E_d - E_x}\right) \\ &= \frac{g_a g_b}{2E_x}\cdot\left(\frac{1}{E_a - E_c - E_x} - \frac{1}{E_a - E_c + E_x}\right) \end{aligned}$$

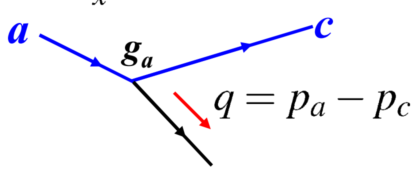
Energy conservation:
$(E_a + E_b = E_c + E_d)$

•Which gives

$$\begin{aligned} M_{fi} &= \frac{g_a g_b}{2E_x}\cdot\frac{2E_x}{(E_a - E_c)^2 - E_x^2} \\ &= \frac{g_a g_b}{(E_a - E_c)^2 - E_x^2} \end{aligned}$$

•From 1st time ordering $E_x^2 = \vec{p}_x^{\,2} + m_x^2 = (\vec{p}_a - \vec{p}_c)^2 + m_x^2$

giving
$$\begin{aligned} M_{fi} &= \frac{g_a g_b}{(E_a - E_c)^2 - (\vec{p}_a - \vec{p}_c)^2 - m_x^2} \\ &= \frac{g_a g_b}{(p_a - p_c)^2 - m_x^2} \end{aligned}$$



(end of non-examinable section)

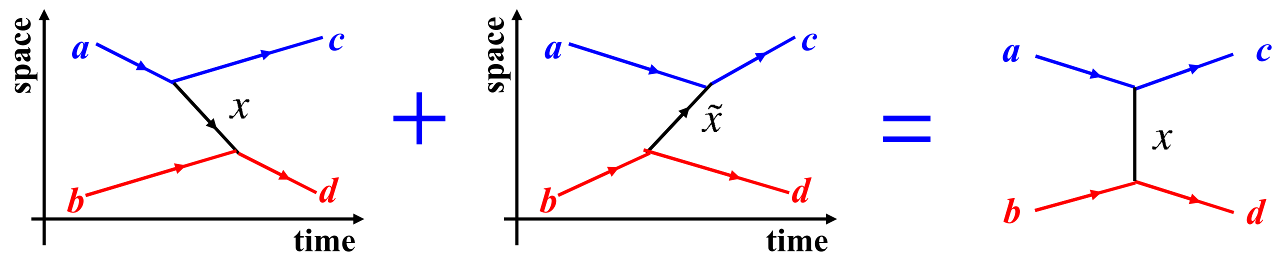
$$\Rightarrow \quad \boxed{M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}}$$

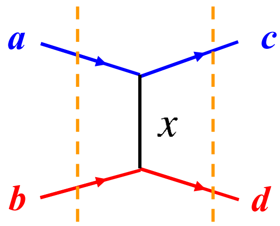
• After summing over all possible time orderings, M_{fi} is (as anticipated) **Lorentz invariant**. This is a remarkable result – the sum over all time orderings gives a frame independent matrix element.

•Exactly the same result would have been obtained by considering the annihilation process

Feynman Diagrams

- The sum over all possible time-orderings is represented by a **FEYNMAN diagram**





In a Feynman diagram:

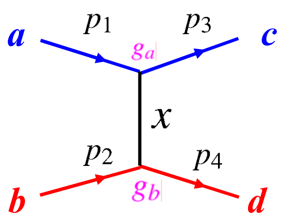
- the LHS represents the initial state
- the RHS is the final state
- everything in between is “how the interaction happened”

- It is important to remember that **energy and momentum** are conserved at each interaction vertex in the diagram.

- The factor $1/(q^2 - m_x^2)$ is the propagator; it arises naturally from the above discussion of interaction by particle exchange

★ The matrix element: $M_{fi} = \dfrac{g_a g_b}{q^2 - m_x^2}$ depends on:

- The fundamental strength of the interaction at the two vertices $g_a,\ g_b$
- The four-momentum, q, carried by the (virtual) particle which is determined from energy/momentum conservation at the vertices. Note q^2 can be either positive or negative.

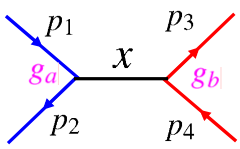


Here $q = p_1 - p_3 = p_4 - p_2 = t$

For **elastic scattering**: $p_1 = (E, \vec{p}_1); \quad p_3 = (E, \vec{p}_3)$

$$q^2 = (E - E)^2 - (\vec{p}_1 - \vec{p}_2)^2$$

$\boxed{q^2 < 0}$ termed “space-like”



Here $q = p_1 + p_2 = p_3 + p_4 = s$

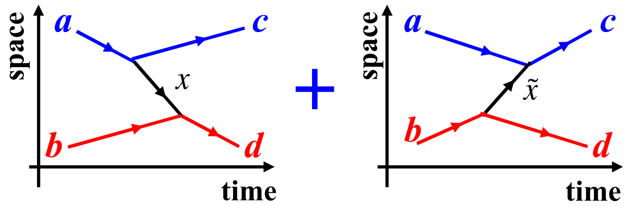
In CoM: $p_1 = (E, \vec{p}); \quad p_2 = (E, -\vec{p})$

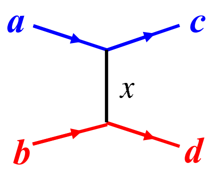
$$q^2 = (E + E)^2 - (\vec{p} - \vec{p})^2 = 4E^2$$

$\boxed{q^2 > 0}$ termed “time-like”

Virtual Particles

"Time-ordered QM" | Feynman diagram





$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- Momentum conserved at vertices
- Energy **not** conserved at vertices
- Exchanged particle **"on mass shell"**

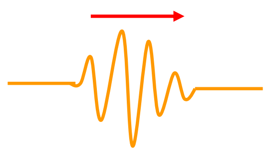
$$E_x^2 - |\vec{p}_x|^2 = m_x^2$$

- Momentum **AND** energy conserved at interaction vertices
- Exchanged particle **"off mass shell"**

$$E_x^2 - |\vec{p}_x|^2 = q^2 \neq m_x^2$$

VIRTUAL PARTICLE

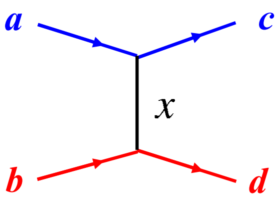
- Can think of observable "on mass shell" particles as propagating waves and unobservable virtual particles as normal modes between the source particles:



Aside: V(r) from Particle Exchange

★ Can view the scattering of an electron by a proton at rest in two ways:

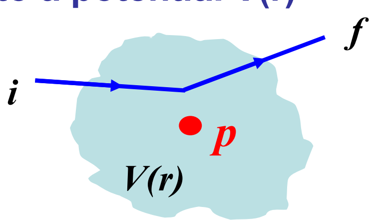
- Interaction by particle exchange in 2nd order perturbation theory.



$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- Could also evaluate the same process in first order perturbation theory treating proton as a fixed source of a field which gives rise to a potential V(r)

$$M = \langle \psi_f | V(r) | \psi_i \rangle$$



Obtain same expression for M_{fi} using

$$V(r) = g_a g_b \frac{e^{-mr}}{r}$$

YUKAWA potential

★ In this way can relate potential and forces to the particle exchange picture

★ However, scattering from a fixed potential $V(r)$ is not a relativistic invariant view

Quantum Electrodynamics (QED)

★Now consider the interaction of an electron and tau lepton by the exchange of a photon. Although the general ideas we applied previously still hold, we now have to account for the spin of the electron/tau-lepton and also the spin (polarization) of the virtual photon.

(Non-examinable)

•The basic interaction between a photon and a charged particle can be introduced by making the minimal substitution (part II electrodynamics)

$$\vec{p} \to \vec{p} - q\vec{A}; \quad E \to E - q\phi$$

(here $q =$ charge)

In QM: $$\vec{p} = -i\vec{\nabla}; \qquad E = i\partial/\partial t$$

Therefore make substitution: $i\partial_\mu \to i\partial_\mu - qA_\mu$

where $A_\mu = (\phi, -\vec{A}); \qquad \partial_\mu = (\partial/\partial t, +\vec{\nabla})$

•The Dirac equation:

$$\gamma^\mu \partial_\mu \psi + im\psi = 0 \;\Rightarrow\; \gamma^\mu \partial_\mu \psi + iq\gamma^\mu A_\mu \psi + im\psi = 0$$

$$(\times i) \;\Rightarrow\; i\gamma^0 \frac{\partial \psi}{\partial t} + i\vec{\gamma}.\vec{\nabla}\psi - q\gamma^\mu A_\mu \psi - m\psi = 0$$

$$i\gamma^0 \frac{\partial \psi}{\partial t} = \gamma^0 \hat{H}\psi = m\psi - i\vec{\gamma}.\vec{\nabla}\psi + q\gamma^\mu A_\mu \psi$$

$$\times\gamma^0: \quad \hat{H}\psi = \underbrace{(\gamma^0 m - i\gamma^0\vec{\gamma}.\vec{\nabla})}_{\text{Combined rest mass + K.E.}}\psi + \underbrace{q\gamma^0\gamma^\mu A_\mu \psi}_{\text{Potential energy}}$$

•We can identify the potential energy of a charged spin-half particle in an electromagnetic field as:

$$\boxed{\hat{V}_D = q\gamma^0\gamma^\mu A_\mu}$$

(note the A_0 term is just: $q\gamma^0\gamma^0 A_0 = q\phi$)

•The final complication is that we have to account for the photon polarization states.

$$A_\mu = \varepsilon_\mu^{(\lambda)} e^{i(\vec{p}.\vec{r} - Et)}$$

e.g. for a real photon propagating in the z direction we have two orthogonal transverse polarization states

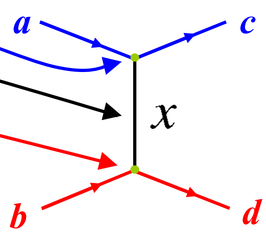
$$\varepsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \varepsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Could equally have chosen circularly polarized states

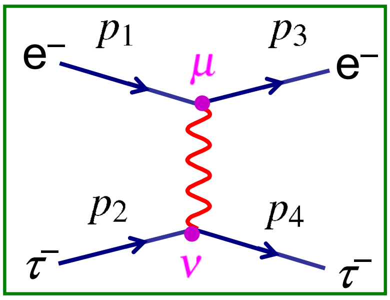
•Previously with the example of a simple spin-less interaction we had:

$$M = \langle\psi_c|V|\psi_a\rangle \frac{1}{q^2 - m_x^2} \langle\psi_d|V|\psi_b\rangle$$

$$\langle\psi_c|V|\psi_a\rangle = g_a \qquad \langle\psi_d|V|\psi_b\rangle = g_b$$



★In QED we could again go through the procedure of summing the time-orderings using Dirac spinors and the expression for $\hat{V}_D$. If we were to do this, remembering to sum over all photon polarizations, we would obtain:



$$M = \underbrace{\left[u_e^\dagger(p_3) q_e \gamma^0 \gamma^\mu u_e(p_1)\right]}_{\text{Interaction of } e^- \text{ with photon}} \underbrace{\sum_\lambda \frac{\varepsilon_\mu^\lambda (\varepsilon_\nu^\lambda)^*}{q^2}}_{\text{Massless photon propagator summing over polarizations}} \underbrace{\left[u_\tau^\dagger(p_4) q_\tau \gamma^0 \gamma^\nu u_\tau(p_2)\right]}_{\text{Interaction of } \tau^- \text{ with photon}}$$

•All the physics of **QED** is in the above expression !

•The sum over the polarizations of the **VIRTUAL** photon has to include longitudinal and scalar contributions, i.e.

$$\varepsilon^{(0)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \varepsilon^{(1)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \varepsilon^{(2)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad \varepsilon^{(3)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

and gives: $$\sum_\lambda \varepsilon_\mu^\lambda (\varepsilon_\nu^\lambda)^* = -g_{\mu\nu}$$

This is not obvious – for the moment just take it on trust

and the invariant matrix element becomes:

(end of non-examinable section)

$$M = \left[u_e^\dagger(p_3) q_e \gamma^0 \gamma^\mu u_e(p_1)\right] \frac{-g_{\mu\nu}}{q^2} \left[u_\tau^\dagger(p_4) q_\tau \gamma^0 \gamma^\nu u_\tau(p_2)\right]$$

•Using the definition of the adjoint spinor $\overline{\psi} = \psi^\dagger \gamma^0$

$$\boxed{M = \left[\overline{u}_e(p_3) q_e \gamma^\mu u_e(p_1)\right] \frac{-g_{\mu\nu}}{q^2} \left[\overline{u}_\tau(p_4) q_\tau \gamma^\nu u_\tau(p_2)\right]}$$

★ This is a remarkably simple expression ! It is shown in Appendix V of Handout 3 that $\overline{u}_1 \gamma^\mu u_2$ transforms as a four vector. Writing

$$j_e^\mu = \overline{u}_e(p_3) \gamma^\mu u_e(p_1) \qquad j_\tau^\nu = \overline{u}_\tau(p_4) \gamma^\nu u_\tau(p_2)$$

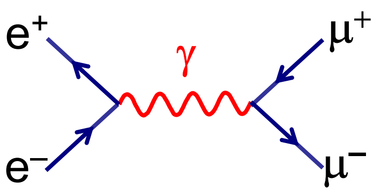
$$M = -q_e q_\tau \frac{j_e \cdot j_\tau}{q^2}$$ showing that M is **Lorentz Invariant**

Feynman Rules for QED

• It should be remembered that the expression

$$M = [\overline{u}_e(p_3) q_e \gamma^\mu u_e(p_1)] \frac{-g_{\mu\nu}}{q^2} [\overline{u}_\tau(p_4) q_\tau \gamma^\nu u_\tau(p_2)]$$

hides a lot of complexity. We have summed over all possible **time-orderings** and summed over all **polarization states** of the virtual photon. If we are then presented with a new Feynman diagram we don't want to go through the full calculation again. Fortunately this isn't necessary – can just write down matrix element using a set of simple rules



Basic Feynman Rules:

- Propagator factor for each internal line
  (i.e. each internal virtual particle)
- Dirac Spinor for each external line
  (i.e. each real incoming or outgoing particle)
- Vertex factor for each vertex

Basic Rules for QED

- **External Lines**

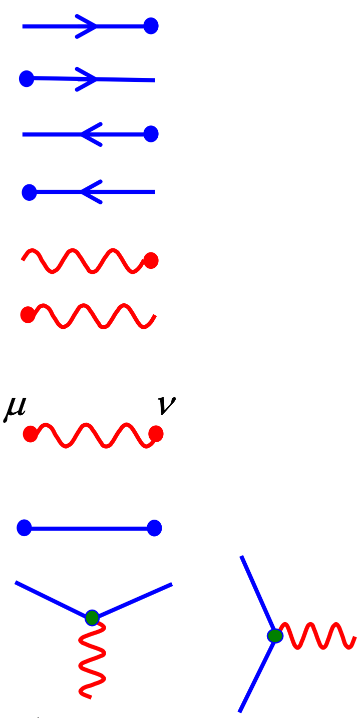
spin 1/2	incoming particle	$u(p)$
	outgoing particle	$\overline{u}(p)$
	incoming antiparticle	$\overline{v}(p)$
	outgoing antiparticle	$v(p)$
spin 1	incoming photon	$\varepsilon^\mu(p)$
	outgoing photon	$\varepsilon^\mu(p)^*$

- **Internal Lines (propagators)**

spin 1	photon	$-\frac{ig_{\mu\nu}}{q^2}$
spin 1/2	fermion	$\frac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2}$

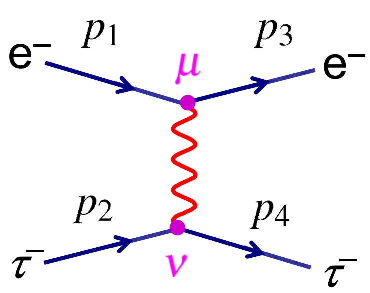
- **Vertex Factors**

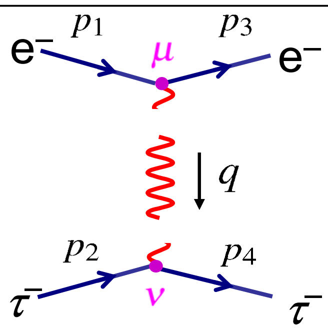
spin 1/2	fermion (charge *-e*)	$ie\gamma^\mu$



- **Matrix Element** $-iM$ = **product of all factors**

e.g.





$$\overline{u}_e(p_3)[ie\gamma^\mu]u_e(p_1)$$

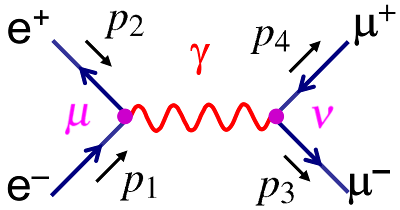
$$\frac{-ig_{\mu\nu}}{q^2}$$

$$\overline{u}_\tau(p_4)[ie\gamma^\nu]u_\tau(p_2)$$

$$-iM = [\overline{u}_e(p_3)ie\gamma^\mu u_e(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}_\tau(p_4)ie\gamma^\nu u_\tau(p_2)]$$

- **Which is the same expression as we obtained previously**

e.g.



$$-iM = [\overline{v}(p_2)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_3)ie\gamma^\nu v(p_4)]$$

Note:
- At each vertex the adjoint spinor is written first
- Each vertex has a different index
- The $g_{\mu\nu}$ of the propagator connects the indices at the vertices

Summary

★ Interaction by particle exchange naturally gives rise to **Lorentz Invariant Matrix Element of the form**

$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

★ Derived the basic interaction in **QED** taking into account the spins of the fermions and polarization of the virtual photons:

$$-iM = [\overline{u}(p_3)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_4)ie\gamma^\nu u(p_2)]$$

★ **We now have all the elements to perform proper calculations in QED !**