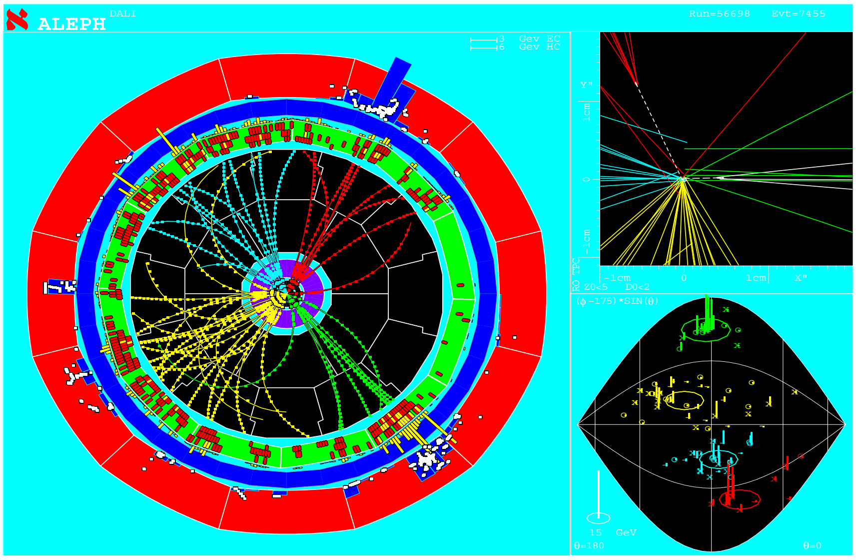
Particle Physics

Michaelmas Term 2009
Prof. Mark Thomson



Handout 1 : Introduction

Cambridge Particle Physics Courses

PART II

"Particle and Nuclear Physics"
Prof Ward/Dr Lester

Introductory course

PART III

Major Option
"Particle Physics"
Prof Thomson

Covering most <u>Standard Model</u> physics, both experiment and underlying theory

Minor Option
"Gauge Field Theory"
Dr Batley

The theoretical principles behind the SM

Minor Option
"Particle Astrophysics"
Profs Efstathiou & Parker

The connection between particle physics and cosmology

Course Synopsis

Handout 1:	Introduction, Decay Rates and Cross Sections
Handout 2:	The Dirac Equation and Spin
Handout 3:	Interaction by Particle Exchange
Handout 4:	Electron – Positron Annihilation
Handout 5:	Electron – Proton Scattering
Handout 6:	Deep Inelastic Scattering
Handout 7:	Symmetries and the Quark Model
Handout 8:	QCD and Colour
Handout 9:	V-A and the Weak Interaction
Handout 10:	Leptonic Weak Interactions
Handout 11:	Neutrinos and Neutrino Oscillations
Handout 12:	The CKM Matrix and CP Violation
Handout 13:	Electroweak Unification and the W and Z Bosons
Handout 14:	Tests of the Standard Model
Handout 15:	The Higgs Boson and Beyond

★ Will concentrate on the modern view of particle physics with the emphasis on how theoretical concepts relate to recent experimental measurements

★ Aim: by the end of the course you should have a good understanding of both aspects of particle physics

Preliminaries

Web-page: www.hep.phy.cam.ac.uk/~thomson/partIIIparticles/

- All course material, old exam questions, corrections, interesting links etc.
- Detailed answers will posted after the supervisions (password protected)

Format of Lectures/Handouts:

- I will derive almost all results from first principles (only a few exceptions).
- In places will include some additional theoretical background in non-examinable appendices at the end of that particular handout.
- Please let me know of any typos: thomson@hep.phy.cam.ac.uk

Books:

★ The handouts are fairly complete, however there a number of decent books:
- "Particle Physics", Martin and Shaw (Wiley): fairly basic but good.
- "Introductory High Energy Physics", Perkins (Cambridge): slightly below level of the course but well written.
- "Introduction to Elementary Physics", Griffiths (Wiley): about right level but doesn't cover the more recent material.
- "Quarks and Leptons", Halzen & Martin (Wiley): good graduate level textbook (slightly above level of this course).

★ Before we start in earnest, a few words on units/notation and a very brief "Part II refresher"…

Preliminaries: Natural Units

- S.I. UNITS: **kg m s** are a natural choice for "everyday" objects
  e.g. M(Prescott) ~ 250 kg
- not very natural in particle physics
- instead use **Natural Units** based on the language of particle physics
  - **From Quantum Mechanics** - the unit of action : $\hbar$
  - **From relativity** - the speed of light: c
  - **From Particle Physics** - unit of energy: GeV (1 GeV ~ proton rest mass energy)

★ **Units become (i.e. correct dimensions):**

Energy	GeV	Time	$(\mathrm{GeV}/\hbar)^{-1}$
Momentum	GeV/c	Length	$(\mathrm{GeV}/\hbar c)^{-1}$
Mass	GeV/c^2	Area	$(\mathrm{GeV}/\hbar c)^{-2}$

★ **Simplify algebra by <u>setting</u>:** $\boxed{\hbar = c = 1}$

- Now all quantities expressed in powers of GeV

Energy	GeV	Time	GeV^{-1}
Momentum	GeV	Length	GeV^{-1}
Mass	GeV	Area	GeV^{-2}

To convert back to S.I. units, need to restore missing factors of $\hbar$ and c

Preliminaries: Heaviside-Lorentz Units

- **Electron charge <u>defined</u> by Force equation:** $F = \dfrac{e^2}{4\pi\varepsilon_0 r^2}$
- In Heaviside-Lorentz units set $\boxed{\varepsilon_0 = 1}$

  and $F \rightarrow \dfrac{e^2}{4\pi r^2}$ **NOW:** electric charge has dimensions $[FL^2]^{\frac{1}{2}} = [EL]^{\frac{1}{2}} = [\hbar c]^{\frac{1}{2}}$

- Since $c = (\varepsilon_0\mu_0)^{-\frac{1}{2}} = 1$ ➡ $\boxed{\mu_0 = 1}$

$$\boxed{\hbar = c = \varepsilon_0 = \mu_0 = 1}$$

★ Unless otherwise stated, Natural Units are used throughout these handouts, $E^2 = p^2 + m^2$, $\vec{p} = \vec{k}$, etc.

Preliminaries: Relativity and 4-Vector Notation

- Will use 4-vector notation with p^0 as the time-like component, e.g.

$$p^\mu = \{E, \vec{p}\} = \{E, p_x, p_y, p_z\} \qquad \text{(contravariant)}$$

$$p_\mu = g_{\mu\nu} p^\mu = \{E, -\vec{p}\} = \{E, -p_x, -p_y, -p_z\} \qquad \text{(covariant)}$$

with

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- In particle physics, usually deal with relativistic particles. <u>Require</u> all calculations to be Lorentz Invariant. L.I. quantities formed from 4-vector scalar products, e.g.

$$p^\mu p_\mu = E^2 - p^2 = m^2 \qquad \text{Invariant mass}$$

$$x^\mu p_\mu = Et - \vec{p}.\vec{r} \qquad \text{Phase}$$

- A few words on **NOTATION**

  Four vectors written as either: p^μ or p

  Four vector scalar product: $p^\mu q_\mu$ or $p.q$

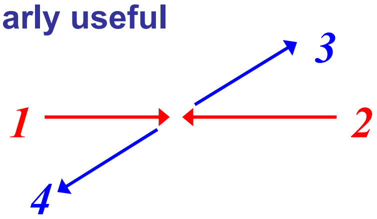
  Three vectors written as: $\vec{p}$

  Quantities evaluated in the **centre of mass frame**: $\vec{p}^*, p^*$ etc.

Example: Mandelstam s, t and u

★ In particle scattering/annihilation there are three particularly useful Lorentz Invariant quantities: s, t and u

★ Consider the scattering process $1+2 \rightarrow 3+4$



- Define three kinematic variables: s, t and u from the following four vector scalar products

$$\boxed{s = (p_1+p_2)^2, \quad t = (p_1-p_3)^2, \quad u = (p_1-p_4)^2}$$

Note: $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$ (Question 1)

★ **e.g.** Centre-of-mass energy, s:

$$s = (p_1+p_2)^2 = (E_1+E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

- This is a scalar product of two four-vectors ➡ Lorentz Invariant

- Since this is a L.I. quantity, can evaluate in any frame. Choose the most convenient, i.e. the centre-of-mass frame:

$$p_1^* = (E_1^*, \vec{p}^*) \quad p_2 = (E_2^*, -\vec{p}^*)$$

$$\Rightarrow \boxed{s = (E_1^* + E_2^*)^2}$$

★Hence $\sqrt{s}$ is the total energy of collision in the centre-of-mass frame

Review of The Standard Model

Particle Physics is the study of:

- ★ MATTER: the fundamental constituents of the universe
  - the elementary particles
- ★ FORCE: the fundamental forces of nature, i.e. the interactions between the elementary particles

Try to categorise the **PARTICLES** and **FORCES** in as simple and fundamental manner possible

★Current understanding embodied in the **STANDARD MODEL:**

- **Forces** between particles due to exchange of **particles**
- Consistent with <u>all</u> current experimental data !
- But it is just a **"model"** with many unpredicted parameters, e.g. particle masses.
- As such it is not the ultimate theory (if such a thing exists), there are many mysteries.

Matter in the Standard Model

★ In the Standard Model the fundamental "matter" is described by **point-like spin-1/2** fermions

	LEPTONS			QUARKS		
		q	m/GeV		q	m/GeV
First Generation	e^-	–1	0.0005	d	–1/3	0.3
	ν_1	0	≈0	u	+2/3	0.3
Second Generation	μ^-	–1	0.106	s	–1/3	0.5
	ν_2	0	≈0	c	+2/3	1.5
Third Generation	τ^-	–1	1.77	b	–1/3	4.5
	ν_3	0	≈0	t	+2/3	175

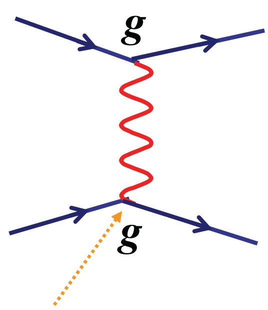
The masses quoted for the quarks are the "constituent masses", i.e. the effective masses for quarks confined in a bound state

- In the SM there are <u>three generations</u> – the particles in each generation are copies of each other differing <u>only</u> in mass. (not understood why three).
- The neutrinos are much lighter than all other particles (e.g. ν_1 has m<3 eV) – we now know that neutrinos have non-zero mass (don't understand why so small)

Forces in the Standard Model

★Forces mediated by the exchange of spin-1 Gauge Bosons

Force	Boson(s)	J^P	m/GeV
EM (QED)	Photon γ	1^-	0
Weak	$W^\pm$ / Z	1^-	80 / 91
Strong (QCD)	8 Gluons g	1^-	0
Gravity (?)	Graviton?	2^+	0



- Fundamental interaction strength is given by charge g.
- Related to the **dimensionless** coupling "constant" α

e.g. QED $$g_{em} = e = \sqrt{4\pi\alpha\varepsilon_0\hbar c}$$

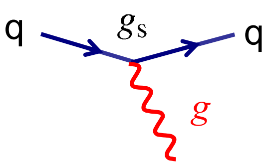
★ In Natural Units $$g = \sqrt{4\pi\alpha}$$ (both g and α are dimensionless, but g contains a "hidden" $\hbar c$)

★ Convenient to express couplings in terms of α which, being genuinely dimensionless does not depend on the system of units (this is not true for the numerical value for e)

Standard Model Vertices

★Interaction of gauge bosons with fermions described by SM vertices
★Properties of the gauge bosons and nature of the interaction between the bosons and fermions determine the properties of the interaction

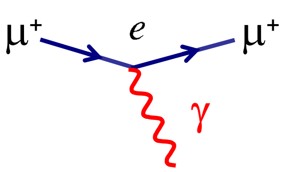
STRONG



Only quarks
Never changes flavour

$$\alpha_S \sim 1$$

EM



All charged fermions
Never changes flavour

$$\alpha \simeq 1/137$$

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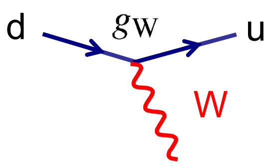

All fermions
Always changes flavour

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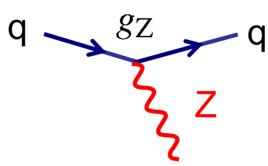
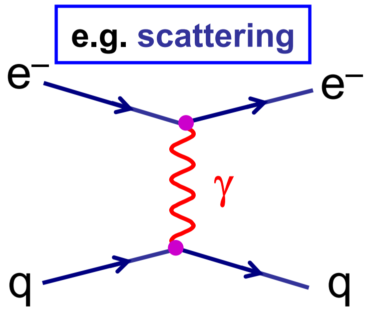


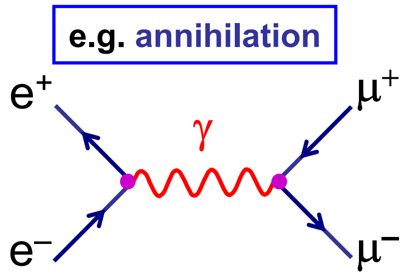
All fermions
Never changes flavour

$$\alpha_{W/Z} \sim 1/40$$

Feynman Diagrams

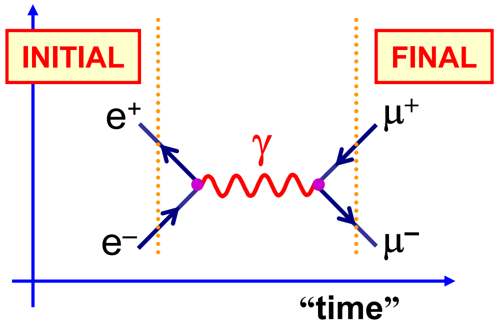
★ Particle interactions described in terms of Feynman diagrams





★ **IMPORTANT POINTS TO REMEMBER:**

- "time" runs from left – right, **only** in sense that:
  - LHS of diagram is initial state
  - RHS of diagram is final state
  - Middle is "how it happened"
- anti-particle arrows in –ve "time" direction
- Energy, momentum, angular momentum, etc. conserved at **all interaction vertices**
- All intermediate particles are "virtual" i.e. $E^2 \neq |\vec{p}|^2 + m^2$ (handout 3)



From Feynman diagrams to Physics

<u>**Particle Physics = Precision Physics**</u>

★ Particle physics is about building fundamental theories and testing their predictions against precise experimental data

- Dealing with fundamental particles and can make **very precise theoretical predictions** – not complicated by dealing with many-body systems
- Many beautiful experimental measurements
  ➡ precise theoretical predictions challenged by precise measurements
- For all its flaws, the Standard Model describes all experimental data ! This is a (the?) remarkable achievement of late 20th century physics.

<u>**Requires understanding of theory and experimental data**</u>

★ **Part II :** Feynman diagrams mainly used to describe how particles interact

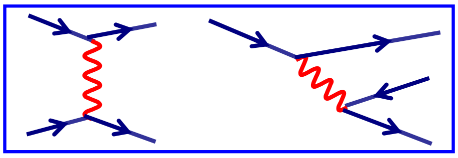
★ **Part III:**
- will use Feynman diagrams and associated Feynman rules to **perform calculations** for many processes
- hopefully gain a fairly deep understanding of the Standard Model and how it explains all current data

<u>**Before we can start, need calculations for:**</u>

- **Interaction cross sections**;
- **Particle decay rates**;

Cross Sections and Decay Rates

- In particle physics we are mainly concerned with particle **interactions** and **decays**, i.e. transitions between states



  - these are the experimental observables of particle physics
- Calculate transition rates from Fermi's Golden Rule

$$\boxed{\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)}$$

Γ_{fi} is number of transitions per unit time from initial state $|i\rangle$ to final state $\langle f|$ – not Lorentz Invariant !

T_{fi} is Transition Matrix Element

$$T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{j\neq i} \frac{\langle f|\hat{H}|j\rangle\langle j|\hat{H}|i\rangle}{E_i - E_j} + \ldots$$

$\hat{H}$ is the perturbing Hamiltonian

$\rho(E_f)$ is density of final states

★ Rates depend on MATRIX ELEMENT and DENSITY OF STATES

the ME contains the fundamental particle physics

just kinematics

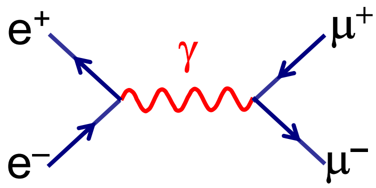
The first five lectures

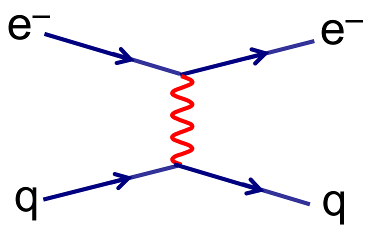
★ Aiming towards a proper calculation of decay and scattering processes

Will concentrate on:

- $e^+e^- \to \mu^+\mu^-$
- $e^-q \to e^-q$

($e^-q \to e^-q$ to probe proton structure)





- Need relativistic calculations of particle decay rates and cross sections:

$$\sigma = \frac{|M_{fi}|^2}{\text{flux}} \times (\text{phase space})$$

- Need relativistic treatment of spin-half particles:

  Dirac Equation

- Need relativistic calculation of interaction Matrix Element:

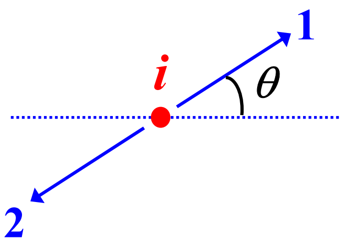
  Interaction by particle exchange and Feynman rules

+ and a few mathematical tricks along, e.g. the Dirac Delta Function

Particle Decay Rates

- Consider the two-body decay

$$i \rightarrow 1+2$$



- Want to calculate the decay rate in first order perturbation theory using plane-wave descriptions of the particles (Born approximation):

$$\psi_1 = Ne^{i(\vec{p}.\vec{r}-Et)} \qquad (\vec{k}.\vec{r} = \vec{p}.\vec{r} \quad \text{as} \quad \hbar = 1)$$
$$= Ne^{-ip.x}$$

where N is the normalisation and $p.x = p^{\mu}x_{\mu}$

For decay rate calculation need to know:

- Wave-function normalisation
- Transition matrix element from perturbation theory
- Expression for the density of states

All in a Lorentz Invariant form

★ First consider wave-function normalisation

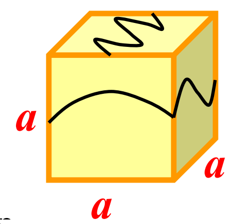
- Previously (e.g. part II) have used a non-relativistic formulation
- Non-relativistic: normalised to one particle in a cube of side a

$$\int \psi\psi^* \mathrm{d}V = N^2 = 1/a^3$$

Non-relativistic Phase Space (revision)

- Apply boundary conditions $(\vec{p} = \hbar\vec{k})$:
- Wave-function vanishing at box boundaries

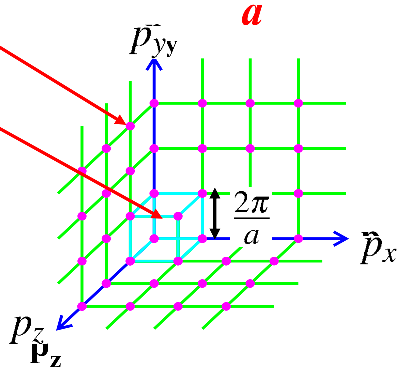
➡ quantised particle momenta:



$$p_x = \frac{2\pi n_x}{a};\; p_y = \frac{2\pi n_y}{a};\; p_z = \frac{2\pi n_z}{a}$$

- Volume of single state in momentum space:

$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$



- Normalising to **one particle/unit volume** gives number of states in element: $\mathrm{d}^3\vec{p} = \mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z$

$$\mathrm{d}n = \frac{\mathrm{d}^3\vec{p}}{\frac{(2\pi)^3}{V}} \times \frac{1}{V} = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}$$

- Therefore density of states in Golden rule:

$$\rho(E_f) = \left|\frac{\mathrm{d}n}{\mathrm{d}E}\right|_{E_f} = \left|\frac{\mathrm{d}n}{\mathrm{d}|\vec{p}|}\frac{\mathrm{d}|\vec{p}|}{\mathrm{d}E}\right|_{E_f}$$

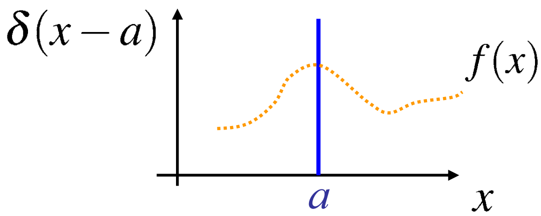
with $p = \beta E$

- Integrating over an elemental shell in momentum-space gives $(\mathrm{d}^3\vec{p} = 4\pi p^2\mathrm{d}p)$

$$\rho(E_f) = \frac{4\pi p^2}{(2\pi)^3} \times \beta$$

Dirac δ Function

- In the relativistic formulation of decay rates and cross sections we will make use of the Dirac δ function: **"infinitely narrow spike of unit area"**



$$\int_{-\infty}^{+\infty} \delta(x-a)\mathrm{d}x = 1$$

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)\mathrm{d}x = f(a)$$

- Any function with the above properties can represent $\delta(x)$

e.g. $$\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x^2}{2\sigma^2}\right)}$$ (an infinitesimally narrow Gaussian)

- In relativistic quantum mechanics delta functions prove extremely useful for integrals over phase space, e.g. in the decay $a \to 1+2$

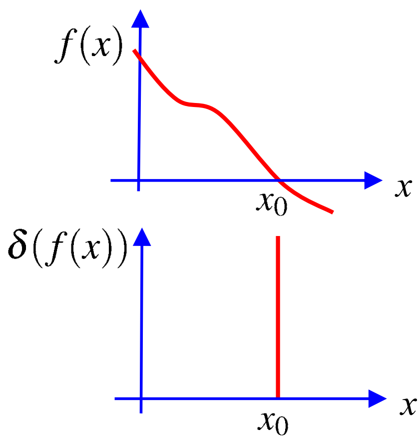
$$\int \ldots \delta(E_a - E_1 - E_2)\mathrm{d}E \quad \text{and} \quad \int \ldots \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\mathrm{d}^3\vec{p}$$

express energy and momentum conservation

★ We will soon need an expression for the **delta function of a function** $\delta(f(x))$

- Start from the definition of a delta function

$$\int_{y_1}^{y_2} \delta(y)\mathrm{d}y = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases}$$



- Now express in terms of $y = f(x)$ where $f(x_0) = 0$ and then change variables

$$\int_{x_1}^{x_2} \delta(f(x))\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

- From properties of the delta function (i.e. here only non-zero at x_0)

$$\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0} \int_{x_1}^{x_2} \delta(f(x))\mathrm{d}x = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

- Rearranging and expressing the **RHS** as a delta function

$$\int_{x_1}^{x_2} \delta(f(x))\mathrm{d}x = \frac{1}{|\mathrm{d}f/\mathrm{d}x|_{x_0}} \int_{x_1}^{x_2} \delta(x-x_0)\mathrm{d}x$$

➡ $$\boxed{\delta(f(x)) = \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0}^{-1} \delta(x-x_0)} \qquad \text{(1)}$$

The Golden Rule revisited

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

- Rewrite the expression for density of states using a delta-function

$$\rho(E_f) = \left|\frac{dn}{dE}\right|_{E_f} = \int \frac{dn}{dE}\delta(E - E_i)dE \qquad \text{since } E_f = E_i$$

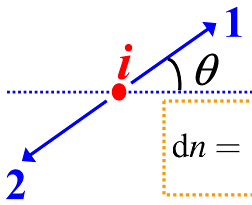
Note : integrating over all final state energies but energy conservation now taken into account explicitly by delta function

- Hence the golden rule becomes: $\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)dn$

the integral is over **all** "allowed" final states of any energy

- For dn in a two-body decay, only need to consider one particle : mom. conservation fixes the other

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E_1 - E_2)\frac{d^3\vec{p}_1}{(2\pi)^3}$$



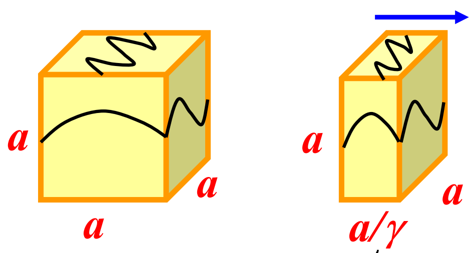
$$dn = \frac{d^3\vec{p}}{(2\pi)^3}$$

- However, can include momentum conservation explicitly by integrating over the momenta of both particles and using another δ-fn

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \underbrace{\delta(E_i - E_1 - E_2)}_{\text{Energy cons.}}\underbrace{\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)}_{\text{Mom. cons.}}\underbrace{\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}}_{\text{Density of states}}$$

Lorentz Invariant Phase Space

- In non-relativistic QM normalise to one particle/unit volume: $\int \psi^*\psi dV = 1$
- When considering relativistic effects, volume contracts by $\gamma = E/m$



- Particle density therefore increases by $\gamma = E/m$

★ **Conclude that a relativistic invariant wave-function normalisation needs to be proportional to E particles per unit volume**

- Usual convention: **Normalise to 2E particles/unit volume** $\int \psi'^*\psi' dV = 2E$
- Previously used ψ normalised to 1 particle per unit volume $\int \psi^*\psi dV = 1$
- Hence $\psi' = (2E)^{1/2}\psi$ is normalised to $2E$ per unit volume
- Define Lorentz Invariant Matrix Element, M_{fi}, in terms of the wave-functions normalised to $2E$ particles per unit volume

$$\boxed{M_{fi} = \langle \psi'_1.\psi'_2...|\hat{H}|...\psi'_{n-1}\psi'_n\rangle = (2E_1.2E_2.2E_3....2E_n)^{1/2}T_{fi}}$$

- For the two body decay

$$i \rightarrow 1+2$$

$$\begin{aligned} M_{fi} &= \langle \psi_1' \psi_2' | \hat{H}' | \psi_i' \rangle \\ &= (2E_i.2E_1.2E_2)^{1/2} \langle \psi_1 \psi_2 | \hat{H}' | \psi_i \rangle \\ &= (2E_i.2E_1.2E_2)^{1/2} T_{fi} \end{aligned}$$

★ Now expressing T_{fi} in terms of M_{fi} gives

$$\boxed{\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}}$$

Note:

- M_{fi} uses relativistically normalised wave-functions. It is Lorentz Invariant
- $\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E}$ is the Lorentz Invariant Phase Space for each final state particle; the factor of $2E$ arises from the wave-function normalisation (prove this in Question 2)
- This form of Γ_{fi} is simply a rearrangement of the original equation **but** the integral is now frame independent (i.e. L.I.)
- Γ_{fi} is inversely proportional to E_a, the energy of the decaying particle. This is exactly what one would expect from time dilation ($E_a = \gamma m$).
- Energy and momentum conservation in the delta functions

Decay Rate Calculations

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}$$

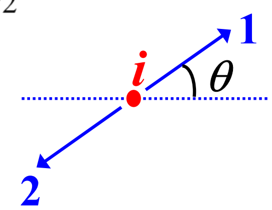
★ Because the integral is Lorentz invariant (i.e. frame independent) it can be evaluated in any frame we choose. The C.o.M. frame is most convenient

- In the C.o.M. frame $E_i = m_i$ and $\vec{p}_i = 0$ ⇨

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2) \delta^3(\vec{p}_1 + \vec{p}_2) \frac{\mathrm{d}^3\vec{p}_1}{2E_1} \frac{\mathrm{d}^3\vec{p}_2}{2E_2}$$

- Integrating over $\vec{p}_2$ using the δ-function:

$$\Rightarrow \quad \Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2) \frac{\mathrm{d}^3\vec{p}_1}{4E_1E_2}$$



now $E_2^2 = (m_2^2 + |\vec{p}_1|^2)$ since the δ-function imposes $\vec{p}_2 = -\vec{p}_1$

- Writing $\mathrm{d}^3\vec{p}_1 = p_1^2 \mathrm{d}p_1 \sin\theta \mathrm{d}\theta \mathrm{d}\phi = p_1^2 \mathrm{d}p_1 \mathrm{d}\Omega$

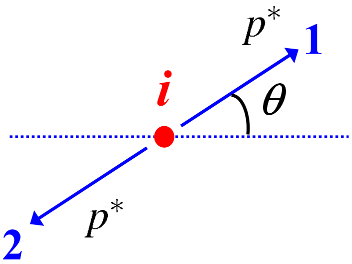
For convenience, here $|\vec{p}_1|$ is written as p_1

$$\Rightarrow \quad \Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right) \frac{p_1^2 \mathrm{d}p_1 \mathrm{d}\Omega}{E_1E_2}$$

- Which can be written in the form
$$\Gamma_{fi}=\frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 g(p_1)\delta(f(p_1))\mathrm{d}p_1\mathrm{d}\Omega \qquad (2)$$

where $g(p_1)=p_1^2/(E_1E_2)=p_1^2(m_1^2+p_1^2)^{-1/2}(m_2^2+p_1^2)^{-1/2}$

and $f(p_1)=m_i-(m_1^2+p_1^2)^{1/2}-(m_2^2+p_1^2)^{1/2}$



Note:
- $\delta(f(p_1))$ imposes energy conservation.
- $f(p_1)=0$ determines the C.o.M momenta of the two decay products

i.e. $f(p_1)=0$ for $p_1=p^*$

★ Eq. (2) can be integrated using the property of $\delta-$ function derived earlier (eq. (1))
$$\int g(p_1)\delta(f(p_1))\mathrm{d}p_1=\frac{1}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}\int g(p_1)\delta(p_1-p^*)\mathrm{d}p_1=\frac{g(p^*)}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}$$

where p^* is the value for which $f(p^*)=0$

- All that remains is to evaluate $\mathrm{d}f/\mathrm{d}p_1$
$$\frac{\mathrm{d}f}{\mathrm{d}p_1}=-\frac{p_1}{(m_1^2+p_1^2)^{1/2}}-\frac{p_1}{(m_2^2+p_1^2)^{1/2}}=-\frac{p_1}{E_1}-\frac{p_1}{E_2}=-p_1\frac{E_1+E_2}{E_1E_2}$$

giving:
$$\Gamma_{fi}=\frac{1}{32\pi^2E_i}\int |M_{fi}|^2\left|\frac{E_1E_2}{p_1(E_1+E_2)}\frac{p_1^2}{E_1E_2}\right|_{p_1=p^*}\mathrm{d}\Omega$$
$$=\frac{1}{32\pi^2E_i}\int |M_{fi}|^2\left|\frac{p_1}{E_1+E_2}\right|_{p_1=p^*}\mathrm{d}\Omega$$

- But from $f(p_1)=0$, i.e. energy conservation: $E_1+E_2=m_i$
$$\Gamma_{fi}=\frac{|\vec{p}^*|}{32\pi^2E_im_i}\int |M_{fi}|^2\mathrm{d}\Omega$$

In the particle's rest frame $E_i=m_i$

$$\boxed{\frac{1}{\tau}=\Gamma=\frac{|\vec{p}^*|}{32\pi^2m_i^2}\int |M_{fi}|^2\mathrm{d}\Omega} \qquad (3)$$

VALID FOR ALL TWO-BODY DECAYS !

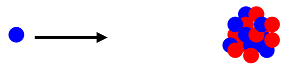
- p^* can be obtained from $f(p_1)=0$
$$(m_1^2+p^{*2})^{1/2}+(m_2^2+p^{*2})^{1/2}=m_i \qquad \text{(Question 3)}$$
$$\Rightarrow\ p^*=\frac{1}{2m_i}\sqrt{\left[(m_i^2-(m_1+m_2)^2\right]\left[m_i^2-(m_1-m_2)^2\right]}$$
(now try Questions 4 & 5)

Cross section definition

$$\sigma = \frac{\text{no of interactions per unit time/per target}}{\text{incident flux}}$$

Flux = number of incident particles/ unit area/unit time

- The "cross section", σ, can be thought of as the effective cross-sectional area of the target particles for the interaction to occur.
- In general this has nothing to do with the physical size of the target although there are exceptions, e.g. neutron absorption



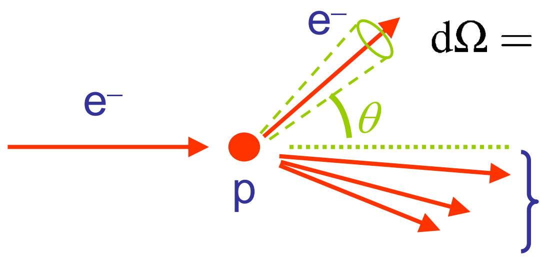
here σ is the projective area of nucleus

Differential Cross section

$$\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}$$

or generally

$$\frac{d\sigma}{d\ldots}$$



$$d\Omega = d(\cos\theta)d\phi$$

integrate over all other particles

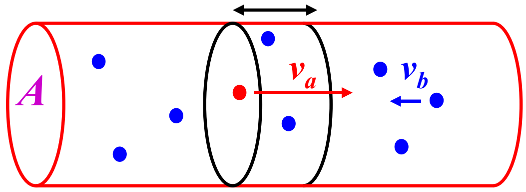
with $\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$

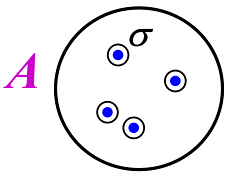
example

- Consider a single particle of type a with velocity, v_a, traversing a region of area A containing n_b particles of type b per unit volume

In time δt a particle of type a traverses region containing $n_b(v_a+v_b)A\delta t$ particles of type b

$(v_a+v_b)\delta t$





★ Interaction probability obtained from effective cross-sectional area occupied by the $n_b(v_a+v_b)A\delta t$ particles of type b

- Interaction Probability = $\frac{n_b(v_a+v_b)A\delta t\sigma}{A} = n_b v\delta t\sigma$ $\quad [v = v_a+v_b]$

➡ Rate per particle of type a = $n_b v \sigma$

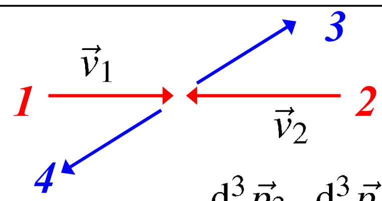
- Consider volume V, total reaction rate = $(n_b v\sigma).(n_a V) = (n_b V)(n_a v)\sigma$ = $N_b\phi_a\sigma$

- As anticipated: **Rate = Flux x Number of targets x cross section**

Cross Section Calculations

- Consider scattering process

$$1+2 \rightarrow 3+4$$



- Start from Fermi's Golden Rule:

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3}$$

where T_{fi} is the transition matrix for a normalisation of 1/unit volume

- Now

$$\begin{aligned} \text{Rate/Volume} &= (\text{flux of 1}) \times (\text{number density of 2}) \times \sigma \\ &= n_1(v_1+v_2) \times n_2 \times \sigma \end{aligned}$$

- For 1 target particle per unit volume $\text{Rate} = (v_1+v_2)\sigma$

$$\sigma = \frac{\Gamma_{fi}}{(v_1+v_2)}$$

$$\sigma = \frac{(2\pi)^4}{v_1+v_2} \int |T_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3}$$

the parts are not Lorentz Invariant

- To obtain a Lorentz Invariant form use wave-functions normalised to $2E$ particles per unit volume

$$\psi' = (2E)^{1/2}\psi$$

- Again define L.I. Matrix element $M_{fi} = (2E_1\, 2E_2\, 2E_3\, 2E_4)^{1/2} T_{fi}$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

- The integral is now written in a Lorentz invariant form
- The quantity $F = 2E_1 2E_2(v_1+v_1)$ can be written in terms of a four-vector scalar product and is therefore also Lorentz Invariant (the Lorentz Inv. Flux)

$$F = 4\left[(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2\right]^{1/2}$$

(see appendix I)

- Consequently cross section is a Lorentz Invariant quantity

Two special cases of Lorentz Invariant Flux:

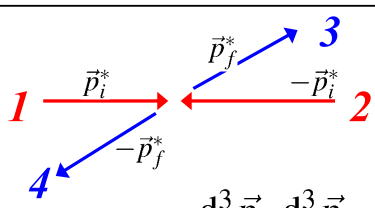
- Centre-of-Mass Frame

$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1E_2(|\vec{p}^*|/E_1 + |\vec{p}^*|/E_2) \\ &= 4|\vec{p}^*|(E_1+E_2) \\ &= 4|\vec{p}^*|\sqrt{s} \end{aligned}$$

- Target (particle 2) at rest

$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1 m_2 v_1 \\ &= 4E_1 m_2(|\vec{p}_1|/E_1) \\ &= 4m_2|\vec{p}_1| \end{aligned}$$

2→2 Body Scattering in C.o.M. Frame



- We will now apply above Lorentz Invariant formula for the interaction cross section to the most common cases used in the course. First consider 2→2 scattering in C.o.M. frame
- Start from

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)}\int |M_{fi}|^2\delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

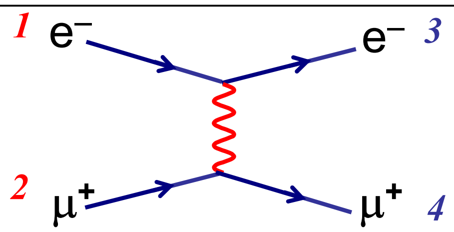
- Here $\vec{p}_1+\vec{p}_2=0$ and $E_1+E_2=\sqrt{s}$

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}}\int |M_{fi}|^2\delta(\sqrt{s}-E_3-E_4)\delta^3(\vec{p}_3+\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

★The integral is exactly the same integral that appeared in the particle decay calculation but with m_a replaced by $\sqrt{s}$

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}}\frac{|\vec{p}_f^*|}{4\sqrt{s}}\int |M_{fi}|^2\mathrm{d}\Omega^*$$

$$\boxed{\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}\int |M_{fi}|^2\mathrm{d}\Omega^*}$$



- In the case of elastic scattering $|\vec{p}_i^*| = |\vec{p}_f^*|$

$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s}\int |M_{fi}|^2\mathrm{d}\Omega^*$$

- For calculating the total cross-section (which is Lorentz Invariant) the result on the previous page (eq. (4)) is sufficient. However, it is not so useful for calculating the differential cross section in a rest frame other than the C.o.M:

$$\mathrm{d}\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}|M_{fi}|^2\mathrm{d}\Omega^*$$

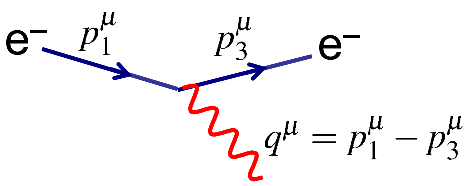
because the angles in $\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^*$ refer to the C.o.M frame

- For the last calculation in this section, we need to find a L.I. expression for $\mathrm{d}\sigma$

★ Start by expressing $\mathrm{d}\Omega^*$ in terms of Mandelstam t i.e. the square of the four-momentum transfer

$$\boxed{t = q^2 = (p_1 - p_3)^2}$$

Product of four-vectors therefore L.I.

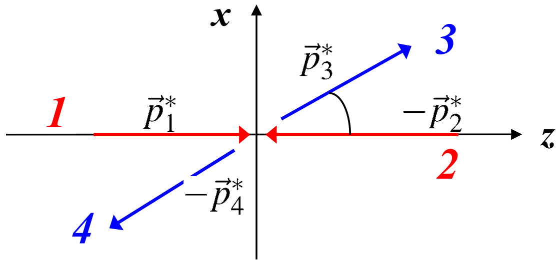


- Want to express $\mathrm{d}\Omega^*$ in terms of Lorentz Invariant $\mathrm{d}t$

 where $t \equiv (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1.p_3 = m_1^2 + m_3^2 - 2p_1.p_3$

- **In C.o.M. frame:**

$$\begin{aligned} p_1^{*\mu} &= (E_1^*, 0, 0, |\vec{p}_1^*|) \\ p_3^{*\mu} &= (E_3^*, |\vec{p}_3^*|\sin\theta^*, 0, |\vec{p}_3^*|\cos\theta^*) \\ p_1^{\mu} p_{3\mu} &= E_1^* E_3^* - |\vec{p}_1^*||\vec{p}_3^*|\cos\theta^* \\ t &= m_1^2 + m_3^3 - E_1^* E_3^* + 2|\vec{p}_1^*||\vec{p}_3^*|\cos\theta^* \end{aligned}$$



giving $\mathrm{d}t = 2|\vec{p}_1^*||\vec{p}_3^*|\mathrm{d}(\cos\theta^*)$

therefore $\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^* = \dfrac{\mathrm{d}t\,\mathrm{d}\phi^*}{2|\vec{p}_1^*||\vec{p}_3^*|}$

hence $$\mathrm{d}\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_3^*|}{|\vec{p}_1^*|}|M_{fi}|^2\mathrm{d}\Omega^* = \frac{1}{2\cdot 64\pi^2 s|\vec{p}_1^*|^2}|M_{fi}|^2\mathrm{d}\phi^*\mathrm{d}t$$

- Finally, integrating over $\mathrm{d}\phi^*$ (assuming no ϕ^* dependence of $|M_{fi}|^2$) gives:

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2}$$

Lorentz Invariant differential cross section

- All quantities in the expression for $\mathrm{d}\sigma/\mathrm{d}t$ are Lorentz Invariant and therefore, it applies to any rest frame. It should be noted that $|\vec{p}_i^*|^2$ is a constant, fixed by energy/momentum conservation

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]$$

- As an example of how to use the invariant expression $\mathrm{d}\sigma/\mathrm{d}t$ we will consider elastic scattering in the laboratory frame in the limit where we can neglect the mass of the incoming particle $E_1 \gg m_1$

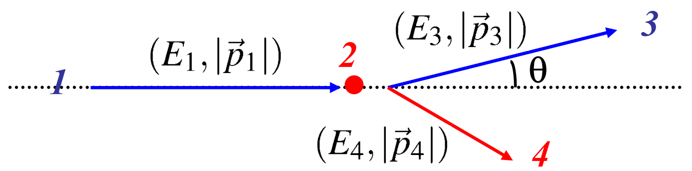
e.g. electron or neutrino scattering

In this limit $$|\vec{p}_i^*|^2 = \frac{(s - m_2)^2}{4s}$$

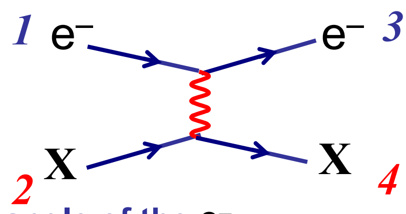
$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi(s - m_2^2)^2}|M_{fi}|^2} \qquad (m_1 = 0)$$

2→2 Body Scattering in Lab. Frame

- The other commonly occurring case is scattering from a fixed target in the Laboratory Frame (e.g. electron-proton scattering)
- First take the case of elastic scattering at high energy where the mass of the incoming particles can be neglected: $m_1 = m_3 = 0, \quad m_2 = m_4 = M$



e.g.



- Wish to express the cross section in terms of scattering angle of the e⁻

$$\mathrm{d}\Omega = 2\pi \mathrm{d}(\cos\theta)$$

therefore
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$$

Integrating over $\mathrm{d}\phi$

- The rest is some rather tedious algebra…. start from four-momenta

$$p_1 = (E_1, 0, 0, E_1), \;\; p_2 = (M, 0, 0, 0), \;\; p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta), \;\; p_4 = (E_4, \vec{p}_4)$$

so here $\quad t = (p_1 - p_3)^2 = -2p_1.p_3 = -2E_1E_3(1-\cos\theta)$

But from (E,p) conservation $\quad p_1 + p_2 = p_3 + p_4$

and, therefore, can also express t in terms of particles 2 and 4

$$\begin{aligned} t &= (p_2 - p_4)^2 = 2M^2 - 2p_2.p_4 = 2M^2 - 2ME_4 \\ &= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3) \end{aligned}$$

Note E_1 is a constant (the energy of the incoming particle) so

$$\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2M\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}$$

- Equating the two expressions for t gives $\quad \boxed{E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}}$

so
$$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{E_1 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M\left(\frac{E_3^2}{E_1 M}\right)^2 = \frac{E_3^2}{M}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{2\pi}2M\frac{E_3^2}{M}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{1}{16\pi(s-M^2)^2}|M_{fi}|^2$$

using $\quad s = (p_1 + p_2)^2 = M^2 + 2p_1.p_2 = M^2 + 2ME_1$

Particle 1 massless ➡ $(p_1^2 = 0)$

gives $\quad (s - M^2) = 2ME_1$

➡ $$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2}$$

In limit $m_1 \to 0$

In this equation, E_3 is a function of θ:

$$E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}$$

giving
$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M+E_1-E_1\cos\theta}\right)^2 |M_{fi}|^2} \qquad (m_1 = 0)$$

General form for 2→2 Body Scattering in Lab. Frame

★The calculation of the differential cross section for the case where m_1 can not be neglected is longer and contains no more "physics" (see appendix II). It gives:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{p_1 m_1}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2) - E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2}$$

Again there is **only one independent variable**, θ, which can be seen from conservation of energy

$$E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{\underbrace{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta}_{\vec{p}_4 = \vec{p}_1 - \vec{p}_3} + m_4^2}$$

i.e. $|\vec{p}_3|$ is a function of θ

Summary

★ Used a Lorentz invariant formulation of Fermi's Golden Rule to derive decay rates and cross-sections in terms of the Lorentz Invariant Matrix Element (wave-functions normalised to 2E/Volume)

Main Results:

★Particle decay:

$$\boxed{\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 d\Omega}$$

Where p^* is a function of particle masses

$$p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2 - (m_1+m_2)^2\right]\left[m_i^2 - (m_1-m_2)^2\right]}$$

★Scattering cross section in C.o.M. frame:

$$\boxed{\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}\int |M_{fi}|^2 d\Omega^*}$$

★Invariant differential cross section (valid in all frames):

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2} \qquad |\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

Summary cont.

★Differential cross section in the lab. frame ($m_1=0$)

$$\boxed{\frac{d\sigma}{d\Omega}=\frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2|M_{fi}|^2} \Longleftrightarrow \boxed{\frac{d\sigma}{d\Omega}=\frac{1}{64\pi^2}\left(\frac{1}{M+E_1-E_1\cos\theta}\right)^2|M_{fi}|^2}$$

★Differential cross section in the lab. frame ($m_1\neq 0$)

$$\boxed{\frac{d\sigma}{d\Omega}=\frac{1}{64\pi^2}\cdot\frac{1}{|\vec{p}_1|m_1}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2)-E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2}$$

with $E_1+m_2=\sqrt{|\vec{p}_3|^2+m_3^2}+\sqrt{|\vec{p}_1|^2+|\vec{p}_3|^2-2|\vec{p}_1||\vec{p}_3|\cos\theta+m_4^2}$

Summary of the summary:

- ★Have now dealt with **kinematics** of particle decays and cross sections
- ★The **fundamental particle physics** is in the **matrix element**
- ★The above equations are the basis for all calculations that follow

Appendix I : Lorentz Invariant Flux

NON-EXAMINABLE

- Collinear collision: $a \longrightarrow$ $v_a, \vec{p}_a$ $\longleftarrow b$ $v_b, \vec{p}_b$

$$\begin{aligned} F=2E_a 2E_b(v_a+v_b) &= 4E_aE_b\left(\frac{|\vec{p}_a|}{E_a}+\frac{|\vec{p}_b|}{E_b}\right) \\ &= 4(|\vec{p}_a|E_b+|\vec{p}_b|E_a) \end{aligned}$$

To show this is Lorentz invariant, first consider

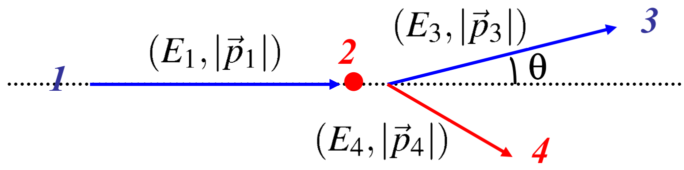
$$p_a.p_b=p_a^\mu p_{b\mu}=E_aE_b-\vec{p}_a.\vec{p}_b=E_aE_b+|\vec{p}_a||\vec{p}_b|$$

Giving

$$\begin{aligned} F^2/16-(p_a^\mu p_{b\mu})^2 &= (|\vec{p}_a|E_b+|\vec{p}_b|E_a)^2-(E_aE_b+|\vec{p}_a||\vec{p}_b|)^2 \\ &= |\vec{p}_a|^2(E_b^2-|\vec{p}_b|^2)+E_a^2(|\vec{p}_b|^2-E_b^2) \\ &= |\vec{p}_a|^2m_b^2-E_b^2m_b^2 \\ &= -m_a^2m_b^2 \\ F &= 4\left[(p_a^\mu p_{b\mu})^2-m_a^2m_b^2\right]^{1/2} \end{aligned}$$

Appendix II : general 2→2 Body Scattering in lab frame

NON-EXAMINABLE



$$p_1 = (E_1, 0, 0, |\vec{p}_1|),\;\; p_2 = (M, 0, 0, 0),\;\; p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta),\;\; p_4 = (E_4, \vec{p}_4)$$

again

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

But now the invariant quantity *t*:

$$\begin{aligned} t &= (p_2 - p_4)^2 = m_2^2 + m_4^2 - 2p_2.p_4 = m_2^2 + m_4^2 - 2m_2 E_4 \\ &= m_2^2 + m_4^2 - 2m_2(E_1 + m_2 - E_3) \end{aligned}$$

$$\Rightarrow \quad \frac{dt}{d(\cos\theta)} = 2m_2 \frac{dE_3}{d(\cos\theta)}$$

Which gives $$\frac{d\sigma}{d\Omega} = \frac{m_2}{\pi}\frac{dE_3}{d(\cos\theta)}\frac{d\sigma}{dt}$$

To determine $dE_3/d(\cos\theta)$, first differentiate $E_3^2 + |\vec{p}_3|^2 = m_3^2$

$$2E_3\frac{dE_3}{d(\cos\theta)} = 2|\vec{p}_3|\frac{d|\vec{p}_3|}{d(\cos\theta)} \qquad \textbf{(AII.1)}$$

Then equate $t = (p_1 - p_3)^2 = (p_4 - p_2)^2$ **to give**

$$m_1^2 + m_3^2 - 2(E_1E_3 - |\vec{p}_1||\vec{p}_3|\cos\theta) = m_4^2 + m_2^2 - 2m_2(E_1 + m_2 - E_3)$$

Differentiate wrt. $\cos\theta$

$$(E_1 + m_2)\frac{dE_3}{d\cos\theta} - |\vec{p}_1|\cos\theta\frac{d|\vec{p}_3|}{d\cos\theta} = |\vec{p}_1||\vec{p}_3|$$

Using (1) $\Rightarrow$

$$\frac{dE_3}{d(\cos\theta)} = \frac{|\vec{p}_1||\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \qquad \textbf{(AII.2)}$$

$$\frac{d\sigma}{d\Omega} = \frac{m_2}{\pi}\frac{dE_3}{d(\cos\theta)}\frac{d\sigma}{dt} = \frac{m_2}{\pi}\frac{dE_3}{d(\cos\theta)}\frac{1}{64\pi s|\vec{p}_i^{\,*}|^2}|M_{fi}|^2$$

It is easy to show $|\vec{p}_i^*|\sqrt{s} = m_2|\vec{p}_1|$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \frac{m_2}{64\pi^2 m_2^2 |\vec{p}_1|^2} |M_{fi}|^2$$

and using (AII.2) obtain

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{p_1 m_1} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |M_{fi}|^2}$$